Evaluating a formula, again

$$\top \wedge \ldots \wedge \top \;\Rightarrow\; \top$$

$$\bot \wedge A_1 \wedge \ldots \wedge A_n \;\Rightarrow\; \bot$$

$$\top \vee A_1 \vee \ldots \vee A_n \;\Rightarrow\; \top$$

$$\bot \vee \ldots \vee \bot \;\Rightarrow\; \bot$$

$$\neg\top \;\Rightarrow\; \bot$$

$$\neg\bot \;\Rightarrow\; \top$$

$$A \rightarrow \top \;\Rightarrow\; \top$$

$$\bot \rightarrow A \;\Rightarrow\; \top$$

$$\top \rightarrow \bot \;\Rightarrow\; \bot$$

$$\top \leftrightarrow \top \;\Rightarrow\; \top$$

$$\top \leftrightarrow \bot \;\Rightarrow\; \bot$$

$$\bot \leftrightarrow \top \;\Rightarrow\; \bot$$

$$\bot \leftrightarrow \bot \;\Rightarrow\; \top$$

Algorithm for evaluating a formula

<u>procedure</u> $evaluate(G, I)$

<u>input</u>: formula G, interpretation I

<u>output</u>: a boolean value

<u>begin</u>

  <u>forall</u> atoms p occurring in G

    <u>if</u> $I(p) = 1$

      <u>then</u> replace all occurrences of p in G by $\top$;

      <u>else</u> replace all occurrences of p in G by $\bot$;

  rewrite G into a normal form using the rewrite rules

  <u>if</u> $G = \top$ <u>then</u> <u>return</u> 1 <u>else</u> <u>return</u> 0

<u>end</u>

Russian spy puzzle

There are three persons: Stirlitz, Müller, and Eismann. It is known that exactly one of them is Russian, while the other two are Germans. Moreover, every Russian must be a spy.

When Stirlitz meets Müller in a corridor, he makes the following joke: “you know, Müller, you are as German as I am Russian”. It is known that Stirlitz always says the truth when he is joking.

We have to establish that Eismann is not a Russian spy.

Formalisation

$$(1)\quad (RS \wedge GM \wedge GE) \vee (GS \wedge RM \wedge GE) \vee (GS \wedge GM \wedge RE)$$

$$(2)\quad (RS \longrightarrow SS) \wedge (RM \longrightarrow SM) \wedge (RE \longrightarrow SE)$$

$$(3)\quad RS \leftrightarrow GM$$

$$(4)\quad (RS \leftrightarrow \neg GS)$$

$$(5)\quad (RM \leftrightarrow \neg GM)$$

$$(6)\quad (RE \leftrightarrow \neg GE)$$

$$(7)\quad RE \wedge SE$$

Truth tables

	subformula	I_0	I_1	I_2	I_3	I_4	I_5	I_6	I_7
1	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	1	1	1	1	1	1	1	1
2	$p \rightarrow r$	1	1	1	1	0	1	0	1
3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	1	1	1	1	0	0	0	1
4	$p \wedge q \rightarrow r$	1	1	1	1	1	1	0	1
5	$p \rightarrow q$	1	1	1	1	0	0	1	1
6	$p \wedge q$	0	0	0	0	0	0	1	1
7	p p p	0	0	0	0	1	1	1	1
8	q q	0	0	1	1	0	0	1	1
9	r r	0	1	0	1	0	1	0	1

Observation

	subformula	I_0	I_1	I_2	I_3	I_4	I_5	I_6	I_7
1	$(p \to q) \land (p \land q \to r) \to (p \to r)$	1	1	1	1	1	1	1	1
2	$p \to r$	1	1	1	1	0	1	0	1
3	$(p \to q) \land (p \land q \to r)$	1	1	1	1	0	0	0	1
4	$p \land q \to r$	1	1	1	1	1	1	0	1
5	$p \to q$	1	1	1	1	0	0	1	1
6	$p \land q$	0	0	0	0	0	0	1	1
7	p p p	0	0	0	0	1	1	1	1
8	q q	0	0	1	1	0	0	1	1
9	r r	0	1	0	1	0	1	0	1

Compact truth table

	subformula	I_0	I_1	I_2	I_3
1	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$	1	1	1	1
2	$p \rightarrow r$	1	0	0	1
3	$(p \rightarrow q) \wedge (p \wedge q \rightarrow r)$	1	0	0	
4	$p \wedge q \rightarrow r$	1	1	0	1
5	$p \rightarrow q$	1	0	1	
6	$p \wedge q$	0	0	1	
7	$p \quad p \quad p$	0	1	1	
8	$q \quad q$		0	1	
9	$r \quad r$	0	0	0	1

Note: depends on the order of atoms!

Partial interpretations

- Partial interpretation I: partial mapping from relation symbols to boolean values.
- A is valid (respectively, satisfiable) in I if A is valid in every (respectively, satisfiable in some) interpretation extending I.
- Notation $I \models A$: means A is valid in I.
- A and B are equivalent in I, if $I \models A \leftrightarrow B$.

Splitting: the theoretical basis

$A_p^\perp$ and $A_p^\top$: the formulas obtained by replacing in A all occurrences of p by $\perp$ and $\top$, respectively.

Lemma. Let p be an atom, A be a formula, and I be a partial interpretation.

1. If $I \models \neg p$, then A is equivalent to $A_p^\perp$ in I.
2. If $I \models p$, then A is equivalent to $A_p^\top$ in I. ❑

Simplification rules for $\top$ *and* $\bot$

Simplification rules for $\top$:

$$\neg\top \;\Rightarrow\; \bot$$

$$\top \wedge A_1 \wedge \ldots \wedge A_n \;\Rightarrow\; A_1 \wedge \ldots \wedge A_n$$

$$\top \vee A_1 \vee \ldots \vee A_n \;\Rightarrow\; \top$$

$$A \rightarrow \top \;\Rightarrow\; \top \qquad \top \rightarrow A \;\Rightarrow\; A$$

$$A \leftrightarrow \top \;\Rightarrow\; A \qquad \top \leftrightarrow A \;\Rightarrow\; A$$

Simplification rules for $\bot$:

$$\neg\bot \;\Rightarrow\; \top$$

$$\bot \wedge A_1 \wedge \ldots \wedge A_n \;\Rightarrow\; \bot$$

$$\bot \vee A_1 \vee \ldots \vee A_n \;\Rightarrow\; A_1 \vee \ldots \vee A_n$$

$$A \rightarrow \bot \;\Rightarrow\; \neg A \qquad \bot \rightarrow A \;\Rightarrow\; \top$$

$$A \leftrightarrow \bot \;\Rightarrow\; \neg A \qquad \bot \leftrightarrow A \;\Rightarrow\; \neg A$$

Splitting algorithm

procedure $split(G)$

parameters: functions $select\_atom,\ select\_pair$

input: formula G

output: partial interpretation I such that $I \models G$ or “unsatisfiable”

Splitting algorithm, continued

begin

G := $simplify(G)$

if ($G = \top$) return $\emptyset$ (the empty partial interpretation)

let N be a set of pairs (partial interpretation,formula)

N := $\{(\emptyset, G)\}$

while ($N \neq \emptyset$) do

(I, A) := $select\_pair(N)$; N := $N - (I, A)$

p := $select\_atom(I, A)$; A_1 := $simplify(A_p^\perp)$; I_1 := $I + (p \mapsto 0)$

if $A_1 = \top$ then return I_1

if $A_1 \neq \perp$ then N := $N + (I_1, A_1)$

I_2 := $I + (p \mapsto 1)$; A_2 := $simplify(A_p^\top)$

if $A_2 = \top$ then return I_2

if $A_2 \neq \perp$ then N := $N + (I_2, A_2)$

return “unsatisfiable”

end