Integrating Linear Arithmetic into Superposition Calculus

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Abstract. We present a method of integrating linear rational arithmetic into superposition calculus for first-order logic. One of our main results is completeness of the resulting calculus under some finiteness assumptions.

1 Introduction

In this paper we consider superposition calculus extended with rules for rational linear arithmetic such as Gaussian Elimination for reasoning with equality and Fourier-Motzkin Elimination for reasoning with inequalities. These rules are similar to superposition and ordered chaining rules in first-order reasoning.

There are a number of approaches to integrate arithmetical reasoning into superposition calculus. Most of these approaches are based on approximation of arithmetical reasoning by considering an axiomatisable theory such as Abelian groups or divisible Abelian groups [4, 12–14]. Although this provides a sound approximation it is generally not complete w.r.t. reasoning in usual arithmetical structures such as rational numbers $\mathbb{Q}$. In our approach we consider $\mathbb{Q}$ as a fixed theory sort in the signature containing theory symbols $+, >, =$ together with non-theory sorts and function symbols. We present a sound Linear Arithmetic Superposition Calculus (LASCA) for this language based on a standard superposition calculus extended with rules for linear arithmetic. As we show, the validity problem for first-order formulas of linear arithmetic extended with non-theory function symbols is Π^1_1-complete even in the case when there are no variables over the theory sort. Therefore, there is no sound and complete calculus for this logic. Nevertheless, one of the main results of this paper is that under some finiteness assumptions it is possible to show completeness of our calculus. In particular, we can show that a finite saturated set of clauses (with variables over non-theory sorts) S is satisfiable if and only if S does not contain the empty clause. For this, we need to assume that a simplification ordering we use in our calculus is finite-based (a notion defined later in the paper). In this paper we also show how to construct such an ordering.

Our calculus LASCA is closely related to [4, 13], but here we are dealing directly with the structure $\mathbb{Q}$ rather than with axiomatisations. One of the differences with [13] is that we do not apply abstraction for theory terms. Such abstraction introduces new variables and can increase the number of inferences. On the other hand, in order to show our completeness result we impose additional restrictions on the ordering and variable occurrences. In our completeness proof we adapt the model generation technique

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(see [2, 9]). We use some ideas from normalised rewriting, symmetrisation [4, 7, 8] and many-sorted reasoning [3, 5].

2 Preliminaries

We consider a many-sorted language. Let Σ be a signature consisting of a non-empty set of sorts $\mathcal{S}$, a set of function symbols $\mathcal{F}$, a set of predicate symbols $\mathcal{P}$ and an arity function $arity : \mathcal{F} \cup \mathcal{P} \to \mathcal{S}^+$, where $\mathcal{S}^+$ denotes the set of finite non-empty sequences of sorts. For a function symbol f with arity $arity(f) = \langle s_0, \ldots, s_n \rangle$, we call $s_0, \ldots, s_{n-1}$ *argument sorts* and s_n the *value sort* of f. In this paper we are mainly dealing with extensions of rational arithmetic. We write $\Sigma_{\mathbb{Q}}$ for a signature such that $\mathcal{S}_{\mathbb{Q}}$ consists of a designated *theory sort* $s_{\mathbb{Q}}$ of rationals, *theory predicate symbols* $\mathcal{P}_{\mathbb{Q}} = \{>, =\}$, and *theory function symbols* $\mathcal{F}_{\mathbb{Q}} = \{+\} \cup \{q, \cdot_q | q \in \mathbb{Q}\}$ where $\mathbb{Q}$ is the set of rationals. We assume that Σ extends $\Sigma_{\mathbb{Q}}$ with non-theory sorts and non-theory function symbols (note that non-theory functions can have arguments and values of the theory sort $s_{\mathbb{Q}}$). We assume that the only non-theory predicates in Σ are equalities on non-theory sorts, denoted as $\simeq_s$, we also write $\simeq$ if there is no confusion, and we use $=$ for equality over the theory sort $s_{\mathbb{Q}}$. Variables, terms, atoms, literals, clauses and first-order formulas are defined in the standard way. We use the standard semantics for many-sorted logic: a Σ-structure consists of a disjoint union of domains indexed by sorts with defined functions and predicates respecting their arities. In addition, we always assume that the domain of the theory sort $s_{\mathbb{Q}}$ is the rational numbers $\mathbb{Q}$ with the usual interpretation of $>, =, +$ and where elements of $\mathbb{Q}$ are also constants in our language and $\cdot_q$ is a unary function symbol interpreted as multiplication by q for each $q \in \mathbb{Q}$. We use convenient abbreviations qt for $\cdot_q(t)$ where t is a term of sort $s_{\mathbb{Q}}$ and $-t$ for $-1t$. We use $\bowtie$ to denote one of theory predicates $>$ or $=$.

We are interested in the question of whether a given first-order formula is (un)satisfiable in a Σ-structure. This question can be reformulated in a standard way as a question of (un)satisfiability of sets of clauses in a Herbrand interpretation which is defined later.

A non-variable term is called a *theory term* (*non-theory term*) if its top function symbol is a theory symbol (non-theory symbol respectively) and similarly for atoms. We assume that $>$ and $=$ occur only positively in clauses (for example $\neg(t > s)$ can be replaced by $s > t \vee s = t$ and $\neg(t = s)$ by $t > s \vee s > t$).

$\mathbb{Q}$-Normalised terms. Define a relation $=_{AC}$ on terms, called AC-congruence, as the least congruence relation generated by associativity and commutativity axioms for $+$. We assume $+$ to be variadic and define $\mathbb{Q}$-normalised terms as follows.

Definition 1. A term t is $\mathbb{Q}$-*normalised* if t is either:

1. a theory constant q, or
2. a non-theory term $f(t_1, \ldots, t_n)$ where $t_1, \ldots, t_n$ are $\mathbb{Q}$-normalised, or
3. $q_1 t_1 + \cdots + q_n t_n$ where $n \geq 1$, and for each $1 \leq i \leq n$, the term t_i is a $\mathbb{Q}$-normalised non-theory term, $q_i \neq 0$ and $t_i \neq_{AC} t_j$ for $i \neq j$, and
4. $q_1 t_1 + \cdots + q_n t_n + q$ where n and q_i, t_i for $1 \leq i \leq n$ are as in 3 above and $q \neq 0$.

It is not hard to argue that for every ground term t there is a unique, up to AC-congruence, $\mathbb{Q}$-equivalent term which is $\mathbb{Q}$-normalised. This term is called a $\mathbb{Q}$-*normal form* of t and denoted by $t\downarrow_{\mathbb{Q}}$. We say that s is an AC-subterm of a $\mathbb{Q}$-normalised term t if either: (i) $t =_{AC} s$, or (ii) $t = f(t_1, \ldots, t_n)$ and s is an AC-subterm of t_i for some $1 \leq i \leq n$, where f is a non-theory function symbol, or (iii) $t = qt'$ and s is an AC-subterm of t', or (iv) $t =_{AC} u + v$ and s is an AC-subterm of u or v. For example, $3d + 5a$ is an AC-subterm of $4f(5a + 2b + 3d)$.

In this paper we deal with orderings satisfying several properties defined below.

Definition 2. Let $\succ$ be an ordering on $\mathbb{Q}$-normalised terms. It is said to have a *subterm property* if $s[t] \succ t$ whenever t is a proper AC-subterm of $s[t]$. We say that $\succ$ is *AC-compatible* if it satisfies the following property: if $s \succ t$, $s =_{AC} s'$ and $t =_{AC} t'$, then $s' \succ t'$. We say that $\succ$ is $\mathbb{Q}$-*monotone* if for any $\mathbb{Q}$-normal form $t[s]$ where s is a non-theory term, from $s \succ s'$, it follows that $t[s] \succ (t[s'])\downarrow_{\mathbb{Q}}$.

An ordering $\succ$ is called $\mathbb{Q}$-total, if for all ground $\mathbb{Q}$-normal forms s, t, if $s \neq_{AC} t$, then either $s \succ t$ or $t \succ s$.

We say that an ordering $\succ$ has a *sum property* if for any non-theory term t and any finite family of non-theory terms $s_1, \ldots, s_n$ of sort $\mathbb{Q}$, such that $t \succ s_i$ for $1 \leq i \leq n$, it follows that $t \succ (q_1 s_1 + \cdots + q_n s_n + q)\downarrow_{\mathbb{Q}}$ for any coefficients $q_1, \ldots, q_n, q \in \mathbb{Q}$. $\Box$

From now on $\succ$ will denote an AC-compatible, $\mathbb{Q}$-monotone, $\mathbb{Q}$-total and well-founded ordering on $\mathbb{Q}$-normal forms which has sum and subterm properties. We show an example of such an ordering in Section 5. We use $\succeq$ for $\succ \cup =_{AC}$.

Let t be a $\mathbb{Q}$-normalised ground term of sort $\mathbb{Q}$, then the *leading monomial* m of t is defined as follows: if t is a theory constant then $m = t$, otherwise m is the greatest w.r.t. $\succ$ non-theory subterm of t. Let $\top$ denote the literal $0 = 0$ and $\bot$ the literal $0 > 1$. We call a ground literal L $\mathbb{Q}$-*normalised* if L is of one of the forms $l = s$, $l > s$, $-l > s$, $l \simeq r$, $l \not\simeq r$, $\top$, $\bot$ where l is a non-theory term and $l \succ r$, we also call l the *leading term* of L. A clause is $\mathbb{Q}$-normalised if all of its literals are $\mathbb{Q}$-normalised and the leading term of a clause is the greatest leading term of its literals. It is easy to see that every ground clause can be $\mathbb{Q}$-normalised into an equivalent clause. From now on we consider only $\mathbb{Q}$-normalised ground terms, literals and clauses.

In order to extend the ordering $\succ$ to literals we represent literals as multisets as follows $m(t = s) = \{t, s\}, m(t > s) = \{t, t, s, s\}, m(t \simeq s) = \{t, s\}, m(t \not\simeq s) = \{t, t, s, s\}$. Now we define $L \succ L'$ iff $m(L) \succ_m m(L')$ where $\succ_m$ is the multiset extension of $\succ$. We compare clauses in the two-fold multiset extension of $\succ$.

Herbrand Interpretation. An *evaluation function* is a mapping from ground non-theory terms of sort $s_{\mathbb{Q}}$ into $\mathbb{Q}$. Let ν be an evaluation function, then define $\bar{\nu}$ to be an extension of ν to the theory terms as follows: $\bar{\nu}(q_1 t_1 + \cdots + q_n t_n + q) = q_1 \nu(t_1) + \cdots + q_n \nu(t_n) + q$.

In order to define a Herbrand interpretation we need a congruence relation $\sim$ on ground $\mathbb{Q}$-normalised terms and an evaluation function ν, such that the following compatibility conditions are satisfied.
Compatibility Conditions:

1. If $t \sim s$ then $\nu(t) = \nu(s)$, for any non-theory terms t, s of sort $s_{\mathbb{Q}}$.
2. If $\bar{\nu}(u) = \bar{\nu}(v)$ then $u \sim v$, for any terms u, v of sort $s_{\mathbb{Q}}$.

We call a pair $\langle \nu, \sim \rangle$, satisfying Compatibility Conditions above, a *Herbrand interpretation*. A theory atom $t \bowtie s$ is true in $\langle \nu, \sim \rangle$ if $\mathbb{Q} \models \bar{\nu}(t) \bowtie \bar{\nu}(s)$, and otherwise false in $\langle \nu, \sim \rangle$. A non-theory atom $t \simeq s$ is true in $\langle \nu, \sim \rangle$ if $t \sim s$, and otherwise false in $\langle \nu, \sim \rangle$.

3 The calculus for ground clauses

The inference rules of our Linear Arithmetic Superposition Calculus (LASCA) are presented in Table 1 (page 15). We assume that all inference rules are applied to $\mathbb{Q}$-normalised clauses and after application of an inference rule we implicitly $\mathbb{Q}$-normalise the conclusion. Note that if we write, e.g., $C \vee l = r$ then implicitly $l \succ r$, since the clause is assumed to be $\mathbb{Q}$-normalised. For a term t, we write $t \succeq C$ ($t \succ C$)if $t \succeq s$ ($t \succ s$) for any term s in C and similarly for literals. For a non-theory term l of sort $s_{\mathbb{Q}}$, we use $\pm l$ to denote l or $-l$, and assume that the choice of the sign is the same for a context, e.g., a rule and its conditions (we use $\mp$ to refer to the opposite sign).

Theorem 1. *Linear Arithmetic Calculus is sound: if the empty clause is derivable in LASCA from S then S is unsatisfiable.*

We say that a set of clauses S is saturated (w.r.t. LASCA) if S is closed under all inferences in LASCA. As we will see in Section 6 there is no sound and complete calculus for Linear Arithmetic extended with non-theory functions. Hence, our LASCA calculus is also incomplete in general: a saturated set of clauses S such that $\square \notin S$ can still be unsatisfi able. Let us characterise some cases when from the fact that the set S is saturated and $\square \notin S$ it follows that S is indeed satisfi able.

Definition 3. Let M be a set of terms or clauses. We say that M satisfi es *Finiteness of Coefficients* condition if the following holds. There exists a fi nite set of coeffi cients P such that if a term qt or q is a subterm of a term in M then $q \in P$. $\square$

In the sequel we impose the following assumption on sets of clauses.

Assumption 1 *Let S be a set of clauses. We assume that S satisfies Finiteness of Coefficients condition.*

Let us note that under Assumption 1, the number of occurrences of a non-theory term (or a theory constant) in S can be infi nite. In Section 4 we show that the set of all ground instances of a fi nite set of clauses with variables over variable-safe sorts, satisfi es Assumption 1. This will be used to show that if a fi nite set S of (possibly non-ground clauses) is saturated, then S is satisfi able if and only if $\square \notin S$ (Theorem 3).

Definition 4. Consider a fi nite set of coeffi cients P, then $\mathbb{T}^P$ denotes the set of all $\mathbb{Q}$-normalised terms t such that any non-theory subterm of t of sort $s_{\mathbb{Q}}$ occurs in t with coeffi cients from P. An ordering on $\mathbb{Q}$-normalised terms is called *finite-based* if for any fi nite set of coeffi cients P and any ground term t the set of all terms in $\mathbb{T}^P$ less than t is fi nite. $\square$

Assumption 2 *The ordering $\succ$ is finite-based.*

In Section 5 we show how to construct an appropriate ordering satisfying Assumption 2. Now we will show how to construct a candidate model $\langle \nu, \sim \rangle$ for a set of clauses S such that under Assumptions (1,2) if S is saturated and $\square \notin S$ then S is true in $\langle \nu, \sim \rangle$.

Model Construction. For simplicity of exposition we consider the case when all functions have arguments and values in $\mathbb{Q}$. Let S be a set of ground clauses satisfying Finiteness of Coefficients Assumption 1. We consider terms modulo AC-congruence, and in particular all rewrite rules are implicitly applied modulo AC. Denote T_S the set of all AC-subterms of terms occurring in S and T_S^{nth} all non-theory AC-subterms of terms in S. Note that T_S and T_S^{nth} satisfy Finiteness of Coefficients condition. An equation $l = r$, where $l \succ r$ and l is a non-theory term, can be seen as a rewrite rule $l \to r$, replacing l with r (and applying $\mathbb{Q}$-normalisation to the resulting term). Any system R of such rules is terminating, and if the left-hand sides of any two rules in R are not overlapping then the system is also convergent. Let us construct a rewriting system R and an evaluation function ν for all terms in T_S^{nth}. The evaluation function ν will be represented via a convergent term rewriting system Υ such that the following holds: (i) Υ consists of rules of type $f(q_1, \ldots, q_n) \to q$, where f is a non-theory function symbol $q_1, \ldots, q_n, q \in \mathbb{Q}$, (ii) $R \cup \Upsilon$ is a convergent term rewriting system. We say that a term t is *evaluated* by Υ if $t \downarrow_\Upsilon \in \mathbb{Q}$. We construct R and Υ by induction on terms in T_S^{nth} ordered by $\succ$ as follows. For each term $l \in T_S^{nth}$ we define a set of rewrite rules ϵ_l and a set of evaluation rules δ_l. We define $R_l = \cup_{l \succ t \in T_S^{nth}} \epsilon_t$; $R^l = R_l \cup \epsilon_l$; $\Upsilon_l = \cup_{l \succ t \in T_S^{nth}} \delta_t$; $\Upsilon^l = \Upsilon_l \cup \delta_l$.

Consider a term l in T_S^{nth}. We inductively assume that we have constructed ϵ_t, δ_t for every $t \prec l, t \in T_S^{nth}$ such that the following invariants hold.

Invariants (Inv):

1. either $\epsilon_t = \emptyset$, or $\epsilon_t = \{t \to r\}$ where $t \succ r, r \in T_S$, and
2. either $\delta_t = \emptyset$, or $\delta_t = \{f(q_1, \ldots, q_n) \to q\}$ and $t = f(t_1, \ldots, t_n)$, where $t_i \in T_S, q, q_i \in \mathbb{Q}$ for $1 \leq i \leq n$, $0 \leq n$, and
3. R^t, Υ^t and $R^t \cup \Upsilon^t$ are convergent term rewriting systems, and
4. t is evaluated by Υ^t, and
5. if t is R_t-irreducible then t is not evaluated by Υ_t, and
6. if t is R^t-irreducible then for any $u, v \in T_S$ such that t is the leading monomial of u, u is R^t-irreducible and $u \succ v$, we have $u \downarrow_{\Upsilon^t} \neq v \downarrow_{\Upsilon^t}$ (note $u \downarrow_{\Upsilon^t}, v \downarrow_{\Upsilon^t} \in \mathbb{Q}$ by *Inv* 4).

Let us note that since $\succ$ is finite-based, there are only a finite number of terms less than l in T_S^{nth}. Therefore $R_l = R^{l'}$ and $\Upsilon_l = \Upsilon^{l'}$ for some $l' \prec l$. We also have that R_l and Υ_l are finite. Now we show how to define ϵ_l, δ_l.

Consider the case when l can be reduced by R_l. If l is evaluated by Υ_l then we define $\epsilon_l = \delta_l = \emptyset$. If l is not evaluated by Υ_l, then $l = f(t_1, \ldots, t_n)$ for a non-theory symbol f. We have $f(t_1 \downarrow_{\Upsilon_l}, \ldots, t_n \downarrow_{\Upsilon_l}) = f(q_1, \ldots, q_n)$ for some $q_i \in \mathbb{Q}$, $1 \leq i \leq n$, (since $l \succ t_i$ we have that all t_i are evaluated by Υ_l). Let us show that $f(q_1, \ldots, q_n)$ does not occur in the left-hand sides of rules in R_l. Indeed, otherwise, $f(q_1, \ldots, q_n) \in T_S^{nth}$ and $l \succ f(q_1, \ldots, q_n)$, therefore $f(q_1, \ldots, q_n)$ and l would be evaluated by Υ_l. Now we define $\epsilon_l = \emptyset$ and $\delta_l = \{f(q_1, \ldots, q_n) \to q\}$ where $q \in \mathbb{Q}$ is selected arbitrary. It is straightforward to check that ϵ_l and δ_l satisfy all invariants above.

Now we assume that l is irreducible by R_l.

Claim. Let us show that l is not evaluated by Υ_l. Let $l = f(t_1, \ldots, t_n)$, then $f(t_1 \downarrow_{\Upsilon_l}, \ldots, t_n \downarrow_{\Upsilon_l}) = f(q_1, \ldots, q_n)$. Assume that l is evaluated, then $f(q_1, \ldots, q_n) \to q \in \Upsilon_l$ for some $q \in \mathbb{Q}$. Consider $s \in T_S^{nth}$, such that $l \succ s$ and $\delta_s = \{f(q_1, \ldots, q_n) \to q\}$. We have $s = f(s_1, \ldots, s_n)$ for some terms $s_i \in T_S$, $1 \leq i \leq n$ (see *Inv* 2). Since $l \succ s$, from monotonicity of $\succ$ it follows that $t_i \succ s_i$ for some $1 \leq i \leq n$. Let $t_i = \alpha_1 u_1 + \ldots + \alpha_k u_k + \alpha_{k+1}$ and $s_i = \beta_1 v_1 + \ldots + \beta_m v_m + \beta_{m+1}$ where we assume summands are ordered in a descending order (w.r.t. $\succ$). Let j be the smallest index such that $\alpha_j u_j \neq_{AC} \beta_j v_j$. If $j = k + 1$ then $m = k$ and $\alpha_{k+1} \neq \beta_{m+1}$, we obtain a contradiction: $0 = t_i \downarrow_{\Upsilon_l} - s_i \downarrow_{\Upsilon_l} = \alpha - \beta \neq 0$. If $j \leq k$ then $\alpha_j u_j \succ \beta_p v_p$ for $j \leq p \leq m + 1$. Since u_j is irreducible w.r.t. R_l (and therefore w.r.t. R^{u_j}) from *Inv* 6 it follows that $(\alpha_j u_j + \ldots + \alpha_k u_k + \alpha_{k+1}) \downarrow_{\Upsilon^{u_j}} \neq (\beta_j v_j + \ldots + \beta_m v_m + \beta_{m+1}) \downarrow_{\Upsilon^{u_j}}$. But then $t_i \downarrow_{\Upsilon_l} \neq s_i \downarrow_{\Upsilon_l}$, which is a contradiction.

We say that a literal $\pm l' \bowtie t$ with the leading term $l' \prec l$ is *true* w.r.t. Υ_l if $\mathbb{Q} \models \pm l' \downarrow_{\Upsilon_l} \bowtie t \downarrow_{\Upsilon_l}$ and *false* otherwise (note that $l' \downarrow_{\Upsilon_l}, t \downarrow_{\Upsilon_l} \in \mathbb{Q}$).

Let S^l be the set of all clauses in S with the leading term l. For a clause $C \in S^l$ define V_C^l, D_C^l such that $C = V_C^l \vee D_C^l$ and V_C^l consists of all literals in C with the leading term l (note D_C^l can be empty). We say that a clause $C \in S^l$, $C = C' \vee l = r$ *weakly produces* a rewrite rule $l \to r$, if the following holds.

- $l = r$ is a strictly maximal literal in C, and
- D_C^l is false w.r.t. Υ_l, and
- there is no $l = r' \in C'$ such that $\mathbb{Q} \models r \downarrow_{\Upsilon_l} = r' \downarrow_{\Upsilon_l}$.

If there is a clause in S^l weakly producing a rewrite rule then we take the smallest (w.r.t. $\succ$) such clause C. Let $l \to r$ be the rewrite rule weakly produced by C, then we say that $l \to r$ is *produced* by C. We define $\epsilon_l = \{l \to r\}$ and $\delta_l = \{l \downarrow_{\Upsilon_l} \to r \downarrow_{\Upsilon_l}\}$. Now we check that all *Inv* are satisfied. It follows immediately from the construction that *Inv* (1,2,4,5,6) are satisfied. Let us show that $R^l \cup \Upsilon^l$ is convergent. First we note that there are no critical pairs between $l \to r$ and R_l. Indeed, l is irreducible by R_l and l is greater (w.r.t. $\succ$) than all left-hand sides of rules in R_l. Likewise, from the Claim above it follows that that l is not evaluated by Υ_l and therefore there are no critical pairs between $l \downarrow_{\Upsilon_l} \to r \downarrow_{\Upsilon_l}$ and rules in Υ_l. The only new critical pairs possible are between $l \to r$ and rules in Υ^l, but they are joinable since $l \downarrow_{\Upsilon^l} = r \downarrow_{\Upsilon^l}$.

Now we assume that there is no clause in S^l producing a rewrite rule. We define $\epsilon_l = \emptyset$, and now we need to find an appropriate evaluation for l. Let us fix a numerical variable x_l. We say that a clause $C \in S^l$, $C = C' \vee \pm l > r$ *weakly produces* a bound $\pm x_l > r \downarrow_{\Upsilon_l}$, if the following holds.

- $\pm l > r$ is a strictly maximal literal in C, and
- D_C^l is false in Υ_l, and
- there is no literal $\pm l > r'$ in C', and
- if there is a literal $\mp l > r'$ in C', then $\mathbb{Q} \models r \downarrow_{\Upsilon_l} \geq -r' \downarrow_{\Upsilon_l}$.

Let B^l be the set of all bounds weakly produced by clauses in S^l, (B^l can be the empty set). It is not difficult to see that Assumptions 1 and 2 imply that B^l is finite. Let B_-^l be the set of lower bounds in B^l (i.e. bounds of the type $x_l > q$) an B_+^l be the set of upper bounds in B^l (i.e. bounds of the type $-x_l > q$). We have $B^l = B^l_- \cup B^l_+$. Since B^l is finite we have that each B^l_- and B^l_+ are satisfiable. Let $x_l > q_{glb}$ be the greatest w.r.t. $>$ lower bound in B^l_-, (since B^l_- is finite such a bound always exists). Let U^l be the set of upper bounds $-x_l > q$ in B^l_+ such that $-q_{glb} > q$. Define $B^l_\pm = B^l_- \cup U^l$. We have $B^l_\pm$ is satisfiable and the set of solutions to $B^l_\pm$ is an open interval. Moreover, if $B^l_+ \neq U^l$ then B^l_- together with any bound from $B^l_+ \setminus U^l$ is unsatisfiable. Clauses weakly producing bounds in $B^l_\pm$ are called *productive*.

In order to satisfy *Inv* 6 we impose additional constraints on evaluation of l defined below. We say that a pair of terms $u, v \in T_S$, such that l is the leading monomial of u and $u \succ v$ produces a disequality constraint d_{uv} if the following holds. Assume that $u = \alpha l + u'$, $\alpha \neq 0$. If l is not a subterm of v and therefore $l \succ v$, then $d_{uv} = \{x_l \neq (v \downarrow_{\Upsilon_l} - u' \downarrow_{\Upsilon_l})/\alpha\}$ (note that $u' \downarrow_{\Upsilon_l}, v \downarrow_{\Upsilon_l} \in \mathbb{Q}$). If l is a subterm of v then $v = \beta l + v'$ and we need to consider the following possible cases. Case (i): $\beta = \alpha$. Then we have $u' \succ v'$ and we can apply *Inv* 6 to the leading term of u', obtaining $u' \downarrow_{\Upsilon_l} \neq v' \downarrow_{\Upsilon_l}$. In this case we have that under any evaluation of l, evaluation of u will be different from evaluation of v and therefore we define $d_{uv} = \emptyset$. Case (ii): $\beta \neq \alpha$. Then we define $d_{uv} = \{x_l \neq (v' \downarrow_{\Upsilon_l} - u' \downarrow_{\Upsilon_l})/(\alpha - \beta)\}$. We define D^l to be the union of all d_{uv} where $u, v \in T_S$, l is the leading monomial of u and $u \succ v$. From Assumptions (1, 2) it follows that D^l is finite, therefore D^l is satisfied by all but possible a finite number of rationals. We have $B^l_\pm \cup D^l$ is satisfiable. Define $\delta_l = \{l \downarrow_{\Upsilon_l} \rightarrow q\}$, where q is any rational satisfying $B^l_\pm \cup D^l$. It is straightforward to check that all *Inv* are satisfied by ϵ_l, δ_l.

We have shown how to construct ϵ_l, δ_l for every $l \in T^{nth}_S$. Now we define $R_S = \cup_{l \in T^{nth}_S} \epsilon_l$ and $\Upsilon_S = \cup_{l \in T^{nth}_S} \delta_l$. We have $R_S \cup \Upsilon_S$ is a convergent term rewriting system such that every term in T^{nth}_S is evaluated by Υ_S. Finally we need to extend evaluation Υ_S to all non-theory terms. We can do it by induction over all non-theory terms as follows. For each term t we define a set of evaluation rules κ_t as follows. Assume, by induction, that we have defined κ_s for non-theory terms $s \prec t$. Define $\Lambda_t = \Upsilon_S \bigcup \cup_{t \succ s} \kappa_s$. If t is evaluated by Λ_t then we define $\kappa_t = \emptyset$, otherwise we define $\kappa_t = \{t \downarrow_{\Lambda_t} \rightarrow q\}$ where $q \in \mathbb{Q}$ is selected arbitrary. Define $\Lambda^t = \Lambda_t \cup \kappa_t$ and $\Lambda_S = \cup \Lambda^t$. It is not difficult to check that $R_S \cup \Lambda_S$ is a convergent term rewriting system such that every non-theory term is evaluated by Λ_S.

Let us define a Herbrand interpretation $\langle \nu, \sim \rangle$, where $\nu(t) = t \downarrow_{\Lambda_S}$ and $t \sim s$ iff $t \downarrow_{\Lambda_S} = s \downarrow_{\Lambda_S}$. We call $\langle \nu, \sim \rangle$ the *candidate model* for S. □

Lemma 1. *In the Model Construction above if a clause C is productive then C is true in the candidate model $\langle \nu, \sim \rangle$.*

Proof. Immediately follows from the Model Construction. □

Lemma 2. *In the Model Construction above if a clause $C = C' \vee \pm l \bowtie r$ produces a rule or a bound $\pm l \bowtie r$ then C' is false in the candidate model $\langle \nu, \sim \rangle$.*

Proof. Consider first when $\pm l \bowtie r$ is $l = r$ and C generates the rule $l \rightarrow r$. We have $C' = V^l_{C'} \vee D^l_{C'}$ where $V^l_{C'}$ consists of all literals in C' with the leading term l. From the conditions on productiveness of C we have that $D^l_{C'}$ is false in $\langle \nu, \sim \rangle$. From the definition of the ordering on atoms $V^l_{C'}$ does not contain any atoms with $>$. If $V^l_{C'}$

contains an atom $l = r'$ then we have $r \downarrow_{\Upsilon_S} \neq r' \downarrow_{\Upsilon_S}$ and $l \downarrow_{\Upsilon_S} = r \downarrow_{\Upsilon_S}$ therefore $l = r'$ is false in $\langle \nu, \sim \rangle$.

Now we consider the case when $\pm l \bowtie r$ is $\pm l > r$ and C produces the bound $\pm x_l > r \downarrow_{\Upsilon_S}$. We have that $D^l_{C'}$ is false in $\langle \nu, \sim \rangle$. If $V^l_{C'}$ contains an atom $l = r'$, then by construction l is irreducible w.r.t. R^l. Since $l \succ r'$, *Inv* 6 implies that $l \downarrow_{\Upsilon_S} \neq r' \downarrow_{\Upsilon_S}$. If $V^l_{C'}$ contains an atom $\mp l > r'$ then we have $\pm l \downarrow_{\Upsilon_S} > r \downarrow_{\Upsilon_S} \geq -r' \downarrow_{\Upsilon_S}$ and therefore $r' \downarrow_{\Upsilon_S} > \mp l \downarrow_{\Upsilon_S}$ implying that $\mp l > r'$ is false. Also, by conditions on productiveness, there is no atom $\pm l > r'$ in $V^l_{C'}$. We have shown that all atoms in $V^l_{C'}$, and therefore in C', are false in $\langle \nu, \sim \rangle$. □

Theorem 2. *LASCA is complete under Assumptions (1,2). Let S be a set of ground clauses such that Assumptions (1,2) are satisfied. If S is saturated and $\square \notin S$ then S is true in the candidate model $\langle \nu, \sim \rangle$.*

Proof. Let S be a saturated set of clauses satisfying Assumption 1. We apply Model Construction above to obtain $R_S, \Upsilon_S, \Lambda_S$ and the candidate model $\langle \nu, \sim \rangle$. In order to show that $\langle \nu, \sim \rangle$ satisfies all clauses in S it is sufficient to show that Υ_S satisfies all clauses in S. Assume otherwise. Let C be the smallest clause in S that is false under Υ_S. Let $C = C' \vee \pm l \bowtie r$, where $\pm l \bowtie r$ be a maximal literal in C. First we show that l is irreducible by R_S. Indeed, assume that $l[l']$ is reducible by a rule $l' \to r'$. Consider the clause $D = D' \vee l' = r'$ producing $l' \to r'$. Then, there is an inference by Gaussian Elimination with the premise C, D and the conclusion $G = D' \vee C' \vee \pm l[r'] \bowtie r$. We have that $C \succ G$ and from Lemma 2 it follows that G is false in Υ_S. This contradicts minimality of C.

By Lemma 1 all productive clauses are true in Υ_S, therefore we assume that C is not productive. Consider possible cases.

Case (1): $C = C' \vee l = r$. If C is not weakly productive then $C' = C'' \vee l = r'$ and $r \downarrow_{\Upsilon_S} = r' \downarrow_{\Upsilon_S}$. Therefore, inference rule Theory Equality Factoring is applicable to C with the conclusion $D = C'' \vee r > r' \vee r' > r \vee l = r'$. We have $C \succ D$ and D is false in Υ_S, contradicting minimality of C. Now assume that C is weakly productive, then there is a clause $C' \preceq C$ which produces a rule $l \to r'$ to R_S. This contradicts that l is irreducible by R_S, which is shown above.

Case (2): $C = C' \vee -l > r$. If C is not weakly productive then either (i) there exists $D = D' \vee l = r'$ and D produces $l \to r'$, but this contradicts that l is irreducible by R_S, or (ii) there is a literal $-l > r'$ in C', or (iii) there is a literal $l > r'$ in C' such that $-r' \downarrow_{\Upsilon_S} > r \downarrow_{\Upsilon_S}$.

Case (2.ii). Assume that $C' = C'' \vee -l > r'$. Then, inference rules InF 1 and InF 2 are applicable to C with the conclusions $D_1 = C'' \vee r > r' \vee -l > r$ and $D_2 = C'' \vee r' > r \vee -l > r'$, respectively. Note that $D_1 \prec C$ and $D_2 \prec C$. Consider possible cases. If $r' \geq r$ is true in Υ_S then D_1 is false in Υ_S. If $r > r'$ is true in Υ_S then D_2 is false in Υ_S. In both cases we obtain a contradiction to the minimality of C.

Case (2.iii). Let us assume that there is a literal $l > r'$ in C' such that $-r' \downarrow_{\Upsilon_S} > r \downarrow_{\Upsilon_S}$. Since $l > r'$ and $-l > r$ are false in Υ_S we have $r' \downarrow_{\Upsilon_S} \geq l \downarrow_{\Upsilon_S}$ and $r \downarrow_{\Upsilon_S} \geq -l \downarrow_{\Upsilon_S}$, therefore $r \downarrow_{\Upsilon_S} \geq -r' \downarrow_{\Upsilon_S}$ which is a contradiction.

Case (2.iv). Now we assume that C is weakly productive. Let C weakly produces a bound $(-x_l > r \downarrow_{\Upsilon_S}) \in B^l_+$. If $(-x_l > r \downarrow_{\Upsilon_S}) \in U^l$ then $(-x_l > r \downarrow_{\Upsilon_S}) \in B^l_\pm$

implying C is productive which is a contradiction. If $(-x_l > r \downarrow_{\Upsilon_S}) \in B^l_+ \setminus U^l$ then we have the following. Let $D = D' \vee l > r_{glb}$ be the clause producing the greatest lower bound (w.r.t. $>$) into B^l_-. Then, the Fourier-Motzkin inference rule is applicable to C and D with the conclusion $K = C' \vee D' \vee -r_{glb} > r$. Let us show that K is false in Υ_S. Indeed, D' is false since D is productive (see Lemma 2), and $-r_{glb} > r$ is false in Υ_S since $(-x_l > r \downarrow_{\Upsilon_S}) \notin U^l$ (see definition of U^l). Now we show that $C \succ K$. Indeed, $(l > r_{glb}) \succ D'$ therefore $(-l > r) \succ D'$ and $(-l > r) \succ (-r_{glb} > r)$. These imply that $C \succ K$, obtaining a contradiction to the minimality of C.

Case (3): $C = C' \vee l > r$. Subcases (3.i-iii) are similar to (2.i-iii).

Case (3.iv). We assume that C is weakly productive. Since C weakly produces a bound $(l > r \downarrow_{\Upsilon_S}) \in B^l_- \subseteq B^l_\pm$ we have that C is also productive, which is a contradiction.

We have considered all possible cases arriving at a contradiction under the assumption that C is false in $\langle \nu, \sim \rangle$. Therefore all clauses in S are true in the candidate model $\langle \nu, \sim \rangle$. □

Let us note that our proof of the completeness theorem is based on the model generation technique, and therefore it is not difficult to adapt redundancy notions from the standard superposition calculus. For details we refer to [6].

4 Lifting

We now consider clauses with variables over variable-safe sorts defined below. It is convenient to define first the set of *variable-unsafe sorts* $\hat{S}$ (w.r.t. Σ) as the minimal set of sorts such that (i) $s_{\mathbb{Q}} \in \hat{S}$ and (ii) if there is a function symbol f in $\mathcal{F}$ with an argument of a sort in $\hat{S}$ then the value sort of f is also in $\hat{S}$. We define the set of *variable-safe sorts* as $\bar{S} = S \setminus \hat{S}$, (see Examples (1,2)).

Assumption 3 *For a set of clauses S, all variables in S are of variable-safe sorts.*

It is easy to see that if a finite set of clauses S satisfies Assumption 3, then the set of all ground instances of S satisfies Finiteness of Coefficients Assumption 1.

Our LASCA calculus for ground clauses works on $\mathbb{Q}$-normalised clauses. In order to lift LASCA calculus into non-ground case we need additional normalisation rules. In formulation of Normalisation rule below we assume that non-ground theory literals are in one-sided form $t \bowtie 0$.

For a pair of terms t, t' let $mgu_{AC}(t, t')$ be a minimal complete set of AC-unifiers.

Normalisation Rule:

$$\frac{C[qt + q't']}{C[(q + q')t]\sigma} \qquad \text{where } \sigma \in mgu_{AC}(t, t').$$

Equality Resolution:

$$\frac{C \vee t \not\simeq t'}{C\sigma} \qquad \text{where } \sigma \in mgu_{AC}(t, t').$$

Now lifting of LASCA calculus is straightforward and we show the corresponding rules only for Gaussian and Fourier-Motzkin elimination rules. We assume that $\succ$ is lifted to non-ground terms, literals and clauses, in such a way that $\succ$ is preserved under substitutions. As in the ground case we assume that before applying the LASCA rules, literals are represented in one of the form $l = r$, $l > r$, $-l > r$, $l \simeq r$, $l \not\simeq r$, $\top$, $\bot$ where there exists a grounding substitution σ such $l\sigma \succ r\sigma$ (there can be several choices for l and r in one literal).

Gaussian Elimination:

$$\frac{C \vee l = r \quad L[l']_p \vee D}{(C \vee D \vee L[r]_p)\sigma}$$

(i) $\sigma \in mgu_{AC}(l, l')$,
(ii) $(l = r)\sigma\theta \succ C\sigma\theta$ for some grounding θ.

Fourier-Motzkin Elimination:

$$\frac{C \vee l > r \quad -l' > r' \vee D}{(C \vee D \vee -r' > r)\sigma}$$

(i) $\sigma \in mgu_{AC}(l, l')$,
and for some grounding substitution θ:
(ii) $(l > r)\sigma\theta \succ C\sigma\theta$,
(iii) there is no $l'' > r'' \in C$ such that $l''\sigma\theta =_{AC} l\sigma\theta$
(iv) $(-l' > r')\sigma\theta \succ D\sigma\theta$,
(v) there is no $-l'' > r'' \in D$ such that $l''\sigma\theta =_{AC} l\sigma\theta$.

Note that from the Assumption 3 it follows that l and l' are not variables in Gaussian and Fourier-Motzkin elimination rules.

Example 1. Let s be a non-theory sort. Let $f : \langle s, s_{\mathbb{Q}}\rangle$; $e : \langle s, s\rangle$; $g, h : \langle s_{\mathbb{Q}}, s_{\mathbb{Q}}\rangle$, assume that $g(x) \succ h(x)$. Consider set of clauses:

$$2g(f(e(x))) + h(f(e(x))) = 0 \qquad (1)$$
$$g(g(f(x)) + 1/2h(f(x))) > 2g(0) \qquad (2)$$
$$g(0) > 0 \qquad (3)$$

We can prove unsatisfiability of the set of clauses $\{(1), (2), (3)\}$ by applying Gaussian Elimination between (1) and (2) obtaining $g(-1/2h(f(e(x)))+1/2h(f(e(x)))) > 2g(0)$, then applying Normalisation obtaining $-g(0) > 0$ and Fourier-Motzkin with (3) obtaining $\bot$. Let us note that our next Theorem 3 implies that the set of clauses $\{(1), (2)\}$ is satisfiable, since the saturation process terminates.

Now we are ready to prove the following completeness theorem.

Theorem 3. *Let $\succ$ be finite-based and S be a finite saturated set of clauses satisfying Assumption 3. Then S is satisfiable if and only if $\Box \notin S$.*

Proof. Let S be a finite saturated set such that $\Box \notin S$. Let us show that S is satisfiable. Let S_{gr} be the set of all ground instances of clauses in S which are $\mathbb{Q}$-normalised. Since S is finite we have that S_{gr} satisfies Finiteness of Coefficients Assumption 1. Let $\langle \nu, \sim\rangle$ be the candidate model for S_{gr} (see Model Construction). Assume that $\langle \nu, \sim\rangle$ is not a model for S and let $C\sigma$ be the minimal w.r.t. $\succ$ instance of S false in $\langle \nu, \sim\rangle$. We can assume that $C\sigma$ is normalised. Indeed, if $C\sigma$ is not normalised, then we can apply Normalisation, or Equality Resolution rule to obtain a smaller clause false in $\langle \nu, \sim \rangle$. Now we can proceed as in Theorem 2. The only additional case to consider is when $x\sigma = t$ is reducible by $R_{S_{gr}}$ for some variable in C. Let $l \rightarrow r$ in $R_{S_{gr}}$ and $t|_p = l$. Then $t \succ t[r]_p$. Define σ' to be a substitution such that $x\sigma' = t[r]_p$ and $y\sigma' = y\sigma$ for variables different from x. Then $C\sigma \succ C\sigma'$ and $C\sigma'$ is false in $\langle \nu, \sim \rangle$, contradicting minimality of $C\sigma$. □

Example 2. Let s be a non-theory sort. Consider $f : \langle s, s \rangle$, $h : \langle s, s_{\mathbb{Q}} \rangle$ and $c : \langle s_{\mathbb{Q}} \rangle$.

$$\begin{aligned} h(f(x)) &> h(x) \\ c &> h(y) \end{aligned}$$

This set of clauses is saturated and therefore is satisfiable by Theorem 3. Note that after grounding this set of clauses we obtain an infinite number of inequalities, and the term c has infinitely many occurrences in different ground inequalities.

5 Finite-Based Ordering

In this section we present an ordering $\succ$ which satisfies all required properties: it is AC-compatible, $\mathbb{Q}$-monotone, $\mathbb{Q}$-total, finite-based, well-founded and satisfies sum and subterm properties. Without the condition to be finite-based, such orderings are well-known to exist by modifying the lexicographic path ordering (see e.g. [11]). Unfortunately these orderings are not finite-based which is a crucial condition for our completeness theorems. Here we show how to modify the Knuth-Bendix ordering to satisfy all requirements. Let $\succ_c$ be any well-founded total ordering on rationals such that $q \succ_c 1 \succ_c 0$ for any q different from $0, 1$. Let Σ_{nth} consists of all non-theory symbols. For an ordering $\succ$, let $\succ^{mul}$ denotes the multiset extension of $\succ$ and $\succ^{lex}$ denotes the lexicographic extension of $\succ$ defined over tuples of the same length.

Denote the set of natural numbers by $\mathbb{N}$. We call a *weight function* on Σ_{nth} any function $w : \Sigma_{nth} \rightarrow \mathbb{N}$ such that $w(e) > 0$ for every constant, or unary function symbol e. A *precedence relation* on Σ_{nth} is any linear order $\gg$ on Σ_{nth}. We call $w(g)$ the *weight* of g. The *weight* of any ground term t over signature Σ_{nth}, denoted $|t|$, is defined as follows: for any constant c we have $|c| = w(c)$ and for any function symbol g of a positive arity $|g(t_1, \dots, t_n)| = w(g) + |t_1| + \dots + |t_n|$.

First we define the Knuth-Bendix order on terms over Σ_{nth} in a usual way. The *Knuth-Bendix order induced by* w and $\gg$ is the binary relation $\succ_{KBO}$ on ground terms over Σ_{nth} defined as follows. For any ground terms $t = g(t_1, \dots, t_n)$ and $s = h(s_1, \dots, s_k)$ we have $t \succ_{KBO} s$ if one of the following conditions holds:

1. $|t| > |s|$;
2. $|t| = |s|$ and $g \gg h$;
3. $|t| = |s|$, $g = h$ and $(t_1, \dots, t_n) \succ_{KBO}^{lex} (s_1, \dots, s_n)$.

It is known that for every weight function w and precedence relation $\gg$ compatible with w, the Knuth-Bendix order induced by w and $\gg$ is a simplification order total on ground terms (see e. g. [1]). Let us note that for any term t there are only a finite number of terms less that t w.r.t. $\succ_{KBO}$. This property is crucial for defining a finite-based ordering.

Now we define an abstraction $abstr$ of terms over Σ into terms over Σ_{nth} as follows. Let $c_m \in \Sigma_{nth}$ be the least constant w.r.t. $\succ_{KBO}$. Then $abstr$ is defined by a structural induction on terms as follows: (i) $abstr(c) = c$ for a constant $c \in \Sigma_{nth}$, (ii) $abstr(f(t_1, \ldots, t_n)) = f(abstr(t_1), \ldots, abstr(t_n))$, for $f \in \Sigma_{nth}$, (iii) $abstr(q) = c_m$ for any constant $q \in \Sigma_{\mathbb{Q}}$, (iv) $abstr(q_1 t_1 + \ldots + q_n t_n + q) = abstr(t_j)$ where $abstr(t_j)$ is the greatest w.r.t. $\succ_{KBO}$ term among $abstr(t_1), \ldots, abstr(t_n)$, $1 \leq n$.

Given an ordering $\succ$ on non-theory terms we denote $\succ'$ an ordering extending $\succ$ to terms of the form qt where t is a non-theory term as follows. For non-theory terms t, s, we say that $qt \succ' q's$ iff either (i) $t \succ s$ or (ii) $t =_{AC} s$ and $q \succ_c q'$. Likewise we say $qt \succ' s$ iff $t \succeq s$, and $t \succ' qs$ iff $t \succ s$.

Finite-Based $\mathbb{Q}$-KBO. Now we define a $\mathbb{Q}$-Knuth-Bendix ordering ($\mathbb{Q}$-KBO) $\succ_{QKBO}$ on general terms as follows. Define $t \succ_{QKBO} s$ if one of the following conditions holds:

1. $abstr(t) \succ_{KBO} abstr(s)$, or
2. $abstr(t) = abstr(s)$ and
   (a) $t = f(t_1, \ldots, t_n)$ and $s = f(s_1, \ldots, s_n)$ for $f \in \Sigma_{nth}$ and $(t_1, \ldots, t_n) \succ^{lex}_{QKBO} (s_1, \ldots, s_n)$, or
   (b) $t = q_1 t_1 + \ldots + q_n t_n + q$ and $s = q'_1 s_1 + \ldots + q'_m s_m + q'$, and
      i. $\{t_1, \ldots, t_n\} \succ^{mul}_{QKBO} \{s_1, \ldots, s_m\}$ or,
      ii. $\{t_1, \ldots, t_n\} =_{AC} \{s_1, \ldots, s_m\}$ and $\{q_1 t_1, \ldots, q_n t_n, q\} \succ'^{\,mul}_{QKBO} \{q'_1 s_1, \ldots, q'_m s_m, q'\}$

Theorem 4. *$\mathbb{Q}$-Knuth-Bendix ordering is an AC-compatible, $\mathbb{Q}$-monotone, $\mathbb{Q}$-total, well-founded and finite-based ordering which satisfies sum and subterm properties.*

6 Negative Results

In this section we remark on complexity of the first-order theories for $\mathbb{Q}$ extended with non-theory function symbols. First we consider the structure $\mathbb{N}$ with theory symbols $\langle 0, S \rangle$ where S is interpreted as the successor function. Now, if we consider validity of sentences in an extended signature with non-theory function symbols then we are in the universal fragment of second-order arithmetic. Indeed, validity of a first-order sentence $\varphi(\bar{f})$ is equivalent to whether the second-order universal sentence $\forall \bar{f} \varphi(\bar{f})$ is true in $\mathbb{N}$. Therefore checking validity of formulas over $\mathbb{N}$ with non-theory function symbols is of the same complexity as checking validity of second-order universal sentences over $\mathbb{N}$ which is a Π^1_1-complete problem [10]. (Usually $\mathbb{N}$ is considered in the signature $\langle +, \cdot, 0, 1 \rangle$, but $+$ and $\cdot$ can be defined via S using standard inductive definitions.) It is easy to see that if $\mathbb{N}$ can be defined (up to isomorphism) in a language then the validity problem for such language extended with non-theory symbols is at least Π^1_1-hard (we can relativise formulas to $\mathbb{N}$).

Now we show that even if we consider formulas without quantifiers over variables of sort $s_{\mathbb{N}}$, still the validity problem is of the same complexity of being Π^1_1-complete. Indeed, consider a non-theory sort s and functions $0_s : \langle s \rangle$, $S_s : \langle s, s \rangle$, and $h : \langle s, s_{\mathbb{N}} \rangle$. Then, $\forall xy\ (h(x) = h(y) \to x \simeq y)$ axiomatises that h is an embedding of the domain of sort s into $\mathbb{N}$. Formulas $h(0_s) = 0$ and $\forall x\ (h(S_s(x)) = h(x) + 1)$, define $\mathbb{N}$ in the non-theory domain. Note that all variables in the above definition are of sort s.

Now we show that $\mathbb{N}$ is definable in $\mathbb{Q}$ extended with non-theory function symbols. Indeed, the following axioms define $\mathbb{N}$ in $\mathbb{Q}$, (for simplicity we consider N as a non-theory predicate symbol, but trivially $\mathbb{N}$ can be redefined using only function symbols).

$$
\begin{array}{l}
N(0) \\
\forall x\ (N(x) \to x = 0 \lor x > 0) \\
\forall x\ (N(x) \to N(x+1)) \\
\forall xy\ ((N(x) \land N(x+1) \land x + 1 > y > x) \to \neg N(y)) \\
\forall x\ (S(x) = x + 1)
\end{array}
$$

Since $\mathbb{Q}$ can be trivially coded in $\mathbb{N}$, we conclude that the validity problem for formulas in $\mathbb{Q}$ extended with non-theory function symbols is Π^1_1-complete.

Similar to the case of $\mathbb{N}$, we show that even if we consider formulas without quantifiers over variables of sort $\mathbb{Q}$, still the validity problem is Π^1_1-complete. The functions $h, 0_s, S_s$ are defined in the same way as for $\mathbb{N}$, but now h defines an embedding into $\mathbb{Q}$ rather than into $\mathbb{N}$. In order to define natural numbers N_s in the domain of sort s we need additional binary predicate $>_s$ over sort s and additional axioms:

$$
\begin{array}{l}
N_s(0_s) \\
\forall xy\ (x >_s y \leftrightarrow h(x) > h(y)) \\
\forall x\ (N_s(x) \to x \simeq 0_s \lor x >_s 0_s) \\
\forall x\ (N_s(x) \to N_s(S_s(x))) \\
\forall xy\ ((N_s(x) \land N_s(S_s(x)) \land S_s(x) >_s y >_s x) \to \neg N_s(y))
\end{array}
$$

It is easy to check that these axioms define $\mathbb{N}$ in the domain of sort s. Therefore the validity problem for formulas without quantifiers over variables of sort $\mathbb{Q}$ is Π^1_1-complete. We summarise these results in the following theorem.

Theorem 5. *Consider $\mathbb{Q}$ in the signature $\langle 0, 1, +, > \rangle$ and $\mathbb{N}$ in the signature $\langle 0, S \rangle$. Then, the following problems are Π^1_1-complete.*

- *Unsatisfiability of sets of clauses, in a signature extending $\mathbb{Q}$ ($\mathbb{N}$) with non-theory function symbols.*
- *Unsatisfiability of sets of clauses with variables ranging over a non-theory sort s, in a signature extending $\mathbb{Q}$ ($\mathbb{N}$) with a non-theory sort s and non-theory function symbols.*

In particular, Theorem 5 implies that there is no sound and complete calculus for linear arithmetic extended with non-theory function symbols.

7 Conclusions

In this paper we have presented an extension of superposition calculus for first-order logic with rules for linear arithmetic. One of our main results is completeness of the resulting calculus under some finiteness assumptions. One of the possible applications of our results is to obtain new decision procedures for fragments of first-order logic extended with rational arithmetic.

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Table 1. Linear Arithmetic Superposition Calculus (LASCA) for ground clauses

Ordered Paramodulation:

$$\frac{C \vee l \simeq r \quad L[l']_p \vee D}{C \vee D \vee L[r]_p}$$

(i) $l =_{AC} l'$,
(ii) $(l \simeq r) \succ C$.

Equality Factoring:

$$\frac{C \vee t' \simeq s' \vee t \simeq s}{C \vee s \not\simeq s' \vee t \simeq s'}$$

(i) $t =_{AC} t'$,
(ii) $(t \simeq s) \succeq C \vee t' \simeq s'$.

Gaussian Elimination:

$$\frac{C \vee l = r \quad L[l']_p \vee D}{C \vee D \vee L[r]_p}$$

(i) $l =_{AC} l'$,
(ii) $(l = r) \succ C$.

Theory Equality Factoring:

$$\frac{C \vee l' = r' \vee l = r}{C \vee r > r' \vee r' > r \vee l = r'}$$

(i) $l =_{AC} l'$,
(ii) $(l = r) \succeq C \vee l' = r'$.

Fourier-Motzkin Elimination:

$$\frac{C \vee l > r \quad -l' > r' \vee D}{C \vee D \vee -r' > r}$$

(i) $l =_{AC} l'$,
(ii) $(l > r) \succ C$,
(iii) there is no $l'' > r'' \in C$ such that $l'' =_{AC} l$
(iv) $(-l' > r') \succ D$
(v) there is no $-l'' > r'' \in D$ such that $l'' =_{AC} l$.

Inequality Factoring (InF1):

$$\frac{C \vee \pm l' > r' \vee \pm l > r}{C \vee r > r' \vee \pm l > r}$$

(i) $l =_{AC} l'$,
(ii) $(\pm l > r) \succeq C \vee \pm l' > r'$.

Inequality Factoring (InF2):

$$\frac{C \vee \pm l' > r' \vee \pm l > r}{C \vee r' > r \vee \pm l > r'}$$

(i) $l =_{AC} l'$,
(ii) $(\pm l > r) \succeq C \vee \pm l' > r'$.

$\perp$*-Elimination:*

$$\frac{C \vee \perp}{C}$$

(i) C contains only $\top, \perp$ literals.