Translating Graded Modalities into Predicate Logic*

Hans Jürgen Ohlbach, Renate A. Schmidt
and Ullrich Hustadt

ABSTRACT In the logic of graded modalities it is possible to talk about sets of finite cardinality. Various calculi exist for graded modal logics and all generate vast amounts of case distinctions. In this paper we present an optimized translation from graded modal logic into many-sorted predicate logic. This translation has the advantage that in contrast to known approaches our calculus enables us to reason with cardinalities of sets symbolically. In many cases the length of proofs for theorems of this calculus is independent of the cardinalities. The translation is sound and complete.

1 Introduction

From Minsky's early frame systems, which were defined purely operationally, and Brachman's KL-ONE knowledge representation system [4, 35] to the language $\mathcal{ALC}$ of Schmidt-Schauß and Smolka's [28] paper there has been a continuous trend in designing knowledge representation systems more and more according to logical principles with clear syntax and semantics and logical inferences as basic operations. $\mathcal{ALC}$ in particular is a language with the usual logical connectives $\sqcap$, $\sqcup$, $\neg$ and the additional constructs (all R C) and (some R C). For example, the following is an $\mathcal{ALC}$ definition which defines a 'concept' proud-father as a father all of whose children are successful persons.

$$\textsf{proud-father} = \textsf{father} \ \sqcap \ (\textsf{all has-child successful-person}),$$

The fragment of $\mathcal{ALC}$ that includes the operations $\sqcap$, $\sqcup$, $\neg$, all, some is just a variant of the multi-modal logic $\mathbf{K}_{(m)}$ [27]. The concept (all R C) corresponds to $[R]C$ where the relational term R (a 'role' in KL-ONE jargon) is the parameter of the modal operator, and is interpreted as a binary accessibility relation. $\mathcal{ALC}$ is still limited in its expressiveness. In pure $\mathcal{ALC}$ it is not possible to define concepts like, for example, a city as a place with more than, say, 100 000 inhabitants. There are extensions of $\mathcal{ALC}$, like $\mathcal{ALCN}$, with additional operators, called 'number restrictions'.

$$\textsf{city} = \textsf{place} \ \sqcap \ (\textsf{atleast } 100\,001 \textsf{ inhabited-by people}) \qquad (1)$$

is a suitable $\mathcal{ALCN}$ definition. (atleast n R C) and (atmost n R C) restrict the number of so-called 'role fillers', i.e. they restrict the number of elements in the range of the relation R to at least n and at most n, respectively. The corresponding modal logic of $\mathcal{ALCN}$ is the multi-modal version of the system of 'graded modalities', which was introduced by Goble [17] and Fine [13, 14] and which is investigated in Fattorosi-Barnaba and de Caro [12, 11], and van der Hoek [34, 33].

Graded modalities are modal operators indexed with cardinals which fix the number of worlds in which a formula is true. The formula $\Diamond_n\varphi$ is true in a world iff there are more than n accessible worlds in which φ is also true. The dual formula $\Box_n\varphi$, given by $\neg\Diamond_n\neg\varphi$, is then true in a world iff there are at most n accessible worlds in which $\neg\varphi$ is true. More formally, the semantics is defined in terms of one accessibility relation, say R, by

$$\begin{array}{lll} \mathcal{M}, x \models_{\overline{K}} \Diamond_n\varphi & \text{iff} & |\{y \mid R(x,y) \ \&\ \mathcal{M}, y \models_{\overline{K}} \varphi\}| > n \\ \mathcal{M}, x \models_{\overline{K}} \Box_n\varphi & \text{iff} & |\{y \mid R(x,y) \ \&\ \mathcal{M}, y \models_{\overline{K}} \neg\varphi\}| \leq n, \end{array}$$

where $\mathcal{M}$ denotes a model and x, y denote possible worlds. For any set A, $|A|$ denotes the cardinality of A. This semantics is very natural and intuitive, but it has one disadvantage. All inference systems

*Published in Wansing, H. (eds), *Proof Theory of Modal Logic*. Applied Logic Series **2**, Kluwer, 253–291.

based on this semantics, in particular, tableaux systems, deal with these $\diamond_n$-operators by generating a corresponding number of terms explicitly. For example, the formula $\diamond_{100\,000}\mathit{people}$ triggers the generation of 100 001 constant symbols as representatives for the individuals denoting *people*. Except for counting these constant symbols and comparing the length of lists, known tableaux systems do not provide for arithmetical computation. In particular, reasoning with symbolic arithmetic terms is impossible. For example, in tableaux systems the formula $\diamond_{n+1}p \rightarrow \diamond_n p$ which is true for all n can only be verified for concrete values of n, but in general it cannot be verified for arbitrary values of n.

This is not the case for the Hilbert system axiomatizing the graded modalities. It is formulated with arithmetical terms, and in principle, this allows for invoking arithmetical computations. However, Hilbert systems have other disadvantages that makes them unsuitable to form the basis for automated reasoning. For example they do propositional reasoning just with modus ponens and the instantiation rule. Even for trivial theorems one gets large proofs and the search space is very unstructured and enormously big.

A direct translation of formulae with graded modalities into predicate logic requires the axiomatization of finite domains. This is feasible only for small cardinalities. We may translate sentence (1) as follows:

$$\begin{array}{lcll}
\forall x\ \mathit{city}(x) & \leftrightarrow & \mathit{place}(x) \wedge & \\
 & & \exists y_1 \ldots y_{100\,001} & y_1 \neq y_2 \wedge y_1 \neq y_3 \wedge \ldots \wedge y_1 \neq y_{100\,001} \wedge \\
 & & & \qquad y_2 \neq y_3 \wedge \ldots \wedge y_2 \neq y_{100\,001} \wedge \\
 & & & \qquad\qquad \vdots \\
 & & & \qquad\qquad y_{100\,000} \neq y_{100\,001} \wedge \\
 & & \multicolumn{2}{l}{\mathit{inhabited\text{-}by}(x, y_1) \wedge \ldots \wedge \mathit{inhabited\text{-}by}(x, y_{100\,001}) \wedge} \\
 & & \multicolumn{2}{l}{\mathit{people}(y_1) \wedge \ldots \wedge \mathit{people}(y_{100\,001}).}
\end{array}$$

The translation of $\diamond_n$-expressions requires $(n+1)n/2$ equations. Even for small n this is more than current theorem provers can cope with. One immediate alternative is introducing set variables and a cardinality function. For sentence (1) an alternative formulation is:

$$\begin{array}{lcl}
\forall x\ \mathit{city}(x) & \leftrightarrow & \mathit{place}(x) \wedge \exists Y\ (|Y| > 100\,000 \wedge \\
 & & \forall y\ (y \in Y \rightarrow (\mathit{inhabited\text{-}by}(x, y) \wedge \mathit{people}(y)))).
\end{array}$$

This is not really a feasible alternative, for the axiomatization of the cardinality function then requires the above $(n+1)n/2$ equations, and this for every n:

$$\begin{array}{lcll}
\forall Y\ |Y| > n & \leftrightarrow & \exists y_1 \ldots y_{n+1} & y_1 \in Y \wedge \ldots \wedge y_{n+1} \in Y \wedge \\
 & & & y_1 \neq y_2 \wedge y_1 \neq y_3 \wedge \ldots \wedge y_1 \neq y_{n+1} \wedge \\
 & & & \qquad y_2 \neq y_3 \wedge \ldots \wedge y_2 \neq y_{n+1} \wedge \\
 & & & \qquad\qquad \vdots \\
 & & & \qquad\qquad y_n \neq y_{n+1}.
\end{array}$$

In this paper we present a two step translation of graded modal logics into predicate logic. In the first step, we transform graded modal logics into another multi-modal logic with standard interpretation. In particular, we accommodate modal logics with graded modalities in a multi-modal logic with two kinds of modalities:

(i) $\langle n\rangle$, $[n]$ characterized by a relational structure (over a universe U) of infinitely but countably many different relations R_n ($n \in \mathbb{N}_0$), and
(ii) $\diamond$, $\square$ characterized by a designated relation E.

We translate formulae of the form $\diamond_n\varphi$ into $\langle n\rangle\square\varphi$ and the intuitive idea underlying this translation is this: If φ is true in a set Y of worlds with more than n elements then we introduce an accessibility relation R_n that connects the actual world and a world w_Y which we can think of as being a representative for the set Y. This defines the $\langle n\rangle$-operator. $\square\varphi$ and its associated accessibility relation E expresses that φ is true in all the worlds of the set Y. E connects the world w_Y with all the worlds in Y and can be thought of as the membership relation. Thus, $\langle n\rangle\square\varphi$ encodes 'there is a set with more than n elements (encoded by $\langle n\rangle$) and φ is true for all the elements of this set (encoded by

$\Box$)'. Our first problem now is to find a sound and adequate axiomatization of the modalities $\langle n\rangle$, $[n]$, $\Diamond$ and $\Box$ as to capture the graded modalities $\Diamond_n$ and $\Box_n$. It turns out that this is not entirely possible. The axiomatization we present in this paper has some non-standard models which do not reflect our intuition. But this does no harm, as we will see. We show: A formula φ is a theorem of a graded modal logic iff the translation of φ is a theorem in the new logic. This translation is only an intermediary step in a translation to predicate logic.

In the second step, we translate the multi-modal logic into a predicate logic using the functional translation of [23, 24, 10, 18, 2, 36]. The reason for using the functional translation instead of the usual relation translation is this: The multi-modal logic of graded modalities can have frame properties that are not first-order definable in terms of R_n relations. However, the frame properties can be formulated in a weak fragment of second-order logic and it is possible to formulate them in an alternative adapted language as first-order expressions. The alternative language is a *functional language* in which binary relations are encoded as sets AF of functions. The set AF of accessibility functions defining the accessibility relation R is given by:

$$R(x, y) \leftrightarrow \exists f \in AF\, y = f(x).$$

This sequence of translations of a system of numerical modalities first into another multi-modal logic and then into a many-sorted predicate logic (using the functional translation) yields an axiomatization, in particular, an axiomatization of properties of finite sets. Instead of counting symbols this system uses arithmetical reasoning.

This paper is structured as follows. In Section 2 we give a short overview of modal logics with graded modalities. In Section 3 we introduce the normal multi-modal logic for accommodating graded modal logics. We define a translation from logics of graded modalities into the multi-modal logic that we exhibit to be sound and complete. In Section 5 we present the functional translation of the multi-modal logic into predicate logic. We conclude with Section 6 in which we apply the new techniques to the knowledge representation language $\mathcal{ALCN}$.

2 Graded modalities, the system $\overline{\mathbf{K}}$

Normal modal logics like **K**, **T**, **S4** and **S5** have one modal operator, the *necessity* (or *box*) operator $\Box$. The *possibility* (or *diamond*) operator $\Diamond$ is defined as its dual. By definition,

$$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi.$$

In [17] Goble investigates modal logics with more than one modality. His logics have a fixed and finite number of modalities. Each modality represents a different grade of necessity. For example, the formula

$$N_m\varphi \wedge N_n\psi$$

for positive integers $m < n$, is read to mean ψ is more necessary than φ. Kit Fine [13, 14] generalizes this idea and introduces modal logics with *numerical modalities*. These are now commonly referred to as modal logics with *graded modalities*. In a series of papers Fattorosi-Barnaba, de Caro and Cerrato [12, 7, 11, 6] rediscover and analyze various modal logics of graded modalities.

Recent investigations of graded modal logics are by van der Hoek in [34] and [33]. Together with de Rijke he applies graded modalities to linguistics and artificial intelligence. In [31] they show that generalized quantifiers can be modelled with graded modalities. In [32] they also show that certain numerical quantifier operations available in KL-ONE-based knowledge representation languages can be modelled with graded modalities.

In this paper we adopt the definition of the graded modal logic $\overline{\mathbf{K}}$ of van der Hoek [34]. $\overline{\mathbf{K}}$ is an extension of the normal modal logic **K** with graded modalities. Formally, the vocabulary of $\overline{\mathbf{K}}$ consists of the set of propositional symbols $p, p_1, p_2, \ldots, q, q_1, q_2, \ldots$, the constant $\bot$ (falsity), the logical symbol $\rightarrow$ (implication) and the modal operator symbols $\Diamond_n$ for $n \in \mathbb{N}_0$, and the usual punctuation symbols. Formulae of $\overline{\mathbf{K}}$ have the following forms:

$$p, q, \ldots, \quad \bot, \quad \varphi \rightarrow \psi, \quad \Diamond_n\varphi.$$

As usual we define $\neg\varphi$ (negation), $\top$ (truth), $\varphi \vee \psi$ (disjunction), $\varphi \wedge \psi$ (conjunction) and $\varphi \leftrightarrow \psi$ (double-implication) as abbreviations for $\varphi \rightarrow \bot$, $\neg\bot$, $\neg\varphi \rightarrow \psi$ and $\neg\varphi \vee \neg\psi$, respectively. Furthermore, $\Box_n\varphi$ abbreviates $\neg\Diamond_n\neg\varphi$ for $n \geq 0$, $\Diamond!_0\varphi$ abbreviates $\Box_0\neg\varphi$, and $\Diamond!_n\varphi$ is the abbreviation for $\Diamond_{n-1}\varphi \wedge \neg\Diamond_n\varphi$ with $n > 0$.

$\Diamond_n\varphi$	is read to mean	φ is true in more than n accessible worlds,
$\Box_n\varphi$	is read to mean	$\neg\varphi$ is true in at most n accessible worlds, and
$\Diamond!_n\varphi$	is read to mean	φ is true in exactly n accessible worlds.

Definition 2.1 The system $\overline{\mathbf{K}}$ of graded modalities is defined by the following axioms

A1 the axioms of propositional logic

A2 $\vdash_{\overline{\mathrm{K}}} \Diamond_{n+1}\varphi \rightarrow \Diamond_n\varphi$

A3 $\vdash_{\overline{\mathrm{K}}} \Box_0(\varphi \rightarrow \psi) \rightarrow (\Diamond_n\varphi \rightarrow \Diamond_n\psi)$

A4 $\vdash_{\overline{\mathrm{K}}} \Box_0\neg(\varphi \wedge \psi) \rightarrow ((\Diamond!_n\varphi \wedge \Diamond!_m\psi) \rightarrow \Diamond!_{n+m}(\varphi \vee \psi))$

together with the uniform substitution rule, Modus Ponens, and the necessitation rule for $\Box_0$:

US if φ is a theorem so is every substitution instance of φ,

MP if $\vdash_{\overline{\mathrm{K}}} \varphi$ and $\vdash_{\overline{\mathrm{K}}} \varphi \rightarrow \psi$ then $\vdash_{\overline{\mathrm{K}}} \psi$

N if $\vdash_{\overline{\mathrm{K}}} \varphi$ then $\vdash_{\overline{\mathrm{K}}} \Box_0\varphi$.

Observe that $\Diamond_0$ and $\Box_0$ coincide with the standard modal operators $\Diamond$ and $\Box$. **K** is therefore a subsystem of $\overline{\mathbf{K}}$.

Theorem 2.2 *[34] The following are theorems of* $\overline{\mathbf{K}}$.

A5 $\vdash_{\overline{\mathrm{K}}} \Box_0(\varphi \rightarrow \psi) \rightarrow (\Box_n\varphi \rightarrow \Box_n\psi)$

A6 $\vdash_{\overline{\mathrm{K}}} \Diamond_n(\varphi \wedge \psi) \rightarrow (\Diamond_n\varphi \wedge \Diamond_n\psi)$

A7 $\vdash_{\overline{\mathrm{K}}} \Diamond!_n\varphi \wedge \Diamond!_m\varphi \rightarrow \bot$ *for* $n \neq m$

A8 $\vdash_{\overline{\mathrm{K}}} \Box_n\neg\varphi \leftrightarrow (\Diamond!_0\varphi \mathbin{\nabla} \Diamond!_1\varphi \mathbin{\nabla} \ldots \mathbin{\nabla} \Diamond!_n\varphi)$

A9 $\vdash_{\overline{\mathrm{K}}} \neg\Diamond_n(\varphi \vee \psi) \rightarrow \neg\Diamond_n\varphi$

A10 $\vdash_{\overline{\mathrm{K}}} \Diamond_{n+m}(\varphi \vee \psi) \rightarrow (\Diamond_n\varphi \vee \Diamond_m\psi)$

A11 $\vdash_{\overline{\mathrm{K}}} \Diamond!_n\varphi \wedge \Diamond_m\varphi \rightarrow \bot$ *for* $m \geq n$

A12 $\vdash_{\overline{\mathrm{K}}} \Diamond_n(\varphi \wedge \psi) \wedge \Diamond_m(\varphi \wedge \neg\psi) \rightarrow \Diamond_{n+m+1}\varphi$

(∇ *denotes exclusive-or.)*

The semantics of $\overline{\mathbf{K}}$ is given by a *frame* $\mathcal{F} = (W, R)$ consisting of a non-empty set W, called the set of *worlds*, and a binary relation R over W, called the *accessibility relation*. We define a *model* (based on a frame $\mathcal{F}$) of $\overline{\mathbf{K}}$ as a triple $\mathcal{M} = (W, R, V)$. V denotes a *valuation* function mapping propositional variables to subsets of W. *Truth* in a model $\mathcal{M}$ for formulae of $\overline{\mathbf{K}}$ at any world x is defined (in terms of the *satisfiability relation* $\models_{\overline{\mathrm{K}}}$) as follows:

$$\begin{array}{lll} \mathcal{M}, x \models_{\overline{\mathrm{K}}} p & \text{iff} & x \in V(p) \\ \mathcal{M}, x \models_{\overline{\mathrm{K}}} \Diamond_n\varphi & \text{iff} & |\{y \in W \mid R(x,y) \wedge \mathcal{M}, y \models_{\overline{\mathrm{K}}} \varphi\}| > n \\ \mathcal{M}, x \models_{\overline{\mathrm{K}}} \Box_n\varphi & \text{iff} & |\{y \in W \mid R(x,y) \wedge \mathcal{M}, y \models_{\overline{\mathrm{K}}} \neg\varphi\}| \leq n \end{array} \tag{2}$$

and as usual for the other connectives. For any binary relation R, let $R(x)$ denote the set of images of x under R. That is, define $R(x) = \{y \mid R(x,y)\}$. Then, satisfiability of $\Diamond_n\varphi$ and $\Box_n\varphi$ can be reformulated as follows:

$$\begin{array}{l} \mathcal{M}, x \models_{\overline{\mathrm{K}}} \Diamond_n\varphi \\ \quad \text{iff} \quad \exists Y \subseteq R(x) \text{ with } |Y| > n \text{ and } \forall y \in Y : \mathcal{M}, y \models_{\overline{\mathrm{K}}} \varphi \\ \mathcal{M}, x \models_{\overline{\mathrm{K}}} \Box_n\varphi \\ \quad \text{iff} \quad \forall Y \subseteq R(x) \text{ with } |Y| > n,\ \exists y \in Y : \mathcal{M}, y \models_{\overline{\mathrm{K}}} \varphi. \end{array} \tag{3}$$

(The proof is routine.) It is now easy to see that the usual duality for box and diamond also hold for $\Diamond_n$ and $\Box_n$, i.e. we have $\Diamond_n\varphi \leftrightarrow \neg\Box_n\neg\varphi$.

A formula φ is a $\overline{\mathbf{K}}$ thoerem, i.e. $\models_{\overline{\mathrm{K}}} \varphi$, iff φ holds in all worlds of all $\overline{\mathbf{K}}$ frames.

Theorem 2.3 *The axiomatization of $\overline{\mathbf{K}}$ is sound and complete, i.e. for all formulae φ, we have*

$$\vdash_{\overline{\mathrm{K}}} \varphi \quad \textit{iff} \quad \models_{\overline{\mathrm{K}}} \varphi.$$

A proof can be found in [12].

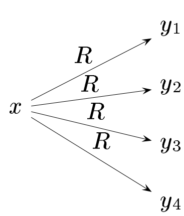
In the remainder of this paper we assume the formulae of $\overline{\mathbf{K}}$ to be in *negation normal form* which can be obtained by systematically applying the following equivalences from left to right.

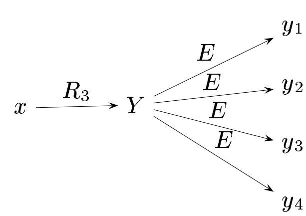
$$\begin{array}{rcl}
\neg(\varphi \vee \psi) & \leftrightarrow & \neg\varphi \wedge \neg\psi \\
\neg(\varphi \wedge \psi) & \leftrightarrow & \neg\varphi \vee \neg\psi \\
\varphi \rightarrow \psi & \leftrightarrow & \neg\varphi \vee \psi \\
\varphi \leftrightarrow \psi & \leftrightarrow & (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \psi) \\
\neg\Box_n\varphi & \leftrightarrow & \Diamond_n\neg\varphi \\
\neg\Diamond_n\varphi & \leftrightarrow & \Box_n\neg\varphi
\end{array}$$

3 From graded modalities to multi-modal logic

In this section we present a new interpretation for $\overline{\mathbf{K}}$ and investigate its logical counterpart.

A formula $\Diamond_n\varphi$ of $\overline{\mathbf{K}}$ is interpreted as an expression in which a subset of the accessible worlds with more than n worlds is selected. More concretely, the formula $\Diamond_n\varphi$ is true in a world x iff there is a set of worlds (a subset of W) accessible by R from x containing more than n worlds in which φ holds. Equivalently, $\Diamond_n\varphi$ is true in x iff there is subset Y of the range of R from x with cardinality strictly greater than n such that φ is true in every world in this subset, see (3). In our alternative interpretation of $\overline{\mathbf{K}}$ we introduce a new class of worlds W_Y, each world representing subsets of accessible worlds of W. That is, we represent the set Y by a designated world in this new class of worlds. Furthermore, instead of having just one accessibility relation R, here, we have for each $n \in \mathbb{N}_0$ a different accessibility relation R_n. Their domain is W and their range is restricted to W_Y. More precisely, for any n, R_n relates worlds in W to those elements in W_Y which represent sets (of worlds in W) of cardinality greater than n. And finally, there is an additional designated accessibility relation, denoted E for 'element of', which relates the new kind of worlds W_Y to the worlds in W again. The relation E represents the element-of relation (strictly speaking the converse of the 'element of' relation) between a subset Y of W and its elements. For example, consider the formula $\Diamond_3\varphi$. According to the definition of the previous section $\Diamond_3\varphi$ is true in a world x iff there are at least 4 worlds to which x is R-related. This definition is depicted in the first picture below. The second picture depicts our new alternative view.





The relation R is replaced by the relational composition of the two new relations R_3 and E. In the process we have introduced a new world which we labelled Y as it is meant to represent the set of worlds y_1, y_2, y_3 and y_4.

The alternative semantics for $\overline{\mathbf{K}}$ sketched above characterizes a new graded modal logic which we describe now. We call the system $\overline{\mathbf{K}}_E$. It is a normal multi-modal logic system with graded modalities. In this logic the operators $\Diamond_n$ are replaced by a combination of two operators. $\overline{\mathbf{K}}_E$ is more expressive than the system $\overline{\mathbf{K}}$. Nevertheless, it has similar properties as $\overline{\mathbf{K}}$ as we will show below.

Our system $\overline{\mathbf{K}}_E$ differs from an alternative translation into a multi-modal logic 'Lcount', developed by [1]. In their system there are n-place operators $\langle n\rangle$ with semantics

$$\mathcal{M}, x \models \langle n\rangle\varphi_1, \ldots, \varphi_n \quad \text{iff} \quad \begin{array}{l}\text{there are distinct } y_1, \ldots, y_n \text{ such that} \\ \mathcal{M}, y_1 \models \varphi_1 \text{ and } \ldots \text{ and } \mathcal{M}, y_n \models \varphi_n\end{array}$$

Calculi based on this semantics, however, seem not to be much different to the calculi based on the original semantics for graded modalities. In a corresponding tableaux system one has to introduce witnesses for the worlds again, but this is what we want to avoid.

3.1 The system $\overline{\mathbf{K}}_E$

The language of $\overline{\mathbf{K}}_E$ is that of $\overline{\mathbf{K}}$ with the graded modalities $\Diamond_n$ and $\Box_n$ replaced by the symbols $\langle n\rangle$, $[n]$, $\Diamond$ and $\Box$. Formulae of $\overline{\mathbf{K}}_E$ have the following forms:

$$p, q, \ldots, \quad \bot, \quad \varphi \to \psi, \quad \langle n\rangle\varphi, \quad \Diamond\varphi.$$

As in Section 2 we define the classical connectives in the usual way. The duals of $\langle n\rangle$ and $\Diamond$ are abbreviated as follows: For $n \in \mathbb{N}_0$, $[n]\varphi$ abbreviates $\neg\langle n\rangle\neg\varphi$ and $\Box\varphi$ abbreviates $\neg\Diamond\neg\varphi$. The intended meaning of $\langle n\rangle\varphi$ is,

φ is true in some world accessible by the binary relation R_n.

The intended meaning of $\Diamond\varphi$ is

φ is true in some world accessible by the binary relation E.

We call $\langle n\rangle$ and $[n]$ the *numerical operators* and $\Diamond$ and $\Box$ the *membership operators*. The relations R_n and E are defined as sketched above. Namely, W forms the domain of the R_n and the co-domain of E and the new class of worlds W_Y forms the co-domain of the R_n and the domain of E. Dually, the intended meaning of $[n]\varphi$ is

φ is true in all worlds accessible by R_n.

And the intended meaning of $\Box\varphi$ is

φ is true in all worlds accessible by E.

So, the syntax $\langle n\rangle\varphi$ (respectively $[n]\varphi$) is the shorthand for $\langle R_n\rangle\varphi$ (respectively $[R_n]\varphi$), $\Diamond\varphi$ (respectively $\Box\varphi$) is the shorthand for $\langle E\rangle\varphi$ (respectively $[E]\varphi$).

$\overline{\mathbf{K}}$-formulae of the form $\Diamond_n\varphi$ and $\Box_n\varphi$ can be formulated as $\overline{\mathbf{K}}_E$-expressions of the form

$$\langle n\rangle\Box\varphi \quad \text{and} \quad [n]\Diamond\varphi,$$

respectively. The logic $\overline{\mathbf{K}}_E$ is more expressive than $\overline{\mathbf{K}}$. It permits arbitrary combinations of modal operators, not only alternate combinations of necessity and possibility operators. For example, $[3][4]\Box\Box\varphi$ is a well-formed formula of $\overline{\mathbf{K}}_E$, although it may not make much sense in our intended semantics. However, there are combinations of modal operators which have no equivalent formulation in $\overline{\mathbf{K}}$, but which have interesting applications. Here is an example of such a formula:

$$[10](\mathit{football\text{-}team} \to \Box\mathit{football\text{-}player}).$$

It says that, if there is a set with more than 10 elements for which the proposition *football-team* holds then the proposition *football-player* must be true for all its elements. In this way we can distinguish between notions like teams which we interpret as sets and notions like players which we interpret as elements.

We now give a Hilbert axiomatization for $\overline{\mathbf{K}}_E$ and investigate its characteristic frames.

Definition 3.1 The axioms and rules of the system $\overline{\mathbf{K}}_E$ are:

N1 the axioms of propositional logic and Modus Ponens

N2 the K-axioms for $[n]$ and $\Box$:
$\vdash_{\overline{\mathbf{K}}_E} [n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi)$ $\qquad$ $\vdash_{\overline{\mathbf{K}}_E} \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

N3 the necessitation rules for $[n]$ and $\Box$:
If $\vdash_{\overline{\mathrm{K}}_E} \varphi$ then $\vdash_{\overline{\mathrm{K}}_E} [n]\varphi$ if $\vdash_{\overline{\mathrm{K}}_E} \varphi$ then $\vdash_{\overline{\mathrm{K}}_E} \Box\varphi$

N4 $\vdash_{\overline{\mathrm{K}}_E} [0]\Diamond\varphi \rightarrow [n]\Box\varphi$

N5 $\vdash_{\overline{\mathrm{K}}_E} \langle n\rangle\Box\varphi \rightarrow \langle n\rangle\Diamond\varphi$

N6 $\vdash_{\overline{\mathrm{K}}_E} [n]\varphi \rightarrow [n+1]\varphi$

N7 $\vdash_{\overline{\mathrm{K}}_E} \langle n+m\rangle\Box(\varphi \vee \psi) \rightarrow (\langle n\rangle\Box\varphi \vee \langle m\rangle\Box\psi)$

N8 $\vdash_{\overline{\mathrm{K}}_E} (\langle n\rangle\Box(\varphi \wedge \psi) \wedge \langle m\rangle\Box(\varphi \wedge \neg\psi)) \rightarrow \langle n+m+1\rangle\Box\varphi$

Although the box operators can be treated as abbreviations in terms of diamond operators, or vice versa, we use both operators and allow for arbitrary conversions back and forth with

$$\begin{array}{rcl} [n]\varphi & \leftrightarrow & \neg\langle n\rangle\neg\varphi \\ \Box\varphi & \leftrightarrow & \neg\Diamond\neg\varphi. \end{array}$$

N1–N3 are the basic axioms for every normal modal logic. N4 captures that, if something holds for every set of worlds with more than zero elements, that is, if it holds for every non-empty set of worlds, then it holds also for all sets with more than n elements. This means, the composition $R_n\,;E$ for n arbitrary is a subrelation of the composition $R_0\,;E$. N5 ensures that no set with more than n elements is empty. A contrapositive version of N6 is $\langle n+1\rangle\Box\varphi \rightarrow \langle n\rangle\Box\varphi$. It captures that sets with more than $n+1$ elements are sets with more than n elements.

N7 corresponds to A10 and is a bit more complicated to explain. As an example suppose $n = 2$ and $m = 4$. For these values N7 is

$$\langle 6\rangle\Box(\varphi \vee \psi) \rightarrow (\langle 2\rangle\Box\varphi \vee \langle 4\rangle\Box\psi)$$

which is equivalent to

$$(\langle 6\rangle\Box(\varphi \vee \psi) \wedge \neg\langle 2\rangle\Box\varphi) \rightarrow \langle 4\rangle\Box\psi.$$

In words, if there are more than 6, say 7, worlds in which the formula $\varphi \vee \psi$ is true, but it is not the case that φ holds in more than two worlds (i.e. $\neg\varphi$ is true in all but possibly two worlds) then in the remaining 5 of 7 worlds ψ is true. The axiom says that every $(n+m)$-element set can be decomposed into an n-element set and an m-element set, but note, the axiom is slightly stronger.

The intuition underlying N8 is the following: Suppose there is a set Y_1 with at least $n+1$ elements where $\varphi \wedge \psi$ holds and there is another set Y_2 with at least $m+1$ elements where $\varphi \wedge \neg\psi$ holds. Since ψ and $\neg\psi$ cannot hold simultaneously in one world, Y_1 and Y_2 must be disjoint. Thus, φ holds in $Y_1 \cup Y_2$ which is of cardinality at least $n+m+2$. Therefore, $\langle n+m+1\rangle\Box\varphi$ is true.

Now we turn to the semantics of $\overline{\mathbf{K}}_E$. The K-axioms and necessitation rules allow us to use the standard Kripke semantics. We choose the Kripke semantics for the multi-modal logic $\mathbf{K}_{(m)}$ where m is the number of modal operators. Note that $\overline{\mathbf{K}}_E$ has infinitely but countably many modal operators. A $\overline{\mathbf{K}}_E$-*frame* is a relational structure

$$\mathcal{F} = (W, \{R_n\}_{n \in \mathbb{N}_0}, E).$$

W is a non-empty set of worlds. The R_n are binary relations over W (each is associated with a modality $\langle n\rangle$) and E is a designated binary relation over W (associated with the modality $\Diamond$). The relations satisfy the properties N4–N8 given below. A *model* of $\overline{\mathbf{K}}_E$ *based on a frame* $\mathcal{F}$ is a pair $\mathcal{M} = (\mathcal{F}, V)$ where V is a function mapping propositional variables to subsets of W. Truth and satisfaction for the propositional fragment of $\overline{\mathbf{K}}_E$ is defined as for the propositional fragment of $\overline{\mathbf{K}}$. See (2) in the previous section. A modal formula is *satisfied* (is *true* or *holds*) in a world x iff depending on its form the following holds:

$$\begin{array}{llll} \mathcal{M}, x \models_{\overline{\mathrm{K}}_E} \langle n\rangle\varphi & \text{iff} & \text{there is a } y \text{ such that } R_n(x,y) \text{ and } \mathcal{M}, y \models_{\overline{\mathrm{K}}} \varphi \\ \mathcal{M}, x \models_{\overline{\mathrm{K}}_E} [n]\varphi & \text{iff} & \text{for all } y \text{ such that } R_n(x,y),\ \mathcal{M}, y \models_{\overline{\mathrm{K}}} \varphi \\ \mathcal{M}, x \models_{\overline{\mathrm{K}}_E} \Diamond\varphi & \text{iff} & \text{there is a } y \text{ such that } E(x,y) \text{ and } \mathcal{M}, y \models_{\overline{\mathrm{K}}} \varphi \\ \mathcal{M}, x \models_{\overline{\mathrm{K}}_E} \Box\varphi & \text{iff} & \text{for all } y \text{ such that } E(x,y),\ \mathcal{M}, y \models_{\overline{\mathrm{K}}} \varphi. \end{array}$$

A formula is a *tautology* if for any frame $\mathcal{F}$ the formula is satisfied in all $\mathcal{F}$-based models.

The following are the characteristic properties of $\overline{\mathbf{K}}_E$-frames that correspond to the axioms N4–N8:

N4′ $\forall xyz\ ((R_n(x,y) \wedge E(y,z)) \rightarrow \exists u\ (R_0(x,u) \wedge \forall v\ (E(u,v) \rightarrow v = z)))$

N5′ $\forall xy\ (R_n(x,y) \rightarrow \exists z\ E(y,z))$

N6′ $\forall xy\ (R_{n+1}(x,y) \rightarrow R_n(x,y))$

N7′
$$\begin{aligned} &\forall xy\ R_{n+m}(x,y) \rightarrow \\ &\quad \forall fg\ \exists uv\ \ (R_n(x,u) \rightarrow E(u,f(u)) \wedge \\ &\qquad\qquad R_m(x,v) \rightarrow E(v,g(v))) \rightarrow \\ &\qquad\qquad (R_n(x,u) \wedge R_m(x,v) \wedge E(y,f(u)) \wedge f(u) = g(v)) \end{aligned}$$

N8′
$$\begin{aligned} &\forall xyz\ (R_n(x,y) \wedge R_m(x,z) \wedge \forall u\ (E(y,u) \rightarrow \neg E(z,u)) \rightarrow \\ &\quad \exists v\ (R_{n+m+1}(x,v) \wedge \forall w\ (E(v,w) \rightarrow E(y,w) \vee E(z,w)))) \end{aligned}$$

for any $n, m \in \mathbb{N}_0$.

We computed these properties with a tool, called SCAN[1], which is an implementation of the quantifier elimination algorithm of Gabbay and Ohlbach [15]. To this end we used the standard Kripke semantics as translation rules for translating the axioms into predicate logic. In this *relational translation* (this is the *standard translation* ST of van Benthem [29, 30]), the formula variables become universally quantified predicate variables. The quantifier elimination algorithm produces for these second-order formulae equivalent formulae without predicate quantifiers. That means ST(N4) $\leftrightarrow$ N4′ etc. This procedure guarantees soundness of the semantics with respect to the axiom system, i.e.

$$\text{if } \vdash_{\overline{\mathrm{K}}_E} \varphi \text{ then } \models_{\overline{\mathrm{K}}_E} \varphi. \tag{4}$$

If all these 'frame axioms' (N4–N8) were first-order then the Sahlqvist Theorem [26, 29, 30] would ensure completeness of this frame class relative to the axioms. Unfortunately, N7′ is again second-order. Therefore we have to prove completeness explicitly. We do this indirectly *for translated* $\overline{\mathbf{K}}$ *formulae* by using the completeness of $\overline{\mathbf{K}}$ and the soundness and completeness of the translation into $\overline{\mathbf{K}}_E$ which is proven below. General completeness for arbitrary formulae is still open, but for the purpose of our translation, this is fortunately not necessary.

The correspondence property N4′ states that all singleton subsets of the set of worlds accessible by R_n are uniquely represented by a world accessible by R_0. N5′ asserts that every world accessible by R_n leads via E to another world. We say E is *weakly serial*. (Recall, a relation R is said to be *serial* (or *total*) iff $\forall x\ \exists y\ R(x,y)$). By N6′ the set $\{R_n\}_{n\in \mathbb{N}_0}$ of R_n relations forms a linear order with R_0 being the largest element, since for any $m > n$, R_m is a subrelation of R_n.

The correspondence property N7′ of N7 expresses intuitively that every set y with more than $n+m$ elements can be decomposed into a set u with more than n elements and a set v with more than m elements, and if y happens to have exactly $n+m+1$ elements then u and v overlap in at least one element.

N8′ expresses, as already mentioned, that for disjoint sets the cardinality of their union is the sum of the cardinalities of the sets.

For a better understanding of the frame properties it is helpful to think of the variable y in $R_n(x,y)$ and $E(y,z)$ as representing a set Y, $R_n(x,y)$ as representing that the cardinality of Y is greater than n, and $E(y,z)$ as representing that z is an element of Y. Then N4′–N8′ represent:

N4″ $\forall Yz\ ((|Y| > n \wedge z \in Y) \rightarrow \{z\} \subseteq Y$

N5″ $\forall y\ (|Y| > n \rightarrow Y \neq \emptyset)$

N6″ $\forall Y\ (|Y| > n+1 \rightarrow |Y| > n)$

[1] SCAN is accessible via World Wide Web at

`http://www.mpi-sb.mpg.de/guide/staff/ohlbach/scan/scan.html.`

This is a WWW interface for activating the program remotely. We invite the reader to use the tool and verify the above correspondence properties for N4–N8.

N7″ $\forall Y\ (|Y| > n+m \rightarrow$
$\forall fg\ \exists UV(|U| > m \wedge |V| > m) \rightarrow$
if f selects from U and g from V then $f(U) \in Y \wedge f(U) = g(V))$

N8″ $\forall YZ\ ((|Y| > n \wedge |Z| > m \wedge Y \cap Z = \emptyset) \rightarrow$
$\exists V\ (|V| > n+m+1 \wedge V \subseteq Y \cup Z))$.

We can show that the standard class of frames associated with $\overline{\mathbf{K}}_E$ have the expected structure, namely that all worlds accessible by R_n have more than n E-successors. However, non-standard $\overline{\mathbf{K}}_E$-frames exist which do not have this intended structure. The problem is, we cannot enforce in a Hilbert system that R_1-accessible worlds have *more than* one E-successor. This may be captured by an axiom like

$$[1](\exists p\ (\Diamond p \wedge \Diamond \neg p)),$$

or a rule similar to Gabbay's irreflexivity rule (but this gives no new theory) [16]. See also [25]. The modal language of $\overline{\mathbf{K}}$ and $\overline{\mathbf{K}}_E$ is not expressive enough to characterize this class of frames. On the other hand, we can show using an inductive argument that whenever an R_1-successor has more than one E-successor, then for any positive integer n every R_n-successor has more than n E-successors. This is to say, the induction step goes through, but unfortunately the base case of the induction cannot be guaranteed. Because the translation of the logic $\overline{\mathbf{K}}$ into the logic $\overline{\mathbf{K}}_E$ is sound and complete (we show this below), we know whenever a translated $\overline{\mathbf{K}}$-formula has a model then it has a model with the expected structure.

We did not investigate the non-standard models further. It may turn out that they are p-morphic images of standard models, in which case they are completely irrelevant because normal modal logics cannot distinguish p-morphic images.

3.2 From $\overline{\mathbf{K}}$ to $\overline{\mathbf{K}}_E$

Next we define a translation function mapping formulae of $\overline{\mathbf{K}}$ into formulae of $\overline{\mathbf{K}}_E$. We show that the translation is sound and complete.

Definition 3.2 The translation function Π maps $\overline{\mathbf{K}}$-formulae into $\overline{\mathbf{K}}_E$-formulae according to the following constraints:

$$\begin{array}{rcl} \Pi(p) & = & p \\ \Pi(\neg\varphi) & = & \neg\Pi(\varphi) \\ \Pi(\varphi\ @\ \psi) & = & \Pi(\varphi)\ @\ \Pi(\psi) \\ \Pi(\Diamond_n\varphi) & = & \langle n\rangle\Box\Pi(\varphi) \\ \Pi(\Box_n\varphi) & = & [n]\Diamond\Pi(\varphi), \end{array}$$

where p denotes any propositional variable and @ denotes any binary logical connective $\wedge$, $\vee$, $\rightarrow$ or $\leftrightarrow$.

Theorem 3.3 (Soundness of Π). *The translation Π from $\overline{\mathbf{K}}$ into $\overline{\mathbf{K}}_E$ is sound. That is, for any formula φ of $\overline{\mathbf{K}}$*

$$\textit{if } \vdash_{\overline{\mathrm{K}}} \varphi \ \textit{ then } \vdash_{\overline{\mathrm{K}}_E} \Pi(\varphi).$$

PROOF. Suppose φ is a theorem in $\overline{\mathbf{K}}$. We proceed by induction on the length of the proof of φ and show that the proof sequence of φ in $\overline{\mathbf{K}}$ determines a proof sequence of $\Pi(\varphi)$ in $\overline{\mathbf{K}}_E$. We are done if we show that the Π-translations of the axioms and the rules of $\overline{\mathbf{K}}$ are $\overline{\mathbf{K}}_E$-theorems.

Π leaves the propositional axioms and Modus Ponens unchanged. The translation of the necessitation rule N is:

$$\vdash_{\overline{\mathrm{K}}_E} \varphi \quad \text{implies} \quad \vdash_{\overline{\mathrm{K}}_E} [0]\Diamond\varphi.$$

If φ holds then, by the necessitation rule for $\Box$ and $[0]$, $[0]\Box\varphi$ holds. Apply modus ponens using the contrapositive instance with $n = 0$ of N5 and get $[0]\Diamond\varphi$[2].

[2]Formally, instead of φ one has to consider $\Pi(\varphi)$. But for the proofs this makes no difference.

The translation of A2 is a contrapositive version of N6. It remains to prove the translations of A3 and A4 can be derived from the axioms of $\overline{\mathbf{K}}_E$ using the rules of $\overline{\mathbf{K}}_E$.

For A3 we prove

$$\Pi(\mathrm{A3}) = [0]\Diamond(\varphi \to \psi) \to (\langle n\rangle\Box\varphi \to \langle n\rangle\Box\psi)$$

is a theorem in $\overline{\mathbf{K}}_E$. Suppose $[0]\Diamond(\varphi \to \psi)$ and $\langle n\rangle\Box\varphi$ hold. Suppose further that $\neg\langle n\rangle\Box\psi$, i.e. $[n]\Diamond\neg\psi$, holds. From $[0]\Diamond(\varphi \to \psi)$ we infer by N4 that $[n]\Box(\varphi \to \psi)$ holds. By the K-axiom for $\Box$, it follows that $[n](\Box\varphi \to \Box\psi)$. This is equivalent to $[n](\neg\Box\psi \to \neg\Box\varphi)$, i.e. $[n](\Diamond\neg\psi \to \Diamond\neg\varphi)$. Using the **K**-axiom for $[n]$ we infer $[n]\Diamond\neg\psi \to [n]\Diamond\neg\varphi$. From $\neg\langle n\rangle\Box\psi$, which is equivalent to $[n]\Diamond\neg\psi$, using Modus Ponens we get $[n]\Diamond\neg\varphi$, or equivalently $\neg\langle n\rangle\Box\varphi$. This contradicts $\langle n\rangle\Box\varphi$. Thus $\Pi(\mathrm{A3})$ is derivable in $\overline{\mathbf{K}}_E$.

For A4: Let

$$\phi = [0]\Diamond\neg(\varphi \wedge \psi) \wedge \langle n \perp 1\rangle\Box\varphi \wedge \neg\langle n\rangle\Box\varphi \wedge \langle m \perp 1\rangle\Box\psi \wedge \neg\langle m\rangle\Box\psi.$$

Then $\Pi(\mathrm{A4})$ is equivalent to

$$\phi \to (\langle n+m \perp 1\rangle\Box(\varphi \vee \psi) \wedge \neg\langle n+m\rangle\Box(\varphi \vee \psi)).$$

We prove this in two steps. First, we prove $\phi \to \langle n+m \perp 1\rangle\Box(\varphi \vee \psi)$. Suppose ϕ holds. It suffices to show

$$\langle n \perp 1\rangle\Box(\varphi \wedge \neg\psi). \tag{5}$$

From $\langle n \perp 1\rangle\Box(\varphi \wedge \neg\psi)$, or equivalently $\langle n \perp 1\rangle\Box((\varphi \vee \psi) \wedge \neg\psi)$, and $\langle m \perp 1\rangle\Box\psi$, or equivalently $\langle m \perp 1\rangle\Box((\varphi \vee \psi) \wedge \psi)$, using axiom N8 we deduce $\langle n+m \perp 1\rangle\Box(\varphi \vee \psi)$.

For proving that (5) follows from ϕ we proceed by contradiction. Suppose that $\neg\langle n \perp 1\rangle\Box(\varphi \wedge \neg\psi)$, i.e. $[n \perp 1]\Diamond(\neg\varphi \vee \psi)$ holds. From $[0]\Diamond\neg(\varphi \wedge \psi)$ using N4 we get $[n \perp 1]\Box\neg(\varphi \wedge \psi)$ Since, in general, in any normal modal logic $\Box$ (and, in particular, $[n \perp 1]$) distributes over conjunction, we obtain

$$[n \perp 1](\Diamond(\neg\varphi \vee \psi) \wedge \Box(\neg\varphi \vee \neg\psi)). \tag{6}$$

The **K**-axiom for $\Box$ is equivalent to $(\Box\varphi \wedge \Diamond\psi) \to \Diamond(\varphi \wedge \psi)$. Thus (6) is equivalent to $[n \perp 1](\Diamond((\neg\varphi \vee \psi) \wedge (\neg\varphi \vee \neg\psi)))$. This in turn is equivalent to $[n \perp 1]\Diamond\neg\varphi$. Thus $\neg\langle n \perp 1\rangle\Box\varphi$ which contradicts $\langle n \perp 1\rangle\Box\varphi$.

Next, we prove $\phi \to \neg\langle n+m\rangle\Box(\varphi \vee \psi)$. Suppose ϕ holds. Then, in particular, $\neg\langle n\rangle\Box\varphi$ and $\neg\langle m\rangle\Box\psi$ hold, and $\neg\langle n+m\rangle\Box(\varphi \vee \psi)$ is derivable by (the contraposition of) N7. Therefore, $\Pi(\mathrm{A4})$ is a theorem in $\overline{\mathbf{K}}_E$. ∎

For proving completeness a semantic proof suffices. We prove (in the next the theorem) that for a translated formula $\Pi(\varphi)$ which is true in all $\overline{\mathbf{K}}_E$-frames, φ is true in all $\overline{\mathbf{K}}$-frames. If $\Pi(\varphi)$ is provable in $\overline{\mathbf{K}}_E$ then the soundness of the $\overline{\mathbf{K}}_E$-semantics guarantees that $\Pi(\varphi)$ is true in all $\overline{\mathbf{K}}_E$-frames, then φ is true in all $\overline{\mathbf{K}}$-frames and then it is provable in $\overline{\mathbf{K}}$ (by the completeness of $\overline{\mathbf{K}}$).

Theorem 3.4 *For any formula φ of $\overline{\mathbf{K}}$,*

$$\textit{if} \ \models_{\overline{\mathrm{K}}_E} \Pi(\varphi) \ \ \textit{then} \ \models_{\overline{\mathrm{K}}} \varphi.$$

PROOF. Our strategy is the following. Suppose $\models_{\overline{\mathrm{K}}_E} \Pi(\varphi)$. (i) For an arbitrary $\overline{\mathbf{K}}$-frame $\mathcal{F}$ we construct a $\overline{\mathbf{K}}_E$-frame $\mathcal{F}'$. $\Pi(\varphi)$ is valid in this particular frame $\mathcal{F}'$. Then we show, (ii) φ is valid in $\mathcal{F}$.

(i): Take any $\overline{\mathbf{K}}$-frame $\mathcal{F} = (W, R)$. We construct a $\overline{\mathbf{K}}_E$-frame $\mathcal{F}'$ as an extension of the frame $\mathcal{F}$ as follows. For any world $x \in W$ let $R(x)$ be the R-image of x. For any finite subset Y of $R(x)$ with $|Y| = n+1$ for n a non-negative integer, we add Y as a new world to $\mathcal{F}$. We call Y a 'set-world'. Note, every Y is non-empty. We define every relation R_m for $m \le n$ to contain the pair (x, Y), and, we define the relation E to contain all pairs (Y, z) for $z \in Y$. Furthermore, we assume the relations R_n and E are the smallest relations satisfying these conditions. Now, define $\mathcal{F}'$ to be the relational structure

$$(W', \{R_n\}_{n \in \mathbb{N}_0}, E)$$

with W' being the set of worlds of $\mathcal{F}'$ that includes the set of worlds W of $\mathcal{F}$ and all set-worlds Y. Note, the set-worlds have no R_n-successors and the worlds in W have no E-successors. We show that $\mathcal{F}'$ is a frame for $\overline{\mathbf{K}}_E$ by showing that $\mathcal{F}'$ satisfies the properties N4′–N8′.

N4′: If $R_n(x,y) \wedge E(y,z)$ holds then y must be a set-world with $|y| > n$ and $z \in y$. For $u = \{z\}$ we obtain $R_0(x,u) \wedge \forall v\ (E(u,v) \to v = z)$.

N5′: If $R_n(x,y)$ then y is a non-empty set-world, i.e. $\exists z\ E(y,z)$ is true.

N6′: If $R_{n+1}(x,y)$ then y is a set-world with $|y| > n+1 > n$, i.e. $R_n(x,y)$ holds as well.

Recall N7′:

$$\begin{aligned}\forall xy\ R_{n+m}(x,y) \to\ & \forall fg\ \exists uv(R_n(x,u) \to E(u,f(u)) \wedge \\ & R_m(x,v) \to E(v,g(v))) \to \\ & (R_n(x,u) \wedge R_m(x,v) \wedge E(y,f(u)) \wedge f(u) = g(v))\end{aligned}$$

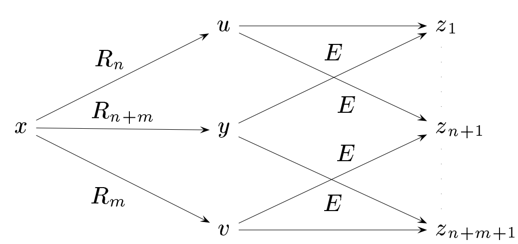
$R_{n+m}(x,y)$ means that y is a set-world with $|y| > n+m$. We distinguish two cases.

Case 1: $|y| = n+m+1$. Let f and g be any functions mapping worlds to worlds. If there is at least one R_n-accessible set-world u with $|u| > n$ and f does not map u to one of its elements ($\neg E(u,f(u))$) or there is at least one R_m-accessible set-world v with $|v| > n$ and g does not map v to one of its elements ($\neg E(v,g(v))$) then we can choose this u or v, respectively. Then the implication

$$\begin{aligned}&(R_n(x,u) \to E(u,f(u)) \wedge R_m(x,v) \to E(v,g(v))) \to \\ &(R_n(x,u) \wedge R_m(x,v) \wedge E(y,f(u)) \wedge f(u) = g(v))\end{aligned} \tag{7}$$

is true because the premiss is false.

Now assume, f chooses for every set-world u with $|u| > n$ some element $f(u) \in u$ and g chooses for every set-world v with $|v| > m$ some element $g(v) \in v$. The key observation for the proof is that for every set with $n+m+1$ elements every decomposition into a set u with $n+1$ elements and a set v with $m+1$ elements overlaps in at least one element. Thus, the situation is as depicted in the following figure.



For finding the right u and v we follow this procedure: we start by choosing a subset $u_1 \subseteq y$ with $n+1$ elements. Suppose $f(u_1) = x_1$. If there is a subset $v \subseteq y$ with $|v| > m$ and $g(v) = x_1$ we are done. Suppose for no such subset we have $g(v) = x_1$. x_1 is marked as 'not an image of g'. Now we choose another $n+1$-element subset u_2 of y which *does not contain* x_1. Suppose $f(u_2) = x_2$. Again, if for some subset v with $|v| > m$ we find $g(v) = x_2$ we are done. If not, we mark x_2 as 'not an image of g'. We continue until we have found a suitable u and v, or until exactly $n+1$ worlds remain which are not marked 'not an image of g'. In the latter case we choose this set for u. Suppose $f(u) = x$. Take $v = y \setminus u \cup \{x\}$. $|v| = m+1$ and $g(v) \neq z$ for all $z \in y \setminus u$. Since g must select some element in v, $g(v) = x$ is the only choice. Thus, suitable u and v exist that satisfy (7).

Case 2: $|y| > n+m+1$. Take any subset $y' \subseteq y$ with $|y'| = n+m+1$. By Case 1, we can find for any f and g subsets $u \subseteq y'$ and $v \subseteq y'$ with the property (7). But these are also subsets of y and therefore the property holds as well.

N8′: This property expresses that the union of two disjoint sets of cardinality $> n$ and $> m$ is a set with cardinality $> n+m+1$, and this is true in $\mathcal{F}'$.

We have proved $\mathcal{F}'$ is a frame for $\overline{\mathbf{K}}_E$.

(ii): Let $\mathcal{M} = (\mathcal{F}, V)$ be any model based on $\mathcal{F}$ with V an arbitrary valuation. Define $\mathcal{M}'$ to be the model $(\mathcal{F}', V)$. (Observe that $V(p)$ does not, and need not contain set-worlds.) (ii) follows from

$$\mathcal{M}', x \models_{\overline{\mathbf{K}}_E} \Pi(\varphi) \quad \text{iff} \quad \mathcal{M}, x \models_{\overline{\mathbf{K}}} \varphi \tag{8}$$

where x is any world in W. We prove (8) by induction on the structure of φ. The base case in which φ is any propositional variable is trivial. The inductive step for the propositional connectives goes through easily. We consider the case φ is of the form $\Diamond_n\psi$. (The case for φ of the form $\Box_n\psi$ is dual.) The inductive hypothesis is:

$$\mathcal{M}', x \models_{\overline{\mathbf{K}}_E} \Pi(\psi) \quad \text{iff} \quad \mathcal{M}, x \models_{\overline{\mathbf{K}}} \psi.$$

Suppose $\mathcal{M}', x \models_{\overline{\mathbf{K}}_E} \Pi(\Diamond_n\psi)$, i.e. $\langle n\rangle\Box\psi$ is true at x in $\mathcal{M}'$. Then, $R_n(x, Y)$ in $\mathcal{F}'$ for some set $Y \subseteq R(x)$ with $n+1$ elements and for all $z \in Y$ we have $\mathcal{M}', z \models_{\overline{\mathbf{K}}_E} \psi$ and by the inductive hypothesis $\mathcal{M}, z \models_{\overline{\mathbf{K}}} \psi$. There are at least n such z, therefore, $\mathcal{M}, x \models_{\overline{\mathbf{K}}} \Diamond_n\psi$.

Conversely, suppose $\mathcal{M}, x \models_{\overline{\mathbf{K}}} \Diamond_n\psi$. This means the world x has more than n successors by R in all of which ψ is true. Consequently, a set Y with cardinality $n+1$ exists that contains R-successors y of x and in all y, ψ is true. This implies, in $\mathcal{F}'$, x and Y are connected by R_n and Y is connected to all its elements by E. Thus, $\langle n\rangle\Box\psi$ is true in x.

This completes the proof. ∎

As consequences we get the following two theorems.

Theorem 3.5 (Completeness of Π). *The translation* Π *from* $\overline{\mathbf{K}}$ *into* $\overline{\mathbf{K}}_E$ *is complete. That is, for any formula* φ *of* $\overline{\mathbf{K}}$*,*

$$\textit{if } \vdash_{\overline{\mathrm{K}}_E} \Pi(\varphi) \textit{ then } \vdash_{\overline{\mathrm{K}}} \varphi.$$

PROOF. Suppose $\Pi(\varphi)$ is a theorem in $\overline{\mathbf{K}}_E$, i.e. $\vdash_{\overline{\mathrm{K}}_E} \Pi(\varphi)$. Then, since $\overline{\mathbf{K}}_E$ is sound (4), $\models_{\overline{\mathrm{K}}_E} \Pi(\varphi)$. By the previous theorem $\models_{\overline{\mathrm{K}}} \varphi$. $\overline{\mathbf{K}}$ is sound and complete (Theorem 2.3). Therefore, it follows that $\vdash_{\overline{\mathrm{K}}} \varphi$. ∎

Now, we can show the completeness of the semantics of $\overline{\mathbf{K}}_E$ with respect to its axiomatization *for translated formulae.*

Theorem 3.6 (Relative completeness of $\overline{\mathbf{K}}_E$). *For any* $\overline{\mathbf{K}}$ *formula* φ

$$\textit{if } \models_{\overline{\mathrm{K}}_E} \Pi(\varphi) \textit{ then } \vdash_{\overline{\mathrm{K}}_E} \Pi(\varphi).$$

PROOF. If $\Pi(\varphi)$ holds in all $\overline{\mathbf{K}}_E$-frames then φ holds in all $\overline{\mathbf{K}}$-frames (by Theorem 3.4), then φ is provable in $\overline{\mathbf{K}}$ (by the completeness of $\overline{\mathbf{K}}$, Theorem 2.3), and then $\Pi(\varphi)$ is provable in $\overline{\mathbf{K}}_E$ (by the soundness of the translation, Theorem 3.3). ∎

4 From multi-modal logic to predicate logic

We aim at making available first-order theorem proving methods for reasoning with graded modal expressions. In the previous section we have embedded the logic $\overline{\mathbf{K}}$ in the multi-modal logic $\overline{\mathbf{K}}_E$. Unfortunately, one of the axioms, namely N7, is not first-order definable in the standard Kripke semantics. Its relational translation N7$'$ is a second-order formula. So, instead of using the standard relational translation we use the functional translation as proposed in Ohlbach and Schmidt [22] for non-first-order axioms like McKinsey's axiom.

The functional translation method was proposed by various authors, for example Ohlbach [23, 24], [10], [18], [2] and [36]. It exploits the fact that every binary relation can be decomposed into a set AF_R of functions, called *accessibility functions.* Any (non-empty) relation R is defined by:

$$R(x,y) \leftrightarrow \exists\gamma \in AF_R \; y = \gamma(x).$$

In the functional translation we quantify over the accessibility functions instead of worlds. For modalities determined by serial (i.e. total) accessibility relations, that is, for modalities satisfying the D-axiom the functional translation rules for modal formulae are:

$$\begin{aligned}\pi_f([R]\psi, x) &= \forall\gamma{:}AF_R \; \pi_f(\psi, \downarrow(\gamma, x)) \\ \pi_f(\langle R\rangle\psi, x) &= \exists\gamma{:}AF_R \; \pi_f(\psi, \downarrow(\gamma, x)).\end{aligned}$$

The target logic is a many-sorted predicate logic, in which AF_R is the sort for accessibility functions defining R and the symbol $\downarrow$ is a function symbol for the 'apply' function (this means, the application of a function f to x, i.e. $f(x)$, is encoded by $\downarrow(f, x)$). For modalities determined by accessibility relations that are not serial the set of accessibility functions AF_R contains partial functions. Accordingly, the functional translation π_f of modal formulae must compensate for partiality by an extra condition involving a predicate de_R. For non-serial modalities π_f is defined by:

$$\neg de_R(x) \to \forall\gamma{:}AF_R\ \pi_f(\psi, \downarrow(\gamma, x)) \quad \text{and} \quad \neg de_R(x) \land \exists\gamma{:}AF_R\ \pi_f(\psi, \downarrow(\gamma, x)).$$

The term $\neg de_R(x)$ is meant to capture that x is not a dead-end in the relation R.

We now give the formal definition of the functional translation Π_f for multi-modal logics of which $\mathbf{K}_{(m)}$ is the weakest. Π_f is a function mapping formulae of $\mathbf{K}_{(m)}$ to formulae of a many-sorted predicate logic PL_M (with predicate variables) with a signature specified by:

(i) sort symbols $\perp$ (for the bottom sort), W (for the world sort) and W^-.
(ii) sort declarations $W \sqsubseteq W^-$ and $\perp \sqsubseteq W^-$.
(iii) sort symbols AF_R for every modality $\langle R\rangle$
(iv) a binary function symbol $\downarrow$ declared by $\downarrow$: $AF_R \times W^- \to W^-$ for all AF_R.
(v) predicate symbols de_R for every modality $\langle R\rangle$ (de is short for dead-end, or in our application, $de_n(x)$ means the set-world x does *not* represent more than n elements) and
(vi) for each propositional variable p there is a unary predicate symbol p (we purposely use the same symbols).

This signature defines a many-sorted predicate logic we refer to as PL_M.

Definition 4.1 (The functional translation). Let π_f be a function that takes two arguments: a modal formula and a 'world term' which are mapped to a formula in PL_M. π_f is defined inductively by

$$\begin{aligned}
\pi_f(p, w) &= p(w) \qquad \text{for } p \text{ a propositional variable}\\
\pi_f([R]\psi, x) &= \neg de_R(x) \to \forall\gamma{:}AF_R\ \pi_f(\psi, \downarrow(\gamma, x))\\
\pi_f(\langle R\rangle\psi, x) &= \neg de_R(x) \land \exists\gamma{:}AF_R\ \pi_f(\psi, \downarrow(\gamma, x))
\end{aligned}$$

and for the propositional connectives π_f is a homomorphism.

The *functional translation* for a multi-modal formula φ with propositional variables $p_1, \ldots, p_n$ is defined by

$$\Pi_f(\varphi) \;=\; \forall p_1, \ldots, p_n \forall w{:}W\ \pi_f(\varphi, w).$$

For a Hilbert rule of the form 'from φ_1 and ... and φ_n infer φ'

$$\Pi_f(\varphi_1) \land \ldots \land \Pi_f(\varphi_n) \to \Pi_f(\varphi)$$

is the functional translation. Π_f maps a set Φ of Hilbert axioms and rules to the conjunction of the functional translation of the members:

$$\Pi_f(\Phi) = \bigwedge_{\varphi\in\Phi} \Pi_f(\varphi).$$

Π_f is called the *functional translation function.*

For serial modalities the $\neg de_R(x)\ldots$ part of the definition in π_f can be omitted.

As with the relational translation, the functional translation of the Hilbert axioms yields second-order formulae with universally quantified predicate variables. This translation is sound, i.e. whatever can be proved from the axioms, holds in the models of these second-order formulae. If these second-order formulae are equivalent to a first-order formula then the Sahlqvist theorem together with the completeness of the transition from the relational to the functional representation (which is easy) guarantees completeness. If the second-order formulae are not equivalent to first-order formulae then completeness is still an open problem. Unfortunately this is the case for $\overline{\mathbf{K}}_E$.

Ohlbach and Schmidt [22] prove the following relativized soundness and completeness result for the functional translation Π_f.

Theorem 4.2 (Relative soundness and completeness of the functional translation) *Let Φ be additional Hilbert axioms in a propositional modal logic $\mathbf{K}_{(m)}$ and φ any modal formula. If the relational second-order translation of Φ is complete then*

$$\varphi \textit{ is a } \Phi\textit{-theorem} \quad \textit{iff} \quad \Pi_f(\Phi) \to \Pi_f(\varphi) \textit{ is a predicate logic theorem in } PL_M.$$

The result of the translation by Π_f is in general not a first-order expression. The translation is useful only if the axioms in $\Pi_f(\Phi)$ can be described by a set Φ' of first-order formulae. If $\Pi_f(\Phi)$ is equivalent to such a set Φ' the implication

$$\Pi_f(\Phi) \rightarrow \Pi_f(\varphi)$$

can be proved by refuting the formula

$$\Phi' \wedge \neg\Pi_f(\varphi).$$

$\Pi_f(\varphi)$ is a monadic second-order formula only with second-order universal quantifiers. In the negation normal form of its negation $\neg\Pi_f(\varphi)$ only existentially quantified second-order predicate variables occur and these are treated as ordinary first-order predicates. Therefore, $\Phi' \rightarrow \Pi_f(\varphi)$ can be proved with the standard first-order procedures.

Not every axiomatization Φ has an equivalent first-order formulation. The theorem above can be strengthened for certain axioms without first-order relational characterizations when we use the following quantifier exchange rule.

Definition 4.3 (Quantifier exchange rule). Let φ be any modal formula in $\mathbf{K}_{(m)}$. Define an operation Υ on PL_M which transforms the functional translation $\Pi_f(\varphi)$ into its prenex normal form according to the rule

$$\exists\gamma{:}AF_R\ \forall\delta{:}AF_R\ \psi \qquad \rightsquigarrow \qquad \forall\delta{:}AF_R\ \exists\gamma{:}AF_R\ \psi. \tag{9}$$

The operation Υ moves existential functional quantifiers inwards thus weakening the original formula. $\Upsilon(\Pi_f(\varphi))$ implies $\Pi_f(\varphi)$, but not conversely. The quantifier exchange rule exploits that one relational frame in general corresponds to many 'functional frames', and there is always one which is rich enough to allow for moving existential quantifiers over universal quantifiers. This is investigated in Ohlbach and Schmidt [22] where a stronger theorem than Theorem 4.2 is proved, namely:

Theorem 4.4 (Relative soundness and completeness of the functional translation with the quantifier exchange rule) *Let Φ be additional Hilbert axioms in a propositional modal logic $\mathbf{K}_{(m)}$ and φ any modal formula. If the relational second-order translation of Φ is complete then*

$$\varphi \textit{ is a } \Phi\textit{-theorem} \quad \textit{iff} \quad \Upsilon(\Pi_f(\Phi)) \rightarrow \Upsilon(\Pi_f(\varphi)) \textit{ is a theorem in } PL_M,$$

provided in $\Upsilon(\Pi_f(\varphi))$ all existential functional quantifiers are moved inward as far as possible.

This theorem says that the original formula φ can be proved to be a theorem in the system Φ by proving its weakened translation $\Upsilon(\Pi_f(\varphi))$ using the weakened forms $\Upsilon(\Pi_f(\Phi))$ of the translations of the axioms in Φ. In $\Upsilon(\Pi_f(\varphi))$ the existential functional quantifiers are pushed inward as far as possible. Negating $\Upsilon(\Pi_f(\varphi))$ we simultaneously replace universal quantifiers by existential quantifiers and existential quantifiers by universal quantifiers. The quantifier prefix of the prenex normal form of $\neg\Upsilon(\Pi_f(\varphi))$ consists of a sequence of existentially quantified predicate variables p_i (ended with an existentially quantified world variable) followed by a sequence of existentially quantified functional variables, followed by a sequence of universally quantified functional variables. In the Skolemized clause form of $\neg\Upsilon(\Pi_f(\varphi))$ no Skolem functions occur, only Skolem constants. This simplifies the translation considerably. More importantly, Υ allows us to move existential quantifiers inward as far as we like. This weakens an axiom like the McKinsey axiom just enough so that we get a first-order translation for the axiom. This operation works for axiom N7 as well, as we will see in the next section.

The functional translation generates nested $\downarrow$-terms as arguments to predicates. We can avoid these by using the world path notation of Ohlbach [23]. To this end we add a new sort symbol AF^* to PL_M and we let $\circ$ be a new binary function symbol. Furthermore, we include the following axioms and sort declarations:

$$\begin{array}{l} AF_R \sqsubseteq AF^* \\ \circ : AF^* \times AF^* \rightarrow AF^* \\ \forall x{:}W\ \forall\gamma,\delta{:}AF^*\ \downarrow(\gamma\circ\delta, x) = \downarrow(\delta, \downarrow(\gamma, x)) \\ \circ \text{ is associative.} \end{array}$$

$\circ$ denotes composition operation of accessibility functions and AF^* denotes the set of all possible compositions of all accessibility functions in the union of AF_R. Instead of nested $\downarrow$-terms, like $\downarrow(\delta, \downarrow(\gamma, x))$, we use a more economic notation and write $\downarrow((\gamma \circ \delta), x)$ or $\downarrow([\gamma\delta], x)$ (omitting $\circ$), instead. The latter uses the world path syntax which we prefer from here on.

Notice that the conditions in the two theorems requiring that the relational translation of the axioms into second-order logic is complete means that all formulae which are valid in the frames characterized by these second-order formulae are provable from the axioms. For $\overline{\mathbf{K}}_E$ we showed this for the translated $\overline{\mathbf{K}}$ formulae only. Since this is sufficient for our purpose, we can assume completeness of these transformations.

5 From $\overline{\mathbf{K}}_E$ to predicate logic

In this section we apply the functional translation method explained in the previous section to the modal logic $\overline{\mathbf{K}}_E$ which we introduced in Section 3.

Recall, $\overline{\mathbf{K}}_E$ is a multi-modal logic with infinitely but countably many numerical modal operators ($\langle n\rangle$ and $[n]$) and two special membership operators $\Diamond$ and $\Box$. In the relational semantics the numerical operators are interpreted by the set $\{R_n\}_{n\in \mathbb{N}_0}$ of binary relations and the membership operators by the special relation E. The functional translation for serial modalities is considerably simpler than for non-serial modalities. The accessibility relations R_n ($n \in \mathbb{N}_0$) and E are not serial. Axiom N5, $\langle n\rangle\Box\varphi \rightarrow \langle n\rangle\Diamond\varphi$, specifies a weak form of seriality for E. Every world accessible by some R_n (i.e. every set-world) has a successor by E.

We do not need the full expressiveness of the language of $\overline{\mathbf{K}}_E$. A subset of formulae with characteristic patterns of modal operators $\langle n\rangle$, $[n]$, $\Diamond$ and $\Box$ will do. For example, in the axiomatization defining $\overline{\mathbf{K}}_E$ the operators $\Diamond$ and $\Box$ do not occur in the scope of $\Diamond$ and $\Box$ operators, they always occur in the scope of $\langle n\rangle$ and $[n]$ operators. This is intentional. Only these patterns make sense in our application of $\overline{\mathbf{K}}_E$. We think of the numeric modalities picking only *sets* and the $\Diamond$ and $\Box$ operators picking only elements of these sets. For this we need a special class of formulae in which the E-successors of worlds accessible by E are irrelevant, only E-successors of worlds accessible by the relations R_n count. We are therefore permitted to assume E is serial (which we prove in Theorem 5.1).

The language of $\overline{\mathbf{K}}_E$ that we will use is restricted to the set of *admissible* formulae. We say a formula φ of $\overline{\mathbf{K}}_E$ is admissible iff all $\Diamond$ and $\Box$ operators appear in the scope of a $\langle n\rangle$ or $[n]$ operator (for some n). Examples of admissible formulae are:

$$\langle n\rangle\Box p, \quad [n]\Diamond p, \quad \langle n\rangle(p \wedge \Diamond q) \quad \text{and} \quad [n](\neg\Box p \rightarrow \Diamond q).$$

The formulae

$$\Diamond p, \quad \text{and} \quad \langle n\rangle\Box\Diamond q$$

on the other hand, are not admissible. We note that the translations of $\overline{\mathbf{K}}$ (presented in Section 3.2) are admissible formulae, since the corresponding modalities for modal operators of $\overline{\mathbf{K}}$ are $\langle n\rangle\Box$ and $[n]\Diamond$. The axioms N4–N8 are also admissible, whenever φ, ψ and ϕ_i are admissible.

For admissible formulae we may assume the relation E is serial.

Theorem 5.1 *Let φ be an admissible formula of $\overline{\mathbf{K}}_E$. If φ is valid in a model then φ is valid in a model in which the relation E (associated with the modalities $\Diamond$ and $\Box$) is serial.*

Proof. Let φ be valid in a model $\mathcal{M} = (\mathcal{F}, V)$ based on the frame

$$\mathcal{F} = (W, \{R_n\}_{n\in \mathbb{N}_0}, E).$$

Define $\mathcal{F}^*$ to be the structure $(W, \{R_n\}_{n\in \mathbb{N}_0}, E^*)$ obtained from $\mathcal{F}$ by replacing E with E^*. E^* includes E and all pairs (x, x) of $x \in W$ for which no $y \in W$ exists such that $E(x, y)$. Then, E^* is serial.

We show that φ is valid in $\mathcal{M}^* = (\mathcal{F}^*, V)$. The only critical case is where φ has a formula equivalent to $\psi = \Box\phi \wedge \Box\neg\phi$ as subformula. For ψ is true at any world x in $\mathcal{M}$ iff x has no E-successor in $\mathcal{M}$. In $\mathcal{M}^*$, however, ψ is false at x whenever ψ is true at x in $\mathcal{M}$(otherwise there is an inconsistency). ψ occurs in the scope of either $\langle n\rangle$ or $[n]$ for some n. First, we consider the case that ψ occurs in the scope of $\langle n\rangle$. Let φ' be the $\langle n\rangle$ subformula of φ with ψ in its scope. We may assume φ' is of the form $\langle n\rangle((\psi \vee \alpha) \wedge \beta)$. Now, suppose φ' is true in a world x of $\mathcal{M}$. Then there is a $y \in W$

such that $R_n(x,y)$. Since E is weakly serial there is a $z \in W$ such that $E(y,z)$, which implies ψ is false in y of $\mathcal{M}$. Hence, φ' is true in x of $\mathcal{M}$ iff it is also true in x of $\mathcal{M}^*$. Next, we consider the case that ψ occurs in the scope of $[n]$. Assume φ' is of the form $[n]((\psi \vee \alpha) \wedge \beta)$ and suppose φ' is true in $x \in W$ of $\mathcal{M}$. Then either there are or there are no y's in W such that $R_n(x,y)$. If there are y's then we argue as above. If there are no y's then φ' is trivially true in x of $\mathcal{M}^*$. We conclude, φ is indeed valid in $\mathcal{M}^*$. ∎

This theorem licenses the translation of the $\Diamond$ and $\Box$ operators without the dead-end predicate de_E. For admissible formulae in $\overline{\mathbf{K}}_E$ the translation function π_f is modified as follows:

$$\begin{aligned}
\pi_f([n]\psi, x) &= \neg de_n(x) \rightarrow \forall\gamma{:}AF_n\ \pi_f(\psi, \downarrow(\gamma, x)) \\
\pi_f(\langle n\rangle\psi, x) &= \neg de_n(x) \wedge \exists\gamma{:}AF_n\ \pi_f(\psi, \downarrow(\gamma, x)) \\
\pi_f(\Box\psi, x) &= \forall\gamma{:}AF_E\ \pi_f(\psi, \downarrow(\gamma, x)) \\
\pi_f(\Diamond\psi, x) &= \exists\gamma{:}AF_E\ \pi_f(\psi, \downarrow(\gamma, x)).
\end{aligned}$$

We are now set to compute the functional translation of the $\overline{\mathbf{K}}_E$-axioms N4–N8. This is a mechanical and tedious task, which we left to an implementation of the general translation procedure and our tool for eliminating second-order quantifiers.

In our listing of the result we use the following notation. $\gamma{:}n$ is the abbreviation for $\gamma{:}AF_n$ and $\gamma{:}E$ for $\gamma{:}AF_E$. Variables without sort declarations are assumed to be of sort W. In the respective clause forms variables and Skolem functions are indexed by their sort. The value in the subscript of a Skolem function, say f_m^n, associates the function with the m-th modality $\langle m\rangle$. AF_m is the sort of the terms formed with f_m^n. The superscript is part of the name of the Skolem function. It is only used to distinguish the different Skolem functions for the different instances of the clauses. (In an actual implementation, these numbers are the objects of symbolic arithemtical manipulations.) γ_n indicates γ is a variable of sort AF_n.

In the following we list for each axiom N4–N8, (i) the functional translation, (ii) the first-order equivalent formulation and (iii) its clause form.

N4: $\vdash_{\overline{\mathrm{K}}_E} [0]\Diamond\varphi \rightarrow [n]\Box\varphi$.

The functional translation $\Pi_f(\mathrm{N4})$ is

$$\begin{aligned}
&\forall\varphi\forall x\ [(\neg de_0(x) \rightarrow \forall\gamma'{:}0\ \exists\delta'{:}E\ \varphi(\downarrow([\gamma'\delta'], x))) \rightarrow \\
&\qquad (\neg de_n(x) \rightarrow \forall\gamma{:}n\ \forall\delta{:}E\ \varphi(\downarrow([\gamma\delta], x)))].
\end{aligned}$$

This has a first-order equivalent formulation, namely

$$\begin{aligned}
&\forall x\ de_0(x) \rightarrow de_n(x) \wedge \\
&\forall x\ [\neg de_n(x) \rightarrow (\forall\gamma{:}n\ \forall\delta{:}E\ \exists\gamma'{:}0\ \forall\delta'{:}E \\
&\quad \downarrow([\gamma\delta], x) = \downarrow([\gamma'\delta'], x))].
\end{aligned} \tag{10}$$

The clause form is:

$$\begin{aligned}
&\neg de_0(x) \vee de_n(x). \\
&de_n(x) \vee \downarrow([\gamma_n\delta_E], x) = \downarrow([f_0^n(x, \gamma_n, \delta_E)\delta'], x).
\end{aligned}$$

N5: This axiom becomes a tautology because we assume E is a serial relation.

N6: $\vdash_{\overline{\mathrm{K}}_E} [n]\varphi \rightarrow [n+1]\varphi$.

$\Pi_f(\mathrm{N6})$ is given by

$$\begin{aligned}
&\forall\varphi\forall x\ [(\neg de_n(x) \rightarrow \forall\delta{:}n\ \varphi(\downarrow(\delta, x))) \rightarrow \\
&\qquad (\neg de_{n+1}(x) \rightarrow \forall\gamma{:}n{+}1\ \varphi(\downarrow(\gamma, x)))],
\end{aligned}$$

the first-order equivalent by:

$$\begin{aligned}
&\forall x\ de_n(x) \rightarrow de_{n+1}(x) \wedge \\
&\forall x\ [\neg de_{n+1}(x) \rightarrow (\forall\gamma{:}n{+}1\ \exists\delta{:}n\ \downarrow(\gamma, x) = \downarrow(\delta, x))],
\end{aligned} \tag{11}$$

and the clause form by:

$$\begin{aligned}
&\neg de_n(x) \vee de_{n+1}(x). \\
&de_{n+1}(x) \vee \downarrow(\gamma_{n+1}, x) = \downarrow(g_n^n(x, \gamma_{n+1}), x).
\end{aligned}$$

N7: $\vdash_{\overline{\mathrm{K}}_E} \langle n+m\rangle\Box(\varphi\vee\psi)\rightarrow(\langle n\rangle\Box\varphi\vee\langle m\rangle\Box\psi)$

is translated to $\Pi_f(\mathrm{N7})$:

$$\begin{aligned}\forall\varphi\psi\forall x\ [&(\neg de_{n+m}(x)\wedge\exists\gamma{:}n{+}m\ \forall\delta{:}E\ (\varphi(\downarrow([\gamma\delta],x))\vee\psi(\downarrow([\gamma\delta],x))))\\ &\rightarrow\ [(\neg de_n(x)\wedge\exists\gamma{:}n\ \forall\alpha{:}E\ \varphi(\downarrow([\gamma\alpha],x)))\vee\\ &\qquad(\neg de_m(x)\wedge\exists\gamma{:}m\ \forall\beta{:}E\ \psi(\downarrow([\gamma\beta],x)))]].\end{aligned}$$

This is equivalent to the formula

$$\begin{aligned}\forall x\ [&de_{n+m}(x)\vee[\neg de_n(x)\wedge\neg de_m(x)\wedge\\ &\forall\gamma{:}n{+}m\ \forall\alpha,\beta{:}E\ \exists\delta{:}E\ \exists\gamma{:}n\ \exists\gamma{:}m\\ &\quad(\downarrow([\gamma_{n+m}\delta],x)=\downarrow([\gamma_n\alpha(\gamma_n)],x)\wedge\\ &\quad\downarrow([\gamma_{n+m}\delta],x)=\downarrow([\gamma_n\beta(\gamma_m)],x))]].\end{aligned}$$

(Note, here we have used the sorts as indices to distinguish three different variables: γ_n, γ_m and γ_{n+m}.) This formula is still second-order (note the $\alpha(\gamma_n)$ and $\beta(\gamma_m)$ terms). We get a first-order equivalent formula if we apply the quantifier exchange rule Υ to $\Pi_f(\mathrm{N7})$. $\Upsilon(\Pi_f(\mathrm{N7}))$ is

$$\begin{aligned}\forall\varphi\psi\forall x\ [&(\neg de_{n+m}(x)\wedge\\ &\quad\exists\gamma{:}n{+}m\ \forall\delta{:}E\ (\varphi(\downarrow([\gamma\delta],x))\vee\psi(\downarrow([\gamma\delta],x))))\rightarrow\\ &\qquad[(\neg de_n(x)\wedge\exists\gamma{:}n\ \forall\alpha{:}E\ \varphi(\downarrow([\gamma\alpha],x)))\vee\\ &\qquad(\neg de_m(x)\wedge\forall\beta{:}E\ \exists\gamma{:}m\ \psi(\downarrow([\gamma\beta],x)))]]\end{aligned}$$

which is equivalent to the first-order formula

$$\begin{aligned}\forall x\ [&de_{n+m}(x)\vee[\neg de_n(x)\wedge\neg de_m(x)\wedge\\ &\forall\gamma{:}n{+}m\ \forall\alpha{:}E\ \exists\delta{:}E\ \exists\gamma{:}m\ \forall\beta{:}E\ \exists\gamma{:}n\\ &\quad(\downarrow([\gamma_{n+m}\delta],x)=\downarrow([\gamma_n\alpha],x)\vee\downarrow([\gamma_{n+m}\delta],x)=\downarrow([\gamma_n\beta],x))]].\end{aligned}$$

The clause form is:

$$\begin{aligned}&de_{n+m}(x)\vee\neg de_n(x).\\ &de_{n+m}(x)\vee\neg de_m(x).\\ &de_{n+m}(x)\vee\ \downarrow([\gamma_{n+m}h1_E^{nm}(x,\gamma_{n+m},\alpha_E)],x)\\ &\qquad\qquad=\downarrow([h2_n^{nm}(x,\gamma_{n+m},\alpha_E,\beta_E)\alpha_E],x).\\ &de_{n+m}(x)\vee\ \downarrow([\gamma_{n+m}h1_E^{nm}(x,\gamma_{n+m},\alpha_E)],x)\\ &\qquad\qquad=\downarrow([h3_n^{nm}(x,\gamma_{n+m},\alpha_E)\beta_E],x).\end{aligned}$$

N8: $\vdash_{\overline{\mathrm{K}}_E} (\langle n\rangle\Box(\varphi\wedge\psi)\wedge\langle m\rangle\Box(\varphi\wedge\neg\psi))\rightarrow\langle n+m+1\rangle\Box\varphi$

is translated to $\Pi_f(\mathrm{N8})$:

$$\begin{aligned}\forall\varphi\psi\forall x\ [[&(\neg de_n(x)\wedge\exists\gamma{:}n\ \forall\alpha{:}E\ (\varphi(\downarrow([\gamma\alpha],x))\wedge\psi(\downarrow([\gamma\alpha],x))))\wedge\\ &(\neg de_m(x)\wedge\exists\gamma{:}m\ \forall\beta{:}E\ (\varphi(\downarrow([\gamma\beta],x))\wedge\neg\psi(\downarrow([\gamma\beta],x))))]\rightarrow\\ &(\neg de_{n+m+1}(x)\wedge\exists\gamma{:}n{+}m{+}1\ \forall\delta{:}E\ \varphi(\downarrow([\gamma\delta],x)))]\end{aligned}$$

This is equivalent to

$$\begin{aligned}\forall x\ [&de_n(x)\vee de_m(x)\vee\\ &\forall\gamma{:}n\ \forall\gamma{:}m\ \exists\gamma{:}n{+}m{+}1\ \forall\delta{:}E\\ &\quad(\neg de_{n+m+1}(x)\vee\exists\alpha,\beta{:}E\ \downarrow([\gamma_n\alpha],x)=\downarrow([\gamma_m\beta],x))\wedge\\ &\quad[\exists\alpha,\beta{:}E\ \downarrow([\gamma_n\alpha],x)=\downarrow([\gamma_m\beta],x)\vee\\ &\quad\exists\alpha{:}E\ \downarrow([\gamma_n\alpha],x)=\downarrow([\gamma_{n+m+1}\delta],x)\vee\\ &\quad\exists\beta{:}E\ \downarrow([\gamma_m\beta],x)=\downarrow([\gamma_{n+m+1}\delta],x)]].\end{aligned}$$

The clause form is

$$\begin{aligned}&de_n(x)\vee de_m(x)\vee\neg de_{n+m+1}(x)\vee\\ &\qquad\downarrow([\gamma_n k1_E^{nm}(x,\gamma_n,\gamma_m)],x)=\downarrow([\gamma_m k2_E^{nm}(x,\gamma_n,\gamma_m)],x).\\ &de_n(x)\vee de_m(x)\vee\\ &\qquad\downarrow([\gamma_n k3_E^{nm}(x,\gamma_n,\gamma_m)],x)=\downarrow([\gamma_m k4_E^{nm}(x,\gamma_n,\gamma_m)],x)\vee\\ &\qquad\downarrow([\gamma_n k5_E^{nm}(x,\gamma_n,\delta_E)],x)=\downarrow([k_{n+m+1}^{nm}(x,\gamma_n,\gamma_m)\delta_E],x)\vee\\ &\qquad\downarrow([\gamma_m k6_E^{nm}(x,\gamma_m,\delta_E)],x)=\downarrow([k_{n+m+1}^{nm}(x,\gamma_n,\gamma_m)\delta_E],x).\end{aligned}$$

The $\overline{\mathbf{K}}_E$-axioms and their translations are schemas. They represent the conjunction of all instances with the n and m taking on concrete non-negative integer values. This can be exploited in certain generalizations. For example, the subformula $\forall x\ de_n(x) \rightarrow de_{n+1}(x)$ of (11) can be generalized to

$$\forall x\ de_n(x) \rightarrow de_m(x) \qquad \text{for all } m > n.$$

This formula subsumes the subformula $\forall x\ de_0(x) \rightarrow de_n(x)$ of the translation (10) of N4. The remaining part of (11) can also be generalized to:

$$\forall x\ [\neg de_m(x) \rightarrow (\forall \gamma{:}m\ \exists \delta{:}n\ \downarrow(\gamma, x) = \downarrow(\delta, x))] \qquad \text{for all } m > n.$$

The clause form is

$$de_m(x) \vee \downarrow(\gamma_m, x) = \downarrow(g_n^{nm}(x, \gamma_m), x) \qquad \text{for all } m > n. \tag{12}$$

Recall the relation translation of N6. We noted N6′ generalizes to

$$R_m \subseteq R_n \qquad \text{for all } m \geq n.$$

This ordering on the accessibility relations $\{R_n\}_{n \in \mathbb{N}_0}$ induces a linear ordering on the set $\{AF_n\}_{n \in \mathbb{N}_0}$ of sets of accessibility functions. We capture this ordering by the subsort declaration

$$AF_m \sqsubseteq AF_n \qquad \text{for all } m \geq n. \tag{13}$$

In a resolution calculus this declaration has the same effect as clause (12). We therefore replace (12) in PL_M by the subsort declaration (13).

The final translation for $\overline{\mathbf{K}}_E$ in PL_M is still to come. We add yet more syntactic sugar and hope this makes the translations more easily readable. Every translated $\overline{\mathbf{K}}_E$-formula φ in negated form, which we aim to *refute*, contains only terms which have the form $\downarrow([s_1 \ldots s_n], x_0)$ (in world-path notation). x_0 is the Skolem constant originating from the $\forall x$ quantifier in $\Pi_f(\varphi)$. Such terms can be replaced by just $[s_1 \ldots s_n]$ or just the empty list $[]$ for formulae not containing modal operators.

In the clause form of the translations of the $\overline{\mathbf{K}}_E$*-axioms* the equations contain terms of the form $\downarrow([t_1 \ldots t_m], x)$ with x a universally quantified variable. We may instantiate x with, say $\downarrow([s_1 \ldots s_n], x_0)$, and get $\downarrow([t_1 \ldots t_m], \downarrow([s_1 \ldots s_n], x_0))$ which is the same as $\downarrow([s_1 \ldots s_n t_1 \ldots t_m], x_0))$. We get the same result if we introduce a new variable w_* of sort AF^* and replace $\downarrow([t_1 \ldots t_m], x)$ with $[w_* t_1 \ldots t_m]$. We further require that w_* can be unified with arbitrary strings $[s_1 \ldots s_n]$.

In this notation the axiomatization of $\overline{\mathbf{K}}_E$ reduces to the following set of PL_M formulae which defines our predicate logic theory for $\overline{\mathbf{K}}_E$.

P1 $\quad de_n(w_*),\ [w_* x_n z] = [w_* f_0^n(w_*, x_n, z) y]$

P2 $\quad AF_m \sqsubseteq AF_n \qquad \text{for all } m > n$

P3 $\quad \neg de_n(w_*),\ de_m(w_*) \qquad \text{for all } m > n$

P4 $\quad de_{n+m}(w_*),$
$\quad\quad [w_* x_{n+m} h1^{nm}(w_*, x_{n+m}, y)] = [w_* h2_n^{nm}(w_*, x_{n+m}, y, z) y]$

P5 $\quad de_{n+m}(w_*),\ [w_* x_{n+m} h1^{nm}(w_*, x_{n+m}, y)] = [w_* h3_n^{nm}(w_*, x_{n+m}, y) z]$

P6 $\quad de_{\max(n,m)}(w_*),\ \neg de_{n+m+1}(w_*),$
$\quad\quad [w_* x_n k1^{nm}(w_*, x_n, y_m)] = [w_* y_m k2^{nm}(w_*, x_n, y_m)]$

P7 $\quad de_{\max(n,m)}(w_*),$
$\quad\quad [w_* x_n k3^{nm}(w_*, x_n, y_m)] = [w_* y_m k4^{nm}(w_*, x_n, y_m)],$
$\quad\quad [w_* x_n k5^{nm}(w_*, x_n, z)] = [w_* k_{n+m+1}^{nm}(w_*, x_n, y_m) z],$
$\quad\quad [w_* y_m k6^{nm}(w_*, y_m, z)] = [w_* k_{n+m+1}^{nm}(w_*, x_n, y_m) z].$

(The variables y and z and the functions $h1^{nm}$ and $k1^{nm}$–$k6^{nm}$ without index are variables and functions of sort AF_E.)

Theorem 5.2 *For any* $\overline{\mathbf{K}}$*-formula* φ,

$$\varphi \textit{ is a } \overline{\mathbf{K}}\textit{-theorem} \quad \textit{iff} \quad (\text{P1–P7}) \rightarrow \Upsilon(\Pi_f(\varphi)) \textit{ is a theorem in } PL_M,$$

By way of examples we illustrate how P1–P7 can be used during inference with a resolution-based theorem prover. Bernhard Nebel provided the following examples.

Example 5.3 The set $A = \{\Diamond_0\Diamond_3\top,\ \Diamond_0\Box_3\bot,\ \Box_1\bot\}$ is an inconsistent set of $\overline{\mathbf{K}}$-formulae. With theory resolution we can show the inconsistency in a single step. The $\overline{\mathbf{K}}_E$ and the PL_M formulations of $\Diamond_0\Diamond_3\top$, $\Diamond_0\Box_3\bot$ and $\Box_1\bot$ are respectively

$$\begin{array}{lll} \langle 0\rangle\Box\langle 3\rangle\Box\top & \text{and} & \neg de_0([]) \wedge \neg de_3([a_0x]), \\ \langle 0\rangle\Box[3]\Diamond\bot & \text{and} & \neg de_0([]) \wedge de_3([b_0y]), \\ [1]\Diamond\bot & \text{and} & de_1([]). \end{array}$$

The set A is represented by the following set of clauses:

$$\begin{array}{llll} C_1 & \neg de_0([]) & C_3 & de_3([b_0y]) \\ C_2 & \neg de_3([a_0x]) & C_4 & de_1([]), \end{array}$$

where x and y are variables and a_0 and b_0 are Skolem constants. Letting $n = 0$ and $m = 0$ in P6 and using the substitution

$$\{x_0 \mapsto a_0, y_0 \mapsto b_0, x \mapsto k1^{00}([], a_0, b_0), y \mapsto k2^{00}([], a_0, b_0)\}$$

P6 simultaneously resolves with C_1–C_4 yielding the empty clause.

The next example is more complicated.

Example 5.4 The set $B = \{\Diamond_0\Diamond_0\Diamond_3\top,\ \Diamond_0\Diamond_0\Box_3\bot,\ \Diamond_0\Box_1\bot,\ \Box_1\bot\}$ of $\overline{\mathbf{K}}$-formulae is inconsistent. B contains the formulae of A (from Example 5.3) prefixed with a $\Diamond_0$ and the formula $\Box_1\bot$. The set of expressions in B is represented by the following clauses (derived as in the previous example via $\overline{\mathbf{K}}_E$ and Π_f translations, which we omit):

$$\begin{array}{llll} C_1 & \neg de_0([]) & C_5 & de_3([c_0x'd_0y']) \\ C_2 & \neg de_0([a_0x]) & C_6 & de_1([e_0x'']) \\ C_3 & \neg de_3([a_0xb_0y]) & C_7 & de_1([]) \\ C_4 & \neg de_0([c_0x']) & & \end{array}$$

For the refutation we use P1 with $n = 0$ and P6 with $n = 0$ and $m = 0$. P1 can immediately be simplified with clause C_1. The instances are:

P1′ $[f_0^0([], x_0, z)y] = [x_0z]$

P6′ $de_0(w_*),\ \neg de_1(w_*),\ [w_*x_0k1^{00}(w_*, x_0, y_0)] = [w_*y_0k2^{00}(w_*, x_0, y_0)]$.

The result of simultaneously resolving P6′, C_1, and C_7 with unifier $\{w_* \mapsto []\}$ is

C_8 $[x_0k1^{00}([], x_0, y_0)] = [y_0k2^{00}([], x_0, y_0)]$.

Paramodulating with C_8 and with unifier $\{x_0 \mapsto a_0, x \mapsto k1^{00}([], a_0, y_0)\}$, C_3 becomes (this means we do equality replacement with unification in C_3 using the equation C_8)

C_9 $\neg de_3([y_0k2^{00}([], a_0, y_0)b_0y])$.

This becomes

C_{10} $\neg de_3([x_0zb_0y])$

when paramodulating with P1′ using the unifier

$$\{y_0 \mapsto f_0^0([], x_0, z), y \mapsto k2^{00}([], a_0, y_0)\}.$$

We resolve P6′ and C_4 using unifier $\{w_* \mapsto [c_0x], x' \mapsto x\}$ to get

C_{11} $\neg de_1([c_0x])$,
$[c_0xx_0'k1^{00}([c_0x], x_0', y_0')] = [c_0xy_0'k2^{00}([c_0x], x_0', y_0')]$.

Now, use the unifier

$$\{x_0 \mapsto c_0, x' \mapsto z \mapsto x, x'_0 \mapsto d_0,\\ y'_0 \mapsto b_0, y' \mapsto k1^{00}([c_0 x], d_0, b_0), y \mapsto k2^{00}([c_0 x], d_0, b_0)\}$$

and apply E-resolution to C_5, C_{10} and C_{11} and get

$C_{12} \quad \neg de_1([c_0 x])$.

(This means we resolve between C_5 and C_{10} using an equation in C_{11}.) Resolving this with C_6 using E-resolution with C_8 yields the empty clause. The unifier is

$$\{x_0 \mapsto e_0, x'' \mapsto k1^{00}([], e_0 c_0), y_0 \mapsto c_0, x'' \mapsto k2^{00}([], e_0 c_0)\}.$$

Example 5.5 In this example we show

$$(\Diamond_n \varphi \wedge \Diamond_m \psi \wedge \Box_0 \neg(\varphi \wedge \psi)) \to \Diamond_{n+m+1}(\varphi \vee \psi) \tag{14}$$

is a theorem in $\overline{\mathbf{K}}$ by showing that the following set of clauses is refutable. The set represents the negation of the theorem.

C_1	$\neg de_n([])$	C_5	$de_0([]),\ \neg\varphi([y_0 c]),\ \neg\psi([y_0 c])$
C_2	$\varphi([a_n x])$	C_6	$de_{n+m+1}([]),\ \neg\varphi([x_{n+m+1} d])$
C_3	$\neg de_m([])$	C_7	$de_{n+m+1}([]),\ \neg\psi([x_{n+m+1} d])$
C_4	$\psi([b_m x'])$		

C_5 can be resolved with P3, letting $n = 0$ and $m = n$, and C_1 yielding

$C'_5 \quad \neg\varphi([y_0 c]),\ \neg\psi([y_0 c])$.

This can be paramodulated using the equation in P1 and using the unifier

$$\{w_* \mapsto [], y_0 \mapsto f_0^n([], x_n, z), y \mapsto c\}.$$

The result is:

$C''_5 \quad \neg\varphi([x_n z]),\ \neg\psi([x_n z])$.

Resolve this with C_2 and get

$C_8 \quad \neg\psi([a_n z])$.

Now, take C_6 and paramodulate with P7 using the second equation and the unifier

$$\{w_* \mapsto [], x_{n+m+1} \mapsto k_{n+m+1}^{nm}([], x_n, y_m), z \mapsto d\}$$

and obtain

$$\begin{aligned} C_9 \quad & de_{n+m+1}([]),\ \neg\varphi([x_n k5^{nm}([], x_n, d)]),\\ & de_{\max(n,m)}([]),\\ & [x_n k3^{nm}([], x_n, y_m)] = [y_m k4^{nm}([], x_n, y_m)],\\ & [y_m k6^{nm}([], y_m, d)] = [k_{n+m+1}^{nm}([], x_n, y_m) d]. \end{aligned}$$

The $de_{\max(n,m)}([])$ literal can be eliminated from C_9 with either C_1 or C_3 in one resolution step. The clause

$$\begin{aligned} C'_9 \quad & de_{n+m+1}([]),\ \neg\varphi([x_n k5^{nm}([], x_n, d)]),\\ & [x_n k3^{nm}([], x_n, y_m)] = [y_m k4^{nm}([], x_n, y_m)],\\ & [y_m k6^{nm}([], y_m, d)] = [k_{n+m+1}^{nm}([], x_n, y_m) d] \end{aligned}$$

remains. Take C_7 and paramodulate with C'_9 and unifier

$$\{x_{n+m+1} \mapsto k_{n+m+1}^{nm}([], x_n, y_m)\}.$$

We obtain

$$C_{10} \quad de_{n+m+1}([]),\ \neg\varphi([x_n k5^{nm}([], x_n, d)]),\ \neg\psi([y_m k6^{nm}([], y_m, d)])$$
$$[x_n k3^{nm}([], x_n, y_m)] = [y_m k4^{nm}([], x_n, y_m)].$$

Use C_2 and C_4 to get rid of the $\neg\varphi$ and the $\neg\psi$ literals. The unifier is

$$\{x_n \mapsto a_n, y_m \mapsto b_m, x \mapsto k5^{nm}([], a_n, d), x' \mapsto k6^{nm}([], b_m, d)\}.$$

C_{10} becomes

$$C_{11} \quad de_{n+m+1}([]),\ [a_n k3^{nm}([], a_n, b_m)] = [b_m k4^{nm}([], a_n, b_m)].$$

This we can use to paramodulate with C_8. The unifier is

$$\{z \mapsto k3^{nm}([], a_n, b_m)\}$$

and the result is:

$$C_{12} \quad de_{n+m+1}([]),\ \neg\psi([b_m k4^{nm}([], a_n, b_m)]).$$

Resolve this with C_4 which yields

$$C_{13} \quad de_{n+m+1}([]).$$

Now we use P6 and get

$$C_{14} \quad de_{\max(n,m)}([]),\ [x_n k1^{nm}([], x_n, y_m)] = [y_m k2^{nm}([], x_n, y_m)].$$

Get rid of the $de_{\max(n,m)}([])$ literal by resolving with either C_1 or C_3. The equation

$$C'_{14} \quad [x_n k1^{nm}([], x_n, y_m)] = [y_m k2^{nm}([], x_n, y_m)].$$

remains. We use C_8 again and paramodulate with C'_{14} substituting with

$$\{x_n \mapsto a_n, z \mapsto k1^{nm}([], x_n, y_m)\}$$

which leaves

$$C_{15} \quad \neg\psi([y_m k2^{nm}([], a_n, y_m)]).$$

In the last step we resolve C_{15} and C_4 with unifier

$$\{y_m \mapsto b_m, x \mapsto k2^{nm}([], x_n, y_m)\}$$

to get the empty clause.

In the functional translation we can prove instances of formulae with concrete values assigned to the n and m in the modal operators. There are examples of formulae for which the proofs with symbolic arithmetic terms instead of concrete values work as well. However, this approach may not always work. The formula (15) below provides an example of a theorem which is true for all n and m (that satisfy the required restriction), but which can be proved in our system only for concrete instances of n and m. The situation may be worse. It may be the case that the proof of a formula for a particular concrete instance n depends on the instance of the formula for $n \perp 1$, and the proof of this instance depends on the formula for $n \perp 2$, and so on, to the formula for 0. Call this 'induction on foot'. We now demonstrate a process of how a schema like (15) can be proved in our system by (ordinary) induction for all values followed by a translation step which yields a lemma we add to our theory.

Suppose there are at least twenty objects in p and at least twenty objects in q and in all thirty objects exist. Then we expect the intersection of p and q to contain at least 10 objects. Our intuition is captured by the following formula

$$\Diamond_n \varphi \wedge \Diamond_m \psi \wedge \Box_j \neg(\varphi \wedge \psi) \rightarrow \Diamond_{n+m+1-j}(\varphi \vee \psi) \tag{15}$$

with $n+m+1 \perp j \geq 0$, if we let $p \equiv \varphi$, $q \equiv \psi$, $n = m = 19$ and $j = 9$. In Example 5.5 we showed (15) for the case that $j = 0$. Unfortunately there is no resolution-based proof for the general case. In the next theorem we use induction to prove (15).

Theorem 5.6 *(15) is a theorem in* $\overline{\mathbf{K}}$.

Proof. The proof is by induction on j. We proved the base case in Example 5.5. Let $j > 0$. As induction hypothesis assume

$$\Diamond_n\varphi \wedge \Diamond_m\psi \wedge \Box_{j-1}\neg(\varphi \wedge \psi) \rightarrow \Diamond_{n+m+1-(j-1)}(\varphi \vee \psi)$$

holds. Assume further $\Diamond_n\varphi$, $\Diamond_m\psi$, and $\Box_j\neg(\varphi \wedge \psi)$ hold. That $\Box_j\neg(\varphi \wedge \psi)$ holds implies $(\neg\Box_{j-1}\neg(\varphi \wedge \psi) \wedge \Box_j\neg(\varphi \wedge \psi)) \vee \Box_j$ holds.

Suppose $\Box_{j-1}\neg(\varphi \wedge \psi)$ holds. Then we apply the induction hypothesis. We get $\Diamond_{n+m+1-(j-1)}(\varphi \vee \psi)$ which implies, by A2, $\Diamond_{n+m+1-j}(\varphi \vee \psi)$ holds.

For the second case assume $\neg\Box_{j-1}\neg(\varphi \wedge \psi) \leftrightarrow \Diamond_{j-1}(\varphi \wedge \psi)$ holds. Let $k = n + m$, $\varphi = \varphi \wedge \neg\psi$ and $\psi = \varphi \wedge \psi$ in A10. Then

$$\Diamond_k\varphi \wedge \Box_k\neg(\varphi \wedge \psi) \rightarrow \Diamond_{k-n}(\varphi \wedge \neg\psi).$$

From $\Diamond_n\varphi$, respectively $\Diamond_m\psi$, and $\Box_j\neg(\varphi \wedge \psi)$ we infer that $\Diamond_{n-j}(\varphi \wedge \neg\psi)$, respectively $\Diamond_{m-j}(\neg\varphi \wedge \psi)$, holds. Hence, by A12,

$$\Diamond_{m+n+1-j-1}((\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi))$$

holds. Using A12 again, this time applied to the formulae

$$\Diamond_{m+n+1-j-1}((\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi)) \quad \text{and} \quad \Diamond_{j-1}(\varphi \wedge \psi),$$

we conclude $\Diamond_{m+n+1-j}(\varphi \vee \psi)$ holds. This proves the theorem. ∎

The next result shows we can replace the axiom N8 in $\overline{\mathbf{K}}_E$ by the corresponding $\overline{\mathbf{K}}_E$-formulation of the formula (15).

Theorem 5.7 *Axiom* N8 *of* $\overline{\mathbf{K}}_E$ *can be replaced by*

$$\langle n\rangle\Box\varphi \wedge \langle m\rangle\Box\psi \wedge [j]\Diamond\neg(\varphi \wedge \psi) \rightarrow \langle n + m + 1 \perp j\rangle\Box(\varphi \vee \psi) \tag{16}$$

for $n + m + 1 \perp j \geq 0$.

Proof. In Theorem 5.6 we proved (15), its $\overline{\mathbf{K}}$-formulation, is a theorem in $\overline{\mathbf{K}}$. Thus, by Theorem 3.3, (16) holds in $\overline{\mathbf{K}}_E$. It remains to show (16) implies N8. This is immediate if we let $j = 0$ and substitute $\varphi \wedge \psi$ for φ and $\varphi \wedge \neg\psi$ for ψ exploiting $[0]\Diamond\top \leftrightarrow \top$ (N4). ∎

Although replacing N8 with (16) does not increase the number of provable formulae, we avoid the induction argument necessary for proving (16) which we would have to provide by hand as we don't have an induction theorem prover at our disposal. Also, we avoid proving instances of (16).

The functional translation of (16) into predicate logic is somewhat more complicated than that of N8. It is given by

$$\begin{aligned}
\forall\varphi\psi\forall x\ [&[(\neg de_n(x) \wedge \exists\gamma{:}n\ \forall\delta{:}E\ \varphi(\downarrow([\gamma\delta], x))) \wedge \\
&(\neg de_m(x) \wedge \exists\gamma{:}m\ \forall\delta{:}E\ \psi(\downarrow([\gamma\delta], x))) \wedge \\
&(\neg de_j(x) \rightarrow \forall\gamma{:}j\ \exists\delta{:}E\ \neg(\varphi(\downarrow([\gamma\delta], x)) \wedge \psi(\downarrow([\gamma\delta], x))))] \rightarrow \\
&[(\neg de_{n+m+1-j}(x) \wedge \\
&\quad \exists\gamma{:}n{+}m{+}1\text{-}j\ \forall\delta{:}E\ (\varphi(\downarrow([\gamma\delta], x)) \vee \psi(\downarrow([\gamma\delta], x))))]].
\end{aligned}$$

Like $\Pi_f(\text{N7})$ this formula cannot be reduced to a first-order formula. We swap the quantifiers $\exists\gamma{:}n{+}m{+}1\text{-}j$ and $\forall\delta{:}E$. The quantification elimination algorithm SCAN produces then for this input the following clauses:

P8 $\quad de_{\max(n,m)}(w_*),\ \neg de_{n+m+1-j}(w_*),$
$\quad\quad [w_* f7_j^{nmj}(w_*, x_n, x_m)u] = [w_* x_n f5^{nmj}(w_*, x_n, u)]$

P9 $\quad de_{\max(n,m)}(w_*),\ \neg de_{n+m+1-j}(w_*),$
$\quad\quad [w_* f7_j^{nmj}(w_*, x_n, x_m)u] = [w_* x_m f6^{nmj}(w_*, x_m, u)]$

P10 $\quad de_{\max(n,m)}(w_*),\ \neg de_j(w_*),$
$\quad\quad [w_* f1_{n+m+1-j}^{nmj}(w_*, v, x_n)v] = [w_* x_n f2^{nmj}(w_*, v, x_n)],$
$\quad\quad [w_* f3_{n+m+1-j}^{nmj}(w_*, v, x_m)v] = [w_* x_m f4^{nmj}(w_*, v, x_m)]$

P11 $de_{\max(n,m)}(w_*)$,
$[w_* f1^{nmj}_{n+m+1-j}(w_*, v, x_n)v] = [w_* x_n f2^{nmj}(w_*, v, x_n)]$,
$[w_* f3^{nmj}_{n+m+1-j}(w_*, v, x_m)v] = [w_* x_m f4^{nmj}(w_*, v, x_m)]$,
$[w_* f7^{nmj}_{j}(w_*, x_n, x_m)u] = [w_* x_n f5^{nmj}(w_*, x_n, u)]$

P12 $de_{\max(n,m)}(w_*)$,
$[w_* f1^{nmj}_{n+m+1-j}(w_*, v, x_n)v] = [w_* x_n f2^{nmj}(w_*, v, x_n)]$,
$[w_* f3^{nmj}_{n+m+1-j}(w_*, v, x_m)v] = [w_* x_m f4^{nmj}(w_*, v, x_m)]$,
$[w_* f7^{nmj}_{j}(w_*, x_n, x_m)u] = [w_* x_m f6^{nmj}(w_*, x_m, u)]$

together with the clause

$$de_n(w),\ de_m(w),\ \neg de_j(w),\ \neg de_{n+m+1-j}(w), \tag{17}$$

which is implicit in P1–P7. We can show that, for any positive integers n, m and j,

$$\exists k\ l\ (k,l) \in \{n,m\} \times \{j, n{+}m{+}1{-}j\} \quad \text{such that} \quad k \geq l.$$

For, suppose not. Suppose n, m and j exist such that for any k and l with $(k,l) \in \{n,m\} \times \{j, n{+}m{+}1{-}j\}$ we have $k < l$. Then, $n < j$, $m < j$ and $n < n + m + 1 \perp j$. Hence, $j < m + 1$, and thus, $m < j < m + 1$, which cannot be for j a positive integer.

If the values n, m and j are such that we can choose k and l with k strictly larger than l then (17) is subsumed by P3. Otherwise, if the values are such that we can choose identical k and l, then (17) is a tautology. In either case (17) is redundant.

We conclude this section with an example (supplied to us by Werner Nutt) in which we exhibit the computational effect of using the clauses P8–P12.

Example 5.8 Suppose the universe consists of at most thirty objects. If there are at least twenty objects in p and there are at least twenty objects in q, then there are at least ten objects in $p \wedge q$. A standard tableaux system for the number operators would generate twenty witnesses for p, twenty witnesses for q and then it would need to identify ten of them in order not to exceed the limit of thirty. But there are combinatorically many ways for identifying ten of them.

In our system we prove the conjecture by showing the following set of $\overline{\mathbf{K}}$-formulae is inconsistent:

$$\{\Diamond_{19} p,\ \Diamond_{19} q,\ \Box_{30}\bot,\ \Box_9 \neg(p \wedge q)\}.$$

We can choose any other suitable combination of numbers. This will not change the structure of the proof at all. The translation into PL_M is:

$$\{\neg de_{19}([]) \wedge p([a_{19}x]),\ \neg de_{19}([]) \wedge q([b_{19}x]),\ de_{30}([]),\\ de_9([]) \vee \neg p([y_9 c]) \vee \neg q([y_9 c])\}.$$

The corresponding set of clauses consists of:

C_1	$\neg de_{19}([])$	C_4	$de_{30}([])$
C_2	$p([a_{19}x])$	C_5	$de_9([]),\ \neg p([y_9 c]),\ \neg q([y_9 c])$
C_3	$q([b_{19}y])$		

Resolve C_5 with P3 and C_1 and eliminate the $de_9([])$ literal from C_5 leaving:

C_5' $\neg p([y_9 c]),\ \neg q([y_9 c])$

We resolve the instance of P9 with $n = m = 19$, $j = 9$, namely

P9' $de_{19}([]),\ \neg de_{30}([])$,
$[f7^{19\,19\,9}_{9}([], x_{19}, x'_{19})u] = [x_{19} f6^{19\,19\,9}([], x'_{19}, u)]$,

with C_1 and C_4 and obtain

C_6 $[f7^{19\,19\,9}_{9}([], x_{19}, x'_{19})u] = [x'_{19} f6^{19\,19\,9}([], x'_{19}, u)]$.

Applying the unifier $\{y_9 \mapsto f7^{19\,19\,9}_{9}([], x_{19}, x'_{19}), u \mapsto c\}$, we can use this in a paramodulation step with C_5' resulting in

$$C_7 \quad \neg p([x'_{19} f6^{19\,19\,9}([], x'_{19}, c)]),\ \neg q([f7_9^{19\,19\,9}([], x_{19}, x'_{19})c])$$

Unify in C_2 and C_7 with $\{x'_{19} \mapsto a_{19}, x \mapsto f6^{19\,19\,9}([], x'_{19}, c)\}$. Resolving C_2 and C_7 yields

$$C_8 \quad \neg q([f7_9^{19\,19\,9}([], x_{19}, a_{19})c]).$$

Now we use the following instance of P8:

$$\text{P8'} \quad de_{19}(w_*),\ \neg de_{30}(w_*), \\ [w_* f7_9^{19\,19\,9}(w_*, x_{19}, x'_{19})u] = [w_* x_{19} f5^{19\,19\,9}(w_*, x_{19}, u)]$$

This can be reduced with C_1 and C_4 to the equation

$$C_9 \quad [f7_9^{19\,19\,9}([], x_{19}, x'_{19})u] = [x_{19} f5^{19\,19\,9}([], x_{19}, u)],$$

which we can now use in a paramodulation step with C_8. We get

$$C_{10} \quad \neg q([x_{19} f5^{19\,19\,9}([], x_{19}, c)]).$$

The empty clause is obtained if we resolve C_{10} with C_3 using the appropriate unifier.

6 From concept description languages to graded modalities

A knowledge representation system in the KL-ONE-style [4] usually consists of a so called T-Box and an A-Box [5]. The T-Box axiomatizes the part of the world that is to be modelled in the system whereas the A-Box is more or less a classical database containing information, in general ground facts, about the actual situation.

Most T-Box (or terminological) languages have as syntactic primitives *concept names* and *rule names*. Concept names denote sets of objects and role names denote binary relations between these objects. Using concept forming connectives, like $\neg$, $\sqcap$, $\sqcup$, some and all, compound concept terms can be built which also denote sets of objects.

A prototypical concept description language is the $\mathcal{ALC}$ language (short for ‘attributive concept description language with complement’). It has a well-defined model-theoretic semantics and its computational behaviour is completely understood. The terminological language of $\mathcal{ALC}$ uses only the concept-forming operators $\neg$, $\sqcap$, $\sqcup$ with the usual meaning (complement, union, intersection) as well as role quantifications (all R C) and (some R C). (all R C) denotes the set of all objects whose R-successors (R-fillers in the KL-ONE terminology) are all in C. (some R C) denotes the set of all objects with some R-successor in C. Typical examples for concept definitions in $\mathcal{ALC}$ are:

$$\begin{aligned} \mathsf{man} &= \mathsf{person} \sqcap (\mathsf{some\ sex\ male}) \\ \mathsf{parent} &= \mathsf{person} \sqcap (\mathsf{some\ child} \top) \\ \mathsf{father} &= \mathsf{parent} \sqcap \mathsf{man} \\ \mathsf{grandfather} &= \mathsf{father} \sqcap (\mathsf{some\ child\ parent}) \\ \mathsf{woman} &= \mathsf{person} \sqcap \neg\mathsf{man} \end{aligned}$$

$\top$ denotes the set of all objects.

Given a set T of concept equations, a concept C is *coherent* if there is a model for T in which C denotes a nonempty set. Furthermore, a concept description C *subsumes* a concept description D in T, if C denotes in *every* model of T a superset of D. Deciding coherence and subsumption is the basic reasoning service of the knowledge representation systems based on $\mathcal{ALC}$. According to the above definitions, for example, it is possible to infer that grandfathers are fathers and persons and men as well, i.e. man subsumes father and grandfather.

In [28], Schmidt-Schauß and Smolka show that deciding coherence and subsumption of concept descriptions is P-SPACE-complete and can be decided with linear space. Many variants and extensions of $\mathcal{ALC}$ have now been investigated [19, 20, 21, 27, 9, 8, 35] and are used in implementations of knowledge representation languages [3]. We focus on a language very much like $\mathcal{ALCN}$ which includes numerical quantification operators atleast and atmost. For example, the concept term (atleast 3 has-child blond) represents the set of individuals who have at least three children who are blond. The term (atmost 2 has-parent $\top$) represents the set of individuals who have at most two parents. The language we consider is slightly more expressive than $\mathcal{ALCN}$. Our version, referred to as $\mathcal{ALCN}^+$, allows for arbitrary concepts to be included in other concepts, whereas in $\mathcal{ALCN}$ only atomic concepts can be included in other concepts.

Now, we define the syntax of $\mathcal{ALCN}^+$. The signature of the terminological language of $\mathcal{ALCN}^+$ consists of a set Σ_R of *role names* and a disjoint set Σ_C of *concept names*. From role names $\mathsf{Q} \in \Sigma_R$ and concept names $\mathsf{A} \in \Sigma_C$ compound concept terms C are formed according to the following rules:

$$\begin{array}{lll} \mathsf{C}, \mathsf{D} & \perp\!\!\rightarrow & \mathsf{A} \mid \neg\mathsf{C} \mid \mathsf{C} \sqcap \mathsf{D} \mid \mathsf{C} \sqcup \mathsf{D} \mid (\mathsf{some\ R\ C}) \mid (\mathsf{all\ R\ C}) \mid \\ & & (\mathsf{atleast}\ n\ \mathsf{R\ C}) \mid (\mathsf{atmost}\ n\ \mathsf{R\ C}) \mid \mathsf{C} \sqsubseteq \mathsf{D} \mid \mathsf{C} = \mathsf{D}. \end{array}$$

n is a non-negative integer. Most authors define the symbols $\sqsubseteq$ and $=$ to be sentential symbols. We define them to be connectives just as $\sqcap$ and $\sqcup$ are. Note, we consider terminological sentences of the form $\mathsf{C} \sqsubseteq \mathsf{D}$ and $\mathsf{C} = \mathsf{D}$ to be concept terms. In $\mathcal{ALCN}$ terminological sentences are constrained to be of the form $\mathsf{A} \sqsubseteq \mathsf{C}$ and $\mathsf{A} = \mathsf{C}$, where A are concept names. A *T-Box* is defined as a set of concept terms.

The semantics of $\mathcal{ALCN}^+$ is specified by an interpretation $\mathcal{I} = (U, V)$ with U a non-empty set U (the domain of interpretation) and a signature assignment V. The signature assignment maps role names to binary relations on U and it maps concept names to subsets of U. The interpretation of concept terms C and D specified by:

$$\begin{aligned} \mathsf{C}^{\mathcal{I}} &= V(\mathsf{C}) \qquad \text{if } \mathsf{C} \text{ is a concept name} \\ (\neg\mathsf{C})^{\mathcal{I}} &= U \setminus \mathsf{C}^{\mathcal{I}} \\ (\mathsf{C} \sqcap \mathsf{D})^{\mathcal{I}} &= \mathsf{C}^{\mathcal{I}} \cap \mathsf{D}^{\mathcal{I}} \\ (\mathsf{C} \sqcup \mathsf{D})^{\mathcal{I}} &= \mathsf{C}^{\mathcal{I}} \cup \mathsf{D}^{\mathcal{I}} \\ (\mathsf{C} \sqsubseteq \mathsf{D})^{\mathcal{I}} &= (U \setminus \mathsf{C}^{\mathcal{I}}) \cup \mathsf{D}^{\mathcal{I}} \\ (\mathsf{C} = \mathsf{D})^{\mathcal{I}} &= (U \setminus (\mathsf{C}^{\mathcal{I}} \cup \mathsf{D}^{\mathcal{I}})) \cup (\mathsf{C}^{\mathcal{I}} \cap \mathsf{D}^{\mathcal{I}})) \\ (\mathsf{some\ R\ C})^{\mathcal{I}} &= \{x \in U \mid \exists y \in U\ \mathsf{R}^{\mathcal{I}}(x,y) \wedge y \in \mathsf{C}^{\mathcal{I}}\} \\ (\mathsf{all\ R\ C})^{\mathcal{I}} &= \{x \in U \mid \forall y \in U\ \mathsf{R}^{\mathcal{I}}(x,y) \rightarrow y \in \mathsf{C}^{\mathcal{I}}\} \\ (\mathsf{atleast}\ n\ \mathsf{R\ C})^{\mathcal{I}} &= \{x \in U \mid |\{y \in \mathsf{C}^{\mathcal{I}} \mid \mathsf{R}^{\mathcal{I}}(x,y)\}| \geq n\} \\ (\mathsf{atmost}\ n\ \mathsf{R\ C})^{\mathcal{I}} &= \{x \in U \mid |\{y \in \mathsf{C}^{\mathcal{I}} \mid \mathsf{R}^{\mathcal{I}}(x,y)\}| \leq n\} \end{aligned}$$

Atomic concept names in a T-Box T are interpreted as the entire domain and are all equivalent to the *top concept* $\top$. $\top$ is the largest element in the subsumption ordering. The complement of $\top$ is $\perp$ and represents the empty set.

An interpretation $\mathcal{I} = (U, V)$ with $\mathsf{C}^{\mathcal{I}} = U$ for all concept terms C in the T-Box T is a *model* of T. A concept term C is *universal* iff $\mathsf{C}^{\mathcal{I}} = U$ for *all* interpretations $\mathcal{I}$. C is *empty* iff $\mathsf{C}^{\mathcal{I}} = \emptyset$ for *all* interpretations $\mathcal{I}$.

The entailment relation $\models$ between concept terms is defined by:

$$\mathsf{C} \models \mathsf{D} \quad \text{iff} \quad \mathsf{D}^{\mathcal{I}} = U \text{ for every interpretation } \mathcal{I} \text{ of } \mathsf{C}.$$

Then $\mathsf{C} \models \mathsf{D}$ iff $\mathsf{C} \sqsubseteq \mathsf{D}$ is universal iff $\mathsf{C} \sqcap \neg\mathsf{D}$ is empty.

We treat sets $\{\mathsf{C}_1, \ldots, \mathsf{C}_n\}$ of concept terms in the same way as the conjunction $\mathsf{C}_1 \sqcap \ldots \sqcap \mathsf{C}_n$. Thus, a given T-Box T will be treated as the conjunction of its elements.

In contrast to other terminological languages the language $\mathcal{ALCN}^+$ includes no role-forming operators. Roles that occur are all atomic. To simplify our presentation, without loss of generality we assume there is one atomic role R.

Now we show that we can embed $\mathcal{ALCN}^+$ in $\overline{\mathbf{K}}$. Define a mapping Π from the language of $\mathcal{ALCN}^+$ to the language of $\overline{\mathbf{K}}$ by:

$$\begin{array}{rcl} \Pi(\mathsf{C}) & = & C \qquad \text{if } \mathsf{C} \text{ is a concept name or } \top \text{ or } \perp \\ \Pi(\neg\mathsf{C}) & = & \neg\Pi(\mathsf{C}) \\ \Pi(\mathsf{C} \sqcap \mathsf{D}) & = & \Pi(\mathsf{C}) \wedge \Pi(\mathsf{D}) \\ \Pi(\mathsf{C} \sqcup \mathsf{D}) & = & \Pi(\mathsf{C}) \vee \Pi(\mathsf{D}) \\ \Pi(\mathsf{C} \sqsubseteq \mathsf{D}) & = & \Pi(\mathsf{C}) \rightarrow \Pi(\mathsf{D}) \\ \Pi(\mathsf{C} = \mathsf{D}) & = & \Pi(\mathsf{C}) \leftrightarrow \Pi(\mathsf{D}) \\ \Pi(\mathsf{some\ R\ C})) & = & \Diamond_0\Pi(\mathsf{C}) \\ \Pi(\mathsf{all\ R\ C}) & = & \Box_0\Pi(\mathsf{C}) \\ \Pi(\mathsf{atleast}\ n\ \mathsf{R\ C}) & = & \Diamond_{n-1}\Pi(\mathsf{C}) \\ \Pi(\mathsf{atmost}\ n\ \mathsf{R\ C}) & = & \Box_n\neg\Pi(\mathsf{C}) \end{array}$$

It is easy to verify that Π is well-defined. The following is the main statement of this section.

Theorem 6.1 (Soundness and Completeness of Π).

$$\textit{A concept term } \mathsf{C} \textit{ is universal} \quad \textit{iff} \quad \Pi(\mathsf{C}) \textit{ is a tautology.}$$

PROOF. Let $\mathcal{I} = (U, V)$ be any interpretation of a T-Box of $\mathcal{ALCN}^+$. Let $\mathcal{M}$ be the modal model $(U, \mathsf{R}^{\mathcal{I}}, V)$. By induction on the structure of C prove, for every $x \in U$:

$$x \in \mathsf{C}^{\mathcal{I}} \quad \text{iff} \quad \mathcal{M}, x \models \Pi(\mathsf{C}).$$

We omit the details. ∎

7 Conclusion

In the logic of graded modalities it is possible to express properties of finite sets. The usual inference calculi for this logic generate for all sets used in the proof at least as many constants (witnesses) as the cardinality of each set. Even for moderate values a vast number of witnesses are generated which are processed by case distinctions in the proof.

In this paper we present an alternative method which avoids case distinctions, instead our method uses limited arithmetical reasoning. It arises in a series of transformation steps. First, we translate the logic of graded modalities $\overline{\mathbf{K}}$ into a new normal multi-modal logic, called $\overline{\mathbf{K}}_E$. Unfortunately, $\overline{\mathbf{K}}_E$ does not reduce by the standard relational translation to first-order logic. One of the axioms of $\overline{\mathbf{K}}_E$ is second-order. We solved this irreducibility problem by, instead of using the relational translation, using a functional translation with a particular optimization which exploits the richer structure of the functional models.

Our method can also be applied in the field of knowledge representation. The terminological logic $\mathcal{ALCN}$ is closely related to the graded modal logic $\overline{\mathbf{K}}$. In fact, there is an exact correspondence between terminological operators and modal operators. Our approach provides a viable alternative inference mechanism to the constraint algorithms commonly used, which also suffer from the overhead of evaluating case distinctions.

Our approach must be viewed as a first step toward efficient reasoning with finite sets. There are a number of open problems which need to be addressed.

(i) A general completeness result for $\overline{\mathbf{K}}_E$ would allow us to use the full expressivity of this system. As long as this is not proved, we can guarantee completeness only for the original $\overline{\mathbf{K}}$ formulae. This is what we wanted from the beginning, but a stronger result would be preferable.

(ii) Our first-order theory is represented by a set of axiom schemas which are understood to be conjunctions of all its instances with the numerical variables instantiated with concrete values. The implementation of the calculus will rely on theory resolution. The axiom schemas will be encoded as inference rules. Since the axiom schemas contain equations the realization will not be easy, but it is certainly solvable.

(iii) The original logic of graded modalities is decidable. Accordingly, we expect a resolution strategy for the translated formulae can be developed that is complete and terminates. This has yet to be done.

(iv) Our calculus is still limited in reasoning with arithmetical terms. It remains to be investigated whether and how this capability can be enhanced.

(v) We have applied our methods to KL-ONE-type reasoning but only for reasoning within the T-Box. This corresponds directly to that in modal logic. We haven't accounted for A-Box reasoning about concrete instantiations of concepts/sets and roles/relations. The functional translation applied to A-Box terms generates many equations. It is not immediate how these can be treated efficiently.

(vi) The correspondence properties for the axioms of $\overline{\mathbf{K}}_E$ except one are first-order. This does not rule out that $\overline{\mathbf{K}}_E$ is complete with respect to a first-order model theory. If this were the case we can get a translation into predicate logic that avoids some equational reasoning.

Acknowledgements

This work was supported by the ESPRIT-Project MEDLAR (Nr. 6471) and by the LOGO-Project ITS 9102 funded by the BMFT.

8 References

[1] H. Andréka, I. Németi, and I. Sain. On interpolation, amalgamation, universal algebra and Boolean algebras with operators. Unpublished manuscript, Univ. of Budapest, 1995.

[2] Y. Auffray and P. Enjalbert. Modal theorem proving: An equational viewpoint. *Journal of Logic and Computation*, 2(3):247–297, 1992.

[3] F. Baader and B. Hollunder. A terminological knowledge representation system with complete inference algorithms. In M. M. Richter and H. Boley, editors, *Proc. of the International Workshop on Processing Declarative Knowledge (PDK'91)*, Univ. of Kaiserslautern, 1991.

[4] R. J. Brachman and J. G. Schmolze. An overview of the KL-ONE knowledge representation system. *Cognitive Science*, 9(2):171–216, 1985.

[5] R. J. Brachman, R. E. Fikes, and H. J. Levesque. KRYPTON: A functional approach to knowledge representation. *IEEE Computer*, 16(10):67–73, 1983.

[6] C. Cerrato. General canonical models for graded normal logics (Graded modalities IV). *Studia Logica*, 49:241–252, 1990.

[7] F. de Caro. Graded modalities II. *Studia Logica*, 47:1–10, 1988.

[8] F. M. Donini, M. Lenzerini, D. Nardi, and W. Nutt. The complexity of concept languages. In J Allen, R Fikes, and E Sandewall, editors, *Proc. of KR'91*, pages 151–162. Morgan Kaufmann, 1991.

[9] F. M. Donini, M. Lenzerini, D. Nardi, and W. Nutt. Tractable concept languages. In *Proc. of IJCAI'91*, pages 458–463, 1991.

[10] L. Fariñas del Cerro and A. Herzig. Quantified modal logic and unification theory. Rapport LSI 293, Languages et Systèmes Informatique, Univ. Paul Sabatier, Toulouse, 1988.

[11] M. Fattorosi-Barnaba and C. Cerrato. Graded modalities III. *Studia Logica*, 47:99–110, 1988.

[12] M. Fattorosi-Barnaba and F. de Caro. Graded modalities I. *Studia Logica*, 44:197–221, 1985.

[13] K. Fine. *For Some Propositions and so Many Possible Worlds.* PhD thesis, Univ. of Warwick, 1969.

[14] K. Fine. In so many possible worlds. *Notre Dame Journal of Formal Logic*, 13(4):516–520, 1972.

[15] D. M. Gabbay and H. J. Ohlbach. Quantifier elimination in second-order predicate logic. *South African Computer Journal*, 7:35–43, 1992. Also in the *Proc. of KR'92*, 425–436. Also available as Technical Report MPI-I-92-231, Max-Planck-Institut für Informatik, Saarbrücken, Germany, July 1992.

[16] D. M. Gabbay. An irreflexivity lemma with applications to axiomatizations of conditions on linear frames. In U. Mönnich, editor, *Aspects of Philosophical Logic*, pages 67–89. Reidel, 1981.

[17] L. F. Goble. Grades of modality. *Logique et Analyse*, 13:323–334, 1970.

[18] A. Herzig. *Raisonnement automatique en logique modale et algorithmes d'unification.* PhD thesis, Univ. Paul-Sabatier, Toulouse, 1989.

[19] H. J. Levesque and R. J. Brachman. Expressiveness and tractability in knowledge representation and reasoning. *Computational Intelligence*, 3:78–93, 1987.

[20] B. Nebel and G. Smolka. Representation and reasoning with attributive descriptions. In K.-H. Bläsius, U. Hedtstück, and C.-R. Rollinger, editors, *Sorts and Types in Artificial Intelligence*, volume 418 of *Lecture Notes in Artificial Intelligence*, pages 112–139. Springer, 1990.

[21] B. Nebel. *Reasoning and Revision in Hybrid Representation Systems*, volume 422 of *Lecture Notes in Artificial Intelligence*. Springer, 1990.

[22] H. J. Ohlbach and R. A. Schmidt. Functional translation and second-order frame properties of modal logics. Technical Report MPI-I-95-2-002, Max-Planck-Institut für Informatik, Saarbrücken, Germany, January 1995.

[23] H. J. Ohlbach. A resolution calculus for modal logics. In E. Lusk and R. Overbeek, editors, *Proc. of CADE'88*, volume 310 of *Lecture Notes in Computer Science*, pages 500–516. Springer, 1988.

[24] H. J. Ohlbach. Semantics based translation methods for modal logics. *Journal of Logic and Computation*, 1(5):691–746, 1991.

[25] A. N. Prior. Egocentric logic. *Nôus*, 2:191–207, 1968.

[26] H. Sahlqvist. Completeness and correspondence in the first and second order semantics for modal logics. In S. Kanger, editor, *Proceedings of the 3rd Scandinavian Logic Symposium, 1973*, pages 110–143. North Holland, 1975.

[27] K. Schild. A correspondence theory for terminological logics: Preliminary report. In *Proc. of IJCAI'91*, pages 466–471, 1991.

[28] M. Schmidt-Schauß and G. Smolka. Attributive concept description with complements. *Artificial Intelligence*, 48:1–26, 1991.

[29] J. van Benthem. *Modal Logic and Classical Logic*. Bibliopolis, Naples, 1983.

[30] J. van Benthem. Correspondence theory. In D. Gabbay and F. Guenther, editors, *Handbook of Philosophical Logic*, volume II, pages 167–247. Reidel, 1984.

[31] W. van der Hoek and M. de Rijke. Generalized quantifiers and modal logic. ITLI Prepublication Series for Logic, Semantics and Philosophy of Language LP-91-01, Institue for Language, Logic and Information, Univ. of Amsterdam, 1991.

[32] W. van der Hoek and M. de Rijke. Counting objects. *Journal of Logic and Computation*, 5(3):325–345, 1995.

[33] W. van der Hoek. *Modalities for Reasoning about Knowledge and Quantities*. PhD thesis, Vrije Univ. Utrecht, 1992.

[34] W. van der Hoek. On the semantics of graded modalities. *Journal of Applied Non-Classical Logics*, 2(1):81–123, 1992.

[35] W. A. Woods and J. G. Schmolze. The KL-ONE family. *Computers and Mathematics with Applications*, 23(2–5):133–177, 1992.

[36] N. K. Zamov. Modal resolutions. *Izvestiya VUZ. Mathematika*, 33(9):22–29, 1989. Also published in *Soviet Mathematics*, Allerton Press.