Abstract Games for Linear Logic
Extended Abstract *

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Abstract

We draw attention to a number of constructions which lie behind many concrete models for linear logic; we develop an abstract context for these and describe their general theory. Using these constructions we give a model of classical linear logic based on an abstract notion of game. We derive this not from a category with built-in computational content but from the simple category of sets and relations. To demonstrate the computational content of the resulting model we make comparisons at each stage of the construction with a standard very simple notion of game. Our model provides motivation for a less familiar category of games (played on directed graphs) which is closely reflected by our notion of abstract game. We briefly indicate a number of variations on this theme and sketch how the abstract concept of game may be refined further.

1 Introduction

This paper presents an illustrative example of a category of abstract games. Games models for linear logic are now extensively used to model intensional features of programming languages [5,6,30,34,38,7,8,37]. The notion of a game is intuitively clear, but mathematical representations can seem complicated: there are positions and moves in a game tree, and strategies have to be composed by some explicit parallel composition plus hiding. An abstract game is a structure obtained by abstracting away from the details of the game tree; typically the structure involves some combination of sets of positions (or outcomes) and sets of strategies. Many categorical models of linear logic allow

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some reading in terms of abstract games [9,10,19,33,35,36,22], and categories for which this reading seems convincing underlie approaches to the Geometry of Interaction [26,25,27,23,2,4].

To give a computationally significant category of abstract games we exploit some general constructions on models of linear logic. Special cases have been known for a long time, but we make precise the general phenomena underlying them. We consider the following.

Self-dualization. Part of the prehistory of category theory: the construction of multiplicatives was known early to Girard and can also be read as a special case of Chu's construction [3] [20].

Comonoid indexing. We exploit the simple properties of the familiar Kleisli category of (free) coalgebras for the comonad induced by an internal comonoid.

Glueing. Again an old idea in category theory: the novelty is glueing to get self-dual categories. The obvious precursor is Loader's category of 'Linear Logical Predicates' [36], but the construction is also an ingredient in Girard's 'Phase Semantics' [24,28] and one approach to his coherence spaces [24,28].

Orthogonality. This is one of the key ideas of linear logic: it is the other ingredient in 'Phase Semantics' and in coherence spaces [40]. It also appears in Loader's 'Totality Spaces' [35].

One way of reading much current work on linear logic is this. One starts with some model of computation (perhaps in the form of a traced monoidal or compact closed category, perhaps with a less clean structure) and uses general techniques from categorical logic to construct a rich mathematical model. Here however we start with a computationally limited model, the category of sets and relations, and show that even using it we arrive at models with definite computational content. At each stage of the construction we compare the category of abstract games with a simple category of standard games. We get closer to this familiar category with each step, and the final resulting category of abstract games motivates a less familiar notion of concrete game. The multiplicative structure of the finite games is exactly reflected by our abstract games. There are extensions of our ideas which take things further but we do not have the space here to develop these. Nonetheless we hope the moral lesson that good models do not require much computational input will be clear. (This point is also effectively made in [10].)

[3] We warn however that this is misleading: consideration of the exponentials reveals a better parallel with Dialectica categories.

2 Preliminaries

Definition 1 *A* (categorical) model of intuitionistic linear logic *consists of a category which is symmetric monoidal closed, has finite products and is equipped with a linear exponential comonad.*

A (categorical) model of classical linear logic *consists of a category which is $*$-autonomous, has finite products and (therefore) finite co-products, and is equipped with a linear exponential comonad and (so) a linear exponential monad.*

The classical case adds nothing more than the duality; but the duality gives the dual of any existing structure. The structures involved are described in the intuitionistic case in [14,15]. For more details and all the required natural transformations and commutative diagrams, see [16,17].

We need to consider functors between models of linear logic. Sometimes we encounter functors which preserve structure, but usually we have a weaker notion.

Definition 2 *Let* $\mathbf{C}$, $\mathbf{D}$ *be models for linear logic. The functor* $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ *is* linearly distributive[4] *if and only if F is monoidal (with structure $n_{\mathbf{I}}, n_{C,C'}$) and is equipped with a distributive law in the sense of Beck* [13] (*see also* [39]) $\lambda\colon {!F} \longrightarrow F!$ *respecting the comonoid structure, in the sense that*

$$\begin{tikzcd}
\mathbf{I} \arrow[d, "n_{\mathbf{I}}"'] & {!F(C)} \arrow[l, "e_{F(C)}"'] \arrow[r, "d_{F(C)}"] \arrow[d, "\lambda_C"] & {!F(C)\otimes !F(C)} \arrow[d, "n_{!C,!C} \circ (\lambda_C \otimes \lambda_C)"] \\
F(I) & F(!C) \arrow[l, "F(e_C)"] \arrow[r, "F(d_C)"'] & F(!C\otimes !C)
\end{tikzcd}$$

commutes.

We make precise the sense in which the categories we describe can be regarded as categories of abstract games by describing linearly distributive functors from a category of games to our categories. We carry out the analysis for the very simple category **Gam** of games described in Section 2 of [31] (see also [1]) though the thrust of the results is pretty insensitive to which category of games we consider.

Like other standard games models, **Gam** is a model for intuitionistic linear

[4] Note that a linearly distributive functor lifts to a functor between the cartesian closed Kleisli categories of (co)free coalgebras.

logic. By self-dualization (see Section 3) one can obtain a category with duality, formally considering 'positive' and 'negative' games; but there is then no relation between the two components of the generalized game.[5] In contrast, our categories of abstract games have a built-in duality. Generally in such cases, an interpretation of the maps as genuinely concurrent processes seems best, and there are many examples of these. However the examples we present here have a strong flavour of sequentiality.

Our starting point is the category **Rel** of sets and relations which is a very degenerate model of classical linear logic. We take as tensor product the product of sets so that **Rel** is compact closed; the disjoint union of sets gives a biproduct. As linear exponential comonad we take the finite multiset comonad W: as explained in Barr [12], this is induced by the (co)free (co)commutative comonoid functor. A concrete description is as follows:

- On objects, $W(A)$ is the set of finite multisets over A.
- For $f\colon A \longrightarrow\!\!\!\!\!\!\!+\;\; B$ a map in **Rel**, $W(f)\colon W(A) \longrightarrow\!\!\!\!\!\!\!+\;\; W(B)$ is defined by considering $f \subseteq A \times B$ and setting $x\ W(f)\ y$ iff there is an element z of $W(f) \subseteq W(A \times B)$ whose first and second projections are x and y, respectively.
- The comonad and comonoid structure maps are the opposites of the usual structure maps for the finite multiset monad in **Sets**.[6]

We now describe a functor $F\colon \mathbf{Gam} \longrightarrow \mathbf{Rel}$. On objects, $F(A)$ is simply the set of positions (or states, or stages) in the game $A \in \mathbf{Gam}$. The action of F on maps is more delicate. By a Player position (P-position) in a game A we mean a position in which Opponent is next to play—the set of all those is denoted by A_P.[7] The remaining positions are Opponent ones (O-positions), collected in the set A_O. Similarly, we occasionally use the abbreviations P-strategy and O-strategy for Player strategies and Opponent strategies. A map $\phi\colon A \longrightarrow B$ in **Gam** is a Player strategy in the game $A \multimap B$, where the dual of A and B are played in parallel. Recall that a position in $A \multimap B$ is given by a sequence of moves (a notion we are abstracting away from) of A and B such that the projections onto A and B respectively are valid positions in the constituent games. Therefore a position r of $A \multimap B$ can be projected to one $r|_A$ of A and one $r|_B$ of B, respectively. Define $F(\phi)$ to be the set

$$\{\langle r|_A, r|_B\rangle \colon r \in \phi \text{ is a P-position}\}$$

[5] A counter instance is the important case of games and history-free strategies [3,5,6]; this provides in the first place a category without products, but still one can dualize. The phenomenon deserves closer study.
[6] In other words, the comonad W is the opposite of a monad obtained by lifting the finite multiset monad from **Sets** to **Rel**.
[7] So in a P-position, Player has just played, including by convention the initial position.

of pairs of positions arising as the projections of a P-position in ϕ. One easily sees that the copy-cat strategy in $A \multimap A$ is mapped by F to the identity relation.

We next consider F applied to a composite $\psi \circ \phi$ of two strategies. Recall that given positions r in $A \multimap B$ and s in $B \multimap C$ such that $r|_B = s|_B$, we can find a unique interleaving of r and s, that is a sequence t of moves from A, B and C, such that the restrictions $t|_{A,B}$ and $t|_{B,C}$ of t to moves from $A \multimap B$ and $B \multimap C$ are r and s respectively. Since

$$\begin{array}{rll} & (t|_{A,C})|_A = t|_A = r|_A & \text{is a position in } A \\ \text{and} & (t|_{A,C})|_C = t|_C = s|_C & \text{is a position in } C \end{array}$$

it follows that $t|_{A,C}$ is a position in $A \multimap C$. This explains why the composite of $\phi\colon A \longrightarrow B$ and $\psi\colon B \longrightarrow C$ in **Gam** is

$$\psi \circ \phi = \{t|_{A,C} : t \text{ sequence of moves in } A,\ B \text{ and } C \text{ with } t|_{A,B} \in \phi, t|_{B,C} \in \psi\}.$$

Now a P-position in the composite $\psi \circ \phi$ arises for the first time via a t as above with both $t|_{A,B}$ and $t|_{B,C}$ P-positions (in $A \multimap B$ and $B \multimap C$ respectively). It follows that $F(\psi \circ \phi)$ is the relational composite $F(\psi) \circ F(\phi)$ so that F is indeed functorial.

Passing from ϕ to $F(\phi)$ appears to lose the information of what interleaving of play in A and B led to a given position in $A \multimap B$. However we can reconstruct ϕ from $F(\phi)$. For different ways of interleaving plays in the constituent games of $A \multimap B$ occur by the choice of Player and hence at P-positions in the game; so these choices are coded in $F(\phi)$. It follows that we can reconstruct ϕ as the set of all positions r in $A \multimap B$ such that for all P-positions $r' \leq r$ we have $\langle r'|_A, r'|_B \rangle \in F(\phi)$. Thus the functor F is faithful.

Theorem 3 *The functor F: **Gam** $\longrightarrow$ **Rel** is faithful and linearly distributive.*

The monoidal structure is given as follows.

- $n_{\mathbf{I}}$ is the unique function from $\mathbf{I}$ to $\mathbf{I} = F(\mathbf{I})$.
- $n_{A,B}\colon F(A) \times F(B) \longrightarrow\!\!\!\!\!\!\!+\;\; F(A \otimes B)$ is the relation

$$\langle r, s \rangle \sim t \quad \text{if and only if} \quad r = t|_A \text{ and } s = t|_B.$$

The distributive map $\lambda_A\colon W(F(A)) \longrightarrow\!\!\!\!\!\!\!+\;\; F(!A)$ is the relation $x \sim \phi$ if and only if x is the multiset of positions arising by projecting ϕ into the active[8] versions of A involved in it. We leave the details to the reader.

[8] This means that the initial position in the first version of A is active initially, but no initial position is active thereafter. The special treatment needed to cope with

Little of the flavour of games is preserved by the functor $F\colon \mathbf{Gam} \longrightarrow \mathbf{Rel}$. In this paper, we aim to develop abstract interpretations with a more game theoretic feel.

3 Self-dualization

Suppose that we are given a category $\mathbf{C}$; then $\mathbf{C}^d = \mathbf{C} \times \mathbf{C}^{op}$ is a category with duality (negation). We have a functor $(\_)^\perp\colon \mathbf{C}^d \longrightarrow (\mathbf{C}^d)^{op}$ with

$$(U, X)^\perp = (X, U),$$

and with the obvious action on morphisms; $(\_)^\perp$ is a self duality on $\mathbf{C}^d$.

It is an important fact that if $\mathbf{C}$ carries enough structure then $\mathbf{C}^d$ is a model of classical Linear Logic. In the presence of a terminal object, $\mathbf{C}^d$ is a degenerate form of Chu's construction [20], so the result for the multiplicatives and additives should be well known. [9]

Proposition 4 *If $\mathbf{C}$ is a symmetric monoidal closed category with finite products, then $\mathbf{C}^d$ is $*$-autonomous. The tensor product is*

$$(U, X) \otimes (V, Y) = (U \otimes V, U \multimap Y \times V \multimap X),$$

the unit $\mathbf{I} = (\mathbf{I}, \mathbf{1})$.

Proposition 5 *If $\mathbf{C}$ has finite products and coproducts, then so does $\mathbf{C}^d$. The products are given as $(U, X) \times (V, Y) = (U \times V, X + Y)$, the unit for the product is $\mathbf{1} = (\mathbf{1}, \mathbf{0})$. Coproducts are formed via $(U, X) + (V, Y) = (U + V, X \times Y)$, the unit for the coproduct is $\mathbf{0} = (\mathbf{0}, \mathbf{1})$.*

Finally we consider the exponentials. We shall assume that $\mathbf{C}$ has a linear exponential comonad; this handles the structure in the first coordinate of $\mathbf{C}^d$ straightforwardly. It is the structure in the second coordinate which presents the challenge.

Definition 6 *Let $\mathbf{C}$ be a model for intuitionistic linear logic. We say that $\mathbf{C}$ has well-adapted monoids just when*

- $\mathbf{C}$ *is equipped with a monad M whose (free) algebras are naturally commutative monoids with respect to product and*

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the initial position will come back to haunt us.

[9] The situation for the exponentials is in any case quite subtle; we give details of our construction and discuss its extension to the general Chu and Dialectica constructions in a companion paper.

- *M is equipped with a strength $\tau_{U,X} \colon !U \otimes M(X) \longrightarrow M(!U \otimes X)$ which respects the monad and monoid structure, and further induces an action of $!$-coalgebras on M-algebras via the linear function space functor $\multimap$.*

Remark 7 *If $\mathbf{C}$ is cartesian closed with $!$ the identity functor this condition amounts to the requirement that M be a strong monad whose algebras are naturally commutative monoids and whose strength is well-behaved with respect to the monoid operations. This case appears in* [22]. *Hyland and de Paiva considered the general definition of the exponential functor and the most obvious structure for it around 1990 but at the time the notion of a linear exponential comonad was not formulated. The details of the full structure and the proofs of the axioms in the symmetric monoidal case are non-trivial and will be described in detail in a companion paper.*

Proposition 8 *Let $\mathbf{C}$ be a model for intuitionistic linear logic with well-adapted monoids. Then $\mathbf{C} \times \mathbf{C}^{op}$ has a linear exponential comonad where $!(U, X) = (!U, !U \multimap M(X))$.*

Putting the above propositions together we have the following.

Theorem 9 *Let $\mathbf{C}$ be a model for intuitionistic linear logic with finite coproducts and well-adapted monoids. Then $\mathbf{C}^d$ is a model for classical linear logic.*

As a category of abstract games, $\mathbf{Rel}^d$ improves on $\mathbf{Rel}$ by representing some aspect of the Player-Opponent dichotomy. If $(R_P, R_O) \in \mathbf{Rel}^d$ we think of $R = R_P + R_O$ as the set of positions of a game, with R_P the P-positions and R_O the O-positions. Clearly there is a functor $+\colon \mathbf{Rel}^d \longrightarrow \mathbf{Rel}$ whose action on objects is $(R_P, R_O) \longmapsto R_P + R_O$.[10] We explain how the functor $F\colon \mathbf{Gam} \longrightarrow \mathbf{Rel}$ from the previous section lifts along this to a functor which we denote again by $F\colon \mathbf{Gam} \longrightarrow \mathbf{Rel}^d$.

For a game A we let $F(A) = (A_P, A_O)$, where A_P is the set of P-positions and A_O is the set of O-positions in A. To give the effect of F on morphisms, we consider a P-position in $A \multimap B$ and its projection onto a pair of positions, one in A and one in B. (Because A occurs contravariantly, the rôles of P-positions and O-position interchange—however, we find it simpler *not* to introduce a dual 'co-game' $A^\perp$ in which the interchange is made explicit.) In a P-position in $A \multimap B$, it is Opponent's turn in precisely one of A and B, so such a position projects down to a pair of the type (P, P) or (O, O). Thus the set of all P-positions in $A \multimap B$ is contained in the disjoint union

$$A_P \times B_P + A_O \times B_O.$$

[10] Special features of $\mathbf{Rel}$ enable one to equip this functor with linearly distributive structure, but this does not appear to be of much importance.

Hence for a map $\phi\colon A \longrightarrow B$ in **Gam**, the set

$$\{\langle t|_A, t|_B\rangle : t \in \phi \text{ a P-position}\}$$

provides relations

$$A_P \longrightarrow\!\!\!\!\!\!\!+\;\; B_P \text{ and } B_O \longrightarrow\!\!\!\!\!\!\!+\;\; A_O,$$

and we let $F(\phi)$ be the corresponding map $(A_P, A_O) \longrightarrow (B_P, B_O)$ in $\mathbf{Rel}^d$. It is easy to see that $F\colon \mathbf{Gam} \longrightarrow \mathbf{Rel}^d$ is functorial and that the composite with $+\colon \mathbf{Rel}^d \longrightarrow \mathbf{Rel}$ from above is the old $F\colon \mathbf{Gam} \longrightarrow \mathbf{Rel}$. Moreover the monoidal and linearly distributive structure lift readily.

Theorem 10 *The functor $F\colon \mathbf{Gam} \longrightarrow \mathbf{Rel}^d$ is faithful and linearly distributive.*

In this reading Player strategies in a function space game contain no information about O-positions of the game. We remedy this in the next section.

4 Comonoid indexing

Assume that (K, e, d) is a comonoid in a monoidal category $\mathbf{C}$; then tensoring with K induces a comonad on $\mathbf{C}$ in the standard way. The induced functor K is $K(C) = K \otimes C$; the co-unit $e \otimes \mathsf{id}_C\colon K \otimes C \longrightarrow \mathbf{I} \otimes C \cong C$, and the comultiplication $d \otimes \mathsf{id}_C\colon K \otimes C \longrightarrow (K \otimes K) \otimes C \cong K \otimes (K \otimes C)$. We consider the (Kleisli) category $\mathbf{C}_K$ of (co)free coalgebras for this comonad, together with the (co)free functor $\mathbf{C} \longrightarrow \mathbf{C}_K$.

Objects of $\mathbf{C}_K$ are objects of $\mathbf{C}$. Morphisms $C \longrightarrow D$ in $\mathbf{C}_K$ are given by morphisms $K \otimes C \longrightarrow D$ in $\mathbf{C}$. Identities are given by the co-unit from above, and composition of $f\colon K \otimes C \longrightarrow D$ and $g\colon K \otimes D \longrightarrow Z$ is

$$K \otimes C \xrightarrow{d \otimes \mathsf{id}_C} K \otimes K \otimes C \xrightarrow{\mathsf{id}_K \otimes f} K \otimes D \xrightarrow{g} Z.$$

Proposition 11 *Let K be a commutative comonoid in a symmetric monoidal category $\mathbf{C}$.*

(1) $\mathbf{C}_K$ is a symmetric monoidal category, and $\mathbf{C} \longrightarrow \mathbf{C}_K$ preserves the structure.
(2) If $\mathbf{C}$ is also closed, then so is $\mathbf{C}_K$, and $\mathbf{C}_K \longrightarrow \mathbf{C}$ preserves the structure.
(3) If $\mathbf{C}$ is $*$-autonomous then so is $\mathbf{C}$ and $\mathbf{C}_K \longrightarrow \mathbf{C}$ preserves the structure.

Proposition 12 *Let K be a commutative comonoid in a symmetric monoidal category $\mathbf{C}$.*

(1) If $\mathbf{C}$ *has products then so has* $\mathbf{C}_K$*, and* $\mathbf{C} \longrightarrow \mathbf{C}_K$ *preserves them.*
(2) Suppose $\mathbf{C}$ *is closed. If* $\mathbf{C}$ *has coproducts, so has* $\mathbf{C}_K$*, and* $\mathbf{C} \longrightarrow \mathbf{C}_K$ *preserves them.*

Definition 13 *Let* $\mathbf{C}$ *be a model for intuitionistic linear logic, and* K *a commutative comonoid in* $\mathbf{C}$*. We say that* K *is an* exponential comonoid *if* K *is a coalgebra for* $!$ *and the comonoid structure on* K *is the one canonically derived from that the* $!$*-coalgebra structure.*

Proposition 14 *Let* $\mathbf{C}$ *be a model for intuitionistic linear logic and let* K *be an exponential comonoid in* $\mathbf{C}$*. Then* $\mathbf{C}_K$ *has a linear exponential comonad.*

Theorem 15 *Let* $\mathbf{C}$ *be a model for classical linear logic and* K *an exponential comonoid in* $\mathbf{C}$*. Then* $\mathbf{C}_K$ *is a model for classical linear logic.*

We identify the comonoid $K = (\mathbf{I}, \mathbf{I})$ in $\mathbf{Rel}^d$, and define the category of 'restricted relations' $\mathbf{RRel}$ to be $\mathbf{Rel}^d_{\mathbf{K}}$. So $\mathbf{RRel}$ has as objects pairs of sets (R_P, R_O); and maps $(R_P, R_O) \longrightarrow (S_P, S_O)$ in $\mathbf{RRel}$ are maps

$$(R_P, R_O + R_P) \longrightarrow (S_P, S_O)$$

in $\mathbf{Rel}^d$ and so can be identified with subsets of

$$(R_P \times S_P) + (R_O \times S_O) + (R_P \times S_O).$$

Thus a map $(R_P, R_O) \longrightarrow (S_P, S_O)$ in $\mathbf{RRel}$ is a relation

$$R = R_P + R_O \longrightarrow\!\!\!\!\!\!\!+\;\; S_P + S_O = S,$$

restricted in that no elements of $(R_O \times S_P)$ appear. This restriction mirrors the usual switching condition in games. By the above discussion, $\mathbf{RRel}$ is a model for classical linear logic. [11]

Adopting these changes, we modify the functor $F\colon \mathbf{Gam} \longrightarrow \mathbf{Rel}^d$ of Section 3 to give $G\colon \mathbf{Gam} \longrightarrow \mathbf{RRel}$. On objects, we have $G(A) = (A_P, A_O)$ as before. For $\phi\colon A \longrightarrow B$ in $\mathbf{Gam}$, the image $G(\phi)\colon (A_P, A_O) \longrightarrow (B_P, B_O)$ in $\mathbf{RRel}$ will be represented by a subset $G(\phi)$ of

$$(A_P \times B_P) + (A_O \times B_O) + (A_P \times B_O).$$

We let $G(\phi) \cap ((A_P \times B_P) + (A_O \times B_O))$ be the projections of P-positions in a play of ϕ as before; the critical issue is the definition of the set $G(\phi) \cap (A_P \times B_O)$ of projections of O-positions. Rather than considering all O-positions in a play according to ϕ, we take only those to which ϕ has no reply; we call these the

[11] There is a monoidal adjunction between $\mathbf{RRel}$ and $\mathbf{Rel}^d$, but we will not dwell on that since we do not exploit the fact in this paper.

final O-positions of ϕ, so $G(\phi) \cap (A_P \times B_O)$ is the set of projections of final O-positions occurring in a play of ϕ.[12]

That this choice is functorial reflects the following feature of composition of strategies. Take $\phi\colon A \longrightarrow B$ and $\psi\colon B \longrightarrow C$ in **Gam** and let t be a sequence of moves over A, B, and C with $t|_{A,B} \in \phi$, $t|_{B,C} \in \psi$ so that $t|_{A,C} \in \psi \circ \phi$. Suppose that $t|_{A,C}$ is a final O-position in $\psi \circ \phi$. Then

$$\begin{array}{ll} \text{either} & \langle t|_A, t_B\rangle \text{ has the type } (P,O) \text{ and } \langle t|_B, t|_C\rangle \text{ the type } (O,O), \\ \text{or} & \langle t|_A, t_B\rangle \text{ has the type } (P,P) \text{ and } \langle t|_B, t|_C\rangle \text{ the type } (P,O). \end{array}$$

(Note that this is true even if t involves further chattering in B.) Moreover, the (P,O) pair is the projection of a final O-position, either of $t|_{A,B}$ final in ϕ, or of $t|_{B,C}$ final in ψ. Conversely, given such a t, $t|_{A,C}$ is a final O-position in $\psi \circ \phi$. It is now not difficult to fill in the details of the monoidal and linearly distributive structure for the functor G.

Theorem 16 $G\colon \mathbf{Gam} \longrightarrow \mathbf{RRel}$ *is a faithful linearly distributive functor.*

Abstract strategies now contain information about O-positions as well as information about P-positions, but otherwise reflect no structure of a game tree. We get this by constraining the possible strategies.

5 Glueing

We describe only the simplest case of a double glueing construction: glueing along the linear element functor $\mathbf{C}(\mathbf{I}, \_)$ on a $*$-autonomous category $\mathbf{C}$.

We construct a new category $\mathbf{G}(\mathbf{C})$, the 'glued category', as follows. Objects of $\mathbf{G}(\mathbf{C})$ are objects R of $\mathbf{C}$ together with sets

$$U \subseteq \mathbf{C}(\mathbf{I}, R) \text{ and } X \subseteq \mathbf{C}(R, \bot) \cong \mathbf{C}(\mathbf{I}, R^{\bot}).$$

Maps in $\mathbf{G}(\mathbf{C})$ from (R, U, X) to (S, V, Y) are maps $f\colon R \longrightarrow S$ such that:

- for all $\mathbf{I} \xrightarrow{u} R$ in U, $\mathbf{I} \xrightarrow{u} R \xrightarrow{f} S$ is in V and
- for all $S \xrightarrow{y} \bot$ in Y, $R \xrightarrow{f} S \xrightarrow{y} \bot$ is in X.

[12] It is a viable option to consider projections of all O-positions in ϕ. The reason for the choice we make emerges in Section 6: in $G(\phi)$ we are encoding the set of final positions which may result from playing ϕ.

We need a notation for generalized composition. Given $h\colon R \otimes S \longrightarrow \perp$ and $v\colon \mathbf{I} \longrightarrow S$, we define $(v|h)_S\colon R \longrightarrow \perp$ to be

$$R \cong R \otimes \mathbf{I} \xrightarrow{\mathsf{id}_R \otimes v} R \otimes S \xrightarrow{h} \perp.$$

We extend this in the obvious way to other 'cuts'.

Proposition 17 *If $\mathbf{C}$ is $*$-autonomous, so is $\mathbf{G}(\mathbf{C})$, and the forgetful functor $\mathbf{G}(\mathbf{C}) \longrightarrow \mathbf{C}$ preserves the structure. The tensor unit is given by*

$$\mathbf{I} = (\mathbf{I}, \{\mathsf{id}_{\mathbf{I}}\}, \mathbf{C}(\mathbf{I}, \perp))$$

and the tensor product

$$(R, U, X) \otimes (S, V, Y) = (R \otimes S, U \otimes V, Z)$$

$$\text{where } U \otimes V = \{\mathbf{I} \cong \mathbf{I} \otimes \mathbf{I} \xrightarrow{u \otimes v} R \otimes S\colon u \in U, v \in V\}$$
$$\text{and } Z = \{R \otimes S \xrightarrow{z} \perp\colon \forall u \in U.\ (u|z)_R \in Y \text{ and } \forall v \in V.\ (v|z)_S \in X\}.$$

Proposition 18 *If $\mathbf{C}$ has finite products, then so has $\mathbf{G}(\mathbf{C})$, and the functor $\mathbf{G}(\mathbf{C}) \longrightarrow \mathbf{C}$ preserves them. The terminal object is $(\mathbf{1}, \mathbf{C}(\mathbf{I}, \mathbf{1}), \emptyset)$, and the product*

$$(R, U, X) \times (S, V, Y) = (R \times S, U \times V, X \oplus Y)$$
$$\text{where } U \times V = \{\langle u, v\rangle\colon \mathbf{I} \longrightarrow R \times S\colon u \in U, v \in V\} \text{ and}$$
$$X \oplus Y = \{R \times S \xrightarrow{\pi_1} R \xrightarrow{x} \perp\colon x \in X\} \cup \{R \times S \xrightarrow{\pi_2} S \xrightarrow{y} \perp\colon y \in Y\}.$$

Dually, if $\mathbf{C}$ has finite coproducts, then so does $\mathbf{G}(\mathbf{C})$ and $\mathbf{G}(\mathbf{C}) \longrightarrow \mathbf{C}$ preserves them.

Note that the functor $\mathbf{C}(\mathbf{I}, \_)\colon \mathbf{C} \longrightarrow \mathbf{Sets}$ is monoidal; and as $\mathbf{Sets}$ is trivially a model for intuitionistic linear logic (with the cartesian closed structure to model the multiplicatives and the identity comonad to take care of linear exponentials), we can ask for a natural transformation

$$\lambda\colon \mathbf{C}(\mathbf{I}, \_) \longrightarrow \mathbf{C}(\mathbf{I}, !(\_))$$

making $\mathbf{C}(\mathbf{I}, \_)$ linearly distributive. With this data we can find linear exponential comonads in $\mathbf{G}(\mathbf{C})$.

Proposition 19 *Let $\mathbf{C}$ be a model for classical linear logic with a linear distribution λ as above.*

(1) We can define an exponential comonad on $\mathbf{G}(\mathbf{C})$ by

$$!(R, U, X) = (!R, \{\lambda_R(u)\colon u \in U\}, \mathbf{C}(!R, \perp)),$$

and then $\mathbf{G}(\mathbf{C}) \longrightarrow \mathbf{C}$ preserves the structure.

(2) We can define an exponential comonad on $\mathbf{G}(\mathbf{C})$ *by*

$$!(R, U, X) = (!R, \{\lambda_R(u) \colon u \in U\}, ?X),$$

where $?X$ *is the smallest subset of* $\mathbf{C}(!R, \perp)$

- *containing* $\{x \circ \epsilon_R \colon x \in X\}$,
- *containing* $\{\chi \circ e_R \colon \chi \colon \mathbf{I} \longrightarrow \perp\}$,
- *and such that whenever for some* $h \colon !R \otimes !R \longrightarrow \perp$, *for all* $u \in U$ *both composites* $(\lambda_R(u)|h)_{!R}$ *are in* $?X$, *then* $h \circ d_R \colon !R \longrightarrow \perp$ *is in* $?X$.

Again $\mathbf{G}(\mathbf{C}) \longrightarrow \mathbf{C}$ *preserves the structure.*

Loader's category of Linear Logical Predicates [36] is essentially $\mathbf{G}(\mathbf{Rel})$, and the cruder of these comonads for the standard power set comonad on **Rel** is described in [36].

Theorem 20 *Let* $\mathbf{C}$ *be a model for classical linear logic equipped with a linear distribution* λ *as above. Then* $\mathbf{G}(\mathbf{C})$ *is a model for classical linear logic, and* $\mathbf{G}(\mathbf{C}) \longrightarrow \mathbf{C}$ *preserves all the structure.*

We now explain how to see $\mathbf{G}(\mathbf{RRel})$ as a category of abstract games. Recall first the functor $G \colon \mathbf{Gam} \longrightarrow \mathbf{RRel}$. As $G(\mathbf{I}) = (\mathbf{I}, \mathbf{0}) = \mathbf{I} \in \mathbf{RRel}$ we can consider for any game A the image

$$U = G(\mathbf{Gam}(\mathbf{I}, A)) \subseteq \mathbf{RRel}(\mathbf{I}, G(A))$$

of the Player strategies in A. If $u = G(\phi)$, where ϕ is a Player strategy in A, then

$$\begin{array}{rll} & u \cap A_P & \text{is the set of P-positions in } \phi \\ \text{and} & u \cap A_O & \text{is the set of final O-positions in } \phi. \end{array}$$

Thus $U \subseteq \mathbf{RRel}(\mathbf{I}, G(A))$ consists of the representations of Player strategies. Clearly if $\phi \colon A \longrightarrow B$ in **Gam** and $U \subseteq \mathbf{RRel}(\mathbf{I}, G(A))$, $V \subseteq \mathbf{RRel}(\mathbf{I}, G(B))$ are the sets of representatives of Player strategies on A and B respectively, then composition with $G(\phi)$ maps U to V.

We wish also to consider representatives of Opponent strategies in a game A. These are in bijective correspondence with Player strategies in $A \multimap \Sigma$ where Σ is the one-move (*ie.* two-position) game.[13] However there is just one position too many in $G(\mathbf{Gam}(A, \Sigma)) \subseteq \mathbf{RRel}(G(A), G(\Sigma))$: the initial position in $A \multimap \Sigma$ does not correspond to a position in A. But since $G(\Sigma) \cong K = (\mathbf{I}, \mathbf{I})$, there is a unique non-zero map $\perp = (\mathbf{0}, \mathbf{I}) \longrightarrow (\mathbf{I}, \mathbf{I}) = G(\Sigma)$ in **RRel** and we

[13] Σ appears as S in [31] and plays the rôle of the object of resumptions in recent work of Laird.

can consider $X \subseteq \mathbf{RRel}(G(A), \bot)$ in the pullback

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d, "\cong"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{RRel}(G(A), \bot) \arrow[d] \\ G(\mathbf{Gam}(A, \Sigma)) \arrow[r, hook] & \mathbf{RRel}(G(A), G(\Sigma)) \end{tikzcd}$$

This removes the unwanted position; for $x \in X$ we have a unique Opponent strategy τ in A with

$$\begin{array}{lll} & x \cap A_O & \text{is the set of O-positions in } \tau \\ \text{and} & x \cap A_P & \text{is the set of final P-positions in } \tau. \end{array}$$

Thus $X \subseteq \mathbf{RRel}(G(A), \bot)$ consists of representatives of Opponent strategies. It easily follows from our definition that if $\phi\colon A \longrightarrow B$ is a morphism in **Gam** and $X \subseteq \mathbf{RRel}(G(A), \bot)$ and $Y \subseteq \mathbf{RRel}(G(B), \bot)$ are the sets of representatives of Opponent strategies, then composition with $G(\phi)$ maps Y to X.

It now follows that we can lift the functor $G\colon \mathbf{Gam} \longrightarrow \mathbf{RRel}$ along the forgetful functor $\mathbf{G}(\mathbf{RRel}) \longrightarrow \mathbf{RRel}$ to a functor $G\colon \mathbf{Gam} \longrightarrow \mathbf{G}(\mathbf{RRel})$ by setting

$$G(A) = (G(A), U, X)$$

where $U \subseteq \mathbf{RRel}(\mathbf{I}, G(A))$ and $X \subseteq \mathbf{RRel}(G(A), \bot)$ consist of the representatives of Player and Opponent strategies in the game A respectively. Obviously, the new G is faithful and the linear logic structure (no matter which of the two exponentials introduced above we choose) lifts readily along the forgetful functor $\mathbf{G}(\mathbf{RRel}) \longrightarrow \mathbf{RRel}$.

But now more is true: the functor G is full. At an intuitive level, we can explain this as follows. We can reach every position r in $A \multimap B$ by playing according to an O-strategy which treats A and B independently (moves in A may not depend on the history in B and vice versa). Such a strategy can be viewed as arising from a pair of strategies, one in A (but since the rôles of Player and Opponent are exchanged in this game as it is played as part of $A \multimap B$, this will be a P-strategy) and one in B. So there is an O-strategy (σ, τ) where τ is an O-strategy on B and σ a P-strategy on A such that r arises when a suitable P-strategy is played against it.

Now suppose $G(A) = (G(A), U, X)$ and $G(B) = (G(B), V, Y)$ as above and suppose $f\colon G(A) \longrightarrow G(B)$ given by a set

$$f \subseteq (A_P \times B_P) + (A_O \times B_O) + (A_P \times B_O)$$

is a map in $\mathbf{G}(\mathbf{RRel})$, that is $[u]f \in V$ and $f[y] \in X$. Arguing inductively we can reconstruct a P-strategy ϕ in $A \multimap B$, that is a map $A \longrightarrow B$ in **Gam**, with $G(\phi) = f$.

The initial stages of the induction are roughly as follows. We start by considering f applied to $\perp_Y$, where $\perp_Y$ is the (representative of the) least O-strategy in B (the one where Opponent refuses to do anything at all). It can be shown that $f[\perp_Y] = \perp_X$, the (representative of the) least O-strategy in A. Then for an initial O-move b in B we consider on the one hand $f[y_b]$, where y_b is the (representative of the) O-strategy generated by the move b, and on the other hand $[\perp_U]f$ where $\perp_U$ is the (representative of the) least P-strategy in A. Then

either	$[\perp_U]f$ contains a reply to b,
or	$f[y_b]$ contains an opening move in A,
or	neither of these.

In the first case ϕ replies to b in B, in the second, ϕ replies to b in A, while in the third, ϕ has no reply to b. The details of the inductive argument will be given in the extended paper.

Theorem 21 G: **Gam** $\longrightarrow$ $\mathbf{G}(\mathbf{RRel})$ *is full and faithful and linearly distributive.*

We have made progress in connecting the concrete category of games with a category of abstract games. Now the last feature which we wish to incorporate is some connection between abstract strategies for Player and for Opponent. We treat this issue in the next section.

6 Orthogonality

In this section we introduce machinery motivated by the following considerations. A Player strategy in a tensor game $A \otimes B$ can use information about what has happened in B to guide play in A and *vice versa*:[14] so there are many more strategies than are given by tensoring a strategy for A and one for B. The simple abstract categories of games do not allow this and our response is to consider the tight orthogonality categories introduced below. The intuition that we are trying to capture is that in a game the Player and Opponent strategies determine each other; and then the multiplicative structure is determined by the maps in the category (that is, by Player strategies in function

[14] Of course this is what visibility is designed to prevent in [29] and [38].

spaces). Thus the Player strategies in $A \otimes B$ are as varied as they can be given the Opponent strategies (that is, the Player strategies in $A \multimap B^\perp \cong B \multimap A^\perp$).

We identify two full subcategories of $\mathbf{G}(\mathbf{C})$ by using the additional structure of an orthogonality on $\mathbf{C}$.

Definition 22 *Let $\mathbf{C}$ be a $*$-autonomous category. An* orthogonality *on $\mathbf{C}$ is a family of relations $\perp_C$ between maps $\mathbf{I} \longrightarrow C$ and $C \longrightarrow \perp$ satisfying the following:*

(1) (Isomorphisms) If $f\colon C \longrightarrow D$ is an isomorphism, then for all maps $u\colon \mathbf{I} \longrightarrow C$ and all maps $x\colon C \longrightarrow \perp$, we have

$$u \perp_C x \textit{ iff } f \circ u \perp_D x \circ f^{-1}$$

(2) (Symmetry) For all $\mathbf{I} \xrightarrow{u} C$ and all $C \xrightarrow{x} \perp$,

$$u \perp_C x \textit{ iff } x^\perp \perp_{C^\perp} u^\perp.$$

(3) (Tensor) Given $\mathbf{I} \xrightarrow{u} C$, $\mathbf{I} \xrightarrow{v} D$ together with $C \otimes D \xrightarrow{h} \perp$, then

$$u \perp_C (v|h)_D \textit{ and } v \perp_D (u|h)_C \textit{ implies } u \otimes v \perp_{C\otimes D} h.$$

(4) (Identity) For all $\mathbf{I} \xrightarrow{u} C$ and all $C \xrightarrow{y} \mathbf{I}$,

$$u \perp_C y \textit{ implies } \mathsf{id}_{\mathbf{I}} \perp_{\mathbf{I}} y \circ u = (u|y)_C.$$

The second condition enables us to regard $\perp$ in lots of different ways. For example, we can consider $u\colon \mathbf{I} \longrightarrow C$ orthogonal to $x^\perp\colon \mathbf{I} \longrightarrow C^\perp$ without ambiguity.

Given $U \subseteq \mathbf{C}(\mathbf{I}, R)$, we set

$$U^\perp = \{x\colon R \longrightarrow \perp : \forall u \in U.\, u \perp_R x\} \subseteq \mathbf{C}(R, \perp).$$

Similarly we define $X^\perp = \{u\colon \mathbf{I} \longrightarrow R\colon \forall x \in X.\, u \perp_R x\}$ for $X \subseteq \mathbf{C}(R, \perp)$. Note that if $U = X^\perp$, then $U^{\perp\perp} = X^{\perp\perp\perp} = X^\perp = U$. We call such sets *closed*.

Definition 23 *An orthogonality is* precise *just when the condition (iii) is an equivalence, that is, $u \perp_C (v|h)$ and $v \perp_D (u|h)$ iff $u \otimes v \perp_{C\otimes D} h$. (Note that the precise form of (iii) implies (iv).)*

An orthogonality is stable *if it is precise and in addition satisfies the condition*

$$(U^{\perp\perp} \otimes V^{\perp\perp})^\perp = (U^{\perp\perp} \otimes V)^\perp.$$

In a precise orthogonality, we have $u \perp_C x$ iff $u \otimes x^\perp \perp_{C \otimes C^\perp} \mathsf{ev}_C$. In case $\mathbf{C}$ is compact closed this means that the precise orthogonality is determined by a family of subsets of $\mathsf{End}(C)$ indexed over the objects C of $\mathbf{C}$.

Definition 24 *The* loose (orthogonality) subcategory $\mathbf{G}^{(\perp)}(\mathbf{C})$ *is the full subcategory of* $\mathbf{G}(\mathbf{C})$ *which contains those objects* (R, U, X) *such that for all* $u \in U$ *and for all* $x \in X$ *we have* $u \perp_R x$*. In other words,* $U \subseteq X^\perp$ *and* $X \subseteq U^\perp$*.*

The tight (orthogonality) subcategory $\mathbf{G}^\perp(\mathbf{C})$ *is the full subcategory of* $\mathbf{G}(\mathbf{C})$ *which comprises those* (R, U, X) *for which* $U = X^\perp$ *and* $X = U^\perp$*.*

Note that if $U \subseteq \mathbf{C}(\mathbf{I}, R)$ is closed, then $(R, U, U^\perp)$ is an object of the tight subcategory. Dual considerations apply to $X \subseteq \mathbf{C}(R, \perp)$.

Proposition 25 *If* $\mathbf{C}$ *is a* $*$*-autonomous category with an orthogonality then so is* $\mathbf{G}^{(\perp)}(\mathbf{C})$*: it is closed under negation and tensor and has the tensor unit* $(\mathbf{I}, \{\mathsf{id}_\mathbf{I}\}, \{\mathsf{id}_\mathbf{I}\}^\perp)$*.*

If the orthogonality is stable then $\mathbf{G}^\perp(\mathbf{C})$ *is* $*$*-autonomous; it is closed under negation, tensor product is given by* $(R \otimes S, (U \otimes V)^{\perp\perp}, (U \otimes V)^\perp)$*, and the new unit is* $(\mathbf{I}, \{\mathsf{id}_\mathbf{I}\}^{\perp\perp}, \{\mathsf{id}_\mathbf{I}\}^\perp)$*.*

For $\mathbf{G}^\perp(\mathbf{C})$ note that $(U \otimes V)^\perp$ is, in fact, the corresponding component of the tensor product in $\mathbf{G}(\mathbf{C})$, so all we are changing is the second component, by 'closing' it. From now on, we will tacitly assume that $\mathbf{C}$ has an orthogonality, but state so explicitly if we assume it to be stable.

To handle the rest of the linear logic structure we need control of the structure maps.

Definition 26 *We say that* $f : C \longrightarrow D$ *is* central *with respect to the orthogonality* $\perp$ *if for all* $u : \mathbf{I} \longrightarrow C$ *and all* $y : D \longrightarrow \perp$ *we have* $f \circ u \perp_D y$ *iff* $u \perp_C y \circ f$*, that is,* $(u|f)_C \perp_D y$ *iff* $u \perp_C (f|y)_D$*. The collection of all central maps is the* centre *of the orthogonality.*

Remark 27 *An orthogonality is* focussed *if and only if there is a set* F *of morphisms from* $\mathbf{I}$ *to* $\perp$ *such that* $u \perp_C x$ *if and only if* $(u|x)_C = x \circ u \in F$*. Such orthogonalities are common, for example the original phase space semantics for Linear Logic is based explicitly on a focussed orthogonality* [24,28]*. Clearly, in this case, all maps are central. Conversely, if we have a central orthogonality, then by setting* $F = \{\chi : \mathbf{I} \longrightarrow \perp : \mathsf{id}_\mathbf{I} \perp_\mathbf{I} \chi\}$ *we obtain* $u \perp_C x$ *iff* $u \circ \mathsf{id}_\mathbf{I} \perp_C x$ *iff* $\mathsf{id}_\mathbf{I} \perp_\mathbf{I} x \circ u$ *iff* $x \circ u \in F$*, so the orthogonality is focussed.*

Proposition 28 *Assume that* $\mathbf{C}$ *has products and projection maps are central. Then the loose subcategory* $\mathbf{G}^{(\perp)}(\mathbf{C})$ *of* $\mathbf{G}(\mathbf{C})$ *is closed under products.*

If additionally the orthogonality on $\mathbf{C}$ *is stable then the tight subcategory* $\mathbf{G}^{\perp}(\mathbf{C})$ *of* $\mathbf{G}(\mathbf{C})$ *has products, given by*

$$(R, U, X) \times (S, V, Y) = (R \times S, U \times V, (U \times V)^{\perp}).$$

In fact, $U \times V = (X \oplus Y)^{\perp}$ *is closed. Dual results holds for coproducts.*

Definition 29 *We say that an exponential comonad on* $\mathbf{C}$ *is* central *with respect to the orthogonality* $\perp$ *if and only if all the structure maps* ϵ, δ, e, d *are central and exponentials* $!f : !R \longrightarrow !S$ *of maps* $f : R \longrightarrow S$ *are central.*

We note that the exponential comonad ! is central if and only if all maps in the category of !-coalgebras are central.

Proposition 30 *Suppose that the structure maps* ϵ, e *and* d *are central. We can define an exponential comonad on* $\mathbf{G}^{(\perp)}(\mathbf{C})$ *by*

$$!(R, U, X) = (!R, \{\lambda_R(u) : u \in U\}, ?X),$$

where $?X$ *is as in* Proposition 19, *but the second clause is replaced by*

$$\{\chi \circ e_a : \mathsf{id}_{\mathbf{I}} \perp_{\mathbf{I}} \chi\} \subseteq ?X.$$

Suppose that the exponential comonad on $\mathbf{C}$ *is central. We can define an exponential comonad on* $\mathbf{G}^{\perp}(\mathbf{C})$ *by*

$$!(R, U, X) = (!R, (\lambda_R[U])^{\perp\perp}, (\lambda_R[U])^{\perp}).$$

Theorem 31 *Suppose* $\mathbf{G}(\mathbf{C})$ *is obtained from* $\mathbf{C}$ *as in* Theorem 20. *If the exponential comonad on* $\mathbf{C}$ *is central, then both the loose category* $\mathbf{G}^{(\perp)}(\mathbf{C})$ *and the tight category* $\mathbf{G}^{\perp}(\mathbf{C})$ *are models for classical linear logic.*

Generally a category will admit many orthogonalities. We recall two of particular importance in the case of **Rel**. One is Loader's *total orthogonality:* for $u : \mathbf{I} \longrightarrow R$ and $x : R \longrightarrow \mathbf{I}$ in **Rel** we set

$$u \perp_R x \quad \text{if and only if} \quad |u \cap x| = 1.$$

Loader's Totality Spaces [35,36] are essentially $\mathbf{G}^{\perp}(\mathbf{Rel})$ for this orthogonality. The other orthogonality is the *partial orthogonality:* for $u : \mathbf{I} \longrightarrow R$ and $x : R \longrightarrow \mathbf{I}$ we have

$$u \perp_R x \quad \text{if and only if} \quad |u \cap x| \leq 1.$$

Girard's Coherence spaces [24,28] are essentially $\mathbf{G}^{\perp}(\mathbf{Rel})$ for this orthogonality. These identifications [15] are exploited in [40].

Our orthogonalities on **Rel** induce orthogonalities on **RRel**, since morphisms $u\colon \mathbf{I} \longrightarrow (R_P, R_O)$ and $x\colon (R_P, R_O) \longrightarrow \mathbf{I}$ in **RRel** correspond to subsets $u \subseteq R = R_P + R_O$ and $x \subseteq R = R_P + R_O$. We have the *total orthogonality* on **RRel**

$$u \perp_R x \quad \text{if and only if} \quad |u \cap x| = 1$$

and *partial orthogonality* on **RRel**

$$u \perp_R x \quad \text{if and only if} \quad |u \cap x| \leq 1.$$

Both these orthogonalities are stable with central exponential comonad and so we find that both the loose category $\mathbf{G}^{(\perp)}(\mathbf{RRel})$ and the tight category $\mathbf{G}^{\perp}(\mathbf{RRel})$ are models for classical linear logic.

We return now to our functor $G\colon \mathbf{Gam} \longrightarrow \mathbf{G}(\mathbf{RRel})$. Take $A \in \mathbf{Gam}$ with $G(A) = ((A_P, A_O), U, X)$, so that we can identify U and X with the sets of Player and Opponent strategies in A, respectively. For $u \in U$ and $x \in X$ there are two possibilities for the play of u against x:

either the play terminates in a position r and $u \cap x = \{r\}$

or the play is infinite and $u \cap x = \emptyset$.

It follows that $G\colon \mathbf{Gam} \longrightarrow \mathbf{G}(\mathbf{RRel})$ factors through $\mathbf{G}^{(\perp)}(\mathbf{RRel})$ when $\perp$ is the partial orthogonality. (That is the largest of the subcategories of $\mathbf{G}(\mathbf{RRel})$ which we have identified.) When we pass from $\mathbf{G}(\mathbf{RRel})$ to the loose category $\mathbf{G}^{(\perp)}(\mathbf{RRel})$ the classical linear logic structure changes, but we still have the following.

Theorem 32 *For the partial orthogonality $\perp$, $G\colon \mathbf{Gam} \longrightarrow \mathbf{G}^{(\perp)}(\mathbf{RRel})$ is full, faithful and linearly distributive.*

To capture the liberal nature of Player strategies in a tensor product which we discussed at the beginning of this section, we need to use tight categories. [16] Using the partial orthogonality category, our coding does not lead to a subcategory of the tight category $\mathbf{G}^{\perp}(\mathbf{RRel})$: The problem is that the zero (empty) map will be in any closed U or X; but (because we make the initial posi-

[15] The very slight mismatch in the case of Totality Spaces need not detain us here, we come back to it in Section 7.

[16] We recall that for the total orthogonality on **Rel** (the case of Loader's Totality Spaces) restricting to the tight category has no effect on the tensor product; but this is rare.

tion explicit) our coding does not identify that with any strategy. From this point of view the total orthogonality is more promising; but we have there the problem of infinite plays for which we have given no explicit representation. However, if we are prepared to forego the exponentials we can restrict to the subcategory $\mathbf{Gam}_{\mathrm{fin}}$ of finite games. The functor $G\colon \mathbf{Gam}_{\mathrm{fin}} \longrightarrow \mathbf{G}(\mathbf{RRel})$ factors through $\mathbf{G}^{(\perp)}(\mathbf{RRel})$ for the total orthogonality as now there are no infinite plays. And now we can make good our motivating intuition as the image lies in $\mathbf{G}^{\perp}(\mathbf{RRel})$ and we have the following

Proposition 33 *For the total orthogonality, $G\colon \mathbf{Gam}_{\mathit{fin}} \longrightarrow \mathbf{G}^{\perp}(\mathbf{RRel})$ is a full, faithful, and monoidal functor.*

While the functor $G\colon \mathbf{Gam}_{\mathrm{fin}} \longrightarrow \mathbf{G}^{\perp}(\mathbf{RRel})$ does not quite preserve the multiplicative structure, we have made an advance. For simple calculations show that in a tensor product

$$((R_P, R_O), U, X) \otimes ((S_P, S_O), V, Y)$$

in $\mathbf{G}^{\perp}(\mathbf{RRel})$, the set $(U \otimes V)^{\perp\perp}$ (which can be read as the representation of Player strategies) is substantially larger than $U \otimes V$. So a process reading of the category seems plausible, and we now give an indication of what this might be.

7 A concrete category of games

In this final section we describe a new category of games and relate it to a category of abstract games which we arrived at in the last section.

Definition 34 *A* graph game *A is given by*

- *a set $A = A_P + A_O$ of* positions *together with an* initial position *$*_A \in A_P$; A_P is the set of* Player positions *(where Opponent is to move) and A_O is the set of* Opponent positions *(where Player is to move);*
- *the structure $a \longrightarrow a'$ on A of an acyclic directed (A_P, A_O)-bipartite graph (so if $a \longrightarrow a'$ then $a \in A_P$ if and only if $a' \in A_O$); if $a \longrightarrow a'$ then* there is a move from a to a'*; and for any $a \in A$ the length of paths from $*_A$ to a is bounded.*[17]

We think of a game A as being played from $*_A$ with Opponent making the first move: positions which are not reachable from $*_A$ play no part in the game and could be deleted.

[17] In fact a well-foundedness condition suffices.

Prima facie the notion of a Player (or Opponent) strategy in a graph game seems clear enough. A Player strategy σ in A will map (some) Opponent positions $a \in A_O$ to Player positions $a' \in A_P$ where $a \longrightarrow a'$ is the Player move according to σ. However the possibilities of arriving at the same position by different paths means that we should consider only conflict-free[18] strategies. A Player strategy σ is *conflict-free* if and only if whenever a'' is reachable from $a \in A_O$, and both are positions occurring in plays according to σ then σ has a response $a \longrightarrow a'$ to a and a'' is reachable from a'. Henceforth by *strategy* we mean *conflict-free strategy.*

In order to describe the category **GGam** of graph games we describe the multiplicative structure.

- The tensor unit $\mathbf{I}$ is the game with just one (initial) position $*_\mathbf{I}$.
- The tensor $A \otimes B$ is the game with
  · P-positions $A_P \times B_P$
  · O-positions $(A_P \times B_O) + (A_O \times B_P)$.
  The initial position is $(*_A, *_B)$ and there are moves $(a, b) \longrightarrow (a', b')$ just when
$$\begin{array}{ll} \text{either} & a \longrightarrow a' \text{ and } b = b' \\ \text{or} & a = a' \text{ and } b \longrightarrow b' \end{array}$$
- The linear function space $A \multimap B$ is the game with
  · P-positions $(A_P \times B_P) + (A_O \times B_O)$
  · O-positions $A_P \times B_O$.
  The initial position is $(*_A, *_B)$ and there are moves $(a, b) \longrightarrow (a', b')$ just when
$$\begin{array}{ll} \text{either} & a \longrightarrow a' \text{ and } b = b' \\ \text{or} & a = a' \text{ and } b \longrightarrow b' \end{array}$$

Now we can define the maps $\phi \colon A \longrightarrow B$ in the category **GGam** of graph games to be the conflict-free Player strategies in $A \multimap B$. Just as with more familiar games these strategies compose associatively and there is an identity (copy-cat) strategy $A \longrightarrow A$.

Proposition 35 **GGam** *is a symmetric monoidal closed category.*

The additive structure on **GGam** is obvious.

- The terminal object $\mathbf{1}$ is (again) the game $\mathbf{I}$ with just one (initial) position $*_\mathbf{I}$.
- The product $A \times B$ is the game with positions $A \vee B$ the 'coalesced sum' of the positions of A and B, identifying $*_A$ with $*_B$ to give the new initial

[18] The terminology hints at a connection with concrete data structures [32] and event structures [41], see also [21].

position. Player and Opponent positions and moves are all inherited from A and B.

In a similar way one gets products for arbitrary families.

Proposition 36 **GGam** *has all products.*

We can define an exponential comonad[19] on **GGam** as follows. The exponential $!A$ is the game with

- P-positions finite multisets in $A_P - \{*_A\}$,
- O-positions finite multisets in $A - \{*_A\}$ containing just one element from A_O.

The initial position is the empty multiset and there are moves $m \longrightarrow m'$ just when

$$\begin{array}{ll} \text{either} & m = n + a,\ a \longrightarrow a' \text{ in } A \text{ and } m' = n + a' \\ \text{or} & m' = m + a \text{ and } *_A \longrightarrow a \text{ in } A. \end{array}$$

We leave the description of comonad and comonoid structure for ! to the reader—it parallels that in **Gam**.

Proposition 37 **GGam** *is a model for intuitionistic linear logic.*

We note in passing relations between the category **GGam** and the more familiar category **Gam**. Every game with game tree is a graph game, and we get a linearly distributive functor $Q\colon \mathbf{Gam} \longrightarrow \mathbf{GGam}$ embedding **Gam** as a full subcategory of **GGam**. On the other hand by taking paths we can derive a game tree from a graph game; there is a functor $P\colon \mathbf{GGam} \longrightarrow \mathbf{Gam}$ which preserves the structure (for models of intuitionistic linear logic).

We briefly indicate some relations between **GGam** and categories of abstract games of the form $\mathbf{G}^{(\perp)}(\mathbf{RRel})$ and $\mathbf{G}^{\perp}(\mathbf{RRel})$. We restrict attention to the total orthogonality. The functor $G\colon \mathbf{Gam} \longrightarrow \mathbf{G}^{\perp}(\mathbf{RRel})$ extends mutatis mutandis to a linearly distributive functor $G\colon \mathbf{GGam} \longrightarrow \mathbf{G}(\mathbf{RRel})$. For finite graph games this will factor through $\mathbf{G}^{(\perp)}(\mathbf{RRel})$ but not (generally) through $\mathbf{G}^{\perp}(\mathbf{RRel})$. However we can find a submodel of **GGam** whose finite members do get mapped into $\mathbf{G}^{\perp}(\mathbf{RRel})$. We say that a graph game is *regulated* if and only if whenever paths diverge at a P-position (O-position) then if they converge again they do so for the first time at a P-position (O-position). Let **RGam** be the full subcategory of **GGam** with objects the regulated games.

Proposition 38 **RGam** *is a model for intuitionistic linear logic.*

[19] Experience with sequential algorithms suggests a more sophisticated alternative.

We have reached our aim since $G\colon \mathbf{GGam} \longrightarrow \mathbf{G}(\mathbf{RRel})$ does restrict to a functor $G\colon \mathbf{RGam}_{\mathrm{fin}} \longrightarrow \mathbf{G}^{\perp}(\mathbf{RRel})$ and the connection between these concrete and abstract games is very close.

To make this precise we need to modify $\mathbf{G}^{\perp}(\mathbf{RRel})$ along familiar lines [12]. Given an object $((R_P, R_O), U, X)$ in $\mathbf{G}^{\perp}(\mathbf{RRel})$ it may well be the case that $\bigcup U$ and $\bigcup X$ are strictly contained in $R = R_P + R_O$; then certainly

$$R' = \bigcup\{u \cap x \mid u \in U, x \in X\}$$

which represents the set of results of plays is strictly contained in R. We wish to consider only $((R_P, R_O), U, X)$ with $R = R'$. Let

- $\mathbf{G}^{\mathrm{l}}(\mathbf{RRel})$ be the full subcategory consisting of objects $((R_P, R_O), U, X)$ with $R = \bigcup X$;
- $\mathbf{G}^{\mathrm{r}}(\mathbf{RRel})$ be the full subcategory consisting of objects $((R_P, R_O), U, X)$ with $R = \bigcup U$;
- $\mathbf{G}^{\sharp}(\mathbf{RRel})$ be the full subcategory consisting of objects $((R_P, R_O), U, X)$ with $R = \bigcup U = \bigcup X = R'$.[20]

There is a left adjoint $L\colon \mathbf{G}^{\perp}(\mathbf{RRel}) \longrightarrow \mathbf{G}^{\mathrm{l}}(\mathbf{RRel})$ to the inclusion functor $\mathbf{G}^{\mathrm{l}}(\mathbf{RRel}) \longrightarrow \mathbf{G}^{\perp}(\mathbf{RRel})$ and dually $R\colon \mathbf{G}^{\perp}(\mathbf{RRel}) \longrightarrow \mathbf{G}^{\mathrm{r}}(\mathbf{RRel})$, a right adjoint to the inclusion $\mathbf{G}^{\mathrm{r}}(\mathbf{RRel}) \longrightarrow \mathbf{G}^{\perp}(\mathbf{RRel})$; and we get the composite $T \cong LR \cong RL\colon \mathbf{G}^{\perp}(\mathbf{RRel}) \longrightarrow \mathbf{G}^{\sharp}(\mathbf{RRel})$. One checks readily that if $A = ((R_P, R_O), U, X)$ and $B = ((S_P, S_O), V, Y)$ are in $\mathbf{G}^{\mathrm{r}}(\mathbf{RRel})$ then so is $A \otimes B$. It follows by routine considerations that $\mathbf{G}^{\sharp}(\mathbf{RRel})$ is $*$-autonomous.

Now we can state the precise connection between concrete and abstract games.

Theorem 39 $G\colon \mathbf{RGam}_{\mathit{fin}} \longrightarrow \mathbf{G}^{\sharp}(\mathbf{RRel})$ *is fully, faithful and preserves the symmetric monoidal closed structure.*

Thus we have shown that the multiplicative structure of a category $\mathbf{RGam}_{\mathrm{fin}}$ of concrete games is exactly represented as a symmetric monoidal closed category within the abstract model $\mathbf{G}^{\sharp}(\mathbf{RRel})$. But $\mathbf{G}^{\sharp}(\mathbf{RRel})$ is $*$-autonomous and so contains by duality representations of the cogames[21] dual to the games of $\mathbf{RGam}_{\mathrm{fin}}$. Hence it is natural to ask how $\mathbf{G}^{\sharp}(\mathbf{RRel})$ handles the problem of composition of strategies when one has both games and cogames together. What happens is that if for example A is a cogame and B is a game then (in contrast with the Blass conventions [18]) there is no map in $\mathbf{G}^{\sharp}(\mathbf{RRel})$ from $G(A)$ to $G(B)$.

[20] The analogous subcategory of $\mathbf{G}^{\perp}(\mathbf{Rel})$ is exactly Loader's category of Totality Spaces [35].
[21] A cogame is just a game in which Player starts.

8 Further directions

The thrust of this paper is that one can arrive at computational models using abstract categorical machinery. We mention some further developments along these lines.

1) Our best results are restricted to finite graph games. One can do better by means of more sophisticated use of orthogonality.

2) The functor $G\colon \mathbf{RGam}_{\mathrm{fin}} \longrightarrow \mathbf{G}^{\sharp}(\mathbf{RRel})$ does not (quite) manage to preserve additives. Intuitively, the reason for this is clear: we have not taken due notice of the initial position in our abstract games. (This issue is already addressed in [10].)

3) One should explore how to arrive at exactly the usual simple category of games and also how to encapsulate more subtle notions of game (and strategy).

4) There are connections with sequential algorithms. In particular more sophisticated exponentials can be studied in an abstract setting.

5) Abstract games lend themselves to clean conceptual proofs of full abstraction and full completeness results.

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It will be clear that we have been influenced in a general way by colleagues who have worked on models for linear logic. Many have given readings of their models as abstract games. However, we would like to mention that a specific precursor of the work described here is a joint project with Robin Cockett on the analysis of sequentiality in the context of yet other categories of abstract games. We hope to report on this in the fullness of time.

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