4 On the Semantics of Graded Modalities

ABSTRACT

We enrich propositional modal logic with operators M_n ($n \in \mathbb{N}$) which are interpreted on Kripke structures as "there are more than n accessible worlds for which ...", thus obtaining a basic graded modal logic $\overline{K}$. We show how some familiar concepts (such as sub-frames, p-morphisms, disjoint unions and filtrations) and techniques from modal model theory can be used to obtain results about expressiveness (like graded modal equivalence, correspondence and definability) for this language. On the basis of the class of linear frames we demonstrate that the expressive power of the language is considerably stronger than that of classical modal logic. We give a class of formulas for which a first order equivalent can be systematically obtained, but also show that the set of formulas for which such an equivalence exists is in some sense a proper subset of the set of so called Sahlqvist formulas, a syntactically defined set of modal formulas for which a corresponding formula is guaranteed to exist. Finally we show how, combining the technique of 'filtration' with a notion of 'copying' worlds - in view of the "more than n" interpretation, one cannot simply collapse worlds - for some graded modal logics ($\overline{K}$, $\overline{T}$,...), the finite model property (and also decidability) is obtained.

1 Introduction

We undertake some investigations in (specifically) the semantics of modal logic $\overline{K}$ ($\overline{T}$, $\overline{KD4}$,...) which is obtained by augmenting the classical modal logic K (T, KD4, ...) with graded modalities M_n ($n \in \mathbb{N}$). $M_n\varphi$ is interpreted on Kripke-structures as "more than n accessible worlds verify φ". This language was already mentioned in [Gob70] and studied in [Kap70]—as an extension of S5—and [Fin72] in the seventies, and rediscovered in [FatCar85] in the eighties, where the main concern of those contributions was to obtain a sound and complete logic for this operator. In [Ben87a] a fixed point theorem for this formalism was proven, which then was used to turn recursive implicit definitions of *finite (tree) automata* into explicit ones. Other applications of this enriched language are to be found in the areas of *epistemic logic* (chapter five) and that of *generalized quantifiers* (chapter six).

Here, we are concerned with the expressibility of the graded modal language and we think that the greater expressive power of the language (over that of modal logic) is appreciated when 'standard' modal techniques are applied to it in order to lay bare and discuss (a-) similarities with modal logic. We present some expressibility results of this logic which are already interesting on their own, but we consider the main contribution of our chapter lies in the use of adapted, or, in some cases new developed techniques that are provided to obtain those results.

In section 2 we define our base logic $\overline{K}$ and derive some properties. We also introduce its proper semantics and give some basic definitions. In the same section we test our graded modal logic (gml) against some so called *reflection*- and *preservation* operations known from modal logic (ml). It appears that, with some adjustments, most of the results remain valid.

We then use those results to establish some expressibility results for gml in sections 4 and 5. Also, we apply classical preservation results to argue that gml can distinguish more (properties of) frames than ml. In section 4, we give some examples of such properties, but we also show that there exist non-isomorphic frames which, graded modally, still cannot be told apart. We illustrate matters using binary frames and linear orders. We show that, even in this simple class of frames, gml is quite stronger that ml: although ml cannot distinguish between strict, weak and ordinary linear orders, the graded modal theory of all those classes are different. We also show that *within* the class of strict linear orders however, the two languages are equally expressive.

Correspondence, yet another theme from classical modal logic, is studied in section 5. We show that more first-order properties become definable in our enriched system. However, 5.1 shows that there are first order properties that remain undefinable in gml. In section 5.2 we present a class of gml-formulas, of which the first-order corresponding formula can be obtained systematically. This class is a subclass of a (syntactically defined) set of ml-formulas for which such a result is known: the so called Sahlqvist formulas. A negative example shows that it is indeed a strict subclass, leaving the open question where the exact borders have to be drawn.

In section 6 we adapt the famous filtration technique of ml for our purposes. We provide a technique to filtrate a model for a given formula φ into a *finite* model for that formula, obtaining the finite model property (and hence, decidability) for several classes of models. Throughout the chapter, we mention some open problems and directions for further research.

2 Language and semantics

In this section we define the language L for our graded modal logic $\overline{K}$ and provide it with a semantics. L resembles very much that of ordinary modal logic. Its semantics is given by means of Kripke structures, which is also standard (cf. [Che80, HugCre68]). The main difference is, that we have a possibility operator M_n for each $n \in \mathbb{N}$, with intended meaning of $M_n\varphi$: "φ is true in more than n possible worlds".

2.1 Definition. The language L for our graded modalities is build according to the following rules:

(i) $P = \{p, q, r, \ldots\} \subseteq L$

(ii) $\varphi \in L \Rightarrow \neg\varphi \in L$

(iii) $\varphi, \psi \in L \Rightarrow \varphi \vee \psi \in L$

(iv) $\varphi \in L \Rightarrow M_n\varphi \in L$, for each $n \in \mathbb{N}$.

We also use brackets and standard abbreviations such as $(\varphi \to \psi) \equiv (\neg\varphi \vee \psi)$, $(\varphi \Delta \psi) \equiv (\varphi \vee \psi) \wedge \neg(\varphi \wedge \psi)$, $\bot \equiv (p \wedge \neg p)$, $\top \equiv \neg\bot$. Moreover, we add $L_n\varphi \equiv \neg M_n\neg\varphi$, $M!_0\varphi \equiv L_0\neg\varphi$ and $M!_n\varphi \equiv M_{n-1}\varphi \wedge \neg M_n\varphi$, $n = 1, 2, 3, \ldots$. With the intended meaning of $M_n\varphi$ given, it is easy verified that the reading of $L_n\varphi$ is "in at most n accessible worlds, ¬φ is the case", and of $M!_n\varphi$ "at exactly n accessible worlds, φ is the case", $n \in \mathbb{N}$. We write $(M_n)^k\varphi$ for $M_n \ldots M_n\varphi$ (k times M_n), and use a similar abbreviation $(L_n)^k\varphi$. We will refer to formulas of L as gml-formulas, as opposed to 'pure' modal formulas (ml-formulas). This terminology is self-explanatory and extends to notions as gml (ml) theorems, gml (ml) properties, etc.

The semantics of L is based on Kripke structures <W,π,R> ([Che80, HugCre68]).

2.2 Definition. For a Kripke structure $\mathcal{M}$ we define the *truth of* φ at w inductively:

(i) $(\mathcal{M},w) \models p$ iff $\pi(s)(p) = tt$, for all $p \in P$

(ii) $(\mathcal{M},w) \models \neg\varphi$ iff not $(\mathcal{M},w) \models \varphi$

(iii) $(\mathcal{M},w) \models \varphi \vee \psi$ iff $(\mathcal{M},w) \models \varphi$ or $(\mathcal{M},w) \models \psi$

(iv) $(\mathcal{M},w) \models M_n\varphi$ iff $|\{w' \in W \mid Rww' \text{ and } (\mathcal{M},w') \models \varphi\}| > n$, $n \in \mathbb{N}$.

2.3 Definition. We say that φ is *true* in $\mathcal{M}$ at w if $(\mathcal{M},w) \models \varphi$. Formula φ is true in $\mathcal{M}$ ($\mathcal{M} \models \varphi$) if $(\mathcal{M},w) \models \varphi$ for all $w \in W$, and φ is called *valid* ($\models \varphi$) if $\mathcal{M} \models \varphi$ for all $\mathcal{M}$. A tuple $\mathcal{F}$ = <W,R> is called a frame. $(\mathcal{F},w) \models \varphi$ ($\mathcal{F} \models \varphi$) means that for all π, $(<\mathcal{F},\pi>,w) \models \varphi$ ($<\mathcal{F},\pi> \models \varphi$, respectively). When a model $\mathcal{M}$ (or frame $\mathcal{F}$) is discussed and we argue about a valuation π or world w, it is assumed that $\mathcal{M}$ = <W,R,π> ($\mathcal{F}$ = <W,R>) for some W and R, and that $w \in W$.

2.4 Remark. Note that $(\mathcal{M},w) \models L_n\varphi$ iff $|\{w' \in W \mid Rww' \text{ and } (\mathcal{M},w') \models \neg\varphi\}| \leq n$. The modal operators M and L (in the literature also written as M and K, or ◊ and □) are special cases of our indexed operators: $M\varphi \equiv M_0\varphi$ and $L\varphi \equiv L_0\varphi$.

2.5 Definition. For a world w we define $R(w) = \{v \mid Rwv\}$. R^nwv is inductively defined to be $(v = w)$ if $n = 0$ and $\exists v'(R^{n-1}wv' \,\&\, Rv'v)$ if $n > 0$. We then say that v is (R-) *reachable* from w. The model *generated* by w, $\langle \vec{w} \rangle$, is the model containing all worlds that are reachable from w, and in which the valuation and accessibility relation are the restriction of the original valuation and relation, respectively, to those worlds.

The system K is known to be the weakest of all common modal systems. We call its graded analogue $\bar{K}$, and give its axioms and rules (from now, n and m $\in \mathbb{N}$):

2.6 Definition. The system $\bar{K}$ has the following axioms and inference rules.

A1 the axioms of propositional logic

A2 $M_{n+1}\varphi \to M_n\varphi$

A3 $L_0(\varphi \to \psi) \to (M_n\varphi \to M_n\psi)$

A4 $L_0\neg(\varphi \wedge \psi) \to ((M!_n\varphi \wedge M!_m\psi) \to M!_{n+n}(\varphi \vee \psi))$

R1 $\vdash \varphi, \vdash \varphi \to \psi \Rightarrow \vdash \psi$

R2 $\vdash \varphi \Rightarrow \vdash L_0\varphi.$

To see the system in action, we derive some of its theorems:

2.7 Proposition. The following are derivable in $\bar{K}$.

(i) $L_0(\varphi \to \psi) \to (L_n\varphi \to L_n\psi)$

(ii) $M_n(\varphi \wedge \psi) \to (M_n\varphi \wedge M_n\psi)$

(iii) $M!_n\varphi \wedge M!_m\varphi \to \bot$ $(n \neq m)$

(iv) $L_n\neg\varphi \leftrightarrow (M!_0\varphi \,\Delta\, M!_1\varphi \,\Delta \ldots \Delta\, M!_n\varphi)$

(v) $\neg M_n(\varphi \vee \psi) \to \neg M_n\varphi$

(vi) $M_{n+m}(\varphi \vee \psi) \to (M_n\varphi \vee M_m\psi)$

(vii) $M!_n\varphi \wedge M_m\varphi \to \bot$ $(m \geq n)$

(viii) $M_n(\varphi \wedge \psi) \wedge M_m(\varphi \wedge \neg\psi) \to M_{n+m+1}\varphi$

Proof. 'sub' denotes the (derivable) rule of substitution: $\vdash \alpha \leftrightarrow \beta \Rightarrow \vdash \varphi \leftrightarrow \varphi[^{\alpha}/_{\beta}]$.

(i) $L_0(\varphi \to \psi) \Rightarrow_{sub} L_0(\neg\psi \to \neg\varphi) \Rightarrow_{A3} M_n\neg\psi \to M_n\neg\varphi \Rightarrow_{A1} \neg M_n\neg\varphi \to \neg M_n\neg\psi$ $\Rightarrow_{defL_n} L_n\varphi \to L_n\psi$

(ii) apply A3 to $L_0((\varphi \wedge \psi) \to \varphi) \wedge L_0((\varphi \wedge \psi) \to \psi)$

(iii) $n \neq m$, say, $0 < n < m$ (other cases are similar): $M!_n\varphi \wedge M!_m\varphi \Rightarrow_{def\,M!} (M_{n-1}\varphi \wedge \neg M_n\varphi \wedge M_{m-1}\varphi \wedge \neg M_m\varphi) \Rightarrow_{A1} \neg M_n\varphi \wedge M_{m-1}\varphi \Rightarrow_{m-1-n \text{ times } A2} \neg M_n\varphi \wedge M_n\varphi \Rightarrow \bot$.

(iv) $\neg(M!_0\varphi \,\Delta\, M!_1\varphi \,\Delta \ldots \Delta\, M!_n\varphi) \Rightarrow_{A1} (\neg M!_0\varphi \wedge \neg M!_1\varphi \wedge \ldots \wedge \neg M!_n\varphi) \Rightarrow_{defM!} (M_0\varphi \wedge (\neg M_0\varphi \vee M_1\varphi) \wedge \ldots \wedge (\neg M_{n-1}\varphi \vee M_n\varphi)) \Rightarrow_{A1} (M_0\varphi \wedge M_1\varphi \wedge \ldots \wedge M_n\varphi) \Rightarrow_{A1} M_n\varphi$ $\Rightarrow_{defL_n} \neg L_n\neg\varphi$; for the converse use $M!_i\varphi \Rightarrow_{defM!} L_i\neg\varphi$ and $L_i\psi \Rightarrow_{A2,defL_i} L_{i+1}\psi$, $i \in \mathbb{N}$.

(v) $\Rightarrow_{A1} L_0(\varphi \to (\varphi \vee \psi)) \Rightarrow_{A1,A3} \neg M_n(\varphi \vee \psi) \to \neg M_n\varphi$.

(vi) $\neg(M_n\varphi \vee M_m\psi) \Rightarrow_{A1} L_n\neg\varphi \wedge L_m\neg\psi \Rightarrow_{(iv)} M!_{n_1}\varphi \wedge M!_{m_1}\varphi$ for some $n_1 \leq n$, $m_1 \leq m$.

For this n_1 and m_1, $\Rightarrow_{sub,defM!} \neg M_{n_1}((\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi)) \wedge \neg M_{m_1}((\varphi \wedge \psi) \vee (\neg\varphi \wedge \psi)) \Rightarrow_{(v)} \neg M_{n_1}(\varphi \wedge \psi) \wedge \neg M_{n_1}(\varphi \wedge \neg\psi) \wedge \neg M_{m_1}(\varphi \wedge \psi) \wedge \neg M_{m_1}(\neg\varphi \wedge \psi) \Rightarrow_{(iv)} M!_{n_{11}}(\varphi \wedge \psi) \wedge M!_{n_{12}}(\varphi \wedge \neg\psi) \wedge M!_{m_{11}}(\varphi \wedge \psi) \wedge M!_{m_{12}}(\neg\varphi \wedge \psi)$ for some $n_{11}, n_{12} \leq n_1$ and $m_{11}, m_{12} \leq m_1$. By (iii), $n_{11} = m_{11}$. Moreover, $n_{11} + n_{12} = n_1$, (applying (iii) to both $M!_{n_1}\varphi$ and $M!_{n_{11}}(\varphi \wedge \psi) \wedge M!_{n_{12}}(\varphi \wedge \neg\psi) \Rightarrow_{A4} M!_{(n_{11}+n_{12})}\varphi$) and, in the same way, $m_{11} + m_{12} = m_1$. So we have $M!_{n_{11}}(\varphi \wedge \psi) \wedge M!_{n_{12}}(\varphi \wedge \neg\psi) \wedge M!_{m_{12}}(\neg\varphi \wedge \psi)$. Using A4, if $h = n_{11} + n_{12} + m_{21}$, we have $M!_h((\varphi \wedge \psi) \vee (\varphi \wedge \neg\psi) \vee (\neg\varphi \wedge \psi)$, which is equivalent to $M!_h(\varphi \vee \psi)$, with $h \leq n_1 + m_1 \leq n + m$. Using the definition of $M!_h$ (which yields $\neg M_h(\varphi \vee \psi)$) and A2 (n+m)-h times, we finally get $\neg M_{n+m}(\varphi \vee \psi)$.

(vii) $M_m\varphi \Rightarrow_{(m \geq n)} M_n\varphi \Rightarrow_{A1} \neg(M_{n-1}\varphi \wedge \neg M_n\varphi) \Rightarrow_{def\,M!} \neg M!_n\varphi$. Similarly if $n = 0$.

(viii) $\neg M_{n+m+1}\varphi \Rightarrow_{sub} \neg M_{n+m+1}((\varphi \wedge \psi) \wedge (\varphi \wedge \neg\psi)) \Rightarrow_{(v)} \neg M_{n+m+1}(\varphi \wedge \psi) \wedge \neg M_{n+m+1}(\varphi \wedge \neg\psi) \Rightarrow_{(iv)} M!_{n_1}(\varphi \wedge \psi) \wedge M!_{m_1}(\varphi \wedge \neg\psi)$ for some $n_1, m_1 \leq n + m + 1$ (1). With $M_n(\varphi \wedge \psi) \wedge M_m(\varphi \wedge \neg\psi)$ and vii, (1) gives $n_1 > n$ and $m_1 > m$, so $n_1 + m_1 \geq n + m + 2$, and $M!_{n_1}(\varphi \wedge \psi) \wedge M!_{m_1}(\varphi \wedge \neg\psi) \Rightarrow_{A3} M!_{(n_1+m_1)}\varphi \Rightarrow_{def\,M!} M_{(n_1+m_1-1)}\varphi \Rightarrow_{A2,(n_1+m_1-n-m-2} M_{n+m+1}\varphi$ (2). Combining (1) and (2) completes the proof.

Note that we can indeed consider $\bar{K}$ to be an extension of K, since we have all propositional tautologies (A1), Necessitation (R2) and the K-axiom (2.7.(i), with n = 0). We end this introduction into graded modalities by stating that the logic and its semantics perfectly fit:

2.8 Theorem (Completeness: [Fin72, FatCar85]). We have, for all gml-formulas φ:

$\bar{K} \vdash \varphi$ iff $\models \varphi$.

3 Elementary model theory: preservation

An important tool when studying elementary equivalences (section 4) and deriving characterisation results (section 5) is the property of *preservation*. It helps us to derive definability results.

3.1 Definition. Let $Th_{gml}(\mathcal{M}) = \{\varphi \in L \mid \mathcal{M} \models \varphi\}$ be the *graded modal theory* of a model $\mathcal{M}$ and $Th_{ml}(\mathcal{M})$ the *modal theory* of $\mathcal{M}$.

3.2 Definition. We say that relation Q between models *preserves gml-validity* if ($Q\mathcal{M}_1\mathcal{M}_2 \Rightarrow Th_{gml}(\mathcal{M}_1) \subseteq Th_{gml}(\mathcal{M}_2)$). If, instead of "$\subseteq$", "$\supseteq$" holds, we have *anti-preservation.*

These definitions easily extend to frames and to preservation of ml-formulas. We start out with an easy example of a preserving and reflecting relation (we omit the proof on the induction of the complexity of formulas).

3.3 Fact. $\cong$ (being isomorphic with) preserves and anti-preserves gml-validity.

An example of an ml-preserving operation is provided by the notion of p-morphism, as defined in section 2.13 of chapter 3. Theorem 2.14 of chapter 3 guarantees that, for the models $\mathcal{M}$ and $\mathcal{M}'$ of figure 4.1, and all modal formulas φ, we have $(\mathcal{M},w) \models \varphi \Leftrightarrow (\mathcal{M}',w') \models \varphi$. That p-morphisms do not preserve *graded* modal formulas, is seen by observing that in w, M_1T is true, whereas it is not a w'.

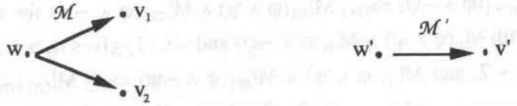

figure 4.1

In [Hoe91d], we generalized this notion of p-morphism to a notion of p⁺-morphism, which *do* preserve truth of graded formulas. Here, we will generalize the notion of *bisimulation* (cf. [Ben83] to the graded language, which is slightly more general than that of p-morphism.

3.4 Definition. For $n \in \mathbb{N}$, we use the following abbreviations:

$\underline{\forall} x_1 \ldots x_n(\phi) \equiv \forall x_1 \ldots x_n(\bigwedge_{1 \le i < j \le n}(x_i \neq x_j) \rightarrow \phi)$

$\underline{\exists} x_1 \ldots x_n(\phi) \equiv \exists x_1 \ldots x_n(\bigwedge_{1 \le i < j \le n}(x_i \neq x_j) \wedge \phi)$

3.5 Definition. Suppose that $B \subseteq W \times V$ is a relation between the models $\mathcal{M} = <W, R_{\mathcal{M}}, \pi_W>$ and $\mathcal{N} = <V, R_{\mathcal{N}}, \pi_V>$.

i B satisfies *n-forth choice* if

$$\forall w, w_0, \ldots w_n \in W\ \forall v \in V((Bwv \wedge \bigwedge_{0 \le i \le n} R_{\mathcal{M}} w w_i) \Rightarrow \underline{\exists} v_0 \ldots v_n \in V(\bigwedge_{0 \le i \le n} R_{\mathcal{N}} v v_i \wedge \bigwedge_{0 \le i \le n} B w_i v_i))$$

ii B satisfies *n-back choice* if

$$\forall w \in W\ \forall v, v_0, \ldots v_n \in V((Bwv \wedge \bigwedge_{0 \le i \le n} R_{\mathcal{N}} v v_i) \Rightarrow \underline{\exists} w_0 \ldots w_n \in W(\bigwedge_{0 \le i \le n} R_{\mathcal{M}} w w_i \wedge \bigwedge_{0 \le i \le n} B w_i v_i))$$

iii B is called a *bisimulation$_n$* between $\mathcal{M}$ and $\mathcal{N}$ if it satisfies n-forth choice and n-back choice for all $n \in \mathbb{N}$, and moreover that $\forall w \in W \forall v \in V(Bwv \Rightarrow \pi_{\mathcal{M}}(w) = \pi_{\mathcal{N}}(v))$. We say that $\mathcal{M}$ and $\mathcal{N}$ *bisimulate$_n$ each other*, $\mathcal{M} \approx \mathcal{N}$, if there exists a bisimulation$_n$ between them (as is easily seen, if B is a bisimulation$_n$, then so is B^{-1}). We then also say that $\mathcal{M}$ and $\mathcal{N}$ *are bisimulated$_n$ by B*. The relation *B* is a *bisimulation$_n$ between the frames* $\mathcal{F} = <W, R_{\mathcal{F}}>$ and $\mathcal{G} = <V, R_{\mathcal{G}}>$ if it satisfies both n-forth and n-back choice for all n.

iv A bisimulation$_n$ B that moreover satisfies that domain(B) = W and range(B) = V, B is called a *zigzag connection$_n$* between $\mathcal{M}$ and $\mathcal{N}$, and we write $\mathcal{M} \rightleftarrows \mathcal{N}$ if such a zigzag connection exists between $\mathcal{M}$ and $\mathcal{N}$. In this dissertation, if B is a function, we will call it a p⁺-morphism (in fact, the p⁺-morphisms of [Hoe91d] are a special case of them). These notions also apply to frames.

The notion of zigzag-connection$_1$ (which is the relevant notion for standard modal logic) is also called a *p-relation* (cf. Ben83). It was Segerberg who introduced the notion of p-morphism for standard modal logic (Seg70a]). If $\mathcal{M}$ and $\mathcal{N}$ are bisimulated$_n$ by B, then the condition on the valuations, together with the fact that for all worlds w and v for which Bwv holds, we can go to the same number of accessible worlds (that are identified by B), yields the following result:

3.6 Theorem. If $\mathcal{M}$ and $\mathcal{N}$ are simulated by B, then, for all graded modal formulas φ and all $w \in W$ and $v \in V$ with Bwv,

$(\mathcal{M},w) \models \varphi$ iff $(\mathcal{N},v) \models \varphi$.

Proof. Suppose Bwv. If $\varphi = p \in P$, the theorem follows immediately from 3.5.ii. For $\varphi = \neg\psi$ or $\varphi = \psi \vee \chi$, the induction hypothesis is applied straightforwardly. Assume $\varphi = M_n\psi$, and the theorem proven for ψ. If $(<\mathcal{M},\pi_{\mathcal{M}}>,w) \models M_n\psi$, then w has more than n $R_{\mathcal{M}}$-successors $w_0, \ldots, w_n$, such that $(<\mathcal{M},\pi_{\mathcal{M}}>,w_i) \models \psi$, $i = 0 \ldots n$. Since B satisfies *n-forth choice*, there are more than n distinct $R_{\mathcal{N}}$-successors $v_0, \ldots v_n$, in which, using induction, ψ is true. So, $(<\mathcal{N},\pi_{\mathcal{N}}>,v) \models M_n\psi$. The '⇐' part is proven in the same way, using *n-back choice*.

3.7 Corollary. Suppose f: $\mathcal{M} \rightarrow \mathcal{N}$ is a p⁺-morphism. Then, for all graded formulas φ:

$\mathcal{M} \models \varphi \Rightarrow \mathcal{N} \models \varphi$.

Proof. If $\mathcal{N} \not\models \varphi$, there is some v such that $(\mathcal{N},v) \models \neg\varphi$. Since f is by definition surjective, we can find a w for which, using 3.6, $(\mathcal{M},w) \models \neg\varphi$. i.e. $\mathcal{M} \not\models \varphi$.

3.8 Example. Let $\mathcal{F}_1 = <\mathbb{N},\{<x,Succ(x)>\}>$ and $\mathcal{F}_2 = <\{0,1\},\{<0,1>,<1,0>\}>$. f, defined by f(n) = n mod 2, is a p⁺-morphism, not an isomorphism. Note that the mapping f of figure 4.1 is (indeed) not a p⁺-morphism.

3.9 Corollary. Suppose f: $\mathcal{F} = <W,R_{\mathcal{F}}> \rightarrow \mathcal{G} = <V,R_{\mathcal{G}}>$ is a p⁺-morphism between the frames $\mathcal{F}$ and $\mathcal{G}$. Then:

i For all $w \in W$: $(\mathcal{F},w) \models \varphi \Rightarrow (\mathcal{G},f(w)) \models \varphi$

ii $\mathcal{F} \models \varphi \Rightarrow \mathcal{G} \models \varphi$.

3.10 Remark. Although not all p^+-morphisms are isomorphisms, on the class of finite transitive generated frames, the two concepts are equivalent. To see this, note that if $\mathcal{F}$ is generated by w, and f: $\mathcal{F} = \langle W,R \rangle \rightarrow \mathcal{F}' = \langle W',R' \rangle$ is a p^+-morphism, then f must be a bijection between w's successors and f(w)'s successors. However, since $\mathcal{F}$ is transitive and generated by w, the set of w's R-successors is just W. We leave it to the reader to verify that W' is transitive and generated by f(w). Hence f is a bijection between $W = \langle \vec{w} \rangle$ and $\langle \overrightarrow{f(w)} \rangle = W'$.

We can use 3.6 to establish related preservation results for the graded language.

3.11 Definition. $\mathcal{M}_1 = \langle W_1,R_1,\pi_1 \rangle$ is a *generated submodel* of $\mathcal{M}_2 = \langle W_2,R_2,\pi_2 \rangle$, $\mathcal{M}_1 \hookrightarrow \mathcal{M}_2$, if $W_1 \subseteq W_2$ and $\forall w \in W_1\ \forall v \in W_2$: $R_2wv \Rightarrow v \in W_1$. Moreover, $\pi_1 = \pi_2|W_1$ (the restriction of π_2 to W_1) and $R_1 = R_2|W_1$. This definition naturally extends to frames.

3.12 Theorem. Suppose $\mathcal{M}_1 = \langle W_1,R_1,\pi_1 \rangle \hookrightarrow \mathcal{M}_2 = \langle W_2,R_2,\pi_2 \rangle$, and $w \in W$, $\varphi \in L$.

i $(\mathcal{M}_1,w) \models \varphi \Leftrightarrow (\mathcal{M}_2,w) \models \varphi$

ii $\mathcal{M}_2 \models \varphi \Rightarrow \mathcal{M}_1 \models \varphi$

Proof. For item i, observe that $W_1 \times W_1$ is a bisimulation$_n$ and use 3.6. Item ii then follows using contraposition.

3.13 Corollary. Let $\mathcal{F}_1 = \langle W_1,R_1 \rangle \hookrightarrow \mathcal{F}_2 = \langle W_2,R_2 \rangle$, $\varphi \in L$ and $w \in W_1$. Then:

(i) $(\mathcal{F}_2,w) \models \varphi \Leftrightarrow (\mathcal{F}_1,w) \models \varphi$

(ii) $\mathcal{F}_2 \models \varphi \Rightarrow \mathcal{F}_1 \models \varphi$.

3.14 Definition. Let $\{\mathcal{M}_i \mid i \in I\}$ be a set of models and $\mathcal{M}_i' = \langle W_i',R_i'\pi_i' \rangle$, with $W_i' = \{\langle i,w \rangle \mid w \in W\}$, $R_i' = \{(\langle i,w \rangle,\langle i,v \rangle) \mid (w,v) \in R_i\}$ and $\pi_i'(\langle i,w \rangle) = \pi_i(w)$. The *disjoint union* $\oplus\{\mathcal{M}_i \mid i \in I\}$ of $\{\mathcal{M}_i \mid i \in I\}$ is the model $\langle \cup_{i\in I} W_i', \cup_{i\in I} R_i', \cup_{i\in I} \pi_i' \rangle$. The *disjoint union* $\oplus\{\mathcal{F}_i \mid i \in I\}$ of the frames $\{\mathcal{F}_i \mid i \in I\}$ is the frame $\langle \cup_{i\in I} W_i', \cup_{i\in I} R_i' \rangle$

3.15 Theorem. Let $\{\mathcal{M}_i \mid i \in I\}$ be as above, and $w \in W_i$, for some $i \in I$ and $\varphi \in L$. Then:

i $(\oplus\{\mathcal{M}_i \mid i \in I\},\langle i,w \rangle) \models \varphi \Leftrightarrow (\mathcal{M}_i,w) \models \varphi$.

ii $\oplus\{\mathcal{M}_i \mid i \in I\} \models \varphi \Leftrightarrow$ for all $i \in I$: $\mathcal{M}_i \models \varphi$

Proof. Observe that, for all $i \in I$, $\mathcal{M}_i \equiv \mathcal{M}_i'$ and $\mathcal{M}_i' \hookrightarrow \oplus\{\mathcal{M}_i \mid i \in I\}$. Now apply 3.3 and 3.12.

3.16 Corollary. $\oplus\{\mathcal{F}_i \mid i \in I\} \models \varphi \Leftrightarrow$ for all $i \in I$: $\mathcal{F}_i \models \varphi$.

Isomorphisms preserve gml-validity, but for modal logic we need not impose such a stringent condition on mappings between frames to achieve this. In the literature, ([Seg71a]), a weakening of isomorphisms to pseudo isomorphisms (or simply p-morphisms) is known which already is sufficient.

In the literature, one more anti-preservation result for modal formulas is known: that of ultrafilter extensions (we refer to [ChaKei73] for the definition of ultrafilter). Although (in [Hoe91d]) we were able, by using some ad hoc arguments, to prove that taking the ultrafilter of the frame ℕ anti-preserves graded modal formulas, as far as we know, the question about anti-preservation in general (for the graded language) has not been settled yet.

4 Expressive power 1: graded modal equivalence

We use the results of section 3 in obtaining expressibility results for $\bar{K}$ in sections 4 and 5, using the following general pattern: suppose we have a frame $\mathcal{G} = \langle W,R \rangle$ satisfying some property ϕ of the accessibility relation R and a frame $\mathcal{G}' = \langle W',R' \rangle$ that is obtained from $\mathcal{G}$ using some validity-preserving operation, and of which R' does not satisfy ϕ. Then there is no way we can define ϕ in our language of gml. For, the assumption

Δ For all $\mathcal{F} = \langle W,R \rangle$: $\mathcal{F} \models \varphi$ iff $\mathcal{F}$ satisfies $\phi(R)$

justifies the following argument: if $\mathcal{G}$ satisfies $\phi(R)$ then, by Δ, $\mathcal{G} \models \varphi$. Since $\mathcal{G}'$ is obtained from $\mathcal{G}$ using some validity preserving operation, we have $\mathcal{G}' \models \varphi$. Using Δ once more, we conclude that $\mathcal{G}'$ satisfies $\phi(R')$—a contradiction.

From the preservation theorem for p-morphisms (cf. theorem 3.14, chapter 3) we know that the existence of a p-morphism f: $\mathcal{F}_1 \rightarrow \mathcal{F}_2$ implies $Th_{ml}(\mathcal{F}_1) \subseteq Th_{ml}(\mathcal{F}_2)$. The frames $\mathcal{F}_1$ and $\mathcal{F}_2$ of based on the models of figure 4.1 then are examples of frames for which $Th_{ml}(\mathcal{F}_1) \subseteq Th_{ml}(\mathcal{F}_2)$, but at the same time $Th_{gml}(\mathcal{F}1) \not\subseteq Th_{gml}(\mathcal{F}_2)$. This implies that using graded modalities, more frames can be distinguished than in traditional modal logic.

4.1 Definition. Two frames $\mathcal{F}$ and $\mathcal{F}'$ are *graded modally equivalent*, $\mathcal{F} \equiv_{gml} \mathcal{F}'$, if $Th_{gml}(\mathcal{F}) = Th_{gml}(\mathcal{F}')$. If $Th_{gml}(\mathcal{F}') \subseteq Th_{gml}(\mathcal{F}')$, we say that $\mathcal{F}$ and $\mathcal{F}'$ are *gml-comparable*. Analogously, we define *modal equivalence* ($\mathcal{F} \equiv_{ml} \mathcal{F}'$ if $Th_{ml}(\mathcal{F}) = Th_{ml}(\mathcal{F}')$) and *ml-comparability* ($Th_{ml}(\mathcal{F}) \subseteq Th_{ml}(\mathcal{F}')$). The definitions can also be applied to models.

So we can rephrase the initial paragraph of this subsection by saying that the two frames $\mathcal{F}1$ and $\mathcal{F}_2$ are ml-comaparable. Since, moreover, $Th_{gml}(\mathcal{F}1) \not\subseteq Th_{gml}(\mathcal{F}_2)$, (this will become obvious in the following subsections) they are not gml-comparable.

Some interesting questions now immediately break surface. For instance, what does it mean for two frames to have the same theory? Are there modally equivalent frames that are not graded modally equivalent? If so, are there finite examples with that property? We will decide on this in this section, showing how the results of section 3 can be fruitfully used here. We start out by comparing ordinary equivalence with graded modal equivalence. Beside some simple finite frames, the class of binary frames is the scenery in which we settle some questions. The restriction to binary frames is not essential, but is rather made because of its conceptual simplicity and mathematical neatness. In fact, in chapter 3 we already gave an example of the expressive power of the graded language. At the end of this section, we will focus on a (mathematically) interesting subclass of all frames: that of linear orders, which, on its turn, is important when studying logics of time (cf. [Ben82]).

On the class of *finite frames*, $\equiv_{gml}$ and even $\equiv_{ml}$ appear to be quite strong properties:

4.2 Theorem ([Ben84a]). Let $\mathcal{F}$ and $\mathcal{F}'$ be finite, and generated by one element. Then:

$\mathcal{F} \cong \mathcal{F}'$ iff $\mathcal{F} \equiv_{ml} \mathcal{F}'$.

4.3 Corollary. If $\mathcal{F}$ and $\mathcal{F}'$ are finite and generated by one element, then

$\mathcal{F} \cong \mathcal{F}'$ iff $\mathcal{F} \equiv_{gml} \mathcal{F}'$.

Proof. '$\Rightarrow$' is obvious; for '$\Leftarrow$', it is sufficient to notice that $\mathcal{F} \equiv_{gml} \mathcal{F}'$ implies $\mathcal{F} \equiv_{ml} \mathcal{F}'$.

We will see that there are non-isomorphic (even generated) frames, which still cannot be told apart, graded modally. Corollary 4.3 implies that they should be infinite. Another consequence of 4.2 and 4.3. is, that if we are searching for possible frames $\mathcal{F}$ and $\mathcal{G}$ which are ml-equivalent, but not gml-equivalent, they have to be either not generated or infinite. It will turn out, that in both cases (4.4 and 4.6, respectively), such frames exist. We start by giving an example of two finite frames.

4.4 Theorem. There exist finite frames $\mathcal{G}$ and $\mathcal{H}$ for which the following hold:
(a) $\mathcal{G} \equiv_{ml} \mathcal{H}$ but also (b) $\mathcal{G} \not\equiv_{gml} \mathcal{H}$.

Proof. Let $\mathcal{X} \to_p \mathcal{Y}$ ($\mathcal{X} \to_{p^+} \mathcal{Y}$) for frames $\mathcal{X}$ and $\mathcal{Y}$ mean that there exists a p morphism (p$^+$-morphism) from $\mathcal{X}$ to $\mathcal{Y}$. We have the following relations between $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ of figure 4.2:

(1) $\mathcal{H} \cong \mathcal{F} \oplus \mathcal{G}$

(2) $\mathcal{F} \hookrightarrow \mathcal{G}$

(3) $\mathcal{F}, \mathcal{G} \hookrightarrow \mathcal{H}$

(4) $\mathcal{G} \to_p \mathcal{F}$, $\mathcal{H} \to_p \mathcal{F}$

Now, (1) together with our preservation theorem about disjoint unions (3.16) and isomorphisms (3.3) implies

(i) $Th_{gml}(\mathcal{H}) \subseteq Th_{gml}(\mathcal{G})$, $Th_{gml}(\mathcal{F}) \cap Th_{gml}(\mathcal{G}) = Th_{gml}(\mathcal{H})$

The preservation result about generated subframes (3.13), together with (2) and (3) yields

(ii) $Th_{gml}(\mathcal{G}) \subseteq Th_{gml}(\mathcal{F})$

(iii) $Th_{gml}(\mathcal{H}) \subseteq Th_{gml}(\mathcal{G})$, $Th_{gml}(\mathcal{H}) \subseteq Th_{gml}(\mathcal{F})$.

Finally, since we have ml-preservation for p-morphisms, we have, using (4)

(iv) $Th_{ml}(\mathcal{G}) \subseteq Th_{ml}(\mathcal{F})$, $Th_{ml}(\mathcal{H}) \subseteq Th_{ml}(\mathcal{F})$.

The relations (i), (ii) and (iii) of course also hold for the modal theory.
We have: $Th_{ml}(\mathcal{G}) \subseteq_{(iv)} Th_{ml}(\mathcal{G}) \cap Th_{ml}(\mathcal{F}) \subseteq_{(i)} Th_{ml}(\mathcal{H})$. Combining this with (iii) (or i), we obtain $\mathcal{G} \equiv_{ml} \mathcal{H}$. Although also $Th_{gml}(\mathcal{H}) \subseteq Th_{gml}(\mathcal{G})$, (from(i)), it is not the case that $Th_{gml}(\mathcal{G}) \subseteq Th_{gml}(\mathcal{H})$. To see the latter, note that $(M!_0\top \vee M!_2\top)$ is valid in $\mathcal{G}$, not in $\mathcal{H}$. Finally, note that from this observation and similar ones about $\mathcal{F}$, instead of (4), we have for p$^+$-morphisms:

(5) $\mathcal{H} \not\to_{p^+} \mathcal{G}$, $\mathcal{G} \not\to_{p^+} \mathcal{F}$, $\mathcal{H} \not\to_{p^+} \mathcal{F}$

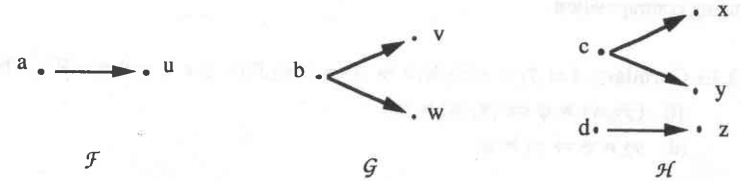

figure 4.2

4.5 Definition. For each $n \in \mathbb{N}$, we define $\mathbf{F}^n = \{\mathcal{F} \mid \mathcal{F} \models M!\top\}$. We call $\mathbf{F}^2$ the class of *binary frames*. A special binary frame $\in \mathbf{F}^2$ is $\mathcal{F}_{bin} = \langle W_{bin}, R_{bin}\rangle$ which can be denoted by a binary tree: it has a generating element *root* (= ε) and each element has one unique predecessor. (See figure 4.3). Formally: $W_{bin} = \{x \mid x \in \{0,1\}^*\}$, i.e. all sequences containing 0's and 1's, including the empty sequence ε. Finally, $R_{bin} = \{(y,z) \mid z = y0 \text{ or } z = y1\}$

First, we use $\mathcal{F}_{bin}$ to discriminate between $\equiv_{ml}$ and $\equiv_{gml}$:

4.6 Theorem. There exist generated frames $\mathcal{F}_1$ and $\mathcal{F}_2$ for which:

$\mathcal{F}_1 \equiv_{ml} \mathcal{F}_2$, but $\mathcal{F}_1 \not\equiv_{gml} \mathcal{F}_2$.

Proof. Let $\mathcal{F}_1$ be $\mathcal{F}_{bin}$ and let $\mathcal{F}_2$ be $\mathcal{F}_{bin}$, preceded by an element w. That is, we extend $\mathcal{F}_{bin}$ with a world w and stipulate that $R_2 = R_{bin} \cup \{(w, root)\}$. Then $Th_{ml}(\mathcal{F}_2) \subseteq Th_{ml}(\mathcal{F}_1)$ since $\mathcal{F}_1 \hookrightarrow \mathcal{F}_2$ and $Th_{ml}(\mathcal{F}_1) \subseteq Th_{ml}(\mathcal{F}_2)$ by the existence of a p-morphism f: $\mathcal{F}_1 \to \mathcal{F}_2$; $f(\varepsilon) = w$, $f(0x) = f(1x) = x$, $x \in \{0,1\}^*$. However, $Th_{gml}(\mathcal{F}_1) \not\subseteq Th_{gml}(\mathcal{F}_2)$, for which $M!_2T$ is a witness (*).

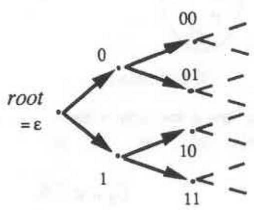

figure 4.3

The latter argument, (*), is easily generalised: if $\mathcal{F}_1 \in \mathbb{F}^n$ and $F_2 \notin \mathbb{F}^n$ for some $n \in \mathbb{N}$, then $Th_{gml}(\mathcal{F}_1) \not\subseteq Th_{gml}(\mathcal{F}_2)$. We cannot reverse this observation. In terms of $\mathbb{F}^2$ again, not all binary frames possess the same graded modal theory; there are $\mathcal{F}_1, \mathcal{F}_2 \in \mathbb{F}^2$ for which $Th_{gml}(\mathcal{F}_1) \not\subseteq Th_{gml}(\mathcal{F}_2)$. This is shown in 4.7.b. Theorem 4.7.a shows that $\mathcal{F}_{bin}$ cannot be graded modally defined. According to c, $\mathcal{F}_{bin}$ is the 'weakest' frame $\in \mathbb{F}^2$

4.7 Theorem. a. There exists a frame $\mathcal{F}_1 \not\equiv \mathcal{F}_{bin}$ with $\mathcal{F}_1 \equiv_{gml} \mathcal{F}_{bin}$

b. There are frames $\mathcal{F}_1$ and $\mathcal{F}_2 \in \mathbb{F}^2$, for which $\mathcal{F}_1 \not\equiv_{gml} \mathcal{F}_2$

c. $\mathcal{F} \in \mathbb{F}^2 \Leftrightarrow Th_{gml}(\mathcal{F}_{bin}) \subseteq Th_{gml}(\mathcal{F})$

Proof. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be as in figure 4.4. The reader is encouraged to verify that there exists a p^+-morphism f: $\mathcal{F}_{bin} \to \mathcal{F}_1$ and that also $\mathcal{F}_{bin} \hookrightarrow \mathcal{F}_1$. Although obviously $\mathcal{F}_1 \not\equiv \mathcal{F}_{bin}$, we can use 3.9 and 3.12.(ii) to conclude $\mathcal{F}_1 \equiv_{gml} \mathcal{F}_{bin}$. To see b, note that $M_0M_1p \to M_1M_0p \in Th_{gml}(\mathcal{F}_2) \setminus Th_{gml}(\mathcal{F}_{bin})$. Finally, we prove c. If $\varphi \notin Th_{gml}(\mathcal{F})$, there is some π and w such that $<\mathcal{F},\pi>, w \models \neg\varphi$. Since $\mathcal{F} \in \mathbb{F}^2$, we can 'unravel' frame $\mathcal{F}$ along $\mathcal{F}_{bin}$, in a way that *root* validates the same formulas as w. Let us denote this unravelling with f. Then, $f(w) = root$, and if $f(v) = x \in \mathcal{F}_{bin}$, we map the R-successors of v onto the R_{bin} successors of x. The valuation π' at $\mathcal{F}_{bin}$ is defined by $\pi'(f(v)) = \pi(v)$. A simple induction shows $(<\mathcal{F}_{bin},\pi'>, root) \models \varphi$.

4.8 Corollary. From 4.7.a we see that 4.3. is generally not true for infinite frames: even for generated frames, '$\equiv_{gml}$' does not imply '$\equiv$'.

A natural question now is, whether the frames $\mathcal{F}$, for which $\mathcal{F} \equiv_{gml} \mathcal{F}_{bin}$, can be characterised.

The positive answer is to be found in [Hoe91d].

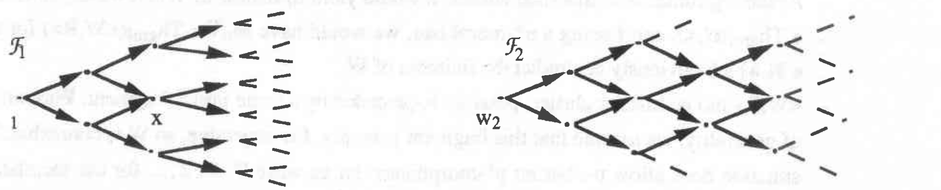

figure 4.4

Now we make a move to the realm of linear orders have an incessant appeal on mathematicians, both for their simplicity and wide applicability. Also in modal logic it makes sense to study them, especially in logics for time. A systematic modal approach to linear orders is to be found in [Ben82]. This section is, although much briefer and sketchier, very much inspired by it. Suggestively, we will write '≤' for the linear order, and '<' if it is strict. Moreover, we will, for instance, write $(\mathbb{N},<)$ instead of $<\mathbb{N},<>$.

4.9 Definition. Let $\mathbb{F}^{lo}$ be the class of frames of which the accessibility relation is a linear order and $\mathbb{F}^{slo}$ the class of strictly linear frames. Finally, $\mathbb{F}^{wlo}$ is the class of weak linear orders: here, the relation need not be anti-symmetric.

Perhaps the most appealing frames in $\mathbb{F}^{lo}$ are $(\mathbb{N},<)$ and $(\mathbb{N},\leq)$. However, they appear to be rather rigid structures, witnessing the following theorems.

4.10 Theorem. ([Hoe91d]). If f: $(\mathbb{N},<) \to (\mathbb{N},<)$ is a p-morphism, then $f \equiv I$(dentity).

Of course, there are frames <W,R> that are a non-trivial p-morphic image of $(\mathbb{N},<)$, like $<\{0\},\{(0,0)\}>$. However, p^+-morphisms even exclude this:

4.11 Theorem. If f: $(\mathbb{N},<) \to$ <W,R> is a p^+-morphism, then <W,R> $\equiv (\mathbb{N},<)$ and $f \equiv I$.

Proof. We recall that transitivity, seriality and linearity are modally definable. Then, by 3.7, R must also have these properties. So, <W,R> is a linear structure, which possibly contains some clusters (in which the relation is symmetric, and whence universal), cf. figure 4.5.

We consider the following a priori possibilities.

- R contains clusters C somewhere 'halfway'. This cluster must be finite, and so there is a greatest $n \in \mathbb{N}$ with $f(n) = c \in C$. By definition of cluster, Rcc, so by definition of p^+-

morphism, there must be an $m \in \mathbb{N}$ with $f(m) = c$ and $m > n$, contradicting our assumption about n. Conclusion: if W contains a cluster, it must be at the end.

- Assuming finitness of this final cluster, it would yield finiteness of W. However, since $M_nT \in Th_{gml}(\mathbb{N},<)$, and f being a p^+-morphism, we would have $M_nT \in Th_{gml}(\langle W,R\rangle)$ for all $n \in \mathbb{N}$ which obviously contradict the finitness of W.
- $\langle W,R\rangle$ has an infinite cluster, possibly R-preceded by a finite initial fragment. Without loss of generality, we assume that this fragment is empty. f is surjective, so W is countable. This situation does allow p-, but no p^+-morphisms: let us write 0', 1', 2',... for the members of W. The function $f: \mathbb{N} \to \langle W,R\rangle$, that assigns 0', 1', 0', 1', 2', 0',... to the natural numbers 0, 1, 2, 3, 4, 5, ... is a p-morphism. However, using 3.9, there cannot be any p^+-morphism: $(M!_1p \to M_0L_0\neg p) \in Th_{gml}(\mathbb{N},<) \setminus Th_{gml}(\langle W,R\rangle)$.

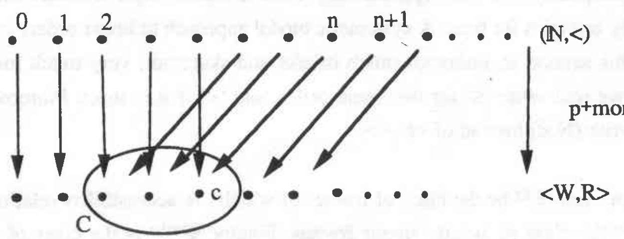

figure 4.5

If we drop the strictness condition on $(\mathbb{N},<)$, 4.10. does not hold for p-morphisms. But still then, p^+-morphisms are more rigid, according to theorem 4.12:

4.12 Theorem a. Not all p-morphisms $f:(\mathbb{N},\leq) \to (\mathbb{N},\leq)$ equal I.
b. If $f: (\mathbb{N},\leq) \to (\mathbb{N},\leq)$ is a p^+-morphism, then $f \equiv I$.

Proof An example for a is $f(x) = x/2$ if x is even, $(x-1)/2$ else. b follows from 4.11.

Let us leave $\mathbb{N}$ as it is and turn to the classes $\mathbf{F}^{wlo}$ and $\mathbf{F}^{slo}$. Using an argument of Segerberg ([Seg71a]), one can 'bulldoze' a symmetric cluster into a-symmetric orders (with the same modal theory), so that:

4.13 Theorem ([HugCre84]). $Th_{ml}(\mathbf{F}^{lo}) \subsetneq Th_{ml}(\mathbf{F}^{wlo})$. And hence, $Th_{ml}(\mathbf{F}^{lo}) = Th_{ml}(\mathbf{F}^{wlo})$.

For gml, this bulldozing technique does not work, because it does not leave numbers of successors the same. Let us illustrate this by a simple example. Consider the frame $\mathcal{F}$ and its

'bulldozed' result $\mathcal{F}'$ of figure 4.6 (R is the transitive closure of the relation denoted with arrows)

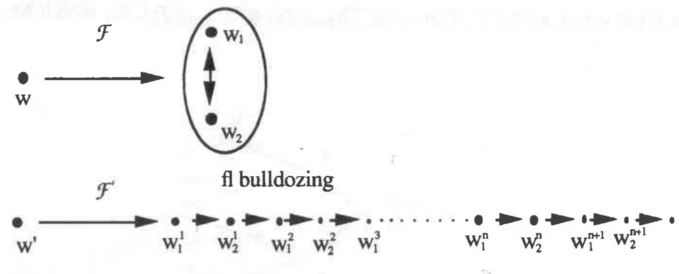

figure 4.6

If the valuations of $\mathcal{F}'$ respect those of $\mathcal{F}$, it is obvious how a modal formula, true in a model on $\mathcal{F}$, can be transported to $\mathcal{F}'$. For instance, if ml-formula $\varphi = L((p \to M\neg p) \wedge (\neg p \to Mp))$ is true at w, we get an alternating sequence of p and $\neg$p worlds on $\mathcal{F}'$. However, it is impossible to do this with the gml-formula $\varphi \wedge M!_2T$, showing that the bulldozing technique is inadequate for gml. Indeed, in section 5.1, we shall show that $Th_{gml}(\mathbf{F}^{lo}) \not\subseteq Th_{gml}(\mathbf{F}^{wlo})$.

Now we will show that in $\mathbf{F}^{slo}$ graded modalities do not supply a greater expressive power than modalities (in section 5 we will see, that gml *is* important in *defining* $\mathbf{F}^{slo}$, though).

4.14 Example. Consider $F = \langle W,R\rangle \in \mathbf{F}^{slo}$ and suppose that $\langle F,\pi\rangle,w \models M_1p$. Then, w is accessible to at least two p-worlds. Since R is anti-symmetric, linear and transitive, it follows that one of those worlds is reached *before* the other, whence $\langle F,\pi\rangle,w \models M_0(p \wedge M_0p)$. Conversely, if $M_0(p \wedge M_0p)$ is true at w, we use transitivity and irreflexivity of R to conclude that $\langle F,\pi\rangle,w \models M_1p$. We thus have a way to translate gml-formulas to equivalent ml-formulas.

4.15 Definition. We define the translation T: gml $\to$ ml as follows:
- $T(p) = p$, for propositional atoms p.
- $T(\varphi \vee \psi) = T(\varphi) \vee T(\psi)$, $T(\neg\varphi) = \neg T(\varphi)$
- $T(M_0\varphi) = M_0T(\varphi)$ and $T(M_n\varphi) = M_0(T(\varphi) \wedge M_{n-1}T(\varphi))$.

4.16 Theorem. Let $\mathcal{F} = \langle W,R\rangle \in \mathbf{F}^{slo}$ and T as above. Then for all gml-formulas φ:

i. for all π and $w \in W$ $(\langle\mathcal{F},\pi\rangle w) \models \varphi$ iff $(\langle\mathcal{F},\pi\rangle w) \models T\varphi$
ii. for all π $\langle\mathcal{F},\pi\rangle \models \varphi$ iff $\langle\mathcal{F},\pi\rangle \models T\varphi$
iii. $\mathcal{F} \models \varphi$ iff $\mathcal{F} \models T\varphi$.

Proof. Immediate, (as in 4.14, note that $TM_1p = M_0(p \wedge M_0p)$) using the definition of translation T and of strict linear order.
We conclude this section by giving two examples of orders on the cartesian product $\mathcal{F}_1 \times \mathcal{F}_2$, where $\mathcal{F}_1$ and $\mathcal{F}_2$ are linear orders. In the first (4.18), gml appears to be richer, in the other (4.19), we use 4.16 to show that ml is as adequate as gml here.

4.17 Definition. Let $\mathcal{F}_1 = \langle W_1,R_1\rangle$ and $\mathcal{F}_2 = \langle W_2,R_2\rangle$ be two linear orders. The *direct product* of $\mathcal{F}_1$ and $\mathcal{F}_2$ is defined as $\mathcal{F}_1 \,\&\, \mathcal{F}_2 = \langle W_1 \times W_2, R\rangle$ where $R(x_1,x_2)(y_1,y_2)$ iff both $R_1x_1y_1$ and $R_2x_2y_2$. The *lexicographical product* of $\mathcal{F}_1$ and $\mathcal{F}_2$ is defined as $\mathcal{F}_1 \odot \mathcal{F}_2 = \langle W_1 \times W_2, R\rangle$ where $R(x_1,y_1)(x_2,y_2)$ iff $R_1x_1y_1$ or ($x_1 = y_1$ and $R_2x_2y_2$).

4.18 Theorem.

i $Th_{ml}(\mathcal{F}_1 \,\&\, \mathcal{F}_2) \subseteq Th_{ml}(\mathcal{F}_1)$ and $Th_{ml}(\mathcal{F}_1 \,\&\, \mathcal{F}_2) \subseteq Th_{ml}(\mathcal{F}_2)$

ii There are $\mathcal{F}_1$ and $\mathcal{F}_2$ for which $Th_{gml}(\mathcal{F}_1 \,\&\, \mathcal{F}_2) \not\subseteq Th_{gml}(\mathcal{F}_1)$.

Proof.

i It is easily verified that the projections π_l: $\mathcal{F}_1 \,\&\, \mathcal{F}_2 \rightarrow \mathcal{F}_1$ and π_r: $\mathcal{F}_1 \,\&\, \mathcal{F}_2 \rightarrow \mathcal{F}_2$ are p-morphisms.

ii We claim that $Th_{gml}(\mathbb{Z} \,\&\, \mathbb{Z}) \not\subseteq Th_{gml}(\mathbb{Z})$. To see this, observe that $(M_0)^3p \rightarrow M_3M_0p$ is valid on $(\mathbb{Z} \,\&\, \mathbb{Z})$, though not on $\mathbb{Z}$. For, if $(M_0)^3p$ is true at some world $(x,y) \in (\mathbb{Z} \,\&\, \mathbb{Z})$, then p is true for some (x',y'), with $x' \geq x + 3$ and $y' \geq y + 3$. But then there are at least four successors of (x,y) for which M_0p is true: (x+1,y+1), (x+1,y+2), (x+2,y+1), (x+2,y+2). Finally, define π on $\mathbb{Z}$ such that $\pi(x)(p) = tt$ iff $x = 4$. Then, at 0, $(M_0)^3p$ is true, but not M_3M_0p, hence $\mathbb{Z} \not\models (M_0)^3p \rightarrow M_3M_0p$.

4.19 Theorem

i $Th_{ml}(\mathbb{Z} \odot \mathbb{Z}) = Th_{ml}(\mathbb{Q} \odot \mathbb{Z})$.

ii $Th_{gml}(\mathbb{Z} \odot \mathbb{Z}) = Th_{gml}(\mathbb{Q} \odot \mathbb{Z})$.

Proof.

i This is proven in [Ben82].

ii $\mathbb{Z} \odot \mathbb{Z}$ and $\mathbb{Q} \odot \mathbb{Z}$ are both strict linear orders, hence we can apply theorem 4.16, yielding $\varphi \in Th_{gml}(\mathbb{Z} \odot \mathbb{Z}) \Leftrightarrow_{4.16} T\varphi \in Th_{ml}(\mathbb{Z} \odot \mathbb{Z}) \Leftrightarrow_{4.19.i} T\varphi \in Th_{ml}(\mathbb{Q} \odot \mathbb{Z}) \Leftrightarrow_{4.16} \varphi \in Th_{gml}(\mathbb{Q} \odot \mathbb{Z})$.

5 Expressive power 2: correspondence

In this section we study first order definability of graded formulas, i.e. the correspondence between gml-formulas on the one side and (first-order) properties of the accessibility relation on

the other. In section 5.1 we give examples of first order (f.o.) properties that can be graded modally defined (the most interesting are of course those, that were not modally definable). Then, negative examples show some limitations on the expressive power of gml. For doing so, the tools developed in sections 3 and 4 appear to be helpful.

Section 5.2. is devoted to deriving f.o. properties from gml-formulas. Then we know already some examples from 5.1, but we will show how for some classes of formulas the corresponding f.o. property can be derived systematically. We show that this is only one side of the picture: there are modal formulas (the so-called 'Sahlqvist-formulas') that correspond to some f.o.-property, but as soon as graded modal operators are plugged in for ordinary modal operators, the correspondence result is smashed up.

5.1 Definability of first order properties

5.1.1 Definition. Let $\varphi \in L$ and $\phi(R)$ be a first-order property of relation R, (possibly using '='). We say that φ *corresponds to* ϕ, iff,

(*) for all frames $\mathcal{F}$: $\mathcal{F} \models \varphi$ iff $\mathcal{F}$ satisfies $\phi(R)$.

We then also say that φ *defines* $\phi(R)$. This definition easily extends to systems with multiple modalities, allowing φ (a gml formula in operators $L^1,\ldots L^n$) to correspond with $\phi(R^1,\ldots R^n)$. We have *relative correspondence* if (*) applies for all frames in some class $\mathbb{F}$ of frames. Then φ defines $\phi(R)$ *in* $\mathbb{F}$.

5.1.2 Example. Graded modalities seem pre-eminently suited to define 'having at least (at most, exactly) n R-successors' (called "seriality" for n = 1). In chapter six of this thesis we explore this feature to show that on models, any first order quantifier is definable. From chapter 3 we also know that the property $R^3 = R^1 \cap R^2$ does not correspond to any modal formula, although it does correspond to a gml-formula—provided that we first add an appropriate rule.

In this section, we will see that there exists a gml-formula that corresponds with transitivity and irreflexivity, viz. $M_0(p \wedge M_0p) \rightarrow M_1p$ (cf. 5.2.7). In section 4 we used the bi-implication $M_0(p \wedge M_0p) \leftrightarrow M_1p$ to 'unfold' gml formulas into equivalent ml-formulas on strict linear orders. It follows that the expressive power of gml is equal to that of ml on $\mathbb{F}^{slo}$. It turns out that this unfolding yields equivalent formulas only on $\mathbb{F}^{slo}$.

5.1.3 Theorem. $\mathcal{F} \in \mathbb{F}^{slo}$ iff $\mathcal{F} \models M_1p \leftrightarrow M_0(p \wedge M_0p)$.

Proof. This follows immediately from our remark that $M_0(p \wedge M_0p) \rightarrow M_1p$ defines irreflexivity and transitivity and the fact that right-linearity is defined by $M_1p \rightarrow M_0(p \wedge M_0p)$. The latter property is also ml-definable, but gml turns out to be more economical here: in ml one cannot do this but by using two propositional atoms ([Ben83]).

Now that $\mathbb{F}^{slo}$ is gml-definable, we can distinguish the strict linear order from the weak linear orders.

5.1.4 Corollary. $Th_{gml}(\mathbb{F}^{slo}) \not\subseteq Th_{gml}(\mathbb{F}^{wlo})$, although $Th_{ml}(\mathbb{F}^{slo}) \subseteq Th_{ml}(\mathbb{F}^{wlo})$.

Proof. $M_1p \leftrightarrow M_0(p \wedge M_0p) \in Th_{gml}(\mathbb{F}^{slo})$, but falsified on the $\mathbb{F}^{wlo}$-frame $\mathcal{F}$ of figure 4.6: make p true only in world w_1, then w satisfies $M_0(p \wedge M_0p)$, but not M_1p. For the second part, cf. 4.13.

5.1.5 Theorem. $Th_{gml}(\mathbb{F}^{lo}) \not\subseteq Th_{gml}(\mathbb{F}^{wlo})$, although $Th_{ml}(\mathbb{F}^{lo}) \subseteq Th_{ml}(\mathbb{F}^{wlo})$.

Proof. $(M!_2p \rightarrow M_0M!_1p) \in Th_{gml}(\mathbb{F}^{lo})$ but it is denied on the $\mathbb{F}^{wlo}$-frame $\mathcal{F}$ of figure 4.6: make p true in both w_1 and w_2.

We summarise the definability results of $\mathbb{F}^{lo}$, $\mathbb{F}^{slo}$ and $\mathbb{F}^{wlo}$ in the following theorem.

5.1.6 Theorem. Let X and Y range over $\{\mathbb{F}^{lo}, \mathbb{F}^{slo}, \mathbb{F}^{wlo}\}$. Then:

a. for all X and Y: $Th_{ml}(X) = Th_{ml}(Y)$
b. for all X and Y: $X \neq Y \Rightarrow Th_{gml}(X) \neq Th_{gml}(Y)$
c. for all X and Y: $Th_{gml}(X) \subseteq Th_{gml}(Y) \Rightarrow Y \subseteq X$.

Apparently, on transitive frames, gml is quite stronger than ml. We give an application of this. It is useful in logics of time, in which, in addition to an operator for the future, there is one for the past. So we assume to have graded modalities F_n and P_n, $n \in \mathbb{N}$. We call the system time-gml.

5.1.7 Definition. A frame $\mathcal{F} = \langle W,R \rangle$ is *connected* if for all w, w' $\in$ W there is a finite sequence $w = w_1,...w_n = w'$ such that, for all $i < n$ either Rw_iw_{i+1} or $Rw_{i+1}w_i$.

5.1.8 Theorem. On the class of connected frames the structure $(\mathbb{Z},<)$ is time-gml definable (up to isomorphism).

Proof. Consider the following axioms, $X \in \{F,P\}$, $\overline{X}_n \equiv \neg X_n \neg$

(1) $X_0\top$
(2) $X_0X_0p \rightarrow X_0p$
(3) $\overline{X}_0(\overline{X}_0p \rightarrow q) \vee \overline{X}_0(\overline{X}_0q \rightarrow p)$

(4) $X_0(p \wedge X_0p) \rightarrow X_1p$
(5) $(X_0(p \wedge \overline{X}_0\neg p) \rightarrow \overline{X}_0\overline{X}_0\neg p) \vee X_0(X_0p \wedge \overline{X}_0\overline{X}_0\neg p)$
(6) $\overline{X}_0(\overline{X}_0p \rightarrow p) \rightarrow (X_0\overline{X}_0p \rightarrow \overline{X}_0p)$

Then, each connected frame $\mathcal{F}$ that satisfies (1) - (6) is isomorphic to $(\mathbb{Z},<)$. For, if (1) - (6) are true at world x, then, by (1), $\langle \overrightarrow{x} \rangle$ is serial (we initially consider the right-successors of x, and use the axioms with $X_n = F_n$, where the case of the past is argued symmetrically), by (2) transitive, by (3) linear. (4), with theorem 5.1.5 guarantees irreflexivity. On strict linear orders, discreteness is defined by (5) (cf. [Ben82]). So $\langle \overrightarrow{x} \rangle$ satisfies the axioms of $(\mathbb{N},<)$. It's non-standard models are excluded by (6). Since $\langle \overleftarrow{x} \rangle$ satisfies the same properties, and $\mathcal{F}$ is connected, we have $\mathcal{F} \cong \mathbb{Z}$.

In ml, $\mathbb{Z}$ is not definable (cf. [Hoe91d]). After this positive result about gml, we have to say something about the limitations of definability (even on transitive frames) in gml now.

5.1.9 Theorem. The following f.o. properties of R do not correspond to any gml-formula

a.	linearity	$\forall xy(x = y \vee Rxy \vee Ryx)$
b.	discreteness	$\forall x(\exists yRxy \rightarrow \exists y(Rxy \wedge \neg\exists z(Rxz \wedge Rzy)))$
c.	left-seriality	$\forall x\exists yRyx$
d.	'selected' reflexivity	$\forall x\exists y(Rxy \wedge Ryy)$

Proof.

a. Suppose linearity corresponds to φ. Then $(\mathbb{Z},<) \models \varphi$, and, by 3.16, $(\mathbb{Z},<) \oplus (\mathbb{Z},<) \models \varphi$ yielding, with our assumption, that the latter is linear, which it is obviously not.
b. The structure $\langle \mathbb{N},\{(x,\text{successor of } x) | x \in \mathbb{N}\} \rangle$ is discrete, its p^+-morphic image $\langle \{0\},\{(0,0)\} \rangle$ is not. Now use 3.3.5. and argue as in a.
c. $\forall x\exists yRyx$ is true in $(\mathbb{Z},<)$ but not in $(\mathbb{N},<)$. Observe that $(\mathbb{N},<) \hookrightarrow (\mathbb{Z},<)$ and use 3.13.
d. Here we use our reflection result about ultrafilter extensions as stated in section 3: the ultrafilter extension $ue(\mathbb{N},<)$ of $(\mathbb{N},<)$ satisfies $\forall x\exists y(Rxy \wedge Ryy)$, whereas $(\mathbb{N},<)$ itself does not.

5.1.10 Corollary. There are f.o. properties that do not have a corresponding gml-formula, even if we restrict ourselves to transitive frames.

Proof. A witness is the d-part of the previous theorem.

We end this section by stating an other negative correspondence result, now adapting a technique that was described for multi modal logic in chapter 3, called unravelling. It appears that this technique is useful also for gml. The definition of how to obtain an unravelled model

$\mathcal{M}^*_w = \langle W^*, R^*, \pi^* \rangle$, given $\mathcal{M} = \langle W, R, \pi \rangle$ is given in 4.4 of chapter 3 of this thesis. Recall that each world of $\mathcal{M}$ is copied in $\mathcal{M}^*_w$ as many times as it is reachable from w: if in $\mathcal{M}$ we have Rwv, Rvu and Rwu, we distinguish in $\mathcal{M}^*_w$ between an u-world that is a successor of v and one that is a successor of w, i.e. we get $R^*\langle w\rangle\langle wv\rangle$, $R^*\langle w\rangle\langle wu\rangle$, $R^*\langle wv\rangle\langle wvu\rangle$ (cf. figure 4.7). We can also consider $\mathcal{M}$ to be the result of an identification of different worlds in $\mathcal{M}^*_w$, according to the following theorem.

5.1.11 Theorem. Let $\mathcal{M}$ be a model with world w and $\mathcal{M}^*_w$ the unravelled result. The function f: $\mathcal{M}^*_w \to \overrightarrow{\langle w\rangle}$ defined by $f(\langle w, w_1, \ldots w_n\rangle) = w_n$ is a p^+-morphism.

Proof. From chapter 3 we know that f is a p-morphism. To show that it satisfies 'n-forth choice' 'and n-back choice', we show that, for each $v \in \mathcal{M}^*_w$, f is a bijection between $R^*(v)$ and $R(f(v))$. Let * denote the concatenation of sequences . f is surjective: for any $y \in R(f(v))$ we have $y = f(v * \langle y\rangle)$, with $(v * \langle y\rangle) \in R^*(v)$. To see that f is injective, take $x \neq x' \in R^*(v)$. By definition of R^*, we get $x = v * \langle x_1\rangle$ and $x' = v * \langle x_2\rangle$ for some $x_1, x_2 \in R(f(v))$. Since $v * \langle x_1\rangle = x \neq x' = v * \langle x_2\rangle$, we have $f(x) = x_1 \neq x_2 = f(x')$.

5.1.12 Corollary. For all models $\mathcal{M}$ and worlds w, $Th_{gml}(\mathcal{M})$, $Th_{gml}(\mathcal{M}^*_w) \subseteq Th_{gml}(\overrightarrow{\langle w\rangle})$.

Proof. From 5.1.11 we know the existence of a p^+-morphism from $\mathcal{M}^*_w$ to $\overrightarrow{\langle w\rangle}$, which on its turn is a generated submodel of $\mathcal{M}$. The corollary then follows from 3.3.5 and 3.12.

5.1.13 Theorem. There are no gml formulas that correspond to *antisymmetry, irreflexivity, asymmetry or intransitivity*.

Proof. Let $\phi(x)$ be any of the four properties, and let $\mathcal{F}$ be a frame that does not satisfy ϕ at w. Then $\overrightarrow{\langle w\rangle}$ does not satisfy $\phi(x)$. If ϕ would correspond to φ, then $\overrightarrow{\langle w\rangle} \not\models \varphi$. By 5.1.12, $\mathcal{M}^*_w \not\models \varphi$. However, $\mathcal{M}^*_w$ *does* satisfy ϕ. For, any world in the unravelled model $\mathcal{M}^*_w$, which is a sequence (of length, say, n) of worlds of $\overrightarrow{\langle w\rangle}$, has only access to sequences of length n + 1.

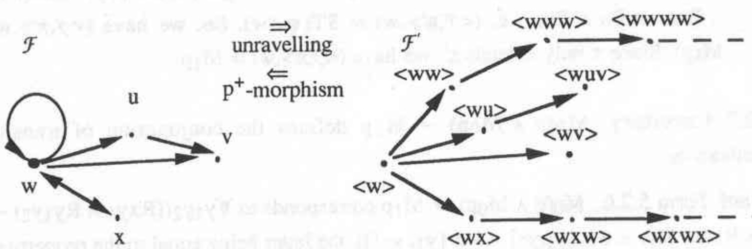

figure 4.7

5.1.14 Example. Figure 4.7 shows how a frame $\mathcal{F}$ is unravelled (from w) into a frame $\mathcal{F}'$. One easily verifies that, although <w> is an irreflexive, intransitive, asymmetric and antisymmetric world, its p^+-morphic image w is not.

5.2 First order definability of modal principles

Each positive result of 5.1. attaches a f.o. property to a given gml-formula. Two questions now emerge: does there always exist such a f.o. property, and, if yes, is there any systematics in deriving them? In order to systematically attach f.o. properties to gml, the following Standard Translation (adapted from [Ben83], where it is introduced for ml) is useful.

5.2.1 Definition. ST translates gml formulas to f.o.properties as follows:

- $ST(p) = Px$
- $ST(\neg\varphi) = \neg\, ST(\varphi)$
- $ST(\varphi \vee \psi) = ST(\varphi) \vee ST(\psi)$
- $ST(L_n\varphi) = \forall y_1, \ldots y_{n+1}((\prod_{i \neq j} y_i \neq y_j \wedge \prod_{i=1 \ldots n} Rxy_i) \to \sum_{i=1 \ldots n} [y_i/x]ST(\varphi))$

with: - no y_i (i = 1 … n) occurs free in ST(φ)
- [y/x]α is α, with each free occurrence of x replaced by y.
- Π is conjunction, Σ is disjunction. In the following, we will leave out the range of such a generalised connective, if this range is clear from context.

5.2.2 Example.

a. $ST(M_n\varphi) = \exists y_1 \ldots y_{n+1}(\Pi(y_i \neq y_j) \wedge \Pi(Rxy_i) \wedge \Pi([y_i/x]ST(\varphi)$

b. $ST(L_0p \to p) = \forall y(Rxy \to Py) \to Px$

c. $ST(M_1M_0p \to M_1p) \equiv \exists y_1y_2(y_1 \neq y_2 \wedge Rxy_1 \wedge Rxy_2 \wedge \exists u_1(Ry_1u_1 \wedge Pu_1) \wedge \exists u_2(Ry_2u_2 \wedge Pu_2) \to \exists z_1z_2(z_1 \neq z_2 \wedge Rxz_1 \wedge Rxz_2 \wedge Pz_1 \wedge Pz_2)$.

5.2.3 Theorem. Let ST be the Standard Translation of 5.2.1. Then for all models $\mathcal{M}$, frames $\mathcal{F}$ worlds w and formulas φ in the propositions $p_1, \ldots p_n$:

i. $(\mathcal{M}, w) \models \varphi$ iff $\mathcal{M} \models ST(\varphi)(w)$
ii. $\mathcal{M} \models \varphi$ iff $\mathcal{M} \models \forall x ST(\varphi)$
iii. $(\mathcal{F}, w) \models \varphi$ iff $\mathcal{F} \models (\forall P_1)(\forall P_2) \ldots (\forall P_n) ST(\varphi)(w)$
iv. $\mathcal{F} \models \varphi$ iff $\mathcal{F} \models \forall x(\forall P_1)(\forall P_2) \ldots (\forall P_n) ST(\varphi)$

Proof. Is an easy generalisation of [Ben83]

Combining 5.2.2.b and 5.2.3.iii we get $\mathcal{F}, w \models L_0p \to p$ iff $\mathcal{F} \models \forall P \forall y(Rwy \to Py) \to Pw$, giving correspondence between a gml-formula and a second order property. We give a sufficient condition for turning this into a f.o. property (theorem 5.2.6.), and illustrate the proof by an example.

5.2.4 Definition. $\pi' \geq_{p,\mathcal{F}} \pi$ if for all $w \in \mathcal{F}$: $(\pi(w)(p) = tt \Rightarrow \pi'(w)(p) = tt)$. For each $k \in \mathbb{N}$, $\pi' \geq_{p_1,\ldots p_k,\mathcal{F}} \pi$ if $\pi' \geq_{p_i,\mathcal{F}} \pi$ for all $i \leq k$. $\varphi \in L$ is called *positive* if it is equivalent with a formula build with $\top$, $\bot$, $\wedge$, $\vee$, M_n, L_n and propositional atoms. φ is *monotonic in* $p_1,\ldots p_k$, if φ's truth is enduring under extended valuations; formally: $<W,R,\pi>,w \models \varphi$ and $\pi' \geq_{p_1,\ldots p_k,<W,R>} \pi$ imply $<W,R,\pi'>,w \models \varphi$.

5.2.5 Lemma ([Ben83]) Any positive formula φ is monotonic in all its propositional atoms.

5.2.6 Theorem. Let φ be a formula build using $p_1,\ldots,p_k$, $\wedge$, $\vee$ and M_n, and ψ positive in $\{p_1,\ldots p_k\}$. Then gml-formula $(\varphi \to \psi)$ corresponds to a f.o. equivalent, which can be systematically obtained.

Proof. We proceed in several steps, and we will exemplify step (n) with the formula $(M_0(p \wedge M_0p) \to M_1p)$ in step n' (5.2.8. provides other examples). The variable s ranges over $\{1,\ldots,k\}$.

1. Obtain $ST(\varphi \to \psi) = ST(\varphi) \to ST(\psi)$, using different variables for each quantifier. Since φ only consists of M_n, $p_1,\ldots,p_k$, $\vee$ and $\wedge$, $ST(\varphi)$ only contains existential quantifiers, which all can be moved to front, giving $\exists y_1\ldots\exists y_n\varphi'$

1' We get $ST(M_0(p \wedge M_0p) \to M_1p) = \exists y_1(Rxy_1 \wedge Py_1 \wedge \exists y_2(Ry_1y_2 \wedge Py_2)) \to \exists uv(u \neq v \wedge Rxu \wedge Rxv \wedge Pu \wedge Pv)$. Here, $ST(\varphi) = \exists y_1\exists y_2(Rxy_1 \wedge Py_1 \wedge (Ry_1y_2 \wedge Py_2))$.

2. Using f.o. predicate logic, we can rewrite $ST(\varphi \to \psi) = ST(\varphi) \to ST(\psi)$ to $\forall y_1\ldots y_n(\varphi' \to ST(\psi))$ $(= \alpha)$.

2' $\alpha = \forall y_1y_2((Rxy_1 \wedge Py_1 \wedge (Ry_1y_2 \wedge Py_2)) \to \exists uv(u \neq v \wedge Rxu \wedge Rxv \wedge Pu \wedge Pv))$

3. Let $P_sY = \{y_i \mid P_sy_i$ occurs in $\varphi'\}$. P_sY is finite, and we abbreviate $\sum_{u \in P_sY}(y = u)$ to $y \in P_sY$. Let f.o. formula $\Phi \to \Psi$ be the result of replacing each occurrence of P_sz in α by $(z \in P_sY)$. For the antecedent, this has only the effect that each occurrence of P_sy_i (in φ') is replaced by 'true'. $(\Phi \to \Psi)(w)$ is the result of replacing the free variable x in $(\Phi \to \Psi)$ by w.

3' $PY = \{y_1, y_2\}$, so $\Phi \to \Psi = \forall y_1y_2((Rxy_1 \wedge Ry_1y_2) \to \exists uv(u \neq v \wedge Rxu \wedge Rxv \wedge u \in \{y_1,y_2\} \wedge v \in \{y_1, y_2\}))$

4. We claim that $\Phi \to \Psi$ corresponds to $(\varphi \to \psi)$, i.e. for all w, $(\mathcal{F},w) \models (\varphi \to \psi)$ iff $(\mathcal{F},w) \models (\Phi \to \Psi)(w)$.

4a $\Rightarrow$: If $(\mathcal{F},w) \models (\varphi \to \psi)$ then, by 5.2.3.iii, $(\mathcal{F},w) \models \forall P_1\ldots P_k\forall y_1\ldots y_n(\varphi' \to ST(\psi))$, and $(\mathcal{F},w) \models \forall y_1\ldots y_n\forall P_1,\ldots P_k(\varphi' \to ST(\psi))$. But $\Phi \to \Psi$ is only an instantiation of this formula: take $P_sz = (z \in P_sY)$. Thus, $(\mathcal{F},w) \models \Phi \to \Psi$.

4a' $\Rightarrow$: If $(\mathcal{F},w) \models M_0(p \wedge M_0p) \to M_1p$, then, by 5.2.3.iii and step 2', $(\mathcal{F},w) \models \forall y_1y_2\forall P((Rxy_1 \wedge Py_1 \wedge (Ry_1y_2 \wedge Py_2)) \to \exists uv(u \neq v \wedge Rxu \wedge Rxv \wedge Pu \wedge Pv))$. An instance of P = 'being equal to either y_1 or y_2', and hence $(\mathcal{F},w) \models \forall y_1y_2((Rxy_1 \wedge Ry_1y_2) \to \exists uv(u \neq v \wedge Rxu \wedge Rxv \wedge u \in \{y_1,y_2\} \wedge v \in \{y_1, y_2\}))$.

4b $\Leftarrow$: Suppose $(\mathcal{F},w) \models \Phi \to \Psi$, and $(<\mathcal{F},\pi>,w) \models \varphi$. w verifies $ST(\varphi)$, giving worlds $w, w_1, \ldots, w_n$ (we denote them together with W_φ) for which φ' is true. $\varphi'(w,w_1,\ldots w_n)$ is a formula using '$\wedge$' '$Rw'w''$' ($w', w'' \in W_\varphi$) and Pw' for some $w' \in W_\varphi$. Let $P_sW = \{w' \in W_\varphi \mid P_sw'$ occurs in $\varphi'\}$. This is the minimal set (in p_s) to make φ' true for the choices of W_φ. We define π' according to this minimal fulfilment: $\pi'(x)(p_s) = tt$ iff $x \in P_sW$. Then also $(<\mathcal{F},\pi'>w) \models \varphi'(w,w_1,\ldots w_n)$ (we did not change the assignment for p at those whitewashes for φ'). Let Φ^o be the result of replacing each occurrence of 'P_sw'' in $\varphi'(w,w_1,\ldots w_n)$ by $(w' \in P_sW)$. By definition of π', $(<\mathcal{F},\pi'>w) \models \Phi^o(w,w_1,\ldots w_n)$. Since Φ^o does not refer to any P_s, we even have $(\mathcal{F},w) \models \Phi^o$. (This amounts to saying that at w, there are witnesses that allow a valuation to make φ' true). Φ^o is an instantiation of Φ, so, by our assumption, we have $(\mathcal{F},w) \models \Psi^o$, where Ψ^o is obtained from Ψ by performing the same substitution. Ψ^o contains subformulas of the form $z \in P_sW$, which was exactly the extension of p_s under π'. If we replace P_sz for each $(z \in P_sW)$ in Ψ^o (giving back $ST(\psi)$), we have $(<\mathcal{F},\pi'>,w) \models ST(\psi)(w)$. With 5.3.2.iii, we conclude $(<\mathcal{F},\pi'>,w) \models \psi$. Since ψ is positive (in p_s) and $\pi \geq_{p_s} \pi'$, we have $(<\mathcal{F},\pi>,w) \models \psi$.

4b' $\Leftarrow$: Suppose $(\mathcal{F},w) \models \forall y_1y_2((Rwy_1 \wedge Ry_1y_2) \to \exists uv(u \neq v \wedge Rwu \wedge Rwv \wedge u \in \{y_1,y_2\} \wedge v \in \{y_1, y_2\}))$ and $(<\mathcal{F},\pi>,w) \models M_0(p \wedge M_0p)$. w verifies $STM_0(p \wedge M_0p)$, giving worlds w_1 and w_2 such that $\varphi'(w,w_1,w_2) = (Rww_1 \wedge Rw_1w_2 \wedge Pw_1 \wedge Pw_2)$ is true. Obviously, $PW = \{w_1, w_2\}$. So, π' makes p true only at w_1 and w_2, and $(<\mathcal{F},\pi'>,w) \models (Rww_1 \wedge Rw_1w_2 \wedge Pw_1 \wedge Pw_2)$. After substitution, we have $(<\mathcal{F},\pi'>,w) \models Rww_1 \wedge Rw_1w_2$. Clearly, $(\mathcal{F},w) \models Rww_1 \wedge Rw_1w_2$. The assumption yields $(\mathcal{F},w) \models \Psi^o$, i.e. $(\mathcal{F},w) \models \exists uv(u \neq v \wedge Rwu \wedge Rwv \wedge u \in \{w_1,w_2\} \wedge v \in \{w_1, w_2\}))$. Since under the minimal assignment π', p is true exactly at PW, we get $(<\mathcal{F},\pi'>,w) \models \exists uv(u \neq v \wedge Rwu \wedge Rwv \wedge Pu \wedge Pv)$, i.e. $(<\mathcal{F},\pi'>,w) \models ST(\psi)(w)$. So, we have $(<\mathcal{F},\pi'>,w) \models \psi$ $(= M_1p)$. Since π only extends π', we have $(<\mathcal{F},\pi>,w) \models M_1p$.

5.2.7 Corollary. $M_0(p \wedge M_0p) \to M_1p$ defines the conjunction of transitivity and irreflexivity.

Proof. Form 5.2.6., $M_0(p \wedge M_0p) \to M_1p$ corresponds to $\forall y_1y_2((Rxy_1 \wedge Ry_1y_2) \to \exists uv(u \neq v \wedge Rxu \wedge Rxv \wedge u \in \{y_1,y_2\} \wedge v \in \{y_1, y_2\}))$, the latter being equal to the property of R being both transitive and irreflexive (at x). The proof of this f.o. property in predicate logic is left to the reader.

5.2.8 Example.

a. $M_1p \rightarrow M_0(p \wedge M_0p)$ corresponds to right-linearity: $\forall yz((y \neq z \wedge Rxy \wedge Rxz) \rightarrow (Ryz \vee Rzy))$.

b. $M_{n-1}p \rightarrow L_kp$ corresponds to having at most (n + k) successors ($n \geq 1$).

Proof. We proceed along the lines of 5.2.6.

a.1 $ST(\varphi \rightarrow \psi) = \exists y_1y_2(y_1 \neq y_2 \wedge Rxy_1 \wedge Rxy_2 \wedge Py_1 \wedge Py_2) \rightarrow \exists u(Rxu \wedge Pu \wedge \exists v(Ruv \wedge Pv))$

a.2 $\forall y_1y_2(\varphi' \rightarrow ST\psi) = \forall y_1y_2((y_1 \neq y_2 \wedge Rxy_1 \wedge Rxy_2 \wedge Py_1 \wedge Py_2) \rightarrow \exists u(Rxu \wedge Pu \wedge \exists v(Ruv \wedge Pv)$.

a.3 $PY = \{y_1,y_2\}$, so $\Phi \rightarrow \Psi$ becomes $\forall y_1y_2((y_1 \neq y_2 \wedge Rxy_1 \wedge Rxy_2) \rightarrow \exists u(Rxu \wedge u \in \{y_1,y_2\} \wedge \exists v(Ruv \wedge v \in \{y_1,y_2\})$.

The latter formula is equivalent to right-linearity (it says that each two different successors of x must be accessible to each other (in at least one direction).

b1 $ST(\varphi \rightarrow \psi) = \exists y_1\ldots y_n(\prod_{i\neq j}(y_i \neq y_j) \wedge \prod_{i\leq n}Rxy_i \wedge \prod_{i\leq n}Py_i) \rightarrow$
$\forall z_1\ldots z_{k+1}(\prod_{i\leq k+1}(z_i \neq z_j) \wedge \prod_{i\leq k+1}Rxz_i) \rightarrow (\sum_{i\leq k+1}Pz_i))$.

b2 $\forall y_1\ldots y_n((\prod_{i\neq j}(y_i \neq y_j) \wedge \prod_{i\leq n}Rxy_i \wedge \prod_{i\leq n}Py_i) \rightarrow$
$\forall z_1\ldots z_{k+1}(\prod_{i\leq k+1}(z_i \neq z_j) \wedge \prod_{i\leq k+1}Rxz_i) \rightarrow (\sum_{i\leq k+1}Pz_i)))$.

b3 $\forall y_1\ldots y_n((\prod_{i\neq j}(y_i \neq y_j) \wedge \prod_{i\leq n}Rxy_i) \rightarrow \forall z_1\ldots z_{k+1}(\prod_{i\leq k+1}(z_i \neq z_j) \wedge \prod_{i\leq k+1}Rxz_i) \rightarrow$
$(\sum_{i\leq k+1}z_i \in \{y_1,\ldots y_n\})))$.

The latter formula expresses that each number n of x-accessible worlds must have a world in common with any number (k+1) of them, which is equivalent to saying that there are at most (k + n) successors of x.

The following theorem shows that we may (at least carefully) allow for negative parts in the antecedent. We omit a formal proof, since, with the substitution we provide, it is is a technical exercise in the spirit of 2.5.6 of [Hoe91d]. An example clarifies some matters.

5.2.9 Theorem. Let ψ be positive in p. Then $M!_n\neg p \rightarrow \psi$ is f.o. definable.

Proof. The key-idea is the observation that, if we have exactly n successors of w that verify $\neg p$, we know that any world satisfying p must be different from them. More formally: $ST(\varphi) = \exists y_1\ldots y_n(\prod_{i\neq j}(y_i \neq y_j) \wedge \prod_{i\leq n}(Rxy_i) \wedge \prod_{i\leq n}(\neg Py_i) \wedge \forall z(\prod_{i\leq n}(z \neq y_i) \rightarrow Pz))$. Now if $Y = \{y_1,\ldots y_n\}$, we replace each occurrence of Pu in $ST(\varphi \rightarrow \psi)$ by $(u \notin Y)$ giving a formula $\Phi \rightarrow \Psi$, with $\Phi \equiv \exists y_1\ldots y_n(\prod_{i\neq j}(y_i \neq y_j) \wedge \prod_{i\leq n}(Rxy_i))$. Now the proof is continued similar to that of 5.2.6, and hence we will not spell it out here. (Note that the valuation π' must be taken the same as π: if

the antecedent φ is true under π at w, this π is immediately minimal with respect to p. cf example 5.2.12).

There is a straightforward way to combine correspondence results to new ones:

5.2.10 Theorem. ([Hoe91d]). Suppose $(\varphi \rightarrow \psi)$ corresponds to α and $(\chi \rightarrow \psi)$ to β. Then $(\varphi \vee \chi) \rightarrow \psi$ is definable by $(\alpha \wedge \beta)$.

5.2.11 Corollary. Let ψ be positive in p. Then $L_np \rightarrow \psi$ is definable.

Proof. L_np is equivalent to $\sum_{i\leq n}(M!_i\neg p)$, and $M!_i\neg p \rightarrow \psi$ is definable by 5.2.9, so $(\sum_{i\leq n}(M!_i\neg p) \rightarrow \psi)$ is by 5.2.10.

5.2.12 Example.

$L_2p \rightarrow M_0p$ is defined by $\exists uvw(u \neq v \wedge u \neq w \wedge v \neq w \wedge Rxu \wedge Rxv \wedge Rxw)$

The properties of 5.2.9 and 5.2.11 suggest that theorem 5.2.6. might be strengthened in that we might freely allow operators $M!_n$ and L_n in the antecedent. It appears that we may not, as we shall show now. In modal logic, a substantial class of formulas is determined from which a corresponding f.o. property can be obtained constructively. The class has been given the name of 'Sahlqvist' formulae, after on of its 'discoverers', i.e. the following theorem was independently proven in [Sah75] and [Ben76]. In 5.2.14, we show that the restriction to 'non-graded' modalities is essential.

5.2.13 Theorem ([Sah75, Ben76]). Let ψ be positive and $\varphi = (M_0)^k(L_0)^np$. Then $(\varphi \rightarrow \psi)$ is f.o. definable.

5.2.14. Theorem There is no f.o. property corresponding to $L_0L_2p \rightarrow M_0M_0L_1p$.

Proof. Central tool here is a theorem of Löwenheim (cf. [ChaKei73]):

LöSk For each frame $\mathcal{F} = \langle W,R\rangle$ and countable $U \subseteq W$, there is a frame $\mathcal{F}' = \langle W',R'\rangle$ s.t.:
- W' is countable and $U \subseteq W'$
- $\mathcal{F}$ and $\mathcal{F}'$ have the same f.o. theory.

Let $\varphi = L_0L_2 \rightarrow M_0M_0L_1p$. To use LöSk, we construct a non-countable $\mathcal{F} = \langle W,R\rangle$ with

① $\mathcal{F} \models \varphi$, and

② indicate a countable $U \subseteq W$, such that for no countable $W' \supseteq U$, $\langle W',R'\rangle, w \models \varphi$.

We define $\mathcal{F} = \langle W,R\rangle$ as follows (cf. figure 4.8). Let the index set $I = \{1, 2, 3\}$ and $FI = \{f \mid f: \mathbb{N} \rightarrow I\}$.

W: $\{w\} \cup \{u, a, b, c\}\} \cup \{y_n \mid n \in \mathbb{N}\} \cup \{y_{n_i} \mid n \in \mathbb{N}, i \in I\} \cup \{z_f \mid f \in FI\}$

R: $\{(w,y_n) \mid n \in \mathbb{N}\} \cup \{(w,u)\} \cup \{(y_n,y_{n_i}) \mid n \in \mathbb{N}, i \in I\} \cup \{(y_{n_i},a), (y_{n_i},b) \mid n \in \mathbb{N}, i \in I\}$ $\cup \{(a,a), (b,b), (c,c)\} \cup \{(u,z_f), (z_f,c), (z_f,y_{nf(n)}) \mid n \in \mathbb{N}, f \in FI\}$.

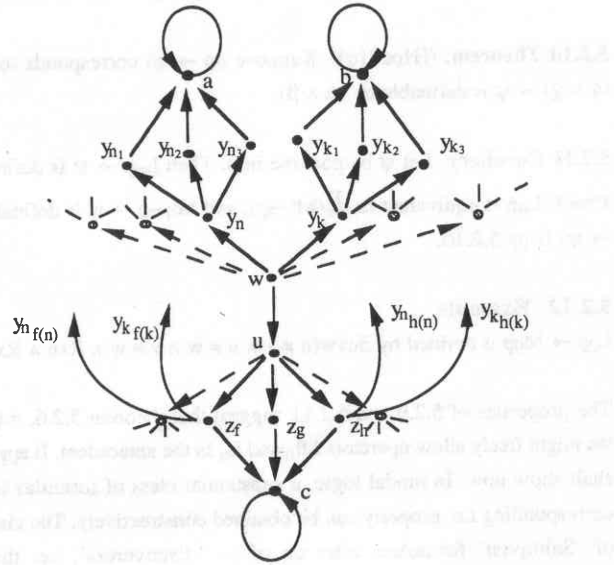

figure 4.8

For this frame we prove ➀ and ➁:

➀ We show $\mathcal{F} \models L_0L_2p \rightarrow M_0M_0L_1p$. Since a, b, and c have only one successor, L_1p is true at those worlds. But then $M_0M_0L_1p$ (and, thus, φ) is true at a, b, c, z_f, u, y_n and y_{n_i} ($f \in FI$, $i \in I$, $n \in \mathbb{N}$). Leaves us to check this at w. If $\mathcal{F},w \models L_0L_2p$ then, for each y_n there must be at least one successor validating p, say $y_{1h(1)}, y_{2h(2)}, \dots y_{nh(n)}, \dots$, for some function h $\in I$. Then, all of z_h's successors that differ from c verify p, so $\mathcal{F},z_h \models L_1p$. Then, since Rwu and Ruz$_h$, also $\mathcal{F},w \models M_0M_0L_1p$, thus $\mathcal{F},w \models L_0L_2p \rightarrow M_0M_0L_1p$.

➁ Let $U = \{w, a, b, c, y_n, y_{n_i} \mid i \in I, n \in \mathbb{N}\}$. Any countable $W' \supseteq U$ lacks at least one element of $\{z_f \mid f \in FI\}$, say z_f. Define $\pi(v,p) = tt$ iff $v \in \{y_{nf(n)}, z_g \mid n \in \mathbb{N}, z_g \in W'\}$.

Then:

i. $\langle W',R'\rangle,w = \mathcal{F}',w \models L_0L_2p$. ($L_0p$ being true at u, L_2p at y_n, $n \in \mathbb{N}$)

ii. $\langle W',R'\rangle,w = \mathcal{F}',w \not\models M_0M_0L_1p$. To see this, we will show that there is no world v that is both an R'-successor of an R'-successor of w and validating L_1p: v cannot be any y_{n_i}, since in each y_{n_i}, $M_1\neg p$ is true (p being false in a and b). v also differs from each z_g that is left in W': we have $R'z_gc$ (and p false at c), but z_g has yet another R'-successor validating $\neg p$. For, since $z_g \in W'$ and $z_f \notin W'$, we have $z_g \neq z_f$, implying that for some $k \in \mathbb{N}$, $g(k) \neq f(k)$. By definition of π, p is false at $y_{kg(k)}$. Conclusion: $M_1\neg p$ is true at z_g.

6 Filtration

We will now discuss a way to obtain a *finite* model for $\overline{K}$-consistent formulas. Not all extensions of $\overline{K}$ satisfy this property. For instance, adding the two schemes M_0T and $M_nT \rightarrow M_{n+1}T$ is sufficient to exclude all finite models. For ml., one way to distil a finite model $\mathcal{M}^*$ that satisfies φ from an arbitrary model $\mathcal{M}$ for φ is to *filter $\mathcal{M}$ through (the subformulas of) φ*. cf [HugCre84]). The gist of this technique is that worlds w of $\mathcal{M}$ are compressed to equivalence classes [w] containing worlds that verify the same subformulas of φ.

6.1 General filtration

6.1.1 Example. Consider $\mathcal{M} = \langle \mathbb{N},<,\pi\rangle$, where π makes p true in all even numbers, and q in all multiples of 3. Suppose we filtrate through $(p \wedge M_0q)$ which is true at, for instance, 2. In essence, there are only four kinds of worlds: each verifying one Boolean combination of p and q (the worlds all verify the same *modal formulas*). Lets denote these worlds with $\underline{1}$, $\underline{2}$, $\underline{3}$ and $\underline{4}$, the worlds of $\mathcal{M}^*$. Since for each of these classes $\underline{x}$ and $\underline{y}$, in $\mathcal{M}$ each x-world has access to a y-world, we take the accessibility relation on $\mathcal{M}^*$ to be universal. If π^* treats $\underline{x}$ as π does x with respect to p and q, it is easily seen that $(p \wedge M_0q)$ is true at $\underline{1}$.

However, if we want to filtrate through a *graded* modal formula, it is obvious that we cannot simply identify worlds with their equivalence classes, because in gml, we may want typically *a number of* some classes. Of course, we may use copies at will, but how many? We give some preliminary definitions before constructing the canonical model $\mathcal{M}^* = \langle W^*,R^*,\pi^*\rangle$. These definitions will be used throughout this section, without explicit reference.

6.1.2 Definition. Let ϕ be the set of all subformulas of φ. The function $S: W \rightarrow \mathcal{P}(\phi)$ defines the equivalence classes on W (through ϕ):

(1) $S(v) = \{\alpha \in \phi \mid \mathcal{M},v \models \alpha\}$

(2) Let $ran(S) = \{S_1, \dots S_m\}$ be the range of S. Then, $m \leq 2^{|\phi|}$. From now, i and j will range over $\{1,\dots m\}$. If $v \in S_i$, we will call v an S_i-world, or of type S_i.

(3) $H = 1 + \max\{i \in \mathbb{N} \mid \text{for some } \beta, M_i\beta \in \phi\}$.

(4) $n_i(v) = \min(H, |\{v' \mid Rvv' \text{ and } S(v') = S_i\}|)$.

(5) $n(v) = \langle n_1(v),\dots n_m(v)\rangle$. (For each $v \in W$, n(v) records the (relevant) numbers of successors of each type.

How many copies of each class S_i must be R^*-accessible from S(v) in our filtrate?. We have to take into account the possibility that $S(v) = S(v') = S_k$, although in the original model v and v' are R-accessible to a different number of S_i- and S_j-worlds. It turns out that we may freely choose a representative in S(v) to determine the number of S_i-successors, but, in order to be able to reason about the original model $\mathcal{M}$ again (cf. 6.1.4), we have to do some bookkeeping about which choice we make.

(6) For each class $S(v) = S_j$ we choose a unique representative $v_j!$.

Finally, to build W^*, the proper number of copies of each class S_j should be available, so we define N_i, the maximum number of S_i-worlds accessible from any relevant world:

(7) $N_i = \max(1, \max\{n_i(v_j!) \mid 1 \leq j \leq m\})$.

6.1.3 Definition. Suppose for a model $\mathcal{M} = \langle W,R,\pi\rangle$, we have $\mathcal{M},w \models \varphi$. We define the filtrate $\mathcal{M}^* = \langle W^*,R^*,\pi^*\rangle$ of $\mathcal{M}$ through ϕ, the set of subformulas of φ, as follows.

W^* of each $S_i \in ran(S)$, W^* contains N_i copies of S_i: $S_i^1,\ldots S_i^{N_i}$

R^* for any $i, j \leq m$, copy S_i^k of S_i ($1 \leq k \leq N_i$) is R^*-related to the first $n_j(w_i!)$ copies of S_j.

π^* For each $S_i^k \in W^*$, we let $\pi^*(S_i^k)(p) = tt$ iff $p \in S_i^k$

Note that all copies of any S_i are R^*-related to the same worlds (but not accessible *from* the same worlds). The number of S_j-copies that are R^*-accessible from any S_i-copy, is completely determined by the number of S_j-type worlds that are R-accessible from $w_i!$ (which may be zero!).

6.1.4 Lemma. Let $\mathcal{M}^*$ be constructed through ϕ as in 6.1.3. Then, for all $\alpha \in \phi$:

$\mathcal{M}^*,w^* \models \alpha$ iff $\alpha \in w^*$.

Proof. The sets $w^* \in W^*$ act like maximal consistent sets with respect to the formulas of ϕ: we have that for all $(\varphi_1 \vee \varphi_2)$ and $\neg\psi \in \phi$,

(i) $(\varphi_1 \vee \varphi_2) \in w^*$ iff $\varphi_1 \in w^*$ or $\varphi_2 \in w^*$, and

(ii) $\neg\psi \in w^*$ iff $\psi \notin w^*$.

Now, the lemma for $\alpha = p$ immediately follows from the definition of π^*. The cases $\alpha = (\alpha_1 \vee \alpha_2)$ and $\alpha = \neg\beta$ are easy with (i) and (ii). We check $\alpha = M_n\beta$ (note that, since $\alpha \in \phi$, $n < H$)

'$\Leftarrow$' First, assume $\alpha \in w^*$, which, in its turn is a copy S_i^k of S_i for some $i \leq m$ and $k \leq N_i$. By definition, $M_n\beta \in S_i$, so $M_n\beta$ must be true at the unique $w_i!$ with $S(w_i!) = S_i$. So $w_i!$ had at least n+1 R-successors at which β was true. We distinguish two cases:

1) For some $j \leq m$, $\beta \in S_j$ and $n_j(w_i!) = H$. Using the definition of R^* and the induction hypothesis on β, we get $\mathcal{M}^*,w^* \models M_H\beta$. Since $H > n$, then $\mathcal{M}^*,w^* \models M_n\beta$.

2) There is no such $j \leq m$. Let $j_1, \ldots j_r$ be such that all β-successors of $w_i!$ are of type $S_{j1}, \ldots S_{jr}$. Since $w_i!$ is of type i and $M_n\beta \in S_i$, $n_{j1}(w_i!) + \ldots n_{jr}(w_i!) > n$, such that, by definition of R^*, each S_i^k ($k \leq N_i$) is accessible to more than n worlds that contain β and thus, using the induction hypothesis, at which β is true. Thus: $\mathcal{M}^*,w^* \models M_n\beta$.

'$\Rightarrow$' If $\mathcal{M}^*,w^* \models M_n\beta$, then (again, let $w^* = S_i^k$) more than n R^*-successors of S_i^k verify β and, by induction, contain β. Let, for $j \leq m$, $r_j = n_j(w_i!)$ if $\beta \in S_j$, 0 else. $r_0 + \ldots + r_m$ is the number of R^*-successors of w^* that contain β, so $r_0 + \ldots + r_m > n$. Using the definition of $n_j(w_i!)$ and that of r_j, we see $\mathcal{M},w_i! \models M_n\beta$. Since $S(w) = S_i = S(w_i)$ and $M_n\beta \in S_i \subseteq \phi$, we conclude to $\mathcal{M},w \models M_n\beta$.

6.1.5 Corollary. Let $\mathcal{M}$ and $\mathcal{M}^*$ be as before. Then $\mathcal{M},w \models \varphi$ iff $\mathcal{M}^*,S(w) \models \varphi$.

Proof. $\mathcal{M}^*,S(w) \models \varphi$ iff $\varphi \in S(w)$ iff $\mathcal{M},w \models \varphi$.

It is important to observe that the filtrate $\mathcal{M}^*$ is constructed through a *finite* set ϕ of formulas. For instance, there is not such a filtrate through $\{M_np \mid n \in \mathbb{N}\}$. Since here, we are interested in gml-properties, rather than in comparing ml with gml, from now on, Th(X) means $Th_{gml}(X)$.

6.1.6 Corollary. Th({$\mathcal{M}$ | $\mathcal{M}$ is a model}) = Th({$\mathcal{M}$ | $\mathcal{M}$ is a finite model});

Th({$\mathcal{F}$ | $\mathcal{F}$ is a frame}) = Th({$\mathcal{F}$ | $\mathcal{F}$ is a finite frame}).

6.1.7 Definition. A (graded) modal system S has the *finite model property* if each non-theorem of S is falsified in a *finite* model.

6.1.8 Corollary. $\bar{K}$ has the finite model property.

Proof. Immediate from 2.10 and 6.1.6.

6.1.9 Theorem. $\bar{K}$ is decidable.

Since $\bar{K}$ is not finitely axiomatised (cf. A2 - A4 of definition 2.8), we cannot immediately apply 6.1.8. However, from the construction of 6.1.2 and 6.1.3, we easily compute an upperbound on the number of models to be considered when seeking a finite model for a consistent φ: this is $2^{|\phi|} \cdot H$, where ϕ and H are defined as in 6.1.2.

6.2 Adding special conditions

6.2.1 Definition. For each class $\mathbb{F}$ of frames we define $\mathbb{F}_{fin} \subseteq \mathbb{F}$ as all of $\mathbb{F}$'s finite elements. We say that $\mathbb{F}$ is *characterised by its finite elements* (c.f.e.) if $Th(\mathbb{F}) = Th(\mathbb{F}_{fin})$.

We can now restate 6.1.6. by saying that the class $\mathbb{F}$ of all frames is c.f.e. We even have $Th(\mathbb{F}) = Th(\mathbb{F}_{fin}) = Th((\mathbb{F}_{fin})^c)$. It is worthwhile to note that this c.f.e.-property is not immediately inherited for subclasses of frames. The filtrate $\mathcal{F}^*$ of $\mathcal{F} \in \mathbb{F}$ need not itself be a member of $\mathbb{F}$ (cf. 6.2.6). We now give some classes for which c.f.e. holds. As we proceed, the application of the filtration-lemma becomes less straightforward.

6.2.2 Theorem. The class $\mathbb{F}^n$ (cf. definition 4.5) is c.f.e.

Proof. We filtrate $\mathcal{M}$ for which $\mathcal{M},w \models \varphi$ through $(\varphi \wedge M!_nT)$.

6.2.3 Theorem. Let $\mathbb{F}_U$ be the class of frames in which R is universal: $\mathbb{F}_U = \{\mathcal{F} = \langle W,R\rangle \mid \forall xy(Rxy \wedge Ryx)\}$. Then $\mathbb{F}_U$ has the finiteness-property.

Proof. We show that the filtrate $\mathcal{M}^*$ of a universal model is itself universal. Take two worlds S_i^k and $S_j^m \in W^*$. Since W^* has at least k copies of S_i, there was a world $w \in W$ for which $n_i(w) \geq k$. Since R was universal, we have for all $w' \in W$ $n_i(w') \geq k$, in particular $n_i(w_j!) \geq k$. By definition of R^*, we see that each copy S_j^m is accessible to at least the first k copies of S_i, implying $R^*S_j^mS_i^k$.

6.2.4 Remark. We thus have $Th(\mathbb{F}_U) = Th((\mathbb{F}_U)_{fin})$. Although we also know that for each universal model $\mathcal{M}$ with $\mathcal{M},w \models \varphi$, there is a finite universal model $\mathcal{M}^*$ with $\mathcal{M}^*,w^* \models \varphi$, the converse is not true. For, if $\mathcal{M}$ is finite, for some n, $\mathcal{M},w \models L_nT$, which obviously fails in any infinite universal model.

6.2.5 Corollary. The system $\overline{S5}$ (which is $\overline{K} + \{(L_0p \rightarrow p), \neg L_np \rightarrow L_0\neg L_np)\}$) has the finite model property and is decidable.

Proof. Each $\overline{S5}$-consistent formula is satisfiable in a model in which R is universal ([Kap70, Fin72]).

In chapter six, system $\overline{S5}$ is studied in more detail. It appears to provide a natural context to study generalised quantifiers. It is shown that it has some very neat normal forms (for instance, embedded modalities are superfluous) and also that the complexity of deciding whether a given gml-formula is satisfiable, is PSPACE. A special kind of semantic normal forms provide a technique with which some questions, familiar from the field of generalised quantifiers, like obtaining a Lyndon theorem (finding a syntactic characterisation for upward-monotonicity), are very easily settled. We think that applying those techniques to systems like $\overline{K}$ and $\overline{T}$, although more complicated by the lack of the mentioned normal forms, is worthwhile studying.

We now consider a class of frames which is not closed under filtrations, but for which we can still prove the finiteness-property:

6.2.6 Theorem. The class $\mathbb{F}_R$ of reflexive frames is cfe. Hence, $\overline{T}$ is decidable.

Proof. We now have $n_i(w_i!) \geq 1$. Suppose S_i^k is R^*-connected to the first t copies of S_i, S_i^1, ... S_i^t, with $t \geq 1$. The problem is that, if S_i^{t+1} exists, it is not accessible to itself. We simply change R^* as follows: for each s > t. we withdraw $R^*S_i^sS_i^1$ and replace it by $R^*S_i^sS_i^s$. Clearly this modified R^* is reflexive and, since we did not change any number of successors of any world, lemma 6.1.4, and hence theorem 6.1.5 still holds.

Ffe is established for one more class of frames:

6.2.7 Theorem ([Hoe91d]) $\mathbb{F}^{lo}$, the class of linear frames, is cfe.

In order to warn the reader for some possible pitfalls, we round off with a negative example.

6.2.8 Theorem. Both the class of transitive irreflexive frames, and that of transitive antisymmetric frames, are not cfe.

Proof. There is no finite transitive irreflexive model in which $(M_0T \wedge L_0M_0T)$ $(= \varphi)$ is true in any world. For, suppose W is finite, and φ true at w. Then w and all its successors must have at least one successor. Let $w_1, w_2, \ldots$ be a sequence such that $w = w_1$ and Rw_iw_{i+1}. By transitivity, Rw_iw_{i+n} for each $i, n \in \mathbb{N}$. Since W is finite there must be n and i such that $Rw_{n+i}w_n$. Transitivity now yields Rw_iw_i, contradicting our assumption about irreflexivity. For transitive antisymmetric frames, we apply a similar argument to $(M_1T \wedge L_0M_1T)$ $(= \varphi)$. We need a higher 'grade' in order to deal with 'reflexive endpoints' now: we find a pair for which Rw_iw_{i+n} and $Rw_{i+n}w_i$ *and* $w_i \neq w_{i+n}$!

So it is an open question which transitive frames are cfe (like those with a universal relation, cf. 6.2.3) and which are not (like those with an irreflexive, or antisymmetric relation, cf. 6.2.8).

A References

[Ack54] W. Ackermann. *Solvable Cases of the Decision Problem.* North Holland, Amsterdam, 1954.

[AieNarSch91] L.C. Aiello, D. Nardi, and M. Schaerf. Reasoning about knowledge: the meta level approach. In B. Mayoh, editor, *Proc. of SCAI-91, Roskilde, Denmark.* IOS Press, Amsterdam, 1991, pages 4–18.

[Aly91a] N.A. Alyoshina. Probabilistic logic 1 (pl1). Report, Institute of Philosophy, Moscow, USSR, 1991.

[AtzPar88] P. Atzeni and D.S. Parker. Set containment inference and syllogisms. *Theoretical Computer Science,* **62**:39–65, 1988.

[Ben76] J.F.A.K. van Benthem. *Modal Correspondence Theory.* PhD thesis, Instituut voor Grondslagenonderzoek, University of Amsterdam, 1976.

[Ben80] J.F.A.K. van Benthem. Some kinds of modal completeness. *Studia Logica,* **39**:125–141, 1980.

[Ben82] J.F.A.K. van Benthem. *The Logic of Time.* Reidel, Dordrecht, 1982.

[Ben83] J.F.A.K. van Benthem. *Modal Logic and Classical Logic.* Bibliopolis, Naples, 1983.

[Ben84a] J.F.A.K. van Benthem. Correspondence theory. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic. Volume 2.* Reidel, Dordrecht, 1984, pages 167–247.

[Ben84b] J.F.A.K. van Benthem. Questions about quantifiers. *The Journal of Symbolic Logic,* **49**:443–466, 1984.

[Ben86a] J.F.A.K. van Benthem. *Essays in Logical Semantics.* Reidel, Dordrecht, 1986.

[Ben87a] J.F.A.K. van Benthem. Towards a computational semantics. In P. Gärdenfors, editor, *Generalized Quantifiers.* Reidel, Dordrecht, 1987, pages 31–71.

[Cer90] C. Cerrato. General canonical models for graded normal logics (Graded modalities IV). *Studia Logica,* **49**:241–252, 1990.

[ChaKei73] C. Chang and H. Keisler. *Model Theory.* North Holland, Amsterdam, 1973.

[Che80] B.F. Chellas. *Modal Logic, an Introduction.* Cambridge University Press, Cambridge, 1980.

[Car88] F. de Caro. Graded modalities II. *Studia Logica,* **47**:1–10, 1988.

[Dom69] Z. Domotor. Probabilistic relational structures and their applications. Psychology Series 144, Stanford University, Stanford, California, 1969.

[Dri87] D. Driankov. Reasoning under uncertainty: Towards a many-valued logic of belief. In *IDA annual reseach report.* Linköping University, 1987, pages 113–120.

[FagHalMeg90] R. F. Fagin, J.Y. Halpern, and N. Megiddo. A logic for reasoning about probabilities. *Information and Computation,* **87**(1/2):78–128, 1990.

[FagHal88a] R.F. Fagin and J.Y. Halpern. Belief, awareness, and limited reasoning. *Artificial Intelligence,* **34**:39–76, 1988.

[FagHal88b] R.F. Fagin and J.Y. Halpern. Reasoning about knowledge and probability. In *Proc. of the 2nd. Conf. on Theoretical Aspects of Reasoning About Knowledge.* Morgan Kaufman, Los Altos, 1988, pages 277–295.

[FagHal89] R.F. Fagin and J.Y. Halpern. Uncertainty, belief, and probability. In *Proc. of the International Joint Conference on Artificial Intelligence,* 1989, pages 1161–1167. Extended version as IBM Research Report RJ 6191 (1990).

[FagHalVar88] R. Fagin, J.Y. Halpern, and M.Y. Vardi. What can machines know? On the properties of knowledge in distributed systems. Research Report RJ 6250, IBM, 1988. To appear in JACM.

[FatAma89] M. Fattorosi-Barnaba and G. Amati. Modal operators with probabilistic interpretations, I. *Studia Logica,* **48**:383–393, 1989.

[FatCar85] M. Fattorosi-Barnaba and F. de Caro. Graded modalities I. *Studia Logica,* **44**:197–221, 1985.

[FatCer88] M. Fattorosi-Barnaba and C. Cerrato. Graded modalities III. *Studia Logica,* **47**:99–110, 1988.

[Fin72] K. Fine. In so many possible worlds. *Notre Dame Journal of Formal Logic,* **13**:516–520, 1972.

[Gab81] D. Gabbay. An irreflexivity lemma with applications to axiomatizations of conditions on linear frames. In U. Monnich, editor, *Aspects of Philosophical Logic.* Reidel, Dordrecht, 1981, pages 67–89.

[Gam82] L.T.F. Gamut. Logica, Taal en Betekenis 2. Het Spectrum, Utrecht/Antwerpen, 1982.

[Gar75] P. Gärdenfors. Qualitative probability as an intensional logic. *Journal of Philosophical Logic,* **4**:171–185, 1975.

[GarGor92] G. Gargov and V. Goranko. Modal logic with names. To appear in *Journal of Philosophical Logic.*

[Gob70] L.F. Goble. Grades of modality. *Logique et Analyse,* **13**:323–334, 1970.

[Gol87] R. Goldblatt. *Logics of Time and Computation.* Number 7 in CSLI Lecture Notes.

Stanford, 1987.

[Gol56] A.J. Goldman. Resolution and separation theorems for polyhedral convex sets. In H.W. Kuhn and A.W. Tucker, editors, *Linear Inequalities and Related Systems.* Number 38 in Annals of Mathematics Studies, Princeton Press, 1956, pages 41–51.

[Gor90] V. Goranko. Modal definability in enriched languages. *Notre Dame Journal of Formal Logic,* **31**(1):81–105, 1990.

[GorPas90] V. Goranko and S. Passy. Using the universal modality: Gains and questions. Preprint, Sofia University, 1990.

[Hal86] J.Y. Halpern, editor. *Proc. of 1st Conf. on Theoretical Aspects of Reasoning about Knowledge.* Morgan Kaufman, Los Altos, 1986.

[Hal87] J.Y. Halpern. Using reasoning about knowledge to analyze distributed systems. *Ann. Rev. Computer Science,* **2**, 1978.

[Hal91a] J.Y. Halpern. The relationship between knowledge, belief and certainty. *Annals of Mathematics and Artificial Intelligence,* **4**:301–322, 1991.

[HalMos84] J.Y. Halpern and Y.O. Moses. Knowledge and common knowledge in a distributed enviornment. In *Proc. 3rd. ACM PODC,* 1984, pages 50–61.

[HalMos85] J.Y. Halpern and Y.O. Moses. A guide to the modal logics of knowledge and belief. *Proceedings IJCAI-85.* Los Angeles, CA, 1985, pages 480–490.

[HalRab87] J.Y. Halpern and Rabin. A logic to reason about likelihood. *Artificial Intelligence,* **32**(3):379–405, 1987.

[HalZuc87] J.Y. Halpern and L.D. Zuck. A little knowledge goes a long way: Simple knowledge-based derivations and correctness proofs for a family of protocols. In *Proc. of 6th PODC,* 1987, pages 269–280.

[Hin62] J. Hintikka. *Knowledge and Belief.* Cornell University Press, Ithaca, NY, 1962.

[Hoe91d] W. van der Hoek. On the semantics of graded modalities. Technical Report IR-246, Free University of Amsterdam, 1991. To appear in *The Journal of Applied Non Classical Logics.*

[Hoe92a] W. van der Hoek. A model-theoretic study of knowledge. Forthcoming.

[HoeMey88] W. van der Hoek and J.-J.Ch. Meyer. Possible logics for belief. Technical Report IR-170, Free University of Amsterdam, 1988. To appear in *Logique et Analyse.*

[HoeMey91a] W. van der Hoek and J.-J.Ch. Meyer. Making some issues of implicit knowledge explicit. To appear in Foundations of Computer Science.

[HoeMey91b] W. van der Hoek and J.-J.Ch. Meyer. Aspecten van impliciete kennis expliciet gemaakt. In J. Treur, editor, *Proc. NAIC'91,* 1991, pages 55–67.

[HoeMey91c] W. van der Hoek and J.-J.Ch. Meyer. Graded modalities for epistemic logic. Technical Report IR 261, Free University, Amsterdam, 1991.

[HoeMeyTanWit92] W. van der Hoek, J.-J.Ch Meyer, Y.H. Tan, and C. Witteveen, editors. *Non-Monotonic Reasoning & Partial Semantics.* Ellis Horwood, Chichester, 1992.

[HoeRij91] W. van der Hoek and M. de Rijke. Modal logic and generalized quantifiers. In J. van der Does and J. van Eijck, editors, *Generalized Quantifier Theory and Ap[plications*. Dutch Network for Language, Logic and Information, 1991, pages 115–142.

[HuaKwa] Z. Huang and K. Kwast. Awareness, negation and logical omniscience. In J. van Eijck, editor, *Logics in AI*. Number 478 in Lecture Notes in Artificial Intelligence, Springer, 1990, pages 282–301.

[HugCre68] G.E. Hughes and M.J. Cresswell. *Introduction to Modal Logic*. Methuen, London, 1968.

[HugCre84] G.E. Hughes and M.J. Cresswell. *A Companion to Modal Logic*. Methuen, London, 1984.

[Jan90a] W. Janssen. Partial order semantics with an application to databases. Master's thesis, Eindhoven University of Technology, 1990.

[Kap70] D. Kaplan. S5 with multiple possibility. *The Journal of Symbolic Logic*, **35**:355–356, 1970.

[Kon86] K. Konolige. *A Deduction Model of Belief*. Research Notes in Artificial Intelligence Series. Pitman, London, 1986.

[KraWol91] M. Kracht and F. Wolter. Properties of independently axiomatizable bimodal logics. *The Journal of Symbolic Logic*, **56**(4):1469–1485, 1991.

[KraPraSei59] C.H. Kraft, J.W. Pratt, and A. Seidenberg. Intuitive probability on finite sets. *Annals of Math. Stat.*, **30**:408–419, 1959.

[KraLeh86] S. Kraus and D. Lehmann. Knowledge, belief and time. In L. Kott, editor, *Proc. ICALP*. Number 226 in Lecture Notes in Computer Science. Springer, 1986. Extended version in: *Theoretical Computer Science*, **58**:155-174, 1988.

[Kri63] S. Kripke. Semantic analysis of modal logic. *Zeitschrift fur Mathematische Logik und Grundlagen der Matematik*, **9**:67–96, 1963.

[Kui89] R. Kuiper. *Combining Linear Time Temporal Logic Descriptions of Concurrent Computations*. PhD thesis, Eindhoven University of Technology, 1989.

[Lak86] G. Lakemeyer. Steps towards a first-order logic of explicit and implicit belief. In J.Y. Halpern, editor, *Proc. of TARK Conf.* Morgan Kaufman, Los Altos, 1986, pages 325–340.

[LehPax69] K. Lehrer and T.D. Paxson. Knowledge: Undefeated justified true belief. *The Journal of Philosophy*, **66**(8):225–237, 1969.

[Len80] W. Lenzen. *Glauben, Wissen und Warscheinlichkeit*. Springer Verlag, Wien, 1980.

[Lev84] H.J. Levesque. A logic of implicit and explicit belief. In *Proc. of NCAI*, 1984, pages 198–202

[LisZil74] B. Liskov and S. Zilles. Programming with abstract data types. In *Proc. ACM SIGPLAN Conf. on Very High Level Languages*. Number 9 in SIGPLAN Notices, 1974, pages 50–60.

[Mak71] D.C. Makinson. Some embedding theorems for modal logic. *Notre Dame Journal of Formal Logic*, **12**:252, 254, 1971.

[Mal90] G. Malcolm. Data structures and program transformation. *Computer Programming*, **14**(2-3):255–280, 1990.

[ManPnu81] Z. Manna and A. Pnueli. Verification of concurrent programs, the temporal framework. In R.S. Boyer and J.S. Moore, editors, *The Correctness Problem in Computer Science*. Academic Press, London, 1981, pages 276–288.

[Mey90] J.-J.Ch. Meyer. Tactatus logico-informaticus. Printed version of inaugural speech (in Dutch), 1990.

[MeyHoe92a] J.-J.Ch. Meyer and W. van der Hoek. A cumulative default logic based on epistemic states. Forthcoming.

[MeyHoe90a] J.-J.Ch. Meyer and W. van der Hoek. Non-monotonic reasoning by monotonic means. In J. van Eijck, editor, *Proc. of Jelia*. Number 478 in Lecture Notes in Artificial Intelligence, Springer, Berlin, 1990, pages 399–411.

[MeyHoeVre91a] J.-J.Ch. Meyer, W. van der Hoek, and G.A.W. Vreeswijk. Epistemic logic for computer science: A tutorial. *EATCS bulletin*, **44**:242–270, 1991. (Part I).

[MeyHoeVre91b] J.-J.Ch. Meyer, W. van der Hoek, and G.A.W. Vreeswijk. Epistemic logic for computer science: A tutorial. *EATCS bulletin*, **45**:256–287, 1991. (Part II).

[MeyWie91] J.-J.Ch. Meyer and R.J. Wieringa, editors. *Proc. of 1st Int. Workshop on Deontic Logic in Computer Science*. Free University, Amsterdam, 1991.

[Moo84] R.C. Moore. Possible-world semantics for autoepistemic logic. In *Proc. Non-Monotonic Reasoning Workshop*. New Paltz NY, 1984, pages 344–354.

[Moo85] R.C. Moore. A formal theory of knowledge and action. In J.R. Hobbs and R.C. Moore, editors, *Formal Theories of the Commonsense World*. Norwood NJ, 1985, pages 319–358.

[Nil86] N.J. Nilson. Probabilistic logic. *Artificial Intelligence*, **28**:71–87, 1986.

[Par90] R. Parikh, editor. *Proc. 3rd. Conf. on Theoretical Aspects of Reasoning about Knowledge*. Morgan Kaufman, Los Altos, 1990.

[Per86] D. Perlis. On the consistency of commonsense reasoning. *Computational Intelligence*, **2**:180–190, 1986.

[Rei80a] R.Reiter. A logic for default reasoning. *Artificial Intelligence*, **13**:81–132, 1986.

[Res76a] N. Rescher. *Plausible Reasoning, an Introduction to the Theory and Practice of Plausibilistic Inference*. Van Gorcum, Assen, 1976.

[Rij91a] M. de Rijke. The modal logic of inequality. To appear in *The Journal of Symbolic Logic*, 1991.

[Rij91b] M. de Rijke. Correspondence theory for extended modal logic. Forthcoming.

[Rij92a] M. de Rijke. A modal logic for updating and contracting. Forthcoming.

[Roo91a] D. Roorda. Dyadic modalities and lambek calculus. In M. de Rijke, editor, *Colloquium on Modal Logic*. Dutch Network for Language, Logic and Information, 1991, pages 187–211.

[Sah75] H. Sahlqvist. Completeness and correspondence in the first- and second-order semantics for modal logic. In S. Kanger, editor, *Proc. of the 3rd. Scandanavian Logic Symposium*. North Holland, Amsterdam, 1975.

[Sch86] A. Schrijver. *Theory of Linear and Integer Programming.*. Interscience series in discrete mathematics and optimization. Wiley, Chicester, 1986.

[Sco64] D. Scott. Measurement structures and linear inequalties. *Journal Math. Psych.*, **1**:233–247, 1964.

[Seg70a] K. Segerberg. Modal logics with linear alternative relations. *Theoria*, **36**:301–322, 1970.

[Seg71a] K. Segerberg. *An Essay in Classical Modal Logic*. Filosofiska Studier **13**, University of Uppsala, 1971.

[Seg71b] K. Segerberg. Qualitative probability in a modal setting. In J.-E. Fenstad, editor, *Proc. of the 2nd Scand. Log. Symp.* North Holland, Amsterdam, 1971.

[ShoMos89] Y. Shoham and Y. Moses. Belief as defeasible knowledge. In *Proc. Eleventh International Joint Conference on Artificial Intelligence (IJCAI)*, 1989, pages 1168–1173.

[SprMeyWie91] P. Spruit, J.-J.Ch. Meyer, and R. Wieringa. Declarative semantics of passive and active updates in logic databases. In J. Göers, A. Heuer, and G. Saake, editors, *Workshop on Foundations of Models and Languages for Data and Objects*. Technische Universität Clausthal, 1991, pages 37–47.

[SupZan76] P. Suppes and M. Zanotti. Necessary and sufficient conditions for existence of a unique measure strictly agreeing with a qualitative probability ordering. *Journal of Philosophical Logic*, **5**:431–438, 1976.

[SupZan82] P. Suppes and M. Zanotti. Necessary and sufficient qualitative axioms for conditional probability. *Zeitschrift für Warscheinlichkeitstheorie und verwandte Gebiete*, **60**:163–169, 1982.

[SupZan89] P. Suppes and M. Zanotti. Conditions on upper and lower probabilities to imply probabilities. *Erkenntnis*, **31**:323–345, 1989.

[TakTomYosVee90] H. Takeda, T. Tomiyama, H. Yoshikawa, and P. J. Veerkamp. Modeling design processes. *AI Magazine*, **11**:13–36, 1990.

[Tan91a] Y.H. Tan. Relating bi-modular systems to default logic and autoepistemic logic. In W. van der Hoek and Y.H. Tan, editors, *Proc. of the Int. Workshop on Non-Monotonic Reasoning and Partial Semantics*. Free University, Amsterdam, 1991, pages 49–87.

[Tan92a] Y.H. Tan. BMS - A meta-level approach to non-monotonic reasoning. Chapter 4 in W. van der Hoek et.al., editors, *Non-Monotonic Reasoning & Partial Semantics*. Series in Artificial Intelligence. Ellis Horwood, Chichester, 1992. Forthcoming.

[TanTre89] Y.H. Tan and H. Treur. A logical descripiton of a bi-modular system for non-monotonic reasoning with dynamic assumptions. Technical Report PE8907, University of Amsterdam, 1989.

[TanTre91] Y.H. Tan and J. Treur. A bi-modular approach to non-monotonic reasoning. In *Proc. of the WOCFAI'91*. CNRS, Paris, 1991, pages 461–476.

[Thi90] E. Thijsse. Partial logic and modal logic: the overall theory. In M. Stokhof and L. Torenvliet, editors, *Proc. of the 7th Amsterdam Colloqium*. ITLI, Amsterdam, 1990, pages 555–579.

[Tur84] R. Turner. *Logics for Artificial Intelligence*. Ellis Horwood/Wiley, Chichester/New York, 1984.

[Tur90] R. Turner. *Truth and Modality for Knowledge Representation*. Pitman, London, 1990.

[Vak91] D. Vakarelov. Modal logics for reasoning about arrows: arrow logics. In *Logic Colloquium '91. Abstracts of the 9th International Congress of Logic. Volume I: Logic. Methodology and Philosophy of Science*. Uppsala, 1991.

[Var88] M.Y. Vardi, editor. *Proc. 2nd Conf. on Theoretical Aspects of Reasoning about Knowledge*. Morgan Kaufman, Los Altos, 1988.

[Vel91a] F. Veltman. Defaults in update semantics. ITLI Prepublication Series LP-91-02, University of Amsterdam, Amsterdam, 1991.

[Ven92a] Y. Venema. *Many Dimensional Modal Logic*. PhD thesis, University of Amsterdam, 1992.

[Voo90a] F. Voorbraak. The logic of objective knowledge and rational belief. In J. van Eijck, editor, *Logics in AI*. Number 478 in Lecture Notes in Artificial Intelligence, Springer, 1990, pages 499–516.

[Wes89] D. Westerståhl. Quantifiers in formal and natural languages. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic. Volume IV*. Reidel, Dordrecht, 1989, pages 1–131.

[WieMeyWei] R.J. Wieringa, J.-J.Ch. Meyer, and H. Weigand. Specifying dynamic and deontic integrity constraints. *Data & Knowledge Engineering*, **4**:157–189, 1989.

[Zwi89] J. Zwiers. *Concurrency and partial correctness*. Number 321 in Lecture Notes in Computer Science. Springer, 1989.