Position and polarity

- ▷ Position is any sequence of positive integers $a_1, \ldots, a_n$, where $n \geq 0$, written as $a_1.a_2.\ \cdots\ .a_n$.
- ▷ Empty position, denoted by ε: when $n = 0$.
- ▷ Polarity: one of the values $-1, 0, 1$.
- ▷ Position in formula, subformula at position. Notation: $A|_\pi$.
- ▷ Polarity of subformula at a position. Notation: $pol(A|_\pi)$.

Monotonic replacement

Monotonic Replacement Lemma. Let A, B, B' be formulas, I be an interpretation, $A|_\pi = B$, $pol(A|_\pi) = 1$ and $I \models B \rightarrow B'$. Let the formula A' be obtained from A by replacing the occurrence of B at the position π by B'. Then $I \models A \rightarrow A'$.

Monotonic Replacement Theorem. Let A, B, B' be formulas such that $B \rightarrow B'$ is valid. Let a formula A' be obtained from A by replacing one or more positive occurrences of B by B'. Then $A \rightarrow A'$ is valid.

Pure Atom

Pure Atom Lemma. Let p be pure in A. Let $I \models A$ and I' be obtained from I as follows:

$$I'(q) \rightleftharpoons \begin{cases} 1, & \text{if } p = q \text{ and } p \text{ occurs in } A \text{ only positively;} \\ 0, & \text{if } p = q \text{ and } p \text{ occurs in } A \text{ only negatively;} \\ I(q), & \text{otherwise.} \end{cases}$$

Then $I' \models A$.

Pure Atom Theorem. Let an atom p has only positive (respectively, only negative) occurrences in A. Then A is satisfiable if and only if so is $A_p^\top$ (respectively, $A_p^\perp$).