Paper:

Modifier Logics Based on Graded Modalities

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Modifier logics are considered as generalizations of "classical" modal logics. Thus modifier logics are so-called multimodal logics. Multimodality means here that the basic logics are modal logics with graded modalities. The interpretation of modal operators is more general, too. Leibniz's motivating semantical ideas (see [8]. p. 20-21) give justification to these generalizations. Semantics of canonical frames forms the formal semantic base for modifier logics. Several modifier systems are given. A special modifier calculus is combined from some "pure" modifier logics. Creating a topological semantics to this special modifier logic may give a basis to some kind of fuzzy topology. Modifier logics of S4-type modifiers will give a graded topological interior operator systems, and thus we have a link to fuzzy topology.

Keywords: graded modality, modifier, modifier logic, modifier calculus

1. Introduction

We suppose that the reader is familiar with usual notation and terminology of logic.

First, consider some fundamental things from history and some motivating things. Lemmon [8, p. 20 - 21] describes Leibniz's basic ideas for motivating the idea of "classical modal logics". He says: "Leibniz's suggestion now becomes: a sentence is necessarily true (in this world) if that sentence is true in all worlds alternative to this world." See Lemmon [8, p. 20] to check the detailed analysis about what the concept "*alternative to this world*" means. Lemmon [8, p.21] continues: "Actually, in many connections it is intuitively simpler to think of world t as *accessible from* world u rather than *alternative* to u. This at least has the merit of avoiding the temptation to suppose that alternativeness is a symmetric relation between worlds - that if t is alternative to u, then u must be alternative to t. Indeed, we shall not assume that each world is accessible from itself, or even that to each world there is at least one accessible world: there may be accessibility-isolated worlds. We shall find that to many such assumptions about the accessibility relation between worlds there correspond distinctive modal sentences which come out valid precisely because we have made those assumptions. If necessity means truth in all accessible worlds, then possibility will mean truth in some accessible world. Thus our remarks about the vagueness of the notion of necessity, and the various more precise accounts of it, may be repeated *mutantis mutandis* for the notion of possibility."

As we see, intuitive ideas for modal logics start from the concepts *necessary* and *possible*. Also these concepts are not truth-functional, because considering truth in a world needs also other worlds accessible from that world. Already Aristotle considered the question of presuppositions for a sentence be necessarily true, or possibly true. He somehow gave probabilistic meanings to these concepts. If we interpret the concept "necessary" to be "certain", the probabilistic meaning would be more clear. But this interpretation is just a special case. The reason for this may be the fact that these modal concepts has closely been related to many-valuedness already in Aristotle's time - and also due to him. Also Lukasiewicz continued this interpretation. He used Aristotle's way to motivate the idea of his three-valued logic with probabilistic examples where the concept "time" (may be accidentally) had the fundamental role in such a way that tomorrow we can see, whether the sentence "*Tomorrow* we will have a naval battle" were true or not *yesterday*. Before yesterday it is uncertain. Anyway, the time is actually not present in the formalism of these logics. Also early considerations of modal logics did not include the deontic aspect. As we have learned later, a deontic logic needs some further operators in addition to the usual ones. As we see above, Lemmon says that vagueness is associated with the notion of necessity. As we know, vagueness does not mean the same as probability. It contains also features of fuzziness. When we consider mathematical results from above mentioned intuitive ideas, i.e. formal semantics of modal logics, especially canonical frames, we see that these results are more general than just the ideas about mathematical models of necessity and possibility. The author has strongly come into the thought that the most general linguistic interpretations for operators resulting from the formal semantics equipped with the considerations above are substantiating operator and weakening operator. We can also call them substantiating modifier and weakening modifier, respectively. The concepts necessity and possibility are corresponding instances of these modifiers. Thus we have for example modifiers of T-style, S4-style, S5-style etc. corresponding modal systems T, S4, S5 etc., respectively. These labels or names do not have any special interpretations, like probabilistic one, as their burdens. The mathematical analysis of the formal semantics does not take any such interpretations for granted.

The term modifier appears in many different connections and in many different meaning. Here we use it in the meaning of either a substantiating or weakening operator having some fuzzy features. For example, some hedges can be modifiers in our purposes.

Example. Consider a situation that illustrates modifiers, accessibility relation, and possible worlds. We describe what kind of spices we can associate with the sentence

(a) "Mr. M can speak Japanese." For example, we can operate with hedges "well" and "more or less well" to this sentence. Thus we have

(b) "Mr. M can speak Japanese well."

This sentence presupposes something more from Mr. M's ability to speak Japanese. The other hedge gives

(c) "Mr. M can speak Japanese more or less well."

This sentence does not presuppose so much from Mr. M as the sentence (a). The hedges "well" and "more or less well" modify the original sentence (a). The first hedge "well" is apparently substantiating and the second one "more or less well" is weakening. If this separation with hedges is too rough, we can make it more dense by adding some further modifiers, say, "rather well", "very well", "extremely not well", and "extremely well". So, we have the set of substantiating modifiers

Mod = {well, very well, extremely well}

And the corresponding set of weakening modifiers

Mod^* = {more or less well, rather well, extremely not well}

Now we have more sentences associated with (a):

(d) "Mr. M can speak Japanese rather well."
(e) "Mr. M can speak Japanese very well."
(f) "Mr. M can speak Japanese extremely well."
(g) "Mr. M can speak Japanese extremely not well."

We may consider the hedges in Mod^* to be the duals of those in Mod, i.e. the dual of "well" is (well)* = not well, that of "more or less well" is (more or less well)* = "very well", that of "extremely well" is (extremely well)* = "extremely not well". We can translate this into propositional modifier language (see the definitions below) as follows. The set of non-weakening modifiers is

$$Mod = \{F_0, F_1, F_2, F_3\}$$

i.e. F_1 stands for "well", F_2 for "very well", and F_3 for "extremely well". F_0 is the identifying operator. We need it for mainly technical reasons. The set of non-substantiating modifiers is

$$Mod^* = \{ F_0^*, F_1^*, F_2^*, F_3^*\}$$

i.e. F_1^* stands for "more or less well", F_2^* for "rather well", and F_3^* for "extremely not well". F_0^* is again the identifying operator. Identifying operators in modifier logic are self-dual, i.e. F_0 is the dual of itself. Thus $F_0 = F_0^*$. See formal bases below. Next. Consider some possible worlds related to this case. These worlds can be sets of different situations, where natural language is needed, and especially, in this case the language is Japanese. Suppose we have a set of possible worlds, say,

$$W = \{w_1, w_2, \dots w_{10}\}.$$

Among these worlds there is a world, say w_3, such that we can manage the situations in w_3 using essential everyday Japanese. We agree, that if Mr. M can do this then the sentence (a) is true in w_3. Let a world, say w_5, be such that we need advanced capability in w_5, and w_5 is accessible from w_3. Thus we say that the sentence (b) is true in w_3, because we can now agree that the sentence (a) is true in w_5, i.e. Mr. M can speak Japanese in w_5, and w_5 is accessible from w_3. The world w_3 need not be a subset of w_5. w_5 may consist of only some "advanced things" without some "usual situations" belonging to w_3. Suppose further that a world, say w_8, is such that we need a lot of special terminology in it, and w_8 is accessible from w_3. If the sentence (a) is true also in this special world w_8, we can say that the sentence (e) is true in w_3. In the same way, if the sentence (a) is true also in w_{10}, and it is accessible from w_5, then apparently the sentence (f) is true in w_3. If the sentence (a) is true in w_5, and w_5 is accessible from w_3, we say that the sentence (c) is true in w_3. If the sentence (a) is true in w_8, and w8 is accessible from w_3, we say that the sentence (d) is true in w_3. If the sentence (a) is true in w10, and w_{10} is accessible from w_3, we say that the sentence (g) is true in w_3. We complete our example by letting a stand for the sentence (a). Thus we have the formalizations of (b), …, (g) as follows:

(*b*) $F_1(\alpha)$
(*c*) $F_1^*(\alpha)$
(*d*) $F_2^*(\alpha)$
(*e*) $F_2(\alpha)$
(*f*) $F_3(\alpha)$
(*g*) $F_3^*(\alpha)$

We usually express the truth of a sentence φ in a given world w by writing $w \models \varphi$. Thus we can give the truth status of the sentences (a), …, (g) in the world w_3 as follows: $w_3 \models F_1(\alpha)$ if $w_3 \models \alpha$ and $w_5 \models \alpha$, $w_3 \models F_2(a)$ if $w_3 \models \alpha$, $w_5 \models \alpha$ and $w_8 \models \alpha$, $w_3 \models F_3(\alpha)$ if $w_3 \models \alpha$, $w_5 \models \alpha$, and $w_8 \models \alpha$ and $w_{10} \models \alpha$. Further $w_3 \models F_1^*(\alpha)$ if $w_5 \models \alpha$, $w_3 \models F_2^*(\alpha)$ if $w_8 \models \alpha$, and $w_3 \models F_3^*(\alpha)$ if $w_{10} \models \alpha$.

The idea illustrated by this example is based on frame semantics of "classical" modal logic. However, we have extended it such that we have graded operators in our system.

Secondly, we consider briefly classical logic as a preliminary thing for modifier logic. Modifier logics are build on the base of propositional logic (i.e. zero order logic) or predicate logic (i.e. first order logic). These classical logics are like the basement and the first floor of a building. Also the first floor stands on the basement, i.e. predicate logic is based on propositional logic.

A propositional language consists of (i) truth-functional connectives '$\rightarrow$', '$\neg$'

(ii) a set Prop of propositional letters $p_0, p_1, \dots, p_k, \dots$

Well formed formulas of a propositional language are defined as