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Generalizing the Goldblatt-Thomason Theorem and Modal Definability

ACADEMIC DISSERTATION
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Abstract

The Goldblatt-Thomason theorem characterizes elementary frame classes that are modally definable to be exactly those that are closed under p-morphic images, generated subframes and disjoint unions, and that in addition reflect ultrafilter extensions. We give variations on this theorem by restricting the frame classes (to finite or image-finite frames) and by generalizing the modal language (with the path quantifier and/or counting modalities).

The second part of this work (Chapter 5) generalizes the concept of definability, which is given in terms of validity of formulas. The validity on the level of frames corresponds to formulas of monadic second order logic, that is, quantification over sets of the universum. We introduce a new concept of validity that allows, from the perspective of second order logic, quantification over (binary) relations.

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Lastly I would like to thank my family. I want to dedicate this thesis to my dear children Asmo and Annu, who have missed their mother many times during this work. I hope they will one day understand at least some parts of what I have written.

Contents

Chapter 1

Introduction

Modal logic has evolved a long way from a syntactically driven, narrow discipline through invention of relational semantics to a versatile field that is mapped extensively to other fields of mathematics and has applications in for example computer science, philosophy and linguistics [6, section 1.7]. As Blackburn et al. so nicely state in their slogan 1 on page xi of [6]: "Modal languages are simple yet expressive languages for talking about relational structures".

Definability issues are part of model theory of modal logic which studies the interplay between modal languages and their models. Goranko and Otto give a very nice view on the topic in Chapter 5: *Model Theory of Modal Logic* [18] of Handbook of Modal logic [5] in which they cover most of the background needed, and also partly introduced, in the preliminaries of this thesis and also many topics left out of this work like definability on the level of Kripke models.

In this thesis we discuss some aspects of modal definability. The key concept we use is the validity on the level of frames. From the perspective of classical logic, this concept leads us to monadic universal second order logic. It is a well-known fact from descriptive complexity that existential second order logic corresponds to the complexity class NP (Fagin's theorem, see for example [22]). Thus in the cases where we restrict ourselves to the class of finite frames, we can think of our results as a way of characterizing modal fragments of coNP. For more background on Descriptive Complexity see for example [12] or [22].

A starting point for this thesis was the Goldblatt-Thomason theorem, which dates back to the seventies. Regardless of its relatively old age, this theorem is still fresh by nature as contemporary modal logic is rich in extensions (see for example [29] and [9]) and for each of these it is natural to ask, whether classical characterizations like the Goldblatt-Thomason theorem still hold. To give an example, Balder ten Cate gives some Goldblatt-Thomason style characterizations for some hybrid logics in his thesis [9].

The Goldblatt-Thomason theorem states that an elementary frame class is modally definable if and only if it is closed under p-morphic images, generated subframes and disjoint unions and in addition reflects ultrafilter extensions. The first three constructions are very natural and correspond to familiar constructions from universal algebra namely subalgebras, homomorphic images and direct products. For more information, see for example Section 5.4. in Blackburn et al. [6]. The ultrafilter extensions are something we want to leave out and see what we could do without them. Also restricting oneself to the first-order definable classes would be a natural choice, if the idea behind this was to find some correspondence between first-order and modal formulas. However, our motivation lies more in descriptive complexity where the extra condition of first-order definability is not so important.

The first idea is to restrict the frame classes and replace the requirement of first-order definability by a certain kind of compactness condition. As every elementary class is also compact, this leads to a potential generalization of the original result. However, we did not manage to construct an example of a frame class that would be non-elementary but still compact, especially when restricting oneself to the class of image-finite frames as we do in Section 4.2.

Finite model theory [12] is an important branch of contemporary model theory as in many practical applications models tend to be finite. It has also been studied in the modal context, see for example [30]. When we restrict ourselves to the class of finite and transitive frames, we can give an easy variant of the original result, which can also be found in van Benthem's article [4]. It states that a class of transitive frames that is closed under taking p-morphic images, generated subframes and disjoint unions is definable in restriction to the class of finite frames [4, p. 29]. The proof is achieved by Jankov-Fine formulas $\psi_{\mathcal{F},w}$ [14] that have two nice properties within this class. Firstly, the Jankov-Fine formula $\psi_{\mathcal{F},w}$ is satisfiable in the state w of the frame $\mathcal{F}$. Secondly, if the Jankov-Fine formula $\psi_{\mathcal{F},w}$ is satisfied in a point v of another frame $\mathcal{G}$, there exists a surjective p-morphism from $\mathcal{G}_v$ to $\mathcal{F}_w$. We extract these properties in definition 3.1.1 and say that a frame class C admits $\mathcal{L}$-description of point-generated subframes up to p-morphisms, if such $\mathcal{L}$-formulas exist for each pair $\mathcal{F}, w$, where $\mathcal{F} \in C$.

In order to get rid of the transitivity requirement in the Benthem's result, we add the path quantifier[1] A that allows us to talk about truth on all paths starting from the point we look at. This gives us the extension MLA of the basic modal logic ML. The path quantifier A gives us a way to look ahead in the Jankov-Fine formulas and this gives us the following result:

> A frame class that is closed under taking p-morphic images, generated subframes and disjoint unions is MLA-definable in restriction to the class of finite frames.

For the other direction we of course get only that if the frame class is definable with respect to some class of frames, then it is also closed in restriction to the same class. This follows from the fact that validity is preserved under the given constructions, but definability in restriction to some frame class can only guarantee closure conditions up to these restrictions. Similar comment holds also in the following variations of this result. As A is definable in modal μ-calculus and in infinitary modal logic, we also get that the above result holds for modal μ-calculus and for infinitary modal logic that allows infinite disjunctions and conjunctions. For more background on μ-calculus, see for example [23, 8] and for infinitary modal logic [18, 3, 31].

Another track that we consider is to look what happens in graded modal logics. Graded modal logics (GML) add to basic modal logic counting operators, *graded modalities* [15, 21], that allow to state that a proposition holds in at least a given number of successor states. This of course affects the concept of bisimulation, whereas the path quantifier left the concept of bisimulation intacted (see also [18, section 5]). So, we talk about g-morphic images instead of p-morphic images. First we restrict our studies to the g-saturated class of image finite frames. There g-saturation allows us to approximate the g-bisimulations by d-g_i-bisimulations where we restrict the degree of the modal formulas (which corresponds to the length of the bisimulation game) to d and the index of the counting operators to i. We define d, g_i-Hintikka formulas for graded modal logic and show that a set of these formulas suffices to get the required g-morphic images. These together give us the following variant of the Goldblatt-Thomason theorem:

> A GML[Φ]-compact frame class K is GML[Φ]-definable in restriction to the class of image-finite frames if it is closed under taking g-morphic images, generated subframes and disjoint unions.

[1] Also known as the master modality, see for example [6, p. 371].

Here we of course require that the set Φ of proposition symbols is large enough. And the compactness assumption is needed because we use a set of formulas to describe the g-morphic images instead of one single formula.

When we add the path quantifier to graded modal logics, we can again define a generalization of Jankov-Fine formulas that allows us to give a single formula to force the surjective g-morphism between the suitable frames. When we restrict ourselves to the class of finite frames we get the following:

> Let $|\Phi| \geq \omega$. A frame class K is GMLA$[\Phi]$-definable in restriction to the class of finite frames if it is closed under taking g-morphic images, generated subframes and disjoint unions.

We get rid of the finiteness condition by allowing infinite disjunctions and conjunctions in the definition of generalized Jankov-Fine formulas. As the path quantifier A is also definable in infinitary modal logic ML_∞ we get the following result.

> Let $|\Phi| \geq \kappa$. A frame class K is $\mathrm{GML}_\infty[\Phi]$-definable with respect to the class K_κ of frames whose sizes are at most κ if it is closed under taking g-morphic images, generated subframes and disjoint unions.

We could leave out the size restriction if we allowed Φ to be a proper class and changed the concept of definability to allow proper classes of defining formulas instead of just sets. Some work on infinitary graded modal logic has been on in [13] and on graded μ-calculus in [24].

The last track we consider in this thesis is to give a new generalization of the concept of validity. If we examine the original concept of validity on the level of models and frames via standard translations, we note that a very natural generalization suggests itself. On the level of models we quantify over states, and on the level of frames we quantify over sets that represent the interpretations of the proposition symbols. What if we allow quantification over relations? When we restrict ourselves to basic modal logic with just one relation, there is not too much we can gain by doing this. We can only talk about the underlying sets and, as an example, we define some cardinality restrictions on them.

Things get much more interesting when we move to multi-modal logics. There we can choose some of the relations to be *the bosses* we want to talk about, and the rest of the relations we name *helpers* and quantify them away. We say that a formula φ of multi-modal logic is τ_H-valid in a frame $\mathcal{F}$, if

$$\mathcal{F} \models \forall H_1 \dots \forall H_m \forall P_1 \dots \forall P_k \forall x St_x(\varphi),$$

where relations $H_1, \dots, H_m$ correspond to the helper operators chosen to be in τ_H and $St_x(\varphi)$ is the standard translation of φ. Note that this resembles a bit *second order propositional modal logic* [16] where some of the propositions are (universally/existentially) quantified away. Balder ten Cate gives an analogue of the Goldblatt-Thomason theorem for second order proposition logic in his PhD thesis [9].

We show that τ_H-validity is preserved under generated subframes and p-morphic images (that are taken with respect to the boss relations) but is, however, *not* preserved under disjoint unions. We can again achieve the other direction when restricting to the class of finite and transivite frames.

> A frame class is ML[τ_H]-definable in restriction to the class of finite and transitive frames if it is closed under generated subframes and p-morphic images.

Again the transitivity condition is handled by adding the path quantifier A, and we get that

> a frame class is MLA[τ_H]-definable in restriction to the class of finite frames if it is closed under generated subframes and p-morphic images.

Here we assume that the path quantifier uses only the boss relations in defining the paths, even though we might also assume otherwise.

In the final section of this thesis we discuss something in between the helpers and the bosses. We name these operators, that have somehow fixed interpretation, *assistants*. We divide assistants into two classes: ones that depend on the bosses and others that are fixed by the set of states only. As an example of *boss-independent* assistant we consider the global diamond E. As an easy analogy to the Goldblatt-Thomason theorem for modal logic with global modality [19], we show that

> a frame class is ML($\Diamond, E$)-definable in restriction to the class of finite frames if it is closed under p-morphic images.

Our results suggest some sort of taxonomy between the three constructions considered. For generalized notion of definability we had an analogue of the Goldblatt-Thomason theorem without disjoint unions. With global modality explicitly added we are left with just p-morphic images. Of course this might be some other way for sother variants of modal logic (see for example [9]) and validity, but at least for our methods (generalization of Jankov-Fine formulas that prove that the frame classes admit certain properties) it seems that construction of disjoint unions is left out first and p-morphic images last.

We also consider one example where we use two *boss-dependent* assistants to define a parity condition. For boss-dependent assistants we need to add invariance conditions to the concept of satisfiability. This seems to open new perspectives, but with the limited time resources allocated for this work we were unable to fully examine the possibilities in this direction.

Chapter 2

Preliminaries

2.1 Basic Definitions

We assume that the reader has at least some familiarity with modal logics. In order to acquaint the reader with the language we use, we, however, present the basic notations used in this work to talk about modal logics.

The formulas of basic modal logic are constructed from the proposition symbols in a set Φ. We denote proposition symbols by p and use indices if there are more distinct symbols needed. However, we often build formulas based on some given frame $\mathcal{F} = \langle W, R \rangle$ in which case we choose $\Phi = \{p_s \mid s \in W\}$. The well-formed formulas are defined inductively by

$$\varphi := p \mid \neg\varphi \mid (\varphi \vee \varphi) \mid \Diamond\varphi$$

This basic modal logic is denoted by $\mathrm{ML}[\Phi]$. Later we examine also other modal languages $\mathcal{L}$, denoted by $\mathcal{L}[\Phi]$, if we want to emphasize the set of proposition symbols in use.

The degree of modal formulas defined inductively:

- $\deg(p) = 0$, for each $p \in \Phi$,
- $\deg(\neg\varphi) = \deg(\varphi)$,
- $\deg(\psi \vee \varphi) = \max\{\deg(\psi), \deg(\varphi)\}$,
- $\deg(\Diamond\varphi) = \deg(\varphi) + 1$.

Other connectives ($\wedge$, $\rightarrow$ and $\leftrightarrow$) can be defined as in propositional logic and the dual of $\Diamond$ is defined by $\Box =_{\text{df}} \neg\Diamond\neg$. Later in this thesis we also talk about multi-modal logics, where there are more than one $\Diamond$-operator. Then it is necessary to give also the similarity type that tells which operators are used for forming modal formulas. For basic modal logic the similarity type[1] is $\tau = \{(\Diamond, 1)\}$, meaning that we have one operator $\Diamond$ and it is furthermore unary. As we use only unary operators in this thesis, we identify the similarity type with the set of modal operators and note that the relations we use to interpret modal operators are always binary.

The truth of modal formulas is evaluated in models. A model for the basic modal logic is a structure $\mathcal{M} = \langle W, R, V\rangle$, where W is a non-empty set of states, $R \subseteq W \times W$ is a binary relation and $V : \Phi \longrightarrow \mathcal{P}(W)$ is a valuation function that assigns to each proposition symbol $p \in \Phi$ the set of states where p is true. The frame $\mathcal{F} = \langle W, R\rangle$ is called the underlying frame of $\mathcal{M} = \langle W, R, V\rangle$ and $\mathcal{M}$ is then a model based on $\mathcal{F}$. We use letters $\mathcal{M}$ and $\mathcal{N}$ for models (with indices if necessary). If we want to emphasize the set of propositions we use, we talk about Φ-models. A special case is the point-separated model $\mathcal{M}^{\mathcal{F}}$ for a given frame $\mathcal{F}$.

Definition 2.1.1 Let $\mathcal{F} = \langle W, R\rangle$ and $\Phi = \{p_s \mid s \in W\}$. The *point-separated model* $\mathcal{M}^{\mathcal{F}}$ is the structure $\langle W, R, V\rangle$ based on $\mathcal{F}$ such that $V(p_s) = \{s\}$ for each $s \in W$.

The truth value of a given formula φ in a state w of a model $\mathcal{M}$ is defined inductively. The cases familiar from propositional logic go as usual and for the $\Diamond$-operator the interpretation is given by

$$\mathcal{M}, w \models \Diamond\varphi \Leftrightarrow \exists w' \in W : ((w, w') \in R \text{ and } \mathcal{M}', w' \models \varphi).$$

If a formula φ is true in each state of a model $\mathcal{M}$, we say it is valid in $\mathcal{M}$ and denote $\mathcal{M} \models \varphi$. As we can see, the truth value of modal formulas depend on the interpretation of the proposition symbols and on the relation of the model. If we are in multi-modal logic with more than one $\Diamond$-operator, there correspondingly has to be a relation R_i for interpreting each diamond $\Diamond_i$.

This work is mostly on the level of frames even though we have to go through models sometimes. On the level of frames we do not care about the proposition symbols, we just have the set of states and some relation(s)

[1]Note that Blackburn et al [6] use a slightly different definition - they give a set of operators and a function that assigns arity to each operator.

between them. Typically we denote frames by $\mathcal{F} = \langle W, R\rangle$ and $\mathcal{G} = \langle W^{\mathcal{G}}, R^{\mathcal{G}}\rangle$ for talking about basic modal logic. In Chapter 5 we move to multi-modal logics, where the frames are of the form $\langle W, R_1, \ldots, R_n\rangle$. On the level of frames we can not directly talk about truth of formulas, but we can still define a form of validity which is based on what happens in any model obtained from a frame by adding some valuation function. If a formula φ is valid on each model based on a frame $\mathcal{F}$, we say that φ is valid in $\mathcal{F}$ and denote $\mathcal{F} \models \varphi$.

One key concept in this work is definability.

Definition 2.1.2 A frame class K is *$\mathcal{L}$-definable in restriction to a frame class C* if there exists a set Γ of $\mathcal{L}$-formulas such that

$$\forall \mathcal{F} \in C : (\mathcal{F} \in K \iff \mathcal{F} \models \Gamma).$$

This notion is very central in the whole thesis. In Chapter 5 we give a slight variation where the validity relation $\models$ is slightly reinforced.

Any formula φ of modal logic (without extra quantifiers or operators) can be translated via so called standard translation into a formula $St_x(\varphi)$ of first-order logic. This gives us a way to view the concept of validity from the perspective of first order model theory.

Validity in a model: $\mathcal{M} \models \varphi \iff \mathcal{M} \models \forall x St_x(\varphi)$.[2]

Validity in a frame: $\mathcal{F} \models \varphi \iff \mathcal{F} \models \forall P_1 \forall P_2 \ldots \forall P_k \forall x St_x(\varphi)$.

From these equivalences it is easy to see that definability on the level of models is inside first-order logic, whereas definability on the level of frames is within monadic universal second-order logic. It has been proven that on the level of frames modal logic can indeed express properties that are beyond first-order logic (see the examples 3.9 and 3.11 in [6]). But can we go beyond monadic second-order logic? We will give some views on this in Chapter 5.

Another central concept is bisimulation. Formally it can be defined as a relation between two models or frames with certain properties.

[2] Note that models of modal logic are slightly different from the models of first order logic. In the models of modal logic we have a valuation whereas first order model gives the corresponding unary predicates. Regardless of this difference, we use the same symbol $\mathcal{M}$ for both models.

Definition 2.1.3 Let $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M}' = \langle W', R', V' \rangle$ be models. A non-empty relation $Z \subseteq W \times W'$ is a *bisimulation between* $\mathcal{M}$ *and* $\mathcal{M}'$ if the following conditions hold:

(i) If $(w, w') \in Z$ then w and w' satisfy the same proposition symbols.

(ii) If $(w, w') \in Z$ and $(w, v) \in R$, there exists $v' \in W'$ such that $(v, v') \in Z$ and $(w', v') \in R'$.

(iii) If $(w, w') \in Z$ and $(w', v') \in R'$, there exists $v \in W$ such that $(v, v') \in Z$ and $(w, v) \in R$.

We denote $\mathcal{M}, w \leftrightarrow \mathcal{M}', w'$, if there is a bisimulation Z between $\mathcal{M}$ and $\mathcal{M}'$ such that $(w, w') \in Z$ and then we say that w and w' are *bisimilar*.

It is easy to show that bisimilar points satisfy the same modal formulas, see for example Theorem 2.20 in Blackburn et al [6]. Later on when we add expressive power to our language, this property is the one we want to maintain. The other direction does not hold in general, but for example for m-saturated models equivalence relation induced by the condition

$$(\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}', w' \models \varphi) \text{ forall } \varphi \in \mathrm{ML}[\Phi] \qquad (*)$$

is in itself a bisimulation (for details see for example [6, p. 92 – 93]). If $(*)$ holds, we write $\mathcal{M}, w \equiv_{\mathrm{ML}} \mathcal{M}', w'$. If it holds only up to degree d, that is, for those ML-formulas whose degree is at most d, we write $\mathcal{M}, w \equiv^{d}_{\mathrm{ML}} \mathcal{M}', w'$.

Another way of looking at bisimulations is to view them as a game (see also [18, section 3.1]) where two players try to reach their own goals. The defending player $\exists$ claims that the two models $\mathcal{M} = \langle W, R, V \rangle$, $\mathcal{M}' = \langle W', R', V' \rangle$ or frames $\mathcal{F} = \langle W, R \rangle$, $\mathcal{F}' = \langle W', R' \rangle$ or some parts of them are similar and the opponent tries to show otherwise. The game starts from given two points $w \in W$ and $w' \in W'$. The opponent $\forall$ always chooses the next state v or v' along the edges $(w, v) \in R$ or $(w', v') \in R'$ and the defending player $\exists$ has to mimic the moves of the opponent $\forall$ along the edges of the other model. This way pairs of points are to be chosen as the game goes on, and each time these pairs have to satisfy the same proposition symbols. If one of the players is unable to move in his/her turn, that player loses. If the defending player has a winning strategy, she has shown that the two models are bisimilar and if the opponent has a winning strategy, they are not. If the game never ends, the opponent has failed in his goal. For bisimulation between frames it is enough to consider just the edges and forget about the proposition symbols.

Sometimes it is enough to consider approximations of this full bisimulation up to some degree d. These d-bisimulations can be defined in terms of sequences $Z_d \subseteq \ldots \subseteq Z_0$ of relations, see for example Blackburn et al [6, p. 74–75]. But we find the game approach (see also [18, section 3.2]) taken here more intuitive.

Definition 2.1.4 Let $\mathcal{M} = \langle W, R, V\rangle$ and $\mathcal{M}' = \langle W', R', V'\rangle$ be Φ-models and $w \in W$, $w' \in W'$. The game $G_d((\mathcal{M}, w), (\mathcal{M}', w'))$ is played by the opponent $\forall$ and the defending player $\exists$, and is defined as follows:

Preface: Set $v_0 = w$, $v'_0 = w'$ and $k = 0$.

Playing the game: Let us assume that the pairs $(v_j, v'_j) \in W \times W'$, $j \leq k$, have been played and $0 \leq k \leq d$. Then

1. If $\mathcal{M}, v_k \models p \not\Leftrightarrow \mathcal{M}', v'_k \models p$ for some $p \in \Phi$, the game is over and $\exists$ loses.
2. If $k = d$, the game is over. Otherwise add one to the value of k.
3. Player $\forall$ chooses $v \in W$ such that $(v_{k-1}, v) \in R$ or $v' \in W'$ such that $(v'_{k-1}, v') \in R'$. If $\forall$ cannot choose, the game is over and $\forall$ loses.
4. Player $\exists$ chooses $v' \in W'$ such that $(v'_{k-1}, v') \in R'$ or $v \in W$ such that $(v_{k-1}, v) \in R$ from the remaining model. If $\exists$ cannot make a choice, the game is over and $\exists$ loses.
5. Set (v_k, v'_k) to be (v, v').

If $\exists$ has not lost in any round, she wins. Otherwise $\forall$ wins.

When the player $\exists$ has a winning strategy in the game $G_d((\mathcal{M}, w), (\mathcal{M}', w'))$, we say that w and w' are *d-bisimilar* and denote it by $\mathcal{M}, w \leftrightarroweq_d \mathcal{M}', w'$.

It is easy to see that the points that are d-bisimilar satisfy the same modal formulas up to degree d and the opposite direction holds when the set of proposition symbols is finite, see for example [6, Proposition 2.31]. Later we generalize this concept of d-bisimulation for graded modal logics where we can also restrict the index of the formulas.

2.2 Frame constructions

In this section we give three central constructions for obtaining new models from given ones, namely generated subframes, disjoint unions and p-morphic images.

Definition 2.2.1 (Generated subframes) Let $\mathcal{F} = \langle W, R\rangle$ be a frame and let $X \subseteq W$. The X*-generated subframe* $\mathcal{F}_X$ is $\langle W_X, R_X\rangle$, where W_X is the smallest set satisfying

$$X \subseteq W_X \text{ and } ((w \in W_X \wedge (w, v) \in R) \Rightarrow v \in W_X)$$

and $R_X = R \cap (W_X \times W_X)$. If $X = \{w\}$ for some w we denote the generated subframe by $\mathcal{F}_w = \langle W_w, R_w\rangle$.

Definition 2.2.2 (Disjoint unions) Let $\mathcal{F}_i = \langle W_i, R_i\rangle$, $i \in I$, be frames. The *disjoint union* $\uplus_i \mathcal{F}_i$ is $\langle \cup W_i', \cup R_i'\rangle$, where

$$W_i' = \{(w, i) \mid w \in W_i\}$$

and

$$R_i' = \{((w, i), (v, i)) \mid (w, v) \in R_i\}.$$

Note that if the $\mathcal{F}_i$'s are disjoint to start with, we can set $\uplus_i \mathcal{F}_i = \langle \cup W_i, \cup R_i\rangle$ as we are interested in frames only up to isomorphisms.

Definition 2.2.3 (p-morphic images) Let $\mathcal{F} = \langle W, R\rangle$ and $\mathcal{F}' = \langle W', R'\rangle$ be frames. A bisimulation between $\mathcal{F}$ and $\mathcal{F}'$ is *p-morphism*, if it is a function. $\mathcal{F}'$ is a *p-morphic image of* $\mathcal{F}$, if there exists a surjective p-morphism from $\mathcal{F}$ to $\mathcal{F}'$.

It is easy to see that these three constructions induce natural bisimulations between the original frames and the constructed frames. For generated subframes the induced bisimulation is identity, for disjoint unions it is inclusion and p-morphisms are bisimulations by definition. The following two propositions are also considered as basic facts about these constructions, see for example [6, Section 3.3].

Proposition 2.2.4 *Every frame can be presented as a p-morphic image of the disjoint union of its point-generated subframes.*

Proposition 2.2.5 *Validity of* ML*-formulas is preserved under generated subframes, disjoint unions and p-morphic images.*

It follows from the latter proposition that ML-definable frame classes are closed under these constructions. For example, for p-morphic images and ML-definable frame class K this would mean that

$$\text{if } \mathcal{G} \text{ is a p-morphic image of } \mathcal{F} \text{ and } \mathcal{F} \in K, \text{ then also } \mathcal{G} \in K. \qquad (*)$$

Later when we restrict the frame classes we look at, we also need a relativized notion of being closed under. For example, for p-morphic images we say that a frame class K is closed in restriction to a frame class C, if $(*)$ holds for all frames $\mathcal{F}, \mathcal{G} \in C$.

A starting point for this thesis was the Goldblatt-Thomason theorem that uses the three constructions we introduced above.

Theorem 2.2.6 (Goldblatt-Thomason Theorem) *An elementary frame class is modally definable if and only if it is closed under generated subframes, disjoint unions and p-morphic images and in addition reflects ultrafilter extensions [17].*

In addition, the theorem above uses the ultrafilter extensions. As this fourth condition is not relevant for the rest of the work, we leave its definition to the background reading, see for example [6, p. 93 –98]. For the proof of the Goldblatt-Thomason theorem see for example [6, Section 3.8].

The original Goldblatt-Thomason theorem talks about elementary classes of frames. We use compactness instead of first order definability in the cases where we need an extra condition. As all first-order definable classes are compact, this is a possible generalization. Before we can define compactness we need to state what it means for a set of formulas to be satisfiable in a frame class.

Definition 2.2.7 Let K be a frame class and Σ a set of $\mathcal{L}$-formulas. The set Σ is *satisfiable in* K, if there exists a frame $\mathcal{F} = \langle W, R \rangle \in K$ and a model $\mathcal{M}$ based on $\mathcal{F}$ such that $\mathcal{M}, w \models \Sigma$ for some $w \in W$. The set Σ is *finitely satisfiable in* K if for each finite $\Delta \subseteq \Sigma$ the set Δ is satisfiable in K.

Definition 2.2.8 Let $\mathcal{L}$ be a modal language. A frame class K is $\mathcal{L}$*-compact*, if every set Σ of $\mathcal{L}$-formulas that is finitely satisfiable in K is also satisfiable in K.

Even though we have not managed to come up with an example of a (image-finite) class that would be compact but non-elementary, we use compactness as it makes some proofs go through neatly without the ultraproducts.

2.3 Characteristic formulas

In this thesis we use two kinds of characteristic formulas to ensure that we find a suitable frame from the given frame class. For finite and transitive frames we can use Jankov-Fine formulas [14] on point-separated models to characterize the structure of finite, transitive frames.

Definition 2.3.1 (Jankov-Fine Formulas) Let $\mathcal{F} = \langle W, R \rangle$ be a finite frame, $w \in W$ and $\Phi = \{p_s \mid s \in W_w\}$. Let φ be the conjunction of the following formulas.

(i) $\bigvee_{s \in W_w} p_s$,

(ii) $\bigwedge_{s,t \in W_w,\ s \neq t} (p_s \to \neg p_t)$,

(iii) $\bigwedge_{(s,t) \in R_w} (p_s \to \Diamond p_t)$,

(iv) $\bigwedge_{(s,t) \notin R_w} (p_s \to \neg \Diamond p_t)$.

Now, the *Jankov-Fine formula* $\psi_{\mathcal{F},w}$ is defined as $p_w \wedge \varphi \wedge \Box \varphi$.

It is easy to show that $\psi_{\mathcal{F},w}$ is satisfiable in $\mathcal{F}, w$, that is, true at w in some model based on $\mathcal{F}$ and if the Jankov-Fine formula $\psi_{\mathcal{F},w}$ of a finite frame $\mathcal{F}$ is satisfied in a point v of a transitive frame $\mathcal{G}$, then $\mathcal{F}_w$ is a p-morphic image of $\mathcal{G}_v$, see for example [6, Lemma 3.20]. Later we go around the transitivity condition by adding a path quantifier or helper modalities. Also the global modality or restricting oneself to frames with uniform bounded depth solves the issue.

What comes to getting rid of the finiteness, allowing infinite disjunctions and conjunctions and an unlimited amount of proposition symbols would of course do the trick. If we do not want to do that, we have to come up with some approximations. For that we use Hintikka formulas, which are defined inductively so that the degree of formulas increases on every step.

Definition 2.3.2 Let $\mathcal{M} = \langle W, R, V\rangle$ be a Φ-model for a finite set Φ of proposition symbols, and let $w \in W$. The *d-Hintikka formula* $\varphi^d_{\mathcal{M},w}$ (see also [18, p. 265]) is defined inductively as follows

$$\varphi^0_{\mathcal{M},w} = \bigwedge\{\psi \mid (\psi = p \text{ or } \psi = \neg p \text{ for some } p \in \Phi) \text{ and } \mathcal{M}, w \models \psi\}$$

$$\varphi^{d+1}_{\mathcal{M},w} = \varphi^d_{\mathcal{M},w} \wedge (\bigwedge\{\Diamond\varphi^d_{\mathcal{M},v} \mid (w,v) \in R\}) \wedge \Box(\bigvee\{\varphi^d_{\mathcal{M},v} \mid (w,v) \in R\}).$$

For possibly empty disjunctions and conjunctions we define $\bigwedge\{\ \} = \top$ and $\bigvee\{\ \} =\perp$.

Note that the disjunctions and conjunctions above are finite, because the set of proposition symbols is finite.

The idea behind these Hintikka formulas is to describe the models up to paths of length d. The 0-Hintikka formula gives the prositional type of a given point and the formulas on the next level $d+1$ say that each type of degree d that is possible in the next state, is possible at least once and no other types of degree d occur. It is easy to see that the degree of a d-Hintikka formula $\varphi^d_{\mathcal{M},w}$ is at most d. Connection between d-Hintikka formulas, formula equivalence up to degree d and d-bisimulation is stated in the following proposition (see also [18, Theorem 32]).

Proposition 2.3.3 *Let $\mathcal{M} = \langle W, R, V\rangle$ and $\mathcal{M}' = \langle W', R', V'\rangle$ Φ-models for a finite set Φ of proposition symbols and let $w \in W$ and $w' \in W'$. Then the following conditions are equivalent [18, Theorem 32].*

(i) $\mathcal{M}, w \leftrightarroweq_d \mathcal{M}', w'$,

(ii) $\mathcal{M}, w \equiv^d_{\mathrm{ML}} \mathcal{M}', w'$,

(iii) $\mathcal{M}', w' \models \varphi^d_{\mathcal{M},w}$.

Later in this thesis we generalize d-bisimulation game and Hintikka formulas for graded modal logic and use the d, g_i-Hintikka formulas for approximating image-finite frames. Then we need an additional compactness condition, as the transfer property of definition 4.2.8 uses a set of formulas instead of a single formula.

2.4 The path quantifier A

When looking at the definition 2.3.1 of the Jankov-Fine formulas, a natural generalization suggests itself as an attempt to get rid of the transitivity requirement — we can try adding a sufficient amount of $\Box$-operators. The problem with this is that even in finite models there can be infinitely long paths, so we could not find a finite n which would be large enough for all frames. Van Benthem [4] goes through this by introducing n-transitive frames and local p-morphic images, but a more direct approach is to extend the modal language by exactly what is needed, that is, the path quantifier. But first we need to specify what we mean by a path.

Definition 2.4.1 Let $\mathcal{M} = \langle W, R, V\rangle$ be a model and $w, v \in W$. There exists a *path from w to v* if there exists $v_0, \ldots, v_k \in W$ such that $v_0 = w$, $v_k = v$ and $(v_i, v_{i+1}) \in R$ for $i : 0 \leq i < k$. We denote $w \rightsquigarrow v$, if there exists a path from w to v. Note that the trivial path from w to w is accepted.

Now we can quantify over all these paths.

Definition 2.4.2 Let $\mathcal{M} = \langle W, R, V\rangle$ a model for basic modal logic and $w \in W$. The *path quantifier A* is defined by the following equivalence.

$$\mathcal{M}, w \models A\varphi \;\Leftrightarrow \mathcal{M}, v \models \varphi \text{ for each } v \text{ such } w \rightsquigarrow v.$$

Remark 2.4.3 Later on in, Chapter 5, we have more relations at our disposal. There we will divide relations of the frames into two disjoint sets (bosses and helpers) and agree that the paths in the definition of the path quantifier only use the boss relations and not the helpers.

The path quantifier gives us a way to put in infinitely many $\Box$-operators by adding just a single operator (see also [2, p. 296]). Similar extensions have been considered in many contexts. In Blackburn et al this master modality is considered in the context of propositional dynamic logics [6, p. 371–373]. They present the master modality as a reflexive transitive closure of union of all relations used in the frames.

In the context of branching time temporal logics considerations over paths (flows of time) arise naturally. There the path quantifier is put into use in languages called CTL* and PCTL* (see for example [20, p. 681 – 682]). The distinction there is that the evaluation models are restricted to trees, which is something we do not do.

2.5 Graded modal logics

It is easy to see that the formula

$$\varphi = \bigwedge_{j=0}^{i} \Diamond p_j \to \Diamond \bigvee_{j \neq k} (p_j \wedge p_k)$$

is valid in a frame if and only if its each state has at most i successors. What if we would like to say that every state has *at least* i successors? This cannot be done in basic modal logic as p-morphisms can diminish the number of successors.

Graded modal logics (GML) are obtained from basic modal logic by adding the possibility of counting successors in some limited sense - we add operators which state that a formula φ has to hold in at least so many successor states. If the formula φ in question happens to be $\top$, this gives a direct restriction on the number of successors.

Definition 2.5.1 The *language of graded modal logic* is obtained from propositional logic by adding the following formation rule.

If φ is a GML-formula and $i \in \mathbf{N} \setminus \{0\}$, then $\triangleleft_{\geq i}\varphi$ is a GML-formula[3].

The *degree of a GML-formula* φ is defined similarly to the definition of ML-degree in page 15 as the maximum nesting of $\triangleleft$-operators occuring in φ. The *index of* φ is the maximum index i occuring in $\triangleleft_{\geq i}$-operators in φ.

For a model $\mathcal{M} = \langle W, R, V\rangle$ and a state $w \in W$

$$\mathcal{M}, w \models \triangleleft_{\geq i}\varphi \iff |\{v \in W \mid (w,v) \in R \wedge (\mathcal{M}, v \models \varphi)\}| \geq i.$$

Note that the basic modal operators $\Diamond$ and $\Box$ are easily definable with $\triangleleft_{\geq 1}$. We can also define operators

$$\triangleleft_{\leq i}\varphi := \neg \triangleleft_{\geq i+1} \varphi$$

and

$$\triangleleft_{= i}\varphi := (\triangleleft_{\geq i}\varphi \wedge \triangleleft_{\leq i}\varphi)$$

for restricting the number of successors from above or to some exact value $i \in \mathbf{N}$.

[3]The $\triangleleft$-notation is from [11].

The reader can find more details in the background reading if needed (see for example [27, 11]). If the reader wants a more application-oriented view, becoming acquainted with the descriptions logics would be the thing to do, see for example [1].

The concept of bisimulation is generalized for GML by de Rijke in [27]. We use the formulation of Conradie in [11]. Note that in the definition we also fix the notation $R(w)$ for the the set of R-successors of w.

Definition 2.5.2 Let $\mathcal{M} = \langle W, R, V \rangle$ and $\mathcal{M}' = \langle W', R', V' \rangle$. A non-empty relation $Z \subseteq W \times W'$ is *g-bisimulation* between $\mathcal{M}$ and $\mathcal{M}'$ if the following conditions hold.

(i) If $(w, w') \in Z$, then w and w' satisfy the same proposition symbols.

(ii) If $(w, w') \in Z$ and $S \subseteq R(w) = \{v \mid (w, v) \in R\}$ is finite, then there exists $S' \subseteq R'(w') = \{v' \mid (w', v') \in R'\}$ such that $|S| = |S'|$, $\forall v \in S : \exists v' \in S' : (v, v') \in Z$ and $\forall v' \in S' : \exists v \in S : (v, v') \in Z$.

(iii) If $(w, w') \in Z$ and $S' \subseteq R'(w')$ is finite, then there exists $S \subseteq R(w)$ such that $|S| = |S'|$, $\forall v \in S : \exists v' \in S' : (v, v') \in Z$ and $\forall v' \in S' : \exists v \in S : (v, v') \in Z$.

We denote

$$Z : \mathcal{M}, w \leftrightarrow_g \mathcal{M}', w',$$

if Z is a g-bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ such that $(w, w') \in Z$. The *g-bisimulation between frames* is defined as usual that is by leaving out the condition (i) concerning the proposition symbols. A g-bisimulation, that is also a function, is called a *g-morphism*. A frame $\mathcal{F}$ is a *g-morphic image* of a frame $\mathcal{G}$, if there exists a surjective g-morphism $\mathcal{G} \to \mathcal{F}$.

2.6 Multi-modal logics

In the first four chapters, we have considered models $\mathcal{M} = \langle W, R, V \rangle$ and frames $\mathcal{F} = \langle W, R \rangle$ of basic (graded) modal logic. In Chapter 5 we move to multi-modal logics. Similarity type $\tau = \{\Diamond_1, \ldots, \Diamond_n\}$ gives the set of modal operators in use and correspondingly frames $\mathcal{F} = \langle W, R_1, \ldots, R_n \rangle$ and models $\mathcal{M} = \langle W, R_1, \ldots, R_n, V \rangle$ based on $\mathcal{F}$ have n relations such that

$$\mathcal{M}, w \models \Diamond_j \varphi \Leftrightarrow \mathcal{M}, v \models \varphi \text{ for some } v \in R_j(w).$$

Thus, multi-modal logics add more relations to the underlying frames. The idea behind this is that in many applications it is too restrictive to allow only one box or diamond and one relation. For example, in knowledge representation it is very useful to have more than one agent, who can all act independently.

Bisimulations are defined in a natural way in the multi-modal logics.

Definition 2.6.1 Let $\mathcal{M} = \langle W, R_1, \ldots, R_n, V\rangle$ and $\mathcal{M}' = \langle W', R_1' \ldots, R_n', V'\rangle$ be Kripke models. A non-empty relation $Z \subseteq W \times W'$ is a (multi-modal) *bisimulation* between $\mathcal{M}$ and $\mathcal{M}'$ if the following conditions hold.

(i) If $(w, w') \in Z$, then w and w' satisfy the same propositional symbols.

(ii) If $(w, w') \in Z$ and $(w, v) \in R_i$, then there exists v' such that $(v, v') \in Z$ and $(w', v') \in R_i'$ for each $i \in \{1, \ldots, n\}$.

(iii) If $(w, w') \in Z$ and $(w', v') \in R_i'$ for some i, then there exists v, such that $(v, v') \in Z$ and $(w, v) \in R_i$.

We denote

$$Z : \ \mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$$

if Z is a bisimulation between multi-modal models $\mathcal{M}$ and $\mathcal{M}'$ such that $(w, w') \in Z$ and say that w and w' are *bisimilar*.

Naturally we can also add the path quantifier of Definition 2.4.2 and show that bisimilarity between multi-modal models implies formula-equivalence for MLA-formulas when the paths are defined with respect to the union of all relations.

Proposition 2.6.2 *Let* $\mathcal{M} = \langle W, R_1, \ldots, R_n, V\rangle$ *and* $\mathcal{M}' = \langle W', R_1' \ldots, R_n', V'\rangle$ *be (multi-modal) Kripke models. If* $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$*, then* $\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}', w' \models \varphi$ *for every (multi-modal)* MLA*-formula.*

Proof. Let us show by induction on an MLA-formula φ that

$$\forall w \forall w' : ((\mathcal{M}, w \leftrightarroweq \mathcal{M}', w') \Rightarrow (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}', w' \models \varphi)). \qquad (*)$$

When φ is a propositional symbol, $(*)$ holds. Assume that $(*)$ holds for ψ_1 and ψ_2. Clearly $(*)$ holds then for all the boolean combinations of ψ_1 and ψ_2.

Assume then that $\varphi = \Diamond_i \psi$ and $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$. If $\mathcal{M}, w \models \varphi$, there exists $v \in W$ such that $(w, v) \in R_i$ and $\mathcal{M}, v \models \psi$. As w and w' are bisimilar, there exists $v' \in W'$ such that $(w', v') \in R'$ and $v \leftrightarroweq v'$. By induction hypothesis $\mathcal{M}', v' \models \psi$. Hence $\mathcal{M}', w' \models \varphi$. Similarly $\mathcal{M}', w' \models \varphi \Rightarrow \mathcal{M}, w \models \varphi$.

Assume then that $\varphi = A\psi$, $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$ and $\mathcal{M}, w \models \varphi$. Choose an arbitrary point $v' \in W'$ and an arbitrary path $w' = w'_0 R'_{j_1} w'_1 \ldots R'_{j_k} w'_k = v'$ from w' to v'. As $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$, there exists a point v and a path $w = w_0 R_{j_1} w_1 \ldots R_{j_k} w_k = v$ from w to v such that $\mathcal{M}, v \leftrightarroweq \mathcal{M}', v'$. Because $\mathcal{M}, w \models A\psi$ and there exists a path from w to v, $\mathcal{M}, v \models \psi$. By induction assumption $\mathcal{M}', v' \models \psi$. As v' was an arbitrary point on arbitrary path starting from w', $\mathcal{M}', w' \models A\psi$. Similarly $\mathcal{M}', w' \models \varphi \Rightarrow \mathcal{M}, w \models \varphi$. $\square$

Remark 2.6.3 The previous proposition also holds for the case, where the paths are defined with respect to the union of certain relations $R_1, \ldots, R_k$, $k < n$, which is what we want to do in Chapter 5. As a special case let $\mathcal{M} = \langle W, R_1, \ldots, R_n, V \rangle$ and $\mathcal{M}' = \langle W', R'_1 \ldots, R'_n, V' \rangle$ be Kripke models. If $\mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$, then $\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}', w' \models \varphi$ holds for every multimodal formula that can in addition use the path quantifier A that is applied only to the relation R_1.

Chapter 3

Generalized perspective

3.1 Basic results

Recall from section 2.3 that Jankov-Fine formula $\psi_{\mathcal{F},w}$ is satisfiable in $\mathcal{F}, w$ and in restriction to finite and transitive frames it holds that if the Jankov-Fine formula $\psi_{\mathcal{F},w}$ is satisfied in a point v of a frame $\mathcal{G}$, then $\mathcal{F}_w$ is a p-morphic image of $\mathcal{G}_v$. Let us extract these properties into a definition.

Definition 3.1.1 Let $\mathcal{L}$ be a modal language. We say that a class C of frames *admits $\mathcal{L}$-description of point-generated subframes up to p-morphisms*, if for every $\mathcal{F} \in C$ and $w \in \mathcal{F}$ there exists an $\mathcal{L}$-formula $\varphi_{\mathcal{F},w}$ such that $\varphi_{\mathcal{F},w}$ is satisfiable in w and if it is satisfiable in a point v of a frame $\mathcal{G}$, then $\mathcal{F}_w$ is a p-morphic image of $\mathcal{G}_v$.

We can formulate a basic definability result for the frame classes that have this property.

Theorem 3.1.2 *Assume that C admits $\mathcal{L}$-description of point-generated subframes up to p-morphisms. Then a frame class that is closed under generated subframes, disjoint unions and taking p-morphic images is $\mathcal{L}$-definable in restriction to the class C.*

Proof. Let K be a frame class that is closed under generated subframes, disjoint unions and taking p-morphic images. We show that

$$\Lambda_K = \{\phi \in \mathcal{L} \mid \forall \mathcal{G} \in K : \mathcal{G} \models \phi\}$$

defines K in restriction to the class C.

Assume first that $\mathcal{F} \models \Lambda_K$ for some frame $\mathcal{F} \in C$. Choose any state w of $\mathcal{F}$. Because C admits $\mathcal{L}$-description of point-generated frames up to p-morphisms, there exists a formula $\varphi_{\mathcal{F},w}$ such that

- $\varphi_{\mathcal{F},w}$ is satisfiable in w and
- if it is satisfiable in a point v of a frame $\mathcal{G}$, then $\mathcal{F}_w$ is a p-morphic image of $\mathcal{G}_v$.

If $\varphi_{\mathcal{F},w}$ is not satisfiable in K, then $\neg\varphi_{\mathcal{F},w}$ is valid in K and $\neg\varphi_{\mathcal{F},w} \in \Lambda_K$. As $\mathcal{F} \models \Lambda_K$ it follows that $\mathcal{F} \models \neg\varphi_{\mathcal{F},w}$ which is a contradiction because $\varphi_{\mathcal{F},w}$ is satisfiable in $\mathcal{F}$. Hence there must exist a frame $\mathcal{G} \in K$ such that $\varphi_{\mathcal{F},w}$ is satisfiable in some state v of $\mathcal{G}$. Now, $\mathcal{F}_w$ is a p-morphic image of $\mathcal{G}_v$ and as K is closed under point-generated subframes and p-morphic images $\mathcal{F}_w \in K$. Now, for each state w of $\mathcal{F}$ we have shown that $\mathcal{F}_w \in K$. By proposition 2.2.4, $\mathcal{F}$ can be presented as a p-morphic image of the disjoint union of its point-generated subframes. As K is closed under these constructions, it follows that $\mathcal{F} \in K$.

Assume then that $\mathcal{F} \in K \cap C$. Then trivially $\mathcal{F} \models \Lambda_K$. Now, we have proven that

$$\forall \mathcal{F} \in C : (\mathcal{F} \models \Lambda_K \Longleftrightarrow \mathcal{F} \in K)$$

and hence K is definable by Λ_K. □

Remark 3.1.3 It is easy to see from the proof that closure under disjoint unions could be replaced here by requirement that K reflects point-generated subframes, that is, if for every $w : \mathcal{F}_w \in K$ then $\mathcal{F} \in K$ (see [9, p. 50]).

In a certain sense the Jankov-Fine formulas describe p-morphisms between point-generated subframes. As we are also interested in the other two frame constructions, it is reasonable to formulate the following questions.

Question 3.1.4 Are there non-trivial classes C and natural modal logics $\mathcal{L}$ such that for every $\mathcal{F} \in C$ and $w \in \mathcal{F}$ there exists an $\mathcal{L}$-formula $\varphi_{\mathcal{F},w}$ such that $\varphi_{\mathcal{F},w}$ is satisfiable in w and if it is satisfiable in a point v of a frame $\mathcal{G}$, then $\mathcal{F}_w$ is isomorphic to a generated subframe $\mathcal{G}_v$?

Question 3.1.5 Are there non-trivial classes C and natural modal logics $\mathcal{L}$ such that for every $\mathcal{F} \in C$ and $w \in \mathcal{F}$ there exists an $\mathcal{L}$-formula $\varphi_{\mathcal{F},w}$ such that $\varphi_{\mathcal{F},w}$ is satisfiable in w and if it is satisfiable in a points v_i of frames $\mathcal{G}^i$ for each $i \in I$, then $\mathcal{F}_w$ is isomorphic to the disjoint union $\biguplus_{i\in I}\mathcal{G}_i$?

The general definability results given here use large sets of formulas (whole theories of frame classes). It is of course good to have these results, but from the applicational point of view the answer to the next question would be even better.

Question 3.1.6 Are there some natural closure conditions under which a frame class is $\mathcal{L}$-definable by a single formula for some natural modal logic $\mathcal{L}$?

It is generally known and easy to show that ML-validity is preserved under p-morphic images, generated subframes and disjoint unions (see for example [6, Theorem 3.14]). We show this in a slightly generalized form, which also applies for example when we add the path quantifier.

Lemma 3.1.7 *Let $\mathcal{L}$ be an arbitrary modal language. If bisimilar states satisfy the same $\mathcal{L}$ formulas, then $\mathcal{L}$-validity is preserved under p-morphic images, generated subframes and disjoint unions.*

Proof. We go through p-morphic images, generated subframes and disjoint unions.

p-morphic images: Assume that f is a surjective p-morphism from $\mathcal{F} = \langle W, R\rangle$ to $\mathcal{F}' = \langle W', R'\rangle$ and φ is valid in $\mathcal{F}$. Assume on the contrary that an $\mathcal{L}$-formula φ is not valid in $\mathcal{F}'$. Then there exists a model $\mathcal{M}' = \langle W', R', V'\rangle$ based on $\mathcal{F}'$ and $w' \in W'$ such that $\mathcal{M}', w' \not\models \varphi$. Define a valuation V on $\mathcal{F}$ by

$$u \in V(p) \Leftrightarrow (f(u) \in V'(p)).$$

As f is a surjection there exists a state $w \in W'$ such that $f(w) = w'$. As f is by definition bisimulation between frames, w and w' are bisimilar. Now, by assumption w and w' satisfy the same $\mathcal{L}$-formulas. Hence $\mathcal{M}, w \not\models \varphi$. But this is a contradiction as we assumed that $\mathcal{F} \models \varphi$. Hence $\mathcal{F}' \models \varphi$ and $\mathcal{L}$-validity is preserved under p-morphic images.

generated subframes: Assume that $\mathcal{F}' = \langle W', R'\rangle$ is a generated subframe of $\mathcal{F} = \langle W, R\rangle$ and φ is valid in $\mathcal{F}$. Assume on the contrary that an

$\mathcal{L}$-formula φ is not valid in $\mathcal{F}'$. Then there exists a model $\mathcal{M}' = \langle W', R', V' \rangle$ based on $\mathcal{F}'$ and $w' \in W'$ such that $\mathcal{M}', w' \not\models \varphi$. Now,

$$Z = \{(u, u') \in W \times W' \mid u' \in W' \text{ and } u = u'\}$$

is a bisimulation between $\mathcal{M}'$ and $\mathcal{M} = \langle W, R, V' \rangle$. By assumption $\mathcal{M}, w' \not\models \varphi$ which is a contradiction as we assumed that $\mathcal{F} \models \varphi$. Hence $\mathcal{F}' \models \varphi$ and $\mathcal{L}$-validity is preserved under generated subframes.

disjoint unions: Assume that $\mathcal{F}_i \models \varphi$ for each $i \in I$ and let $\mathcal{F} = \uplus_{i \in I} \mathcal{F}_i$. Assume on the contrary that there exists a model $\mathcal{M} = \langle W, R, V \rangle$ based on $\mathcal{F}$ and a state $(w, i) \in W$ such that $\mathcal{M}, (w, i) \not\models \varphi$. We define a valuation for $\mathcal{F}_i = \langle W_i, R_i \rangle$ by

$$u \in V_i(p) \Leftrightarrow (v, i) \in V(p).$$

Let $\mathcal{M}_i = \langle W_i, R_i, W_i \rangle$. Now, $Z = \{(u, (u, i) \in W \times W_i\}$ is a bisimulation between $\mathcal{M}_i$ and $\mathcal{M}$ and there exists $w \in W_i$ such that w and (w, i) satisfy the same formulas. But this is a contradiction as we assumed that $\mathcal{F}_i \models \varphi$. Hence $\mathcal{F} \models \varphi$ and $\mathcal{L}$-validity is preserved under disjoint unions. $\square$

Remark 3.1.8 Note that lemma above talks about *bisimulation invariant* languages, that is, about languages in which for every formula φ:

$$(\mathcal{M}, w \leftrightarrows \mathcal{M}', w') \Rightarrow (\mathcal{M}, w \models \varphi \Leftrightarrow \mathcal{M}', w' \models \varphi).$$

Bisimilar states satisfy the same MLA-formulas by Proposition 2.6.2. This proves the following corollary:

Corollary 3.1.9 *An* MLA[Φ]*-definable frame class is closed under taking p-morphic images, generated subframes and disjoint unions.*

Relativized formulation of this corollary goes as follows:

Corollary 3.1.10 *If a frame class is* MLA[Φ]*-definable in restriction to the class K_{fin} of finite frames, it is closed under taking p-morphic images, generated subframes and disjoint unions in restriction to the class K_{fin}.*

3.2 Generalizing Jankov-Fine Formulas

We generalize the Jankov-Fine formulas with the help of the path quantifier in order to get a variant of the Goldblatt-Thomason theorem where the transitivity assumption is not needed over finite models. If we allow the use of infinitary modal logic, we can also give up the finiteness assumption.

Definition 3.2.1 Let $\mathcal{F} = \langle W, R \rangle$ be a (finite) frame and $w \in W$. Let $\Phi = \{p_s \mid s \in W_w\}$. Let φ be the conjunction of the following formulas.

(i) $\bigvee_{s \in W_w} p_s$,

(ii) $\bigwedge_{s,t \in W_w, s \neq t}(p_s \to \neg p_t)$,

(iii) $\bigwedge_{(s,t) \in R_w}(p_s \to \Diamond p_t)$,

(iv) $\bigwedge_{(s,t) \notin R_w}(p_s \to \neg\Diamond p_t)$.

Now, $\psi^A_{\mathcal{F},w}$ is defined as $p_w \wedge A\varphi$.

Van Benthem used a similar technique in [2, p. 295–297], when giving (over finite models) a dynamic logic formula $\beta_{M,w}$, which is true in N, v iff N, v is bisimilar to M, w. He also notes that this can be generalized to arbitrary models by allowing infinite conjunctions and disjunctions. Here we need the bisimulation to also be a surjective function.

Next we want to check that $\psi^A_{\mathcal{F},w}$ is indeed the type of formula that proves that the class of finite frames admits MLA-description of point-generated subframes up to p-morphisms.

Lemma 3.2.2 *Let $\mathcal{F} = \langle W, R \rangle$ and $w \in W$. Then the formula $\psi^A_{\mathcal{F},w}$ is satisfiable in w.*

Proof. Let $\mathcal{M}^{\mathcal{F}_w}$ be the point-separated model for $\mathcal{F}_w$. It is clear that $\mathcal{M}^{\mathcal{F}_w}, w \models \psi^A_{\mathcal{F},w}$. □

Lemma 3.2.3 *Let $\mathcal{F} = \langle W, R \rangle$ and $w \in W$. If $\psi^A_{\mathcal{F},w}$ is satisfied in a point v of $\mathcal{G}$, then $\mathcal{F}_w$ is a p-morphic image of $\mathcal{G}_v$.*

Proof. Let $\mathcal{N} = \langle W', R', V' \rangle$ be a model based on $\mathcal{G}$ such that $\mathcal{N}, v \models \psi^A_{\mathcal{F},w}$. Let us show that

$$f = \{(u', u) \in W'_v \times W_w \mid u' \in V'(p_u)\}$$

is a surjective p-morphism $\mathcal{G}_v \to \mathcal{F}_w$.

Let $u' \in W_v$. Now, $\mathcal{N}, u' \models (\bigvee_{s \in W_w} p_s) \wedge (\bigwedge_{s,t \in W_w, s \neq t}(p_s \to \neg p_t))$. Hence $u' \in V(p_u)$ for exactly one $u \in W_w$. Thus f is a function.

Let $u \in W_w$. Let us show by induction on n that for every point $w_i \in W_w$ in a path $w = w_0 R_w w_1 R_w \dots R_w w_n = u$, there exists a point $v_i \in W_v$ such that $f(v_i) = w_i$. If $n = 0$, the claim holds as $f(w) = v$. Let us assume, that the claim holds for $k < n$. Now, $\mathcal{N}, v_k \models p_{w_k} \wedge (p_{w_k} \to \Diamond p_{w_{k+1}})$. Hence there exists $v_{k+1} \in W_v$ such that $(v_k, v_{k+1}) \in R'_v$ and $\mathcal{N}, v_{k+1} \models p_{w_{k+1}}$. Now $f(v_{k+1}) = w_{k+1}$. By induction principle, for every point $w_i \in W_w$ on the path from w to u there exists a point $v_i \in W_v$ such that $f(v_i) = w_i$. In particular for the point u such a state exists. Thus f is surjective.

Now, it is left to show that f is a bisimulation. For that, we first assume that $(s', t') \in R'_v$ and $f(s') = s$. Let $t = f(t')$. If $(s, t) \notin R_w$, then $\mathcal{N}, s' \models p_s \to \neg\Diamond p_t$. But this contradicts $\mathcal{N}, s' \models p_s \wedge \Diamond p_t$. Hence $(f(s'), f(t')) \in R_w$. Assume then that $f(s') = s$ and $(s, t) \in R_w$. Now, $\mathcal{N}, s' \models p_s \wedge (p_s \to \Diamond p_t)$. Hence there exists $t' \in W'_v$ such that $(s', t') \in R'_v$ and $f(t') = t$. $\square$

From the Theorem 3.1.2 and the above lemmas we get the following corollary.

Corollary 3.2.4 *Let $|\Phi| \geq \omega$. If a frame class is closed under taking p-morphic images, generated subframes and disjoint unions, then it is* MLA$[\Phi]$*-definable in restriction to the class K_{fin} of finite frames.*

Note that the size limit $|\Phi| \geq \omega$ is needed to have enough proposition symbols for all finite point-separated models.

3.3 Corollaries to other modal logics

The path quantifier A is definable in modal μ-calculus [23] by the formula

$$\nu p.(\varphi \wedge \Box p),$$

see for example section 3.5 in [8]. Hence we can use the formulas of Definition 3.2.1 for μ-calculus as well to get the following corollary of Theorem 3.1.2.

Corollary 3.3.1 *Let $|\Phi| \geq \omega$. If a frame class is closed under taking p-morphic images, generated subframes and disjoint unions, then it is* μML-*definable in restriction to the class K_{fin} of finite frames.*

Bisimilar states satisfy the same μML-formulas, see for example Theorem 4 in [8]. It follows by Lemma 3.1.7 that validity of μML-formulas is also preserved under p-morphic images, generated subframes and disjoint unions.

The path quantifier A is also definable in infinitary modal logic ML_∞ (see for example [18, p. 271 – 273]) by the formula

$$\bigwedge\{\Box^n\varphi \mid n \in \mathbf{N}\}.$$

Hence we get the following corollary of Theorem 3.1.2.

Corollary 3.3.2 *Let $|\Phi| \geq \kappa$. If a frame class is closed under taking p-morphic images, generated subframes and disjoint unions then it is* $\mathrm{ML}_\infty[\Phi]$-*definable with respect to the class K_κ of frames whose size is at most κ.*

Note that if we allowed definability by a proper class and used proper classes of proposition symbols instead of sets, we could give up the size restriction in the corollary above. Clearly bisimilar states satisfy the same ML_∞-formulas. By Lemma 3.1.7 the validity of ML_∞-formulas is therefore preserved under p-morphic images, generated subframes and disjoint unions.

Chapter 4

Graded Modal Logics

4.1 Basic results

Any frame can be presented as a p-morphic image of the disjoint union of its point-generated subframes. The same holds also for g-morphic images.

Proposition 4.1.1 *Let $\mathcal{F} = \langle W, R\rangle$ and let $\mathcal{G} = \langle W^{\mathcal{G}}, R^{\mathcal{G}}\rangle$ be the disjoint union of the point-generated frames $\mathcal{F}_w$, $w \in W$. Then there exists a surjective g-morphism from $\mathcal{G}$ to $\mathcal{F}$.*

Proof. We show that

$$Z = \{((v, w), v) \mid W^{\mathcal{G}} \times W \mid v, w \in W\}$$

is a surjective g-morphism $\mathcal{G} \to \mathcal{F}$.

Every point $(v, w) \in W^{\mathcal{G}}$ has a unique image $v \in W$. So Z is a function. Furthermore for each $v \in W$ at least $((v, v), v) \in Z$ and so Z is surjective. For Z to be a g-morphism we have to prove the items (*ii*) and (*iii*) of the definition 2.5.2.

(ii) Assume that $((v, w), v) \in Z$ and $S \subseteq R^{\mathcal{G}}((v, w))$ is finite. Let $S' = \{v' \mid (v', w) \in S\}$. By definition of $R^{\mathcal{G}}$, $(v, v') \in R$ for each $v' \in S'$. Furthermore $|S| = |S'|$, $\forall (v', w) \in S : \exists v' \in S' : ((v', w), v') \in Z$ and $\forall v' \in S' : \exists (v', w) \in S : ((v', w), v') \in Z$.

(iii) Assume that $((v, w), v) \in Z$ and $S' \subseteq R(v)$ is finite. Let $S = \{(v', w) \in W^{\mathcal{G}} \mid v' \in S'\}$. Now, $((v, w), (v', w)) \in R^{\mathcal{G}}$ for each $(v', w) \in S$. Furthermore $|S| = |S'|$, $\forall (v', w) \in S : \exists v' \in S' : ((v', w), v') \in Z$ and $\forall v' \in S' : \exists (v', w) \in S : ((v', w), v') \in Z$.

By items (ii) and (iii) the relation Z is a g-morphism and hence we have shown that $\mathcal{F}$ is a g-morphic image of the disjoint union of its point-generated subframes. $\square$

Analogously to Lemma 3.1.7 we get the following lemma for graded modal logics.

Lemma 4.1.2 *Let $\mathcal{L}$ be a modal language that includes GML. If g-bisimilar states satisfy the same $\mathcal{L}$-formulas, then $\mathcal{L}$-validity is preserved under g-morphic images, generated subframes and disjoint unions.*

Remark 4.1.3 Note that the lemma above talks about counting bisimulation invariant languages.

Lates we use 4.1.2 to extension of GML, but of course it also give the following (known) result.

Proposition 4.1.4 *Validity of* GML*-formulas is preserved under disjoint unions, generated subframes and g-morphic images.*

Proof. By Proposition 3.3 in [27] g-similar points satisfy exactly the same GML-formulas. Hence by Lemma 4.1.2 validity of GML-formulas is preserved under disjoint unions, generated subframes and g-morphic images. $\square$

4.2 Image-finite Frames

The d-bisimulation game and the idea of d-Hintikka formulas can be generalized for graded modal logics. This way we get the following d, i-bisimulation game $G_{d,i}$ and d, g_i-Hintikka formulas $\varphi^{d,i}_{\mathcal{F},w}$ for $\mathcal{F}$ and w.

Definition 4.2.1 Let $\mathcal{M} = \langle W, R, V\rangle$ and $\mathcal{M}' = \langle W', R', V'\rangle$ be Φ-models and $w \in W$, $w' \in W'$. The game $G_{d,i}((\mathcal{M}, w), (\mathcal{M}', w'))$ is played by the opponent $\forall$ and the defending player $\exists$, and is defined as follows:

Preface: Set $v_0 = w$, $v'_0 = w'$ and $k = 0$.

Playing the game: Let us assume that the pairs $(v_j, v'_j) \in W \times W'$, $j \leq k$, have been played and $0 \leq k \leq d$. Then

1. If $\mathcal{M}, v_k \models p \not\Leftrightarrow \mathcal{M}', v'_k \models p$ for some $p \in \Phi$, the game is over and $\exists$ loses.
2. If $k = d$, the game is over. Otherwise add one to the value of k.
3. Player $\forall$ chooses $S \subseteq R(v_{k-1})$ such that $0 < |S| \leq i$ or $S' \subseteq R'(v'_{k-1})$ such that $0 < |S'| \leq i$. If $\forall$ cannot choose, the game is over and $\forall$ loses.
4. Player $\exists$ chooses $S' \subseteq R'(v'_{k-1})$ or $S \subseteq R(v_{k-1})$ from the remaining model and picks a bijection $f : S \to S'$. If $\exists$ cannot make a choice or pick the bijection, the game is over and $\exists$ loses.
5. The player $\forall$ chooses an element $v \in S$ and sets $(v_k, v'_k) = (v, f(v))$.

If $\exists$ has not lost in any round, she wins. Otherwise $\forall$ wins.

When the player $\exists$ has a winning strategy in the game $G_{d,i}((\mathcal{M}, w), (\mathcal{M}', w'))$, we say that w and w' are *d-g_i-bisimilar* (see also [27]) and denote it by $\mathcal{M}, w \leftrightarroweq_{d,g_i} \mathcal{M}', w'$.

If w and w' are d-g_i-bisimilar, they satisfy the same GML-formulas whose index is at most i and whose degree is at most d, denoted by $\mathcal{M}, w \equiv^d_{g_i} \mathcal{M}', w'$. The opposite of this implication holds if we have only finitely many proposition symbols. In the proof we can use the following d, g_i-Hintikka formulas.

Definition 4.2.2 Let Φ be a finite set of proposition symbols. The *d, g_i-Hintikka formula* $\varphi^{d,i}_{\mathcal{M},w}$ for a Φ-model $\mathcal{M} = \langle W, R, V\rangle$ and $w \in W$ is defined inductively as follows:

$$
\begin{aligned}
\varphi^{0,i}_{\mathcal{M},w} &= \bigwedge\{\psi \mid (\psi = p \text{ or } \psi = \neg p), p \in \Phi \text{ and } \mathcal{M}, w \models \psi\} \\
\varphi^{d+1,i}_{\mathcal{M},w} &= \varphi^{d,i}_{\mathcal{M},w} \wedge \\
&\quad \bigwedge\{\lhd_{=j}\varphi^{d,i}_{\mathcal{M},v} \mid \exists^{=j} v' : (w, v') \in R \text{ and } \varphi^{d,i}_{\mathcal{M},v'} = \varphi^{d,i}_{\mathcal{M},v}, 1 \leq j \leq i-1\} \wedge \\
&\quad \bigwedge\{\lhd_{\geq i}\varphi^{d,i}_{\mathcal{M},v} \mid \exists^{\geq i} v' : (w, v') \in R \text{ and } \varphi^{d,i}_{\mathcal{M},v'} = \varphi^{d,i}_{\mathcal{M},v}\} \wedge \\
&\quad \Box\,(\bigvee\{\varphi^{d,i}_{\mathcal{M},v} \mid (w, v) \in R\}).
\end{aligned}
$$

Note that the notation $\exists^{=j} v'$ above means that there exists *exactly* j states v' such that the given property holds and the notation $\exists^{\geq j} v'$ means that there exists *at least* j states v' such that the given property holds. The intuition behind this definition of Hintikka formulas is again to describe the types of the successor. But this time we also want to count the number of types up to index $i-1$. If there are more than i successor of the same type, we only need to know that there are at least so many. The exact number is irrelevant then. Later we define d, g_i-Hintikka formulas also for (image-finite) frames.

De Rijke [27] states in his Proposition 3.6 (without proof) that d-g-bisimilar states satisfy the same GML-formulas of degree at most d. We prove in the following proposition similar result up to given index i. With the help of the d, g_i-Hintikka formulas we show that also the opposite direction holds when there are only finitely many proposition symbols. In general d-g_i-equivalent states are not necessarily d-g_i-bisimilar.

Proposition 4.2.3 *Let $\mathcal{M} = \langle W, R, V\rangle$ and $\mathcal{M}' = \langle W', R', V'\rangle$ Φ-models for a finite set Φ of proposition symbols and let $w \in W$ and $w' \in W'$. Then the following conditions are equivalent*

(i) $\mathcal{M}, w \leftrightarroweq_{d,g_i} \mathcal{M}', w'$,

(ii) $\mathcal{M}, w \equiv^d_{g_i} \mathcal{M}', w'$,

(iii) $\mathcal{M}', w' \models \varphi^{d,i}_{\mathcal{M},w}$.

Proof.

$(i) \Rightarrow (ii)$ We show by induction on d that

$$\forall w \in W : \forall w' \in W' : ((\mathcal{M}, w \leftrightarroweq_{d,g_i} \mathcal{M}', w') \Rightarrow (\mathcal{M}, w \equiv^d_{g_i} \mathcal{M}', w')). \ (*)$$

When $d = 0$, $(*)$ holds for the proposition symbols by the definition of the $G_{d,i}$-game and it is easy to see that then it also holds for their boolean combinations.

Assume that $(*)$ holds for $d = k$ and $\mathcal{M}, w \leftrightarroweq_{k+1,g_i} \mathcal{M}', w'$. We check only the case $\phi = \triangleleft_{\geq j}\psi$, where $\deg(\psi) = k$ and $j \leq i$. For that we assume first that $\mathcal{M}, w \models \phi$. Then there exists states $v_1, \ldots, v_j$ such that $(w, v_l) \in R$ and $\mathcal{M}, v_l \models \psi$ for each l, $1 \leq l \leq j$. Let $S = \{v_1, \ldots, v_j\}$. Because the defending player $\exists$ has a winning strategy in the game $G_{k+1,i}((\mathcal{M}, w), (\mathcal{M}', w'))$, she can choose a set $S' = \{v'_1, \ldots, v'_j\}$ such that $(w', v'_l) \in R'$ and $\mathcal{M}, v_l \leftrightarroweq_{k,g_i} \mathcal{M}', v'_l$ for each l, $1 \leq l \leq j$. As the defending player $\exists$ was using her winning strategy, she has a winning strategy in the rest of the game. Hence by induction assumption

$\mathcal{M}', v'_l \models \psi$ for each l, $1 \leq l \leq j$, and hence $\mathcal{M}', w' \models \phi$. Similary we can show that $\mathcal{M}', w' \models \phi \Rightarrow \mathcal{M}, w \models \phi$. It is furthermore easy to see that now the equivalence also hold for all boolean combinations of the formulas that are of the form ϕ, that is,

$$\mathcal{M}, w \models \phi' \Leftrightarrow \mathcal{M}', w' \models \phi'$$

holds for all formulas ϕ' of degree $k+1$.

$(ii) \Rightarrow (iii)$ Follows directly from the facts that $\mathcal{M}, w \models \varphi^{d,i}_{\mathcal{M},w}$ and that the degree of d, g_i-Hintikka formulas is at most d.

$(iii) \Rightarrow (i)$ We show by induction on $k \leq d$ that

$$\forall w \in W : \forall w' \in W' : ((\mathcal{M}', w' \models \varphi^{k,i}_{\mathcal{M},w}) \Rightarrow (\mathcal{M}, w \leftrightarroweq_{k,g_i} \mathcal{M}', w')). \quad (*)$$

When $k = 0$, $(*)$ holds for every i. Assume that $(*)$ holds up to $k < d$ for every i and $\mathcal{M}', w' \models \varphi^{k+1,i}_{\mathcal{M},w}$. Let us look at the game $G_{k+1,i}((\mathcal{M}, w), (\mathcal{M}', w'))$. By the case $k = 0$, $\exists$ does not lose before $\forall$ makes his choice. There are three things that can happen.

1. Player $\forall$ chooses $S = \{v_1, \ldots, v_n\} \subseteq R(w)$, $n \leq i$. For each $v_h \in S$ we define $j_h = |\{u \in S \mid \varphi^{k,i}_{\mathcal{M},v_h} = \varphi^{k,i}_{\mathcal{M},u}\}|$. Now, for each h, $1 \leq h \leq n$, $\mathcal{M}', w' \models \lhd_{\geq j_h} \varphi^{k,i}_{\mathcal{M},v_h}$. Hence $\exists$ can choose a set $S' = \{v'_1, \ldots, v'_n\} \subseteq R'(w')$ and a bijection $f : S \to S'$ such that $\mathcal{M}', v'_h \models \varphi^{k,i}_{\mathcal{M},v_h}$ for each h, $1 \leq h \leq n$. By induction assumption $\exists$ has a winning strategy in the rest of the game.
2. Player $\forall$ chooses $S' = \{v'_1, \ldots, v'_n\} \subseteq R'(w')$, $n \leq i$. Because $\mathcal{M}', w' \models \Box(\bigvee\{\varphi^{k,i}_{\mathcal{M},v} \mid (w, v) \in R\})$, there exist states $v_1, \ldots, v_n \in R(w)$ such that $\mathcal{M}', v'_h \models \varphi^{k,i}_{\mathcal{M},v_h}$. Now, we have to show that $\exists$ can choose these states in such a way that a suitable bijection exists between S and S'. For that we define for each $v'_h \in S'$: $j'_h = |\{u' \in S' \mid \mathcal{M}', u' \models \varphi^{k,i}_{\mathcal{M},v_h}\}|$. The states for which $j'_h = 1$ do not give any trouble, but assume that for some h, $j'_h > 1$. If $\mathcal{M}, w \models \lhd_{<j'_h} \varphi^{k,i}_{\mathcal{M},v_h}$, then also $\mathcal{M}', w' \models \lhd_{<j'_h} \varphi^{k,i}_{\mathcal{M},v_h}$ as k, g_i-Hintikka formulas count the successor types up to i. But the latter is impossible as $\mathcal{M}', w' \models \lhd_{\geq j'_h} \varphi^{k,i}_{\mathcal{M},v_h}$. Hence there are at least j'_h states in $\{v \mid (w, v) \in R\}$ which satisfy $\varphi^{k,i}_{\mathcal{M},v_h}$. This proves that $\exists$ can choose the set $S = \{v_1, \ldots, v_h\} \subseteq R(w)$ and a bijection

$f : S \to S$' in such a way that $\mathcal{M}', v'_h \models \varphi^{k,i}_{\mathcal{M},v_h}$. By induction assumption $\exists$ has a winning strategy in the rest of the game.

3. $\forall$ cannot choose, in which case the game is over and $\forall$ loses.

By items 1-3 the player $\exists$ has a winning strategy in the game $G_{k+1,i}$ and hence $\mathcal{M}, w \leftrightarrows_{k+1,g_i} \mathcal{M}', w'$. $\square$

Remark 4.2.4 Note that the part $(i) \Rightarrow (ii)$ of Proposition 4.2.3 holds also when there is more than a finite amount of proposition symbols available.

Model $\mathcal{M}$ is m-saturated, if every set Σ of modal formulas, that is, finitely satisfiable in $R(w)$ for some state w in $\mathcal{M}$ is also satisfiable in $R(w)$, see for example [6]. An analogous concept for graded modal logics can be defined as follows.

Definition 4.2.5 Let $\mathcal{M} = \langle W, R, V \rangle$ be a Kripke model. A set of formulas Σ in *n-satisfiable in a set $X \subseteq W$*, if there exists at least n elements of X that satisfy Σ. The set Σ is *finitely n-satisfiable in X*, if every finite $\delta \subseteq \Sigma$ is n-satisfiable in X. $\mathcal{M}$ is *g-saturated*, if for every $n \in \mathbf{N}$ every set of GML-formulas Σ that is finitely n-satisfiable in $R(w)$ for some state w in $\mathcal{M}$ is also n-satisfiable in $R(w)$.

A model $\mathcal{M} = \langle W, R, V \rangle$ is called image-finite, if for each $w \in W$ the set $R(w)$ is finite. We prove here that image-finite models are g-saturated.

Proposition 4.2.6 *Image-finite models are g-saturated.*

Proof. Let $\mathcal{M} = \langle W, R, V \rangle$ be an image-finite model, $w \in W$ and $n \in \mathbf{N}$. Assume that a set Σ of formulas is finitely n-satisfiable in $R(w)$. Assume on the contrary that Σ is not n-satisfiable in $R(w)$. Now, there exists at most $n-1$ points $v \in R(w)$ such that v satisfies Σ. Let $X = \{u_1, \ldots, u_m\} \subseteq R(w)$ be the set of those successors that do not satisfy Σ. Now, for every state $u_j \in X$ there exists $\psi_j \in \Sigma$ such that $\mathcal{M}, u_j \not\models \psi_j$. As $R(w)$ is finite, the collection $\Delta = \{\psi_1, \ldots, \psi_m\}$ is a finite subset of Σ and Δ is not satisfied in any state of X. As there are at most $n-1$ states outside X, this means that Σ is not finitely n-satisfiable in $R(w)$, which is a contradiction. Hence Σ must be n-satisfiable in $R(w)$. $\square$

De Rijke [27, Lemma 4.2] proves that if $\mathcal{M}$ and $\mathcal{M}'$ are ω-saturated and $\mathcal{M}, w \equiv_{\mathrm{GML}} \mathcal{M}', w'$, then $\mathcal{M}, w \leftrightarroweq_g \mathcal{M}', w'$. We prove this for g-saturated models.

Proposition 4.2.7 *If* $\mathcal{M} = \langle W, R, V\rangle$ *and* $\mathcal{M}' = \langle W', R', V'\rangle$ *are g-saturated and* $\mathcal{M}, w \equiv_{\mathrm{GML}} \mathcal{M}', w'$*, then* $\mathcal{M}, w \leftrightarroweq_g \mathcal{M}', w'$*.*

Proof. Assume that $\mathcal{M}, w \equiv_{\mathrm{GML}} \mathcal{M}', w'$. We show that $Z = \{(v, v') \mid v \equiv_{\mathrm{GML}} v'\}$ is a g-bisimulation between $\mathcal{M}, w$ and $\mathcal{M}', w'$.

(i) By the assumption Z is non-empty and by the definition of Z two points v and v' satisfy the same proposition symbols whenever $(v, v') \in Z$.

(ii) Assume that $(v, v') \in Z$ and $S \subseteq R(v)$ is finite. List the GML-types[1] $tp_1, tp_2, \ldots, tp_k$ of points in S and let $i_1, i_2, \ldots, i_k$ be respectively their orders in S (i.e. there are i_j points of type tp_j in S).

For each type tp_j, $1 \leq j \leq k$, and each finite subset δ of tp_j, $\mathcal{M}, v \models \lhd_{\geq i_j} \bigwedge \delta$ holds. As v and v' satisfy the same GML-formulas, also $v' \models \lhd_{\geq i_j} \bigwedge \delta$. Hence tp_j is finitely i_j-satisfiable in $R'(v')$. Because $\mathcal{M}'$ is g-saturared, tp_j is also i_j-satisfiable in $R'(v')$.

Now, we can choose a set $S' \subseteq R'(v')$ such that its types and orders in S' match those of S. Then $|S| = |S'|$, $\forall u \in S : \exists u' \in S' : (u, u') \in Z$ and $\forall u' \in S' : \exists u \in S : (u, u') \in Z$.

(iii) The case (iii) is symmetric to (ii).

By (i)–(iii) we have shown that Z is a g-bisimulation. $\square$

As we do not restrict ourselves to finite frames now, we can't expect to describe frames by only one formula. That is why we need a slight modification of definition 3.1.1.

Definition 4.2.8 We say that a class C of frames *admits $\mathcal{L}$-set description of point-generated subframes up to g-morphisms*, if for every $\mathcal{F} \in C$ and $w \in \mathcal{F}$ there exists a set of $\mathcal{L}$-formulas Δ such that Δ is satisfiable in w and if it is satisfiable in a point v of a frame $\mathcal{G}$, then $\mathcal{F}_w$ is a g-morphic image of $\mathcal{G}_v$.

[1] GML-type of a given state of a given model is the set of all formulas that are true in that state.

When we want to get an analogue of Theorem 3.1.2 for set-descriptions, we have to add a compactness condition.

Theorem 4.2.9 *Assume that C admits $\mathcal{L}$-set description of point-generated subframes up to g-morphisms. Then an $\mathcal{L}$-compact frame class, that is closed under generated subframes, disjoint unions and taking g-morphic images, is $\mathcal{L}$-definable in restriction to the class C.*

Proof. Let K be an $\mathcal{L}$-compact frame class, that is closed under generated subframes, disjoint unions and taking g-morphic images. We show that

$$\Lambda_K = \{\phi \in \mathcal{L} \mid \forall \mathcal{G} \in K : \mathcal{G} \models \phi\}$$

defines K in restriction to the class C.

Assume first that $\mathcal{F} \models \Lambda_K$ for some $\mathcal{F} \in C$. Choose any state w of $\mathcal{F}$. Because C admits $\mathcal{L}$-set description of point-generated subframes up to g-morphisms, there exists a set $\Delta_{\mathcal{F},w}$ of $\mathcal{L}$-formulas such that

- $\Delta_{\mathcal{F},w}$ is satisfiable in w and
- if it is satisfiable in a point v of a frame $\mathcal{G}$, then $\mathcal{F}_w$ is a g-morphic image of $\mathcal{G}_v$.

Choose a finite $\delta \subseteq \Delta_{\mathcal{F},w}$. If δ is not satisfiable in K, then $\neg(\bigwedge \delta)$ is valid in K. As $\mathcal{F} \models \Lambda_K$ it would follow that $\mathcal{F} \models \neg(\bigwedge \delta)$, which would be a contradiction because clearly $\bigwedge \delta$ is satisfiable in $\mathcal{F}$. Hence for each finite $\delta \subseteq \Delta_{\mathcal{F},w}$ there exists a frame $\mathcal{G}_\delta \in K$ such that δ is satisfiable in $\mathcal{G}_\delta$. Because K is an $\mathcal{L}$-compact frame class, there exists a frame $\mathcal{G} \in K$ such that $\Delta_{\mathcal{F},w}$ is satisfiable in a state v of the frame $\mathcal{G}$. Now, $\mathcal{F}_w$ is a g-morphic image of $\mathcal{G}_v$ and as K is closed under generated subframes and g-morphic images, $\mathcal{F}_w \in K$. By the Proposition 4.1.1 $\mathcal{F}$ can be presented as a g-morphic image of the disjoint union of its point-generated subframes. As K is closed under these constructions, it follows that $\mathcal{F} \in K$.

Assume then that $\mathcal{F} \in K \cap C$. Then trivially $\mathcal{F} \models \Lambda_K$. Now, we have proven that

$$\forall F \in C : (\mathcal{F} \models \Lambda_K \iff \mathcal{F} \in K)$$

and hence K is definable by Λ_K. ☐

Next we show that the class of image-finite Kripke frames admits GML-set description of frames up to g-morphisms. For this we define d, g_i-Hintikka formulas for image finite frames.

Definition 4.2.10 Let $\mathcal{F} = \langle W, R \rangle$ be an image-finite frame and $w \in W$. The *d, g_i-Hintikka formula* $\varphi^{d,i}_{\mathcal{F},w}$ for the frame $\mathcal{F}$ and its point w is defined as $\varphi^{d,i}_{\mathcal{N},w}$, where $\mathcal{N}$ is the restriction of the point-separated model $\mathcal{M}^{\mathcal{F}}$ to depth d.

Note that the restriction to depth d above is only needed to cut the possibly infinite set of proposition symbols to a finite set. Clearly each d, g_i-Hintikka formula $\varphi^{d,i}_{\mathcal{F},w}$ is true in the state w of the point-separated model $\mathcal{M}^{\mathcal{F}}$ based on $\mathcal{F}$ and a set Δ of these formulas is what is needed to describe frames up to g-morphic images, as we prove in the following lemma.

Lemma 4.2.11 *Let $\mathcal{G} = \langle W', R' \rangle$ be an image-finite Kripke frame, $w' \in W'$ and let $\Delta_{\mathcal{G},w'} = \{\varphi^{d,i}_{\mathcal{G},w'} \mid d, i \in \mathbf{N}\}$. If a point w of a frame $\mathcal{F}$ satisfies $\Delta_{\mathcal{G},w'}$, then $\mathcal{G}_{w'}$ is a g-morphic image of $\mathcal{F}_w$.*

Proof. Let $\mathcal{M}' = \langle W', R', V' \rangle$ be the point-separated model for $\mathcal{G}$ and let $\mathcal{M} = \langle W, R, V \rangle$ be a model based on $\mathcal{F}$ such that $\mathcal{M}, w \models \Delta_{\mathcal{G},w'}$. Note that by Proposition 4.2.3 w and w' are now GML-equivalent. Let us show that

$$f = \{(u, u') \in W_w \times W'_{w'} \mid \mathcal{M}, u \equiv_{\mathrm{GML}} \mathcal{M}', u'\}$$

is a surjective g-morphism $\mathcal{F}_w \to \mathcal{G}_{w'}$.[2]

First we show that f is a function. For that we choose $v \in W_w$ and an arbitrary $n \in \mathbf{N}$. As v is in the point-generated subframe $\mathcal{F}_w$, there exists an R-path $v_0, \ldots, v_k$ from w to v. Because $\mathcal{M}, w \models \Delta_{\mathcal{G},w'}$, by Proposition 4.2.3 $(\mathcal{M}, w \leftrightarrows_{d,g_i} \mathcal{M}', w')$ holds for each d and i, especially when $d = k+n$. Hence the player $\exists$ has a winning strategy γ in this game $G_{k+n,i}((\mathcal{M}, w), (\mathcal{M}', w'))$, where $\{v_1\}, \ldots, \{v_k\}$ are the first k moves of the player $\forall$. Let $\{v'_1\}, \ldots, \{v'_k\}$ be the corresponding moves given by this strategy γ. We set $v' = v'_k$. As $\exists$ was using her winning strategy, she has also a winning strategy in the rest of the game $\mathcal{G}_{n,i}((\mathcal{M}, v), (\mathcal{M}', v'))$. Because $\mathcal{M}'$ is a point separated model, this v' is unique and furthermore it is the same for any value for n and i. As n and i were arbitrary, it follows by Proposition 4.2.3 that $\mathcal{M}, v \equiv_{\mathrm{GML}} \mathcal{M}', v'$. Hence $(v, v') \in f$ for a unique v' that is f is a function.

Next we show that f is surjective. Let $v' \in W_{w'}$. As v' is in the point-generated subframe $\mathcal{G}_{w'}$ there exists $n \in \mathbf{N}$ such that $\mathcal{M}', w' \models \Diamond^n p_{v'}$. As $\mathcal{M}, w \equiv_{\mathrm{GML}} \mathcal{M}', w'$, also $\mathcal{M}, w \models \Diamond^n p_{v'}$. Now, $\mathcal{M}, v \models p_{v'}$ for some v.

[2] Note that if we could assume that $\mathcal{F}_w$ is also image-finite, both models would be g-saturated and hence the GML-formula equivalence would also be a g-bisimulation.

Because $\mathcal{M}'$ is a point-separated model and f is a function, $f(v) = v'$. Hence f is a surjection.

Now, it is left to show that f is a g-bisimulation.

(i) Assume that $(v, v') \in f$ and $S = \{v_1, \ldots, v_k\} \subseteq R(v)$. Denote the GML-types of points in S by $tp_1, \ldots, tp_k$. As $\mathcal{M}'$ is a point-separated model and $\mathcal{M}, v \equiv_{\mathrm{GML}} \mathcal{M}', v'$, these types are all different. Now, look at the type tp_j of a point $v_j \in S$. As $(v, v_j) \in R$, $\mathcal{M}, v \models \Diamond(\wedge\delta)$ for every finite $\delta \subseteq tp_j$. Because $\mathcal{M}, v \equiv_{\mathrm{GML}} \mathcal{M}', v'$, also $\mathcal{M}', v' \models \Diamond(\wedge\delta)$ for every finite $\delta \subseteq tp_j$. Because $\mathcal{M}'$ is image-finite, it is g-saturated. Thus tp_j is satisfiable among the successors of v'. As the same argument holds for every GML-type of S, it is possible to choose $S' \subseteq R'(v')$ such that $|S| = |S'|$, $\forall u \in S : \exists u' \in S' : (u, u') \in f$ and $\forall u' \in S' : \exists u \in S : (u, u') \in f$.

(ii) Assume that $(v, v') \in f$ and $S' \subseteq R'(v')$ is finite. Let

$$S = \{u \in W_w \mid f(u) \in S'\} \cap R(v).$$

Now we have to show that this S satisfies the conditions in part (iii) of Definition 2.5.2.

First choose $u' \in S'$. Now, $(v', u') \in R'$ and because $\mathcal{M}'$ is a point-separated model, $\mathcal{M}', v' \models p_{v'} \to \triangleleft_{=1} p_{u'}$. Because $(v, v') \in f$, they satisfy the same GML-formulas and also $\mathcal{M}, v \models p_{v'} \to \triangleleft_{=1} p_{u'}$. Hence there exists exactly one u such that $(v, u) \in R$ and $\mathcal{M}, u \models p_{u'}$. As f is a function and $\mathcal{M}'$ is a point-separated model, $f(u) = u'$. Hence $u \in S$, $|S| = |S'|$ and $\forall u' \in S' : \exists u \in S : (u, u') \in f$.

Second, let $u \in S$. By the definition of S there exists $u' \in S'$ such that $(u, u') \in f$. Hence $\forall u \in S : \exists u' \in S' : (u, u') \in f$ and we have shown that the conditions in part (iii) of the definition 2.5.2 hold.

Items (i) and (ii) guarantee that f is a g-bisimulation. Thus we have shown that $\mathcal{G}_{w'}$ is a g-morphic image of $\mathcal{F}_w$. □

By Theorem 4.2.9 and Lemma 4.2.11 we get the following corollary.

Corollary 4.2.12 *Let* $|\Phi| \geq \omega$. *A* GML[Φ]*-compact frame class* K *that is is closed under taking g-morphic images, generated subframes and disjoint unions is* GML[Φ]*-definable in restriction to the class of image-finite frames.*

By Proposition 4.1.4 the validity of GML-formulas is preserved under g-morphic images, generated subframes and disjoint unions. Hence a frame class that is GML[Φ]-definable (with respect to the class of image-finite frames) is also closed under taking g-morphic images, generated subframes and disjoint unions (in restriction to the class of image-finite frames).

4.3 Adding the path quantifier to GML

Here we add the path quantifier A to graded modal logics and generalize Jankov-Fine formulas for this case.

Definition 4.3.1 Let $\mathcal{F} = \langle W, R\rangle$ be a frame, $w \in W$, $\Phi = \{p_s \mid s \in W_w\}$. Let φ be the conjunction of the following formulas

(i) $\bigvee_{s \in W_w} p_s$,

(ii) $\bigwedge_{s,t \in W_w, s \neq t}(p_s \to \neg p_t)$,

(iii) $\bigwedge_{(s,t) \in R_w}(p_s \to \lhd_{=1} p_t)$,

(iv) $\bigwedge_{(s,t) \notin R_w}(p_s \to \neg\Diamond p_t)$.

Now, $\psi^{\mathrm{gA}}_{\mathcal{F},w}$ is defined as $p_w \wedge A\varphi$.

Note that these formulas use infinite disjunctions and conjunctions if the frames are allowed to be infinite.

Next we want to check that $\psi^{\mathrm{gA}}_{\mathcal{F},w}$ is indeed the type of formula that guarantees that the class of (finite) frames admits GMLA-description of point-generated frames up to g-morphisms.

Lemma 4.3.2 *Let $\mathcal{F} = \langle W, R\rangle$ and $w \in W$. Then the formula $\psi^{\mathrm{gA}}_{\mathcal{F},w}$ is satisfiable in w.*

Proof. Let $\mathcal{M}^{\mathcal{F}_w}$ be the point-separated model for $\mathcal{F}_w$. It is clear that $\mathcal{M}^{\mathcal{F}_w}, w \models \psi^{\mathrm{gA}}_{\mathcal{F},w}$. □

Lemma 4.3.3 *Let $\mathcal{G} = \langle W', R'\rangle$ and $w' \in W'$. If a point w of a frame $\mathcal{F}$ satisfies the formula $\psi^{\mathrm{gA}}_{\mathcal{G},w'}$, then $\mathcal{G}_{w'}$ is a g-morphic image of $\mathcal{F}_w$.*

Proof. Let $\mathcal{M}' = \langle W', R', V' \rangle$ be the point-separated model based on $\mathcal{G}$ and $\mathcal{M} = \langle W, R, V \rangle$ be a model based on $\mathcal{F}$ such that $\mathcal{M}, w \models \psi^{\mathrm{gA}}_{\mathcal{G},w'}$. Let us show that

$$f = \{(u, u') \in W_w \times W'_{w'} \mid u \in V(p_{u'})\}$$

is a surjective g-morphism $\mathcal{G}_v \to \mathcal{F}_w$.

First we show that f is a function. For that we choose $v \in W_w$. Because $\mathcal{M}, w \models A\varphi$, we have that $\mathcal{M}, v \models (\bigvee_{s \in W_w} p_s) \wedge (\bigwedge_{s,t \in W_w s \neq t}(p_s \to \neg p_t))$. Hence $v \in V(p_{u'})$ for exactly one $u' \in W'_{w'}$. Thus f is a function.

Let $v' \in W'_{w'}$. Let us show by induction on n that for every point $v'_j \in W'_{w'}$ in a path $w' = v'_0 R' v'_1 R' \ldots R' v'_n = v'$, there exists a point $v_j \in W_w$ such that $f(v_j) = v'_j$. If $n = 0$, the claim holds as $f(w) = w'$. Let us assume, that the claim holds for $k < n$. Now, $\mathcal{M}, v_k \models p_{v'_k} \wedge (p_{v'_k} \to \lhd_{=1} p_{v'_{k+1}})$. Hence there exists $v_{k+1} \in W_w$ such that $(v_k, v_{k+1}) \in R_w$ and $\mathcal{M}, v_{k+1} \models p_{v'_{k+1}}$. Now $f(v_{k+1}) = v'_{k+1}$. By induction principle, for every point v'_j on the path from w' to v' there exists a point $v_j \in W_w$ such that $f(v_j) = v'_j$. In particular for the point v' such a state exists. Thus f is surjective.

Now, it is left to show that f is a g-bisimulation. For that we assume that $(v, v') \in f$ and show that restriction g of f into $R(v)$ is a bijection from $R(v)$ to $R'(v')$. Assume first that there exists $u_1, u_2 \in R(v)$ such that $f(u_1) = f(u_2) = u'$. If $(v', u') \notin R'$, then $\mathcal{M}, v \models p_{v'} \to \neg\Diamond p_{u'}$. But this contradicts the fact that $\mathcal{M}, v \models p_{v'} \wedge \Diamond p_{u'}$ (which follows from $f(v) = v'$, $(v, u_1) \in R$ and $f(u_1) = u'$). Hence $(v', u') \in R'$. By the definition of Jankov-Fine formulas it follows that $\mathcal{M}, v \models p_{v'} \to \lhd_{=1} p_{u'}$. Hence $u_1 = u_2$ and g is injective.

Assume then that $u' \in R'(v')$. By the definition of Jankov-Fine formulas we have that $\mathcal{M}, v \models p_{v'} \to \lhd_{=1} p_{u'}$. Hence there exists u such that $(v, u) \in R$ and $\mathcal{M}, u \models p_{u'}$. As f is a function and $\mathcal{M}'$ a point-separated model, $f(u) = u'$. Thus g is a surjective.

Now, it is easy to see that for any $S \subseteq R(v)$ there exists $S' = f(S) \subseteq R'(v')$ and for any $S' \subseteq R'(v')$ there exists $S = f^{-1}(S') \subseteq R(v)$ such that the conditions in the Definition 2.5.2 hold. Hence f is a g-bisimulation and we have shown that $\mathcal{G}_{w'}$ is a g-morphic image of $\mathcal{F}_w$. $\square$

We get the following corollary of Theorem 3.1.2 and Lemmas 4.3.2 and 4.3.3.

Corollary 4.3.4 *Let* $|\Phi| \geq \omega$. *A frame class* K *is* GMLA$[\Phi]$*-definable in* *restriction to the class of finite frames if it is closed under taking g-morphic images, generated subframes and disjoint unions.*

As g-bisimilar states satisfy the same GML-formulas [27], they also satisfy the same GMLA-formulas. Hence by Lemma 4.1.2 the validity of GMLA-formulas is preserved under g-morphic images, generated subframes and disjoint unions. From this it follows that a frame class K that is GMLA[Φ]-definable (in restriction to the class of finite frames) is also closed under taking g-morphic images, generated subframes and disjoint unions (in restriction to the class of finite frames).

As the path quantifier A is definable in μGML and in GML$_\infty$ we get the following corollaries for these logics.

Corollary 4.3.5 *Let $|\Phi| \geq \omega$. A frame class that is closed under taking g-morphic images, generated subframes and disjoint unions is* μGML[Φ]-*definable in restriction to the class K_{fin} of finite frames.*

Corollary 4.3.6 *Let $|\Phi| \geq \kappa$. A frame class that is closed under taking g-morphic images, generated subframes and disjoint unions is* GML$_\infty$[Φ]-*definable in restriction to the frame class K_κ.*

Clearly g-bisimilar states satisfy the same μGML-formulas and the same GML$_\infty$-formulas. Hence by Lemma 4.1.2 the validity of μGML-formulas and GML$_\infty$-formulas is preserved under g-morphic images, generated subframes and disjoint unions. This gives us that if a frame class is μGML[Φ]-definable or GML$_\infty$[Φ]-definable with respect to a frame class C, it is also closed under taking g-morphic images, generated subframes and disjoint unions in restriction to the class C.

Remark 4.3.7 If we allow Φ to be a proper class and change the definition of definability to allow proper classes of formulas, we can leave out the size restriction. Then a frame class that is closed under taking g-morphic images, generated subframes and disjoint unions is GML$_\infty$[Φ]-definable.

Chapter 5

Generalizing modal definability

5.1 A new perspective on definability

In this chapter we move to talk about multi-modal logics introduced in section 2.6. A natural way to lift the concept of validity up from the level of multi-modal Kripke frames $\mathcal{F} = \langle W, R_1, \ldots, R_n \rangle$ is to allow quantifications over the frame relations $R_1, \ldots, R_n$.

Definition 5.1.1 Let $\tau = \{\Diamond_1, \ldots, \Diamond_n\}$ be a modal similarity type. We say that a τ-formula φ is *τ-valid in a set* W, if it is valid in each frame $\mathcal{F} = \langle W, R_1, \ldots, R_n \rangle$ based on W.

We write $W \models_\tau \varphi$, if φ is τ-valid in W. With the help of the standard translation $St_x(\varphi)$ of φ, in which only proposition symbols $p_1, \ldots, p_k$ occur, this can be formulated as

$$W \models_\tau \varphi \iff W \models \forall R_1 \ldots \forall R_n \forall P_1 \ldots \forall P_k \forall x St_x(\varphi).$$

Let us illuminate this with a simple example

Example 5.1.2 Let $\tau = \{\Diamond\}$. Now

$$W \models_\tau \Diamond p \to \Box p \iff |W| \leq 1.$$

Proof.

$\Rightarrow$ Assume on the contrary that $|W| \geq 2$. We can assume without loss of generality that $\{1, 2\} \subseteq W$. Let $R = \{(1, 1), (1, 2)\}$ and $V(p) = \{2\}$. Now, $\langle W, R, V \rangle, 1 \models \Diamond p$, but $\langle W, R, V \rangle, 1 \not\models \Box p$ which is a contradiction as $\langle W, R, V \rangle$ is a model based on W.

$\Leftarrow$ Let us assume that $|W| \leq 1$. Let R be an arbitrary relation and V an arbitrary valuation on W. If $\langle W, R, V\rangle, w \models \Diamond p$ for some $w \in W$, then $\langle W, R, V\rangle \models p$ and hence $\langle W, R, V\rangle, x \models \Box p$. As R and V were arbitrary, $W \models_\tau \Diamond p \to \Box p$. $\square$

We can also define other cardinality restrictions as in the following example:

Example 5.1.3 Let $\tau = \{\Diamond\}$ and let

$$\varphi = (\Diamond p_1 \wedge \Diamond p_2 \wedge \Diamond p_3) \to \Diamond((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee (p_2 \wedge p_3))$$

Now

$$W \models_\tau \varphi \iff |W| \leq 2.$$

Proof.

$\Rightarrow$ Assume on the contrary that $|W| \geq 3$. We can assume without loss of generality that $\{1, 2, 3\} \subseteq W$. Let $R = \{(1,1), (1,2), (1,3)\}$ and $V(p_i) = \{i\}$. Now, $\langle W, R, V\rangle, 1 \models (\Diamond p_1 \wedge \Diamond p_2 \wedge \Diamond p_3)$, but $\langle W, R, V\rangle, 1 \not\models \Diamond((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee (p_2 \wedge p_3))$ which is a contradiction as $\langle W, R, V\rangle$ is a model based on W.

$\Leftarrow$ Let us assume that $|W| \leq 2$. Let R be an arbitrary relation and V an arbitrary valuation on W. Assume that $\langle W, R, V\rangle, w \models (\Diamond p_1 \wedge \Diamond p_2 \wedge \Diamond p_3)$. Because $|W| \leq 2$ there exists a state v such that $(w, v) \in R$ and $\langle W, R, V\rangle, v \models ((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee (p_2 \wedge p_3))$. Hence $\langle W, R, V\rangle, w \models \Diamond((p_1 \wedge p_2) \vee (p_1 \wedge p_3) \vee (p_2 \wedge p_3))$. As R and V were arbitrary, we have proven that $W \models_\tau \varphi$. $\square$

Similarly, we could define any cardinality restriction $|W| \leq n$. Note that none of the classes $C_n = \{\langle W, R\rangle \mid |W| \leq n\}$ is definable in basic modal logic as they are not closed under disjoint unions, but they are of course definable when we add global modality.

Set validity allows us to say something about the underlying set, but at the same time we lose the ability to talk about relations as we have quantified them away. We get a more general perspective if we set some of the relations to be *helpers* and the other to be *bosses*. We quantify over the helpers and see what we can say about the bosses in this case. Note that as we only use unary operators in this thesis, the helpers we quantify over are actually binary relations. Let us see, where this framework leads us.

Definition 5.1.4 Let $\mathcal{L}$ be a modal language with the similarity type $\tau = \tau_H \cup \tau_B$ and let Φ be a set of proposition symbols. We assume that $\tau_H \cap \tau_B = \emptyset$ and we call the operators in $\tau_H = \{\Diamond_1^H, \dots, \Diamond_m^H\}$ *helpers* and the operators in $\tau_B = \{\Diamond_1, \dots, \Diamond_n\}$ *bosses.*

We say that a $\mathcal{L}[\Phi, \tau]$-formula φ is *τ_H-valid in a frame* $\mathcal{F} = \langle W, R_1, \dots, R_n \rangle$, if

$$\langle W, R_1, \dots, R_n, H_1, \dots, H_m \rangle \models \varphi,$$

for all choices of the helper relation $H_1, \dots, H_m$.

With the help of the standard translation $St_x(\varphi)$ for formulas of basic modal logic this can be formulated as

$$\mathcal{F} \models \forall H_1 \dots \forall H_m \forall P_1 \dots \forall P_k \forall x St_x(\varphi) \tag{5.1}$$

that is we get a formula of universal second order logic. We write $\mathcal{F} \models_{\tau_H} \varphi$, if φ is τ_H-valid in $\mathcal{F}$.

The notion of definability is now naturally defined with respect to this concept of validity.

Definition 5.1.5 Let $\mathcal{L}$ be a modal language with the similarity type $\tau = \tau_H \cup \tau_B$ and let Φ a set of proposition symbols. A frame class K is $\mathcal{L}[\Phi, \tau_H]$-*definable* in restriction to a frame class C if there exists a set Γ of $\mathcal{L}[\Phi, \tau]$-formulas such that

$$\forall \mathcal{F} \in C : (\mathcal{F} \in K \Leftrightarrow \mathcal{F} \models_{\tau_H} \Gamma).$$

A formula is satisfiable, if its negation is not valid. From this we get the following definition.

Definition 5.1.6 Let $\mathcal{L}$ be a modal language with the similarity type $\tau = \tau_H \cup \tau_B$ and let Φ be a set of proposition symbols. An $\mathcal{L}[\Phi, \tau]$-formula φ τ_H*-satisfiable* in a frame $\mathcal{F} = \langle W, R_1, \dots, R_n \rangle$, if

$$\langle W, R_1, \dots, R_n, H_1, \dots, H_m, V \rangle, w \models \varphi$$

for some helper relations $H_1, \dots, H_m$, a valuation V and a state $w \in W$. The model $\langle W, R_1, \dots, R_n, H_1, \dots, H_m, V \rangle$ is a τ-model based on the frame $\langle W, R_1, \dots, R_n \rangle$.

With the help of the standard translation $St_x(\varphi)$ for formulas of basic modal logic this can be formulated as

$$\mathcal{F} \models \exists H_1 \ldots \exists H_m \exists P_1 \ldots \exists P_k \exists x St_x(\varphi)$$

that is we get a formula of existential second order logic.

Let us illuminate this new concept of validity by a simple example.

Example 5.1.7 Let $\tau_B = \{\Diamond\}$, $\tau_H = \{\Diamond^H\}$ and $\mathcal{F} = \langle W, R\rangle$. Now, the formula $\varphi = \Box p \to \Box^H p$ says that R is the universal relation. That is

$$\mathcal{F} \models_{\tau_H} \Box p \to \Box^H p \iff R = W \times W.$$

Proof.

- $\Rightarrow$ Let $\mathcal{F} \models_{\tau_H} \Box p \to \Box^H p$. Assume on the contrary that $R \neq W \times W$. Now, there exists a pair $(w, v) \in W \times W$ such that $(w, v) \notin R$. Let $H = \{(w, v)\}$ and $V(p) = W \setminus \{v\}$. Now, $\langle W, R, H, V\rangle, w \models \Box p$ but $\langle W, R, H, V\rangle, w \not\models \Box^H p$. Hence $\langle W, R, H, V\rangle, w \not\models \Box p \to \Box^H p$ which is a contradiction as $\langle W, R, H, V\rangle$ is a τ_H-model based on $\mathcal{F}$.

- $\Leftarrow$ Now, let us assume that $R = W \times W$. Let H be an arbitrary relation and V an arbitrary valuation on W. If $\langle W, R, H, V\rangle, w \models \Box p$ for some $w \in W$, then $\langle W, R, H, V\rangle \models p$ and hence also $\langle W, R, H, V\rangle, w \models \Box^H p$. As H and V were arbitrary, $\mathcal{F} \models_{\tau_H} \Box p \to \Box^H p$. $\square$

As a corollary to this example we get the following statement.

Corollary 5.1.8 *τ_H-validity of* ML*-formulas is* not *preserved under disjoint unions.*

τ_H-validity of ML-formulas is, however, still preserved under generated subframes and p-morphic images as we can conclude from the following Lemma. Before going into the proof, we note that the frame constructions are naturally made with respect to the relations $R_1, \ldots, R_n$ corresponding to the bosses. Concrete helpers only come into the picture on the level of models, where the concept of satisfiability is as it is in normal multi-modal logic with relations $R_1, \ldots, R_n, H_1, \ldots, H_m$.

Lemma 5.1.9 *Let $\mathcal{L}$ be an arbitrary modal language with similarity type $\tau = \tau_B \cup \tau_H$, where $\tau_B = \{\Diamond_1, \dots, \Diamond_n\}$ and $\tau_H = \{\Diamond_1^H, \dots, \Diamond_m^H\}$. If bisimilar states satisfy the same $\mathcal{L}$-formulas, then τ_H-validity of $\mathcal{L}$-formulas is preserved under generated subframes and p-morphic images.*

Proof. We go through generated subframes and p-morphic images.

generated subframes: Let $\mathcal{F} = \langle W, R_1, \dots, R_n\rangle$ and let $\mathcal{G}$ be a generated subframe of $\mathcal{F}$. Assume that $\mathcal{F} \models_{\tau_H} \varphi$ and suppose on the contrary that $\mathcal{G} \not\models_{\tau_H} \varphi$ that is $\neg\varphi$ is τ_H-satisfiable in $\mathcal{G}$. By definition 5.1.6 there exist some relations $H_1^{\mathcal{G}}, \dots, H_m^{\mathcal{G}}$ and a valuation $V^{\mathcal{G}}$ on $W^{\mathcal{G}}$ and a state $w \in W^{\mathcal{G}}$ such that

$$\langle W^{\mathcal{G}}, R_1^{\mathcal{G}}, \dots, R_n^{\mathcal{G}}, H_1^{\mathcal{G}}, \dots, H_m^{\mathcal{G}}, V^{\mathcal{G}}\rangle, w^{\mathcal{G}} \models \neg\varphi.$$

Let $\mathcal{N} = \langle W^{\mathcal{G}}, R_1^{\mathcal{G}}, \dots, R_n^{\mathcal{G}}, H_1^{\mathcal{G}}, \dots, H_m^{\mathcal{G}}, V^{\mathcal{G}}\rangle$. Clearly $\mathcal{N}$ is a τ_H-model based on $\mathcal{G}$. Let $\mathcal{M} = \langle W, R_1, \dots, R_n, H_1, \dots, H_m, V\rangle, w \models \neg\varphi$ where $H_i = H_i^{\mathcal{G}}$ and $V = V^{\mathcal{G}}$. As relations and valuations based on $W^{\mathcal{G}}$ are also relations and valuations on W, $\mathcal{M}$ is a τ_H-model based on $\mathcal{F}$. It is easy to see that

$$Z : \{(v, v^{\mathcal{G}}) \mid v^{\mathcal{G}} \in W^{\mathcal{G}}, v = v^{\mathcal{G}}\}$$

is a bisimulation such that $\mathcal{M}, w \leftrightarrows \mathcal{N}, w^{\mathcal{G}}$, where $w = w^{\mathcal{G}}$. Because $\mathcal{N}, w^{\mathcal{G}} \models \neg\varphi$, by assumption also $\mathcal{M}, w \models \neg\varphi$. This is a contradiction as $\mathcal{F} \models_{\tau_H} \varphi$ and $\langle W, R_1, \dots, R_n, H_1, \dots, H_m, V\rangle$. This is a contradiction as $\mathcal{M}$ is a τ_H-model based on $\mathcal{F}$. Hence $\mathcal{G} \models_{\tau_H} \varphi$ and we have proven that τ_H-validity of $\mathcal{L}$-formulas is preserved under generated subframes. $\square$

p-morphic images: Let $\mathcal{F} = \langle W, R_1, \dots, R_n\rangle$, $\mathcal{G} = \langle W^{\mathcal{G}}, R_1^{\mathcal{G}}, \dots, R_n^{\mathcal{G}}\rangle$ and let f be a surjective p-morphism $\mathcal{F} \longrightarrow \mathcal{G}$. Assume that $\mathcal{F} \models_{\tau_H} \varphi$ and suppose on the contrary that $\mathcal{G} \not\models_{\tau_H} \varphi$. Now, there exist some relations $H_1^{\mathcal{G}}, \dots, H_m^{\mathcal{G}}$ and a valuation $V^{\mathcal{G}}$ on $W^{\mathcal{G}}$ and a state $v \in W^{\mathcal{G}}$ such

$$\langle W^{\mathcal{G}}, R_1^{\mathcal{G}}, \dots, R_n^{\mathcal{G}}, H_1^{\mathcal{G}}, \dots, H_m^{\mathcal{G}}, V^{\mathcal{G}}\rangle, v \models \neg\varphi.$$

With the help of the surjective p-morphism f we can define relations H_j:

$$(x, y) \in H_j \iff (f(x), f(y)) \in H_j^{\mathcal{G}},$$

a valuation V:

$$v \in V(p) \Longleftrightarrow f(v) \in V^{\mathcal{G}}(p)$$

and pick a state w in W such that

$$w \in f^{-1}(v).$$

Now, it is easy to see that the model

$$\mathcal{N} = \langle W^{\mathcal{G}}, R_1^{\mathcal{G}}, \ldots, R_n^{\mathcal{G}}, H_1^{\mathcal{G}}, \ldots, H_m^{\mathcal{G}}, V^{\mathcal{G}} \rangle$$

is a p-morphic image of the model

$$\mathcal{M} = \langle W, R_1, \ldots, R_n, H_1, \ldots, H_m, V \rangle$$

and hence f is a bisimulation. Because $f(w) = v$, these points satisfy by assumption the same $\mathcal{L}$-formulas. Hence $\mathcal{M}, w \models \neg\varphi$. But this is a contradiction as $\mathcal{F} \models_{\tau_H} \varphi$ and $\mathcal{M}$ is a τ_H-model based on $\mathcal{F}$. Hence $\mathcal{G} \models_{\tau_H} \varphi$ and we have proven that τ_H-validity of $\mathcal{L}$-formulas is preserved under p-morphic images. $\square$

As bisimilar states satisfy the same (multi-modal) ML-formulas, we get the following corollary.

Corollary 5.1.10 *An* ML$[\tau_H]$*-definable frame class is closed generated subframes and taking p-morphic images.*

Note that it follows from Proposition 5.1.10 that we can not define cardinality restrictions of the form $|W| > n$ even with the help of this generalized concept of validity as p-morphisms can reduce the size of frames.

In section 2.3 we defined a concept of a frame class C admitting $\mathcal{L}$-description of point-generated subframes up to p-morphisms. Then we showed that a frame class that is closed under generated subframes, disjoint unions and p-morphic images is $\mathcal{L}$-definable in restriction to a frame class C with this property. Here we can not use disjoint unions, but it turns out that with the helpers we can get to different parts of the frames. This gives us a generating set (and not just one state). Hence we generalize Definition 3.1.1 from describing just point-generated subframes into describing generated subframes in general.

Definition 5.1.11 Let $\mathcal{L}$ be a modal language with similarity type $\tau = \tau_B \cup \tau_H$ and let Φ be a set of proposition symbols. We say that a frame class C *admits* $\mathcal{L}[\Phi, \tau_H]$*-description of generated subframes up to p-morphisms*, if for every $\mathcal{F} \in C$ there exists an $\mathcal{L}[\Phi, \tau]$-formula $\varphi_{\mathcal{F}}$ such that

1. $\varphi_{\mathcal{F}}$ is τ_H-satisfiable in $\mathcal{F}$ and
2. if $\varphi_{\mathcal{F}}$ is τ_H-satisfiable in $\mathcal{G}$, then $\mathcal{F}$ is a p-morphic image of some generated subframe of $\mathcal{G}$.

Theorem 5.1.12 *Assume that a class C admits $\mathcal{L}[\Phi, \tau_H]$-description of frames up to p-morphisms from generated subframes. Then a frame class K that is closed under generated subframes and taking p-morphic images is $\mathcal{L}[\Phi, \tau_H]$-definable in restriction to the class C.*

Proof. Let K be a frame class that is closed under generated subframes and taking p-morphic images. We show that $\Lambda_K = \{\phi \in \mathcal{L} \mid \forall \mathcal{G} \in K : \mathcal{G} \models_{\tau_H} \phi\}$ defines K in restriction to the class C.

Assume first that $\mathcal{F} \models_{\tau_H} \Lambda_K$ for some frame $\mathcal{F} \in C$. Because C admits $\mathcal{L}[\Phi, \tau_H]$-description of frames up to p-morphisms from generated subframes, there exists an $\mathcal{L}[\Phi, \tau]$-formula $\varphi_{\mathcal{F}}$ such that

- $\varphi_{\mathcal{F}}$ is τ_H-satisfiable in $\mathcal{F}$ and
- if $\varphi_{\mathcal{F}}$ is τ_H-satisfiable in $\mathcal{G}$, then $\mathcal{F}$ is a p-morphic image of some generated subframe of $\mathcal{G}$.

If $\varphi_{\mathcal{F}}$ is not τ_H-satisfiable in K, then $\neg\varphi_{\mathcal{F}}$ is valid in K and $\neg\varphi_{\mathcal{F}} \in \Lambda_K$. As $\mathcal{F} \models_{\tau_H} \Lambda_K$ it follows that $\mathcal{F} \models_{\tau_H} \neg\varphi_{\mathcal{F}}$ which is a contradiction because $\varphi_{\mathcal{F}}$ is τ_H-satisfiable in $\mathcal{F}$. Hence there must exist a frame $\mathcal{G} \in K$ such that $\varphi_{\mathcal{F}}$ is τ_H-satisfiable in $\mathcal{G}$. Now, $\mathcal{F}$ is a p-morphic image of some generated subframe of $\mathcal{G}$ and as K is closed under these constructions, $\mathcal{F} \in K$.

Assume then that $\mathcal{F} \in K \cap C$. Then trivially $\mathcal{F} \models_{\tau_H} \Lambda_K$. Now, we have proven that

$$\forall \mathcal{F} \in C : (\mathcal{F} \models_{\tau_H} \Lambda_K \Longleftrightarrow \mathcal{F} \in K)$$

and hence K is definable by Λ_K. □

Next we generalize Jankov-Fine Formulas for this context where helpers are allowed and prove that these generalized formulas satisfy the items 1 and 2 of Definition 5.1.11 in restriction to the class of finite and transitive frames. In these considerations we assume that $\tau = \tau_B \cup \tau_H$, where $\tau_B = \{\Diamond\}$ and $\tau_H = \{\Diamond^H\}$, and that $\mathcal{F} = \langle W, R\rangle$ is a finite τ_B-frame and the set Φ of proposition symbols is $\{p_s \mid s \in W\}$.

Definition 5.1.13 Let ML$[\tau, \Phi]$-formula φ be the conjunction of the following formulas.

(i) $\bigvee_{s\in W} p_s$,

(ii) $\bigwedge_{s,t\in W,\ s\neq t}(p_s \to \neg p_t)$,

(iii) $\bigwedge_{(s,t)\in R}(p_s \to \Diamond p_t)$,

(iv) $\bigwedge_{(s,t)\notin R}(p_s \to \neg\Diamond p_t)$.

We define $\psi_{\mathcal{F}}$ as $(\bigwedge_{s\in W}(\Diamond^H p_s)) \wedge (\Box^H(\varphi \wedge \Box\varphi))$.

Lemma 5.1.14 *The formula $\psi_{\mathcal{F}}$ is τ_H-satisfiable in $\mathcal{F}$.*

Proof. Let $V(p_s) = \{p_s\}$ for every $s \in W$ and let $H = W \times W$. Now, $\langle W, R, H, V\rangle \models \psi_{\mathcal{F}}$. Hence $\psi_{\mathcal{F}}$ is τ_H-satisfiable in $\mathcal{F}$. $\square$

Lemma 5.1.15 *If $\psi_{\mathcal{F}}$ is τ_H-satisfiable in a finite, transitive frame $\mathcal{G} = \langle W^{\mathcal{G}}, R^{\mathcal{G}}\rangle$, then $\mathcal{F}$ is a p-morphic image of some generated subframe $\mathcal{G}_X = \langle W^{\mathcal{G}}_X, R^{\mathcal{G}}_X\rangle$.*

Proof. Let $\mathcal{N} = \langle W^{\mathcal{G}}, R^{\mathcal{G}}, H^{\mathcal{G}}, V^{\mathcal{G}}\rangle$ be a model based on $\mathcal{G}$ such that $\mathcal{N}, v \models \psi_{\mathcal{F}}$ for some $v \in W^{\mathcal{G}}$. Let $X = H^{\mathcal{G}}(v)$. We show that

$$f = \{(v', s) \in W^{\mathcal{G}}_X \times W_{\mathcal{F}} \mid v' \in V^{\mathcal{G}}(p_s)\}$$

is a surjective p-morphism from $\mathcal{G}_X$ (X-generated subframe of $\mathcal{G}$) to $\mathcal{F}$. For this we need to show that

1. f is a function $\mathcal{G}_X \longrightarrow \mathcal{F}$,
2. f is surjective, and
3. f is a bisimulation.

1. Let $v_2 \in W_X$. As v_2 is in the X-generated subframe of $\mathcal{G}$, there exists a point $v_1 \in X$ such that $(v, v_1) \in H^{\mathcal{G}}$ and there is an $R^{\mathcal{G}}$-path from v_1 to v_2. Because $R^{\mathcal{G}}$ is transitive, $(v_1, v_2) \in R^{\mathcal{G}}$. Because $\mathcal{N}, v \models \psi_{\mathcal{F}}$, then $\mathcal{N}, v_1 \models \Box\varphi$ and hence $\mathcal{N}, v_2 \models (\bigvee_{s\in W} p_s) \wedge (\bigwedge_{s,t\in W,\ s\neq t}(p_s \to \neg p_t))$. Therefore $v_2 \in V(p_s)$ for exactly one $s \in W$ and hence f is a function.

2: Let $s \in W$. Because $\mathcal{N}, v \models (\bigwedge_{s\in W}(\Diamond^H p_s))$, there exists a point $v_1 \in W_X$ such that $\mathcal{N}, v_1 \models p_s$. Now, $f(v_1) = s$. Thus f is surjective.

3 a: Assume first that $(v_1, v_2) \in R^{\mathcal{G}}_X$ and $f(v_1) = s$. Now, there exists a point $v' \in X$ such that $(v, v') \in H^{\mathcal{G}}$ and there exists an $R^{\mathcal{G}}$-path from v' to v_1. Because $R^{\mathcal{G}}$ is transitive, $(v', v_1) \in R^{\mathcal{G}}$. Let $t = f(v_2)$. If $(s, t) \notin R$, then $\mathcal{N}, v_1 \models p_s \to \neg\Diamond p_t$. But this contradicts the fact that $\mathcal{N}, v_1 \models p_s \wedge \Diamond p_t$. Hence $(f(v_1), f(v_2)) = (s, t) \in R$.

3 b: Assume then that $f(v_1) = s$ and $(s, t) \in R$. Now, $v_1 \in W_X$ that is there exists v' such that $(v, v') \in H^{\mathcal{G}}$ and there exists an $R^{\mathcal{G}}$-path from v' to v_1. Because $R^{\mathcal{G}}$ is transitive, $(v', v_1) \in R^{\mathcal{G}}$. Hence $\mathcal{N}, v_1 \models p_s \wedge (p_s \to \Diamond p_t)$. Therefore there exists $v_2 \in W_X$ such that $(v_1, v_2) \in R^{\mathcal{G}}_X$ and $f(v_2) = t$.

By items 1, 2, 3a and 3b, f is a surjective p-morphism from $\mathcal{G}_X$ to $\mathcal{F}$ and hence $\mathcal{F}$ is a p-morphic image of $\mathcal{G}_X$. $\Box$

From Lemmas 5.1.14 and 5.1.15 we get the following corollary.

Corollary 5.1.16 *Let $\mathcal{L}$ be a modal language with similarity type $\tau \supseteq \tau_B \cup \tau_H$, where $\tau_B = \{\Diamond\}$ and $\tau_H = \{\Diamond^H\}$ and let $|\Phi| \geq \omega$. Then the class of finite and transitive frames admits $\mathcal{L}[\Phi, \tau_H]$-description of generated subframes up to p-morphisms.*

From this, together with Theorem 5.1.12, we get the following corollary.

Corollary 5.1.17 *A frame class K that is closed under generated subframes and taking p-morphic images is* ML$[\Phi, \tau_H]$*-definable in restriction to the class of finite and transitive frames.*

We can easily get rid of the transitivity requirement by adding again the path quantifier A to the language, but the question of whether this holds without adding the path quantifier remains open.

Question 5.1.18 Is a frame class that is closed under generated subframes and taking p-morphic images definable in ML$[\Phi, \tau_H]$ in restriction to the class of finite frames?

5.2 Adding the path quantifier

In this section, A always refers to the path quantifier of the definition 2.4.2, but paths are taken with respect to the union of boss relations. Let us start with an example of what we can define with the path quantifier.

Example 5.2.1 Let $\tau = \tau_B \cup \tau_H$, where $\tau_B = \{\Diamond\}$ and $\tau_H = \{\Diamond^H\}$ and let $\Phi = \{p\}$. The class of strongly R-connected[1] frames is MLA$[\Phi, \tau_H]$-definable in restriction to the class of all frames.

Proof. Let $\mathcal{F} = \langle W, R\rangle$. We show that

$$\mathcal{F} \models_{\tau_H} \varphi \iff \mathcal{F} \text{ is strongly } R\text{-connected}$$

for $\varphi = (\Diamond^H p \to \neg A \neg p)$.

- $\Rightarrow$ Assume that $\mathcal{F} \models_{\tau_H} \varphi$. Assume on the contrary that $\mathcal{F}$ is not strongly R-connected. Now, there exist points $w, v \in W$ such that there is no R-path from w to v. Choose $H = \{(w, v)\}$ and $V(p) = \{v\}$. Let $\mathcal{M} = \langle W, R, H, V\rangle$. Now, $\mathcal{M}, w \models \Diamond^H p$, but $\mathcal{M}, w \not\models \neg A \neg p$. Hence $\mathcal{M} \not\models \varphi$, which is a contradiction as $\mathcal{M}$ is a τ_H-model based on $\mathcal{F}$.
- $\Leftarrow$ Assume that $\mathcal{F}$ is strongly R-connected. Choose arbitrary relation $H \subseteq W \times W$, valutation V and a state $w \in W$. Set $\mathcal{M} = \langle W, R, H, V\rangle$ and assume that $\mathcal{M}, w \models \Diamond^H p$. Then there exists v such that $(w, v) \in H$ and $v \in V(p)$. Because $\mathcal{F}$ is strongly R-connected, there exists an R-path from w to v. Hence $\mathcal{M}, w \models \neg A \neg p$. As H and V were arbitrary we get that $\mathcal{F} \models_{\tau_H} (\Diamond^H p \to \neg A \neg p)$. $\square$

Remark 5.2.2 Note that the property of being strongly R-connected is not definable in basic MLA without the helpers as this property is not closed under disjoint unions.

[1] A frame $\langle W, R\rangle$ is strongly R-connected, if for each $x \in W$ and $y \in W$ there exists an R-path from x to y.

Question 5.2.3 Is the property of being strongly R-connected definable without the path quantifier in $\mathrm{ML}[\Phi, \tau_H]$?

Next we generalize Jankov-Fine Formulas for MLA with helpers. In this definition and the following lemmas we assume that $\tau = \tau_B \cup \tau_H$, where $\tau_B = \{\Diamond\}$ and $\tau_H = \{\Diamond^H\}$.

Definition 5.2.4 Let $\mathcal{F} = \langle W, R\rangle$ be a finite τ_B-frame and $\Phi = \{p_s \mid s \in W\}$. Let $\mathrm{ML}[\tau, \Phi]$-formula φ be the conjunction of the following formulas.

(i) $\bigvee_{s\in W} p_s$,

(ii) $\bigwedge_{s,t\in W,\ s\neq t}(p_s \to \neg p_t)$,

(iii) $\bigwedge_{(s,t)\in R}(p_s \to \Diamond p_t)$,

(iv) $\bigwedge_{(s,t)\notin R}(p_s \to \neg\Diamond p_t)$.

We define $\psi^A_{\mathcal{F}}$ as $(\bigwedge_{s\in W}(\Diamond^H p_s)) \wedge (\Box^H A\varphi)$.

The proofs of the following lemmas are easy variations of the corresponding lemmas without the path quantifier.

Lemma 5.2.5 *The formula $\psi^A_{\mathcal{F}}$ is τ_H-satisfiable in $\mathcal{F}$.*

Lemma 5.2.6 *If $\psi^A_{\mathcal{F}}$ is τ_H-satisfiable in a finite frame $\mathcal{G}$, then $\mathcal{F}$ is a p-morphic image of some generated subframe on $\mathcal{G}$.*

Corollary 5.2.7 *Let $\tau \supseteq \tau_B \cup \tau_H$, where $\tau_B = \{\Diamond\}$ and $\tau_H = \{\Diamond^H\}$ and let $|\Phi| \geq \omega$. The class of finite frames admits* $\mathrm{MLA}[\Phi, \tau_H]$*-description of generated subframes up to p-morphisms.*

From this together with Theorem 5.1.12 we get the following corollary.

Corollary 5.2.8 *A frame class K that is closed under generated subframes and taking p-morphic images is* $\mathrm{MLA}[\tau_H]$*-definable in restriction to the class of finite frames.*

For the other direction we use proposition 2.6.2 (with note 2.6.3) and lemma 5.1.9 to conclude that τ_H-validity of MLA-formulas is preserved under generated subframes and p-morphic images, which gives us the following corollary.

Corollary 5.2.9 *A frame class K that is* $\mathrm{MLA}[\Phi, \tau_H]$*-definable is closed under generated subframes and taking p-morphic images.*

5.3 Assistants

There are several directions towards which we could develop the ideas presented in the previous sections of this chapter. We could for example

- consider existential quantification on some of the helpers instead of universal quantification, or

- insist helpers to have certain properties like being non-trivial, reflexive or transitive.

In this section we consider replacing helpers with familiar operators like global modality or path quantifier, but we treat them as kind of relational constants whose interpretation is somehow fixed with respect to the underlying frames.

Definition 5.3.1 Let modal similarity type τ be of the form $\tau_B \cup \tau_A$. If the operators in τ_A have fixed interpretation in the models we consider, then we call these operators *assistants*.

Interpretation of an assistant always depends somehow on the frames. We can divide assistants into two classes. We call an assistant

1. *boss-dependent*, if its interpretation depends on the boss-relation(s) on the frame, and

2. *boss-independent*, if its interpretation only depends on the universe of the frame.

The global modality would belong to the second class whereas for example the path quantifier or the transitive closure would be in the first class. In both classes, assistants are fixed in given a frame. For our own convenience, we add the interpreting relations for assistants into the models we use even though their interpretations are fixed.

Remark 5.3.2 Note that the concept of boss-independent assistants corresponds to the notion of build-in relations[2] in descriptive complexity theory.

Example 5.3.3 Let $\tau = \tau_B \cup \tau_A$, where $\tau_B = \{\Diamond\}$ and $\tau_A = \{\Diamond^A\}$. Now, a τ-formula φ is satisfiable in a τ_B-frame $\langle W, R\rangle$, if there exists a τ-model

$$\mathcal{M} = \langle W, R, A\rangle$$

and a state $w \in W$ such that $\mathcal{M}, w \models \varphi$. Note that the relation A is constant with respect to given W and/or R. For example, if $\Diamond^A$ happens to be the global diamond E, it would be the case that $A = W \times W$.

When we define bisimulations, we have to take assistants into account. Let us look at the familiar global diamond E as an example of this. We recall that

$$\mathcal{M}, w \models E\varphi \Leftrightarrow \exists u \in W : \mathcal{M}, u \models \varphi$$

for any $\mathcal{M} = \langle W, R, V\rangle$ and $w \in W$. Defining the corresponding concept of bisimulation was left as an exercise in [6].

Definition 5.3.4 Let $\mathcal{M} = \langle W, R, V\rangle$ and $\mathcal{M}' = \langle W', R', V'\rangle$ be Kripke models and $w \in W$, $w' \in W'$. Let $\tau_A = \{E\}$ and $\tau \supseteq \tau_A$. A τ-bisimulation $Z : \mathcal{M}, w \leftrightarroweq \mathcal{M}', w'$ is an *E-bisimulation* if it is total, that is

$$\forall v \in W : \exists v' \in W' : (v, v') \in Z$$

and

$$\forall v' \in W' : \exists v \in W : (v, v') \in Z.$$

We denote $\mathcal{M}, v \leftrightarroweq_E \mathcal{M}', v'$, if there exists is an E-bisimulation Z such that $(v, v') \in Z$.

It is easy to see that E-bisimilar states satisfy the same MLE-formulas.

Proposition 5.3.5 *If $Z : \mathcal{M}, w \leftrightarroweq_E \mathcal{M}', w'$, then $\mathcal{M}, w \equiv_{\mathrm{MLE}} \mathcal{M}', w'$.*

[2]Build-in relation is a relation that is assumed to be present in all models. Typical examples are linear order, successor relation or some arithmetical relations, see for example [22].

Proof. We only consider the case where $\varphi = E\psi$ and the claim holds for ψ in all states. If $\mathcal{M}, v \models E\psi$, there exits $u \in W : \mathcal{M}, u \models \psi$. Because Z is total, there exists $u' \in W'$ such that $(u, u') \in Z$. By induction hypothesis $\mathcal{M}', u' \models \psi$. Hence $\mathcal{M}', v' \models E\psi$. The other direction goes similarly. □

Surjective p-morphisms are total. Hence this does not effect the concept of p-morphic images and the preservation result (which essentially follows from [19]) is still achieved.

Proposition 5.3.6 MLE-*validity is preserved under p-morphic images.*

Proof. Let $\mathcal{F} = \langle W, R\rangle$, $\mathcal{G} = \langle W^{\mathcal{G}}, R^{\mathcal{G}}\rangle$ and let f be a surjective p-morphism $\mathcal{F} \longrightarrow \mathcal{G}$. Assume that $\mathcal{F} \models \varphi$ and assume on the contrary that $\mathcal{G} \not\models \varphi$. Now, there exists a model $\mathcal{N} = \langle W^{\mathcal{G}}, R^{\mathcal{G}}, V^{\mathcal{G}}\rangle$ based on $\mathcal{G}$ and a state $v' \in W^{\mathcal{G}}$ such that $\mathcal{N}, v' \not\models \varphi$. With the help of the surjective p-morphism f we can define a valuation V for $\mathcal{F}$ and a state $v \in W$ such that $f : \mathcal{M}, v \leftrightarrows_E \mathcal{N}, v'$, where $\mathcal{M} = \langle W, R, V\rangle$. By Proposition 5.3.5 $\mathcal{M}, v \not\models \varphi$. But this is a contradiction as $\mathcal{M}$ was a model based on $\mathcal{F}$ and $\mathcal{F} \models \varphi$. Hence $\mathcal{G} \models \varphi$. □

Goranko and Passy proved in [19] that an elementary frame class is MLE-definable iff it is closed under p-morphic images. With the help of generalized Jankov-Fine formulas we get a similar result for the class of finite frames even without the elementariness assumption.

Definition 5.3.7 Let $\mathcal{F} = \langle W, R\rangle$ be a finite Kripke-frame and $\Phi = \{p_s \mid s \in W\}$. Let φ be the conjunction of the following formulas

(i) $\bigvee_{s\in W} p_s$,

(ii) $\bigwedge_{s,t\in W, s\neq t}(p_s \rightarrow \neg p_t)$,

(iii) $\bigwedge_{(s,t)\in R}(p_s \rightarrow \Diamond p_t)$,

(iv) $\bigwedge_{(s,t)\notin R}(p_s \rightarrow \neg\Diamond p_t)$.

Now, $\psi^E_{\mathcal{F}}$ is defined as $(\bigwedge_{s\in W}(Ep_s)) \wedge (\neg E\neg\varphi)$.

The familiar results hold for these formulas. The proofs are standard.

Lemma 5.3.8 *The formula $\psi^E_{\mathcal{F}}$ is satisfiable in $\mathcal{F}$.*

Lemma 5.3.9 *If $\psi^E_{\mathcal{F}}$ is satisfiable in $\mathcal{G}$, then $\mathcal{F}$ is a p-morphic image of $\mathcal{G}$.*

Theorem 5.3.10 *A frame class K is* MLE*-definable in restriction to the class K_{fin} of finite frames, if it is closed under p-morphic images.*

Proof. Assume that K is closed under p-morphic images and let $\mathcal{F} \in K_{\text{fin}}$ be a frame such that $\mathcal{F} \models \Lambda_K$, where

$$\Lambda_K = \{\phi \in \text{MLE} \mid \forall \mathcal{G} \in K : \mathcal{G} \models \phi\}.$$

If $\psi^E_{\mathcal{F}}$ is not satisfiable in K, $K \models \neg\psi^E_{\mathcal{F}}$ and $\neg\psi^E_{\mathcal{F}} \in \Lambda_K$. Then $\mathcal{F} \models \neg\psi^E_{\mathcal{F}}$ which is a contradiction by Lemma 5.3.8. Hence there must be a frame $\mathcal{G} \in K$ such that $\psi^E_{\mathcal{F}}$ is satisfiable in $\mathcal{G}$. By Lemma 5.3.9 $\mathcal{F}$ is now a p-morphic image Therefore $\mathcal{F} \in K$. $\square$

A similar result due to de Rijke [28] (pointed to me by Antti Kuusisto) states that the complement D of the identity relation, defined by

$$\mathcal{M}, w \models D\varphi \Leftrightarrow \exists v \neq w : \mathcal{M}, v \models \varphi$$

takes us all the way to isomorphisms.

Theorem 5.3.11 *A class K of finite frames is* MLD*-definable iff it is closed under isomorphisms. [28, Proposition 4.3.]*

Question 5.3.12 Could we formulate similar results for generated subframes or disjoint unions, that is, can we find some natural extensions $\mathcal{L}_1$ and $\mathcal{L}_2$ of modal logic such that

- a class of frames is $\mathcal{L}_1$-definable iff it is closed under generated subframes
- a class of frames is $\mathcal{L}_2$-definable iff it is closed under disjoint unions.

Given the definition 5.1.6 of satisfiability, the only boss-independent assistants that come to question are the modal operators whose interpretation is the universal relation, the identity relation, the complement of identity relation and the empty relation. But if we adjust the definition with an invariance condition, we open a whole new branch of possibilities. Let us see first what we mean by invariance.

Definition 5.3.13 Let $\tau_A = \{\Diamond_1^A, \ldots, \Diamond_k^A\}$ and let K_A be a class of τ_A frames that is closed under isomorphisms. Let Φ be a set of proposition symbols and $\tau \supseteq \tau_A$ a modal similarity type. A τ-formula φ is K_A-*invariant* if for all (Φ, τ)-frames $\mathcal{F} = \langle W, R_1, \ldots, R_n, A_1, \ldots, A_k \rangle$ and $\mathcal{F}' = \langle W, R_1, \ldots, R_n, A_1', \ldots, A_k' \rangle$ it holds that

> if there exists model $\mathcal{M} = \langle \mathcal{F}, V \rangle$ and $w \in W$ such that $\mathcal{M}, w \models \varphi$, then there exists a model $\mathcal{M}' = \langle \mathcal{F}', V' \rangle$ and $w' \in W$ such that $\mathcal{M}', w' \models \varphi$

and τ_A-restrictions of $\mathcal{M}$ and $\mathcal{M}'$, namely the τ_A-frames $\langle W, A_1, \ldots A_k, \rangle$ and $\langle W, A_1', \ldots, A_k' \rangle$, belong to K_A.

Now, we define a concept of satisfiability with an invariance condition for assistants.

Definition 5.3.14 Let $\tau = \tau_B \cup \tau_A$ be a modal similarity type, Φ a set of proposition symbols and K_A a class of τ_A frames closed under isomorphisms. An ML$[\Phi, \tau]$-formula φ is K_A-*satisfiable* in a τ_B-frame $\mathcal{F} = \langle W, R_1, \ldots, R_n \rangle$, if φ is K_A-invariant and there exists model $\mathcal{M}$ based on $\mathcal{F}$ and $w \in W$ such that

$$\mathcal{M}, w \models \varphi$$

and the τ_A-restriction of $\mathcal{M}$ belongs to K_A.

With this definition we could allow boss-independent assistant to be for example a linear order or a successor relation, that is, any path with respect to $W \times W$ that visits every node $w \in W$ exactly once.

Example 5.3.15 Let $\tau = \tau_B \cup \tau_A$ be a modal similarity type, where $\tau_A = \{\Diamond_1, \Diamond_2\}$ and let $\Phi = \{p\}$. Let K_A be a class of τ_A-frames such that the first relation is a successor relation and the second relation is its inverse.

Let $\vartheta_1 = (\Box_2 \bot \to p)$, $\vartheta_2 = (p \to \Diamond_1 \neg p)$ and $\vartheta_3 = (\neg p \to (\Diamond_1 p \vee \Box_1 \bot))$. Let

$$\vartheta = \bigwedge_{n=0}^{k} \Box_1^n (\vartheta_1 \wedge \vartheta_2 \wedge \vartheta_3) \wedge \bigwedge_{n=0}^{k} \Box_2^n (\vartheta_1 \wedge \vartheta_2 \wedge \vartheta_3)$$

Now ϑ is K_A-satisfiable in a frame $\mathcal{F} = \langle W, R \rangle$, where $|W| \leq k$, if and only if $|W|$ is even.

Proof. Let $\mathcal{F} = \langle W, R \rangle$, where $|W| \leq k$. Let A_1 be any successor relation on W and A_2 its inverse. Assume that $|W|$ is even and define P and Q as follows:

1. First element of the successor relation A_1 belongs to P.
2. If s belongs to P, then its A_1-successor belongs to Q.
3. If t belongs to Q, then its A_1-successor belongs to P.
4. No element t belongs to both P and Q.

Clearly this can be done in a similar manner for any successor relation A'_1 and its inverse A'_2, whence the invariance condition follows. This definition gives sets P and Q such that

$$|W| \text{ is even } \iff l \in Q,$$

where l is the last element of the successor relation A_1. Then clearly ϑ_1, ϑ_2 and ϑ_3 hold in each state $w \in W$. Hence also ϑ holds in each state of W and we get that

$$\mathcal{F} \models \exists P \exists Q \exists x St_x(\vartheta)$$

that is ϑ is satisfied in the frame $\mathcal{F}$.

Assume then that ϑ is satisfied in the frame $\mathcal{F}$. As $|W| \leq k$ the boxes $\Box_1^n$ and $\Box_2^n$ guarantee that subformulas of ϑ hold in each state of $\mathcal{F}$. Then the formula ϑ_1 says that, the last element of A_2 belongs to P that is the first element of A_1 belong the P. Formulas ϑ_2 and ϑ_3 say that every second element belongs to P and every second element belongs to Q. The formula ϑ_3 says in addition that the last element of the successor relation A_1 belongs to Q. Now, $|W|$ has to be even as otherwise these conditions would be impossible to fill. Hence we have shown that ϑ is K_A-satisfiable in a frame $\mathcal{F} = \langle W, R \rangle$ such that $|W| \leq k$ if and only if $|W|$ is even.

Question 5.3.16 What could be done with only one assistant?

Remark 5.3.17 Note that we can get rid of the the size restriction by adding the global modality. On the other, hand global modality alone is not enough for defining the parity of a set W, as parity is not closed under p-morphic images.

Order-invariant logics have a central role in descriptive complexity theory. If we take assistants to be linear orders, we get a modal logical version of order-invariant logic.

Question 5.3.18 What is the expressive power of order-invariant modal logic?

Chapter 6

Conclusions

This thesis was divided into two parts. First part, consisting of chapters from 2 to 4, gave several model-theoretic characterizations of definable frame classes. This was achieved by modifying the setup in the original Goldblatt-Thomason theorem, namely by

1. restricting the frame classes we looked at to finite or image-finite frames
2. extending the modal language we used by path quantifier (which is definable also in μ-calculus and in infinitary modal logic) and by graded modalities.

Probably the most interesting part of this thesis is Chapter 5, which gave a new generalization of frame validity by universally quantifying away some of the relations of multi-modal logic. Some basic properties for this new notion of validity were explored, but many natural questions were left unaddressed due to limited time resources. Continuing this work by aswering questions like

- what happens if we consider definability with helpers in restriction to elementary frame classes,
- can we add expressive power by adding more helpers,
- can we separate ML[τ_H]-definability from MLA[τ_H]-definability or
- what is the expressive power of order-invariant modal logic

might offer a nice starting point for a researcher interested in developing further model theory of modal logic.

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