A NOTE ON INTERPRETING INTUITIONISTIC HIGHER ORDER LOGIC

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INTRODUCTION. In some unpublished notes, [5], Powell gave a generalisation of the Boolean valued model construction of classical set theory. This generalisation gives models for intuitionistic set theories and use what Powell called complete Heyting filtered algebras (cHfa's for short) instead of Boolean algebras as in the classical construction. cHa's are special cases of cHfa's, and for these the construction was also carried out by Grayson in [2]. But the construction using cHfa's also include, as a special case, extensions to set theory of realisability interpretations of arithmetic. More recently Hyland, Johnstone and Pitts, in [3], have introduced the category theoretic notion of a tripos. This turns out to be closely related to, and at least as general as the notion of a cHfa. They show how to associate a topos with each tripos by generalising the construction of the topos of Ω-sets for any cHa Ω.

In this note I introduce the notion of a frame. Frames are essentially special sorts of cHfa's or triposes. But they are sufficiently general to include all natural examples of cHfa's that are known to me. I show how to construct a topos from a frame via a combination of fairly standard constructions connected with intuitionistic higher order logic. A variety of formulations of intuitionistic higher order logic have been considered in connection with topoi (see [1], [4], [6]). My formulation is essentially a standard many sorted intuitionistic higher order predicate logic without function symbols or other frills. A language for higher order logic specifies the ground sorts and constants. Given a frame Ω, an Ω-interpretation $\mathcal{A}$ of a language determines a semantics and hence a theory $Th(\mathcal{A})$. Each frame gives rise to a canonical interpretation $\mathcal{A}_\Omega$ of the universal language that has a ground sort for every inhabited set and a constant for every object in the set that interprets a sort. Next we give a syntactical version of an inner model construction for forming extensional structures from arbitrary ones. This associates with every theory T an extensional theory T^{ext}. Finally, as in [1], a topos $Top(T)$ is associated with any extensional theory T. Starting from a frame Ω the above constructions can be combined to give the topos $Top(Th(\mathcal{A}_\Omega)^{ext})$.

§1. HIGHER ORDER LOGIC.

1.1. A <u>language</u> for higher order logic consists of collections of <u>sorts</u> and <u>constants</u>, each constant being of a specified sort. The sorts are built up inductively from a collection of <u>ground</u> sorts using the rule: If $\sigma_1, \dots, \sigma_n$ $(n \geq 1)$ are sorts then $[\sigma_1, \dots, \sigma_n]$ is a sort. There is a ground sort $\Box$, a constant $\rightarrow$ of sort $[\Box, \Box]$ and for each sort σ a constant $\forall_\sigma$ of sort $[[\sigma]]$.

1.2. Associated with any language is a collection of <u>variables</u>, each of a specified sort, so that there are infinitely many variables of each sort. The collection of terms is inductively generated by rules (i) – (iii) below. Each term t is of a specified sort denoted $*t$.

(i) Every constant or variable is a term.

(ii) If φ is a term of sort $\Box$ and $x_1, \dots, x_n$ $(n \geq 1)$ is a pairwise distinct list of variables then $\{x_1, \dots, x_n \mid \varphi\}$ is a term of sort $[*x_1, \dots, *x_n]$.

(iii) If $t_1, \dots, t_n$ $(n \geq 1)$ are terms and t is a term of sort $[*t_1, \dots, *t_n]$ then $t(t_1, \dots, t_n)$ is a term of sort $\Box$.

1.3. Free and bound occurrences of variables are defined as usual. A term is <u>closed</u> if no variable occurs free in it. <u>Formulae</u> are terms of sort $\Box$. <u>Sentences</u> are closed formulae. $p, q, \dots$ will denote variables of sort $\Box$. $t[t_1, \dots, t_n / x_1, \dots, x_n]$ is the result of simultaneously substituting t_i for free occurrences of x_i in t for $i = 1, \dots, n$. The notation will only be used when there are no variable clashes, and $*t_i = *x_i$ for $i = 1, \dots, n$.

1.4. In the following abreviations p is not free in φ or in ψ, $\sigma = \# x$, $[\sigma] = \# y$, $\# t_1 = \# t_2 = \sigma$ and y is not free in t_1 or t_2.

Abreviations

$\varphi \to \psi$	$\to(\varphi, \psi)$
$\forall x \varphi$	$\forall_\sigma \{x \mid \varphi\}$
$\perp$	$\forall p\, p$
$\varphi \wedge \psi$	$\forall p[(\varphi \to (\psi \to p)) \to p]$
$\varphi \vee \psi$	$\forall p[(\varphi \to p) \to ((\psi \to p) \to p)]$
$\exists x \varphi$	$\forall p[\forall x(\varphi \to p) \to p]$
$\neg \varphi$	$(\varphi \to \perp)$
$\varphi \leftrightarrow \psi$	$(\varphi \to \psi) \wedge (\psi \to \varphi)$
$t_1 = t_2$	$\forall y\, [y(t_1) \leftrightarrow y(t_2)]$

1.5. Logical axioms

(i) $\varphi \to (\psi \to \varphi)$

(ii) $[\varphi \to (\psi \to \theta)] \to [(\varphi \to \psi) \to (\varphi \to \theta)]$

(iii) $\forall x \varphi \to \varphi[t/x]$

(iv) $\forall x (\theta \to \varphi) \to (\theta \to \forall x \varphi)$, x not free in θ

(v) $\{x_1, \dots, x_n \mid \varphi\}(x_1, \dots, x_n) \leftrightarrow \varphi$

Rules

$$\frac{\varphi \quad \varphi \to \psi}{\psi}$$ modus ponens

$$\frac{\varphi}{\forall x \varphi}$$ universal generalisation

1.6. If T is a set of sentences and there is a proof of the formula φ from sentences in T using the logical axioms and the rules write $T \vdash \varphi$. T is a <u>theory</u> if $T \vdash \varphi$ implies $\varphi \in T$ for every sentence φ. T is <u>consistent</u> if $T \nvdash \bot$.

§2. FRAMES.

2.1. Let ∇ be an upward closed subset of the complete lattice $(\Omega, \leq)$, with infinitary inf and sup operations $\bigwedge$ and $\bigvee$. A binary function $\Rightarrow$ on Ω is a <u>∇-implication</u> if

(i) $(\bigvee X \Rightarrow \bigwedge Y) = \bigwedge\{x \Rightarrow y \mid x \in X, y \in Y\}$
for all $X, Y \subseteq \Omega$.

(ii) If $a, a \Rightarrow b \in \nabla$ then $b \in \nabla$.

(iii) $\bigwedge\{a \Rightarrow (b \Rightarrow a) \mid a, b \in \Omega\} \in \nabla$.

(iv) $\bigwedge\{[a \Rightarrow (b \Rightarrow c)] \Rightarrow [(a \Rightarrow b) \Rightarrow (a \Rightarrow c)] \mid a, b, c \in \Omega\} \in \nabla$.

2.2. $\mathbf{\Omega} = (\Omega, \leq, \nabla, \Rightarrow)$ is a <u>frame</u> if ∇ is an upward closed subset of a complete lattice $(\Omega, \leq)$ and $\Rightarrow$ is a ∇-implication on Ω.

2.3. Frames are usually constructed via the following notion. Given $\Omega, \leq, \nabla$ as before, a perhaps partial binary function $\circ$ on Ω is a <u>∇-application</u> if

(i) $\bigvee X \circ \bigvee Y = \bigvee\{x \circ y \mid x \in X, y \in Y\}$,
if $x \circ y$ is defined for all $x \in X, y \in Y$, and is undefined otherwise, for all $X, Y \subseteq \Omega$.

(ii) If $x \circ y$ is defined and $x, y \in \nabla$ then $x \circ y \in \nabla$.

(iii) For some $k \in \nabla$ $(k \circ x) \circ y$ is defined and $(k \circ x) \circ y \leq x$ for all $x, y \in \Omega$.

(iv) For some $s \in \nabla$ $(s \circ x) \circ y$ is defined and whenever $(x \circ z) \circ (y \circ z)$ is defined then so is $((s \circ x) \circ y) \circ z$ and $((s \circ x) \circ y) \circ z \leq (x \circ z) \circ (y \circ z)$ for all $x, y, z \in \Omega$.

2.4. <u>Theorem</u>. If $\circ$ is a ∇-application then $\Rightarrow$ is a ∇-implication where

$$(a \Rightarrow b) \quad = \bigvee \{c \mid c \circ a \text{ is defined and } c \circ a \leq b\}.$$

Moreover every ∇-implication $\Rightarrow$ can be obtained in this way from the ∇-application $\circ$ where

$$a \circ b \quad = \bigvee \{c \mid a \leq (b \Rightarrow c)\}$$

whenever $a \leq (b \Rightarrow c)$ for some c, and is undefined otherwise.

2.5. The examples of frames treated in [3] or [5] are either cHa-frames or realisability frames. The <u>cHa frames</u> $(\Omega, \leq, \nabla, \Rightarrow)$ are complete Heyting algebras $(\Omega, \leq)$ with a lattice filter ∇ and Heyting algebra implication $\Rightarrow$. This is obtained from the binary inf operation $\wedge$, which is easily seen to be a ∇-application. The <u>realisability frames</u> $(\Omega, \subseteq, \nabla, \Rightarrow)$ are complete lattices $(\Omega, \subseteq)$ where Ω is the power set of a set Ω_0, ordered by inclusion. If the associated ∇-application has the property that $\{a\} \circ \{b\}$ is a singleton set whenever it is defined then it can always be obtained in the following way:

Let ∇ be any upward closed set of subsets of Ω_0, and let $\cdot$ be a possibly partial binary function on Ω_0 such that there are sets $K, S \in \nabla$ satisfying

(i) for all $k \in K$, $x, y \in \Omega_0$, $(k \cdot x) \cdot y$ is defined and $(k \cdot x) \cdot y = x$.

(ii) for all $s \in S$, $x, y \in \Omega_0$, $(s \cdot x) \cdot y$ is defined and whenever $(x \cdot z) \cdot (y \cdot z)$ is defined then so is $((s \cdot x) \cdot y) \cdot z$ and $((s \cdot x) \cdot y) \cdot z = (x \cdot z) \cdot (y \cdot z)$.

Then $\cdot$ determines a ∇-application $\circ$ on Ω given by

$$A \circ B \quad = \{ a \cdot b \mid a \in A, b \in B \}$$

whenever $a \cdot b$ is defined for all $a \in A$, $b \in A$, and is undefined otherwise, for all $A, B \subseteq \Omega_0$.

Ordinary realisability is the case when $\Omega_0 = \mathbb{N}$ and $m \cdot n \simeq \{m\}(n)$, where $\{m\}$ is the m^{th} unary partial recursive function in a standard enumeration.

§3. FRAME INTERPRETATIONS.

3.1. If Ω is a frame an <u>Ω-interpretation</u> $\mathcal{A}$ of a language is an assignment of an inhabited set $\mathcal{A}(\sigma)$ to each sort and an element $\mathcal{A}(c) \in \mathcal{A}(\sigma)$ to each constant c of sort σ. Such an assignment must satisfy

(i) $\mathcal{A}(\Box) = \Omega$.

(ii) $\mathcal{A}([\sigma_1, \ldots, \sigma_n])$ is the set of functions $\mathcal{A}(\sigma_1) \times \ldots \times \mathcal{A}(\sigma_n) \to \Omega$, for any sorts $\sigma_1, \ldots, \sigma_n$.

(iii) $\mathcal{A}(\to) = \Rightarrow$

(iv) $\mathcal{A}(\forall_\sigma) : \Omega^{\mathcal{A}(\sigma)} \to \Omega$ is given by
$$\mathcal{A}(\forall_\sigma)(f) = \bigwedge \{ f(x) \mid x \in \mathcal{A}(\sigma) \} \text{ for } f : \mathcal{A}(\sigma) \to \Omega,$$
for every sort σ.

3.2. If $\mathcal{A}$ is an Ω-interpretation of a language $\mathcal{L}$ then extend $\mathcal{L}$ to $\mathcal{L}\mathcal{A}$ by adding a new constant $\underline{a}^\sigma$ of sort σ to each object $a \in \mathcal{A}(\sigma)$ for each sort σ. Now define $\mathcal{A}(t) \in \mathcal{A}(*t)$

for each closed term t of $\mathcal{L}\mathcal{A}$ as follows:

t	$\mathcal{A}(t)$
c (constant of $\mathcal{L}$)	already defined
$\underline{a}^{\sigma}$ $(a \in \mathcal{A}(\sigma))$	a
$\{x_1,\dots,x_n \mid \varphi\}$ $(\# x_i = \sigma_i$ for $i=1,\dots,n)$	$f : \mathcal{A}(\sigma_1) \times \dots \times \mathcal{A}(\sigma_n) \to \Omega$ where $f(a_1,\dots,a_n) = \mathcal{A}(\varphi[\underline{a}_1^{\sigma_1},\dots,\underline{a}_n^{\sigma_n}/x_1,\dots,x_n])$ for $a_1 \in \mathcal{A}(\sigma_1),\dots,a_n \in \mathcal{A}(\sigma_n)$.
$t_0(t_1,\dots,t_n)$	$\mathcal{A}(t_0)(\mathcal{A}(t_1),\dots,\mathcal{A}(t_n))$.

3.3. A sentence φ of $\mathcal{L}\mathcal{A}$ is <u>true in $\mathcal{A}$</u> if $\mathcal{A}(\varphi) \in \nabla$.

$$\mathrm{Th}(\mathcal{A}) = \{\varphi \mid \varphi \text{ is a sentence of } \mathcal{L} \text{ true in } \mathcal{A}\}.$$

3.4. <u>Theorem.</u> $\mathrm{Th}(\mathcal{A})$ is always a theory.

3.5. Given a frame simultaneously define the <u>universal language</u> $\mathcal{L}_\Omega$ and its <u>canonical</u> Ω-interpretation $\mathcal{A}_\Omega$ as follows:

In addition to $\square$, $\mathcal{L}_\Omega$ has a ground sort σ_A for every inhabited set A, with $\mathcal{A}_\Omega(\sigma_A) = A$. For each sort σ of $\mathcal{L}_\Omega$ and each $a \in \mathcal{A}_\Omega(\sigma)$ $\mathcal{L}_\Omega$ has a constant c_a^σ of sort σ, with $\mathcal{A}_\Omega(c_a^\sigma) = a$.

§4. EXTENSIONALITY.

4.1. A theory T is <u>extensional</u> if each of the infinitely many extensionality axioms, given below, are theorems of T.

$E_\square$ $\quad (p \leftrightarrow q) \to (p = q)$

E_σ $\quad (y \approx z) \to (y = z)$ where $\# y = \# z = \sigma$ is not a ground sort

In E_σ $(y \approx z)$ is $(y \subseteq z) \wedge (z \subseteq y)$ where for terms t_1, t_2 of sort $\sigma = [\sigma_1, \dots, \sigma_n]$ $(t_1 \subseteq t_2)$ is

$$\forall x_1 \dots \forall x_n [t_1(x_1, \dots, x_n) \to t_2(x_1, \dots, x_n)]$$

4.2. Define a formula $(t_1 =_\sigma t_2)$ for each sort σ and terms t_1, t_2 of sort σ:

σ	$t_1 =_\sigma t_2$
ground sort $\neq \square$	$t_1 = t_2$
$\square$	$t_1 \leftrightarrow t_2$
$[\sigma_1, \dots, \sigma_n]$	$\forall \bar{x} \forall \bar{y} [\bar{x} =_{\bar{\sigma}} \bar{y} \to [t_1(\bar{x}) \leftrightarrow t_2(\bar{y})]]$

where $\bar{x} = x_1, \dots, x_n$ and $\bar{y} = y_1, \dots, y_n$ are lists of variables of the appropriate sorts that do not occur free in t_1 or t_2, $\bar{\sigma} = \sigma_1, \dots, \sigma_n$ and $\bar{x} =_{\bar{\sigma}} \bar{y}$ is $(x_1 =_{\sigma_1} y_1) \wedge \dots \wedge (x_n =_{\sigma_n} y_n)$.

4.3. Define a term t^* for each term t:

t	t^*
x	x
c (constant of ground sort)	c
c (constant of sort $[\sigma_1, \dots, \sigma_n]$	$\{\bar{y} \mid \exists \bar{x} (\bar{x} =_{\bar{\sigma}} \bar{y} \wedge c(\bar{x}))\}$
$\{\bar{x} \mid \varphi\}$	$\{\bar{y} \mid \exists \bar{x} (\bar{x} =_{\bar{\sigma}} \bar{y} \wedge \varphi^*)\}$
$t_0(t_1, \dots, t_n)$	$t_0^*(t_1^*, \dots, t_n^*)$

with obvious conditions on the choice of $\bar{y}$.

4.4. <u>Theorem</u>. For any theory T $T^{ext} = \{\varphi \mid \varphi^* \in T\}$ is an extensional theory, which is consistent if T is.

§5. THE TOPOS OF AN EXTENSIONAL THEORY.

5.1. Let T be an extensional theory. Closed terms that are not of ground sort will be called T-sets. For T-sets a, b let:

$$a =_T b \quad \text{iff} \quad T \vdash a = b$$

$$a \subseteq_T b \quad \text{iff} \quad T \vdash a \subseteq b$$

$$a \times b \quad \overline{\overline{\text{def}}} \quad \{\bar{x}, \bar{y} \mid a(\bar{x}) \wedge b(\bar{y})\}$$

$$f : a \to b \quad \text{iff} \quad f \subseteq_T a \times b \quad \text{and}$$

$$T \vdash a(\bar{x}) \to \exists! \bar{y}\, f(\bar{x}, \bar{y})$$

Given $f : a \to b$ and $g : b \to c$ define $g \circ f : a \to c$ by

$$g \circ f \;\overline{\overline{\text{def}}}\; \{\bar{x}, \bar{z} \mid \exists \bar{y}\, [f(\bar{x}, \bar{y}) \wedge g(\bar{y}, \bar{z})]\}$$

Note that $\iota_a : a \to a$ where $\iota_a = \{\bar{x}, \bar{y} \mid a(\bar{x}) \wedge a(\bar{y}) \wedge \bar{x} = \bar{y}\}$.

5.2. The category $\mathcal{T}op(T)$ has as objects the equivalence classes $[a]_T = \{b \mid a =_T b\}$ of T-sets a. A map from $[a]_T$ to $[b]_T$ is a triple $([a]_T, [f]_T, [b]_T)$ such that $f : a \to b$. Composition and identity maps are defined in the obvious way

5.3. Theorem. For every extensional theory T $\mathcal{T}op(T)$ is a topos. Moreover every topos is equivalent to one of the form $\mathcal{T}op(T)$.

§6. CONCLUSION.

Each frame Ω determines the topos $\mathcal{T}op(\mathrm{Th}(\mathcal{A}_\Omega)^{ext})$. I believe that this is essentially the same construction as that of the construction of a topos from a tripos that is carried out in [3].

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