Evaluating a formula

$$I_{10} \models \forall p \exists q (p \leftrightarrow q) \Leftrightarrow \boxed{\begin{array}{ll} I_{00} \models \exists q (p \leftrightarrow q) & \\ & \text{and} \\ I_{10} \models \exists q (p \leftrightarrow q) & \end{array}} \Leftrightarrow \boxed{\begin{array}{ll} \boxed{\begin{array}{ll} I_{00} \models p \leftrightarrow q & \\ & \text{or} \\ I_{01} \models p \leftrightarrow q & \end{array}} & \\ & \text{and} \\ \boxed{\begin{array}{ll} I_{10} \models p \leftrightarrow q & \\ & \text{or} \\ I_{11} \models p \leftrightarrow q & \end{array}} & \end{array}}$$

Evaluating a formula

$$I_{10} \models \forall p \exists q (p \leftrightarrow q)$$

$$\wedge$$

$$I_{00} \models \exists q (p \leftrightarrow q) \qquad I_{10} \models \exists q (p \leftrightarrow q)$$

$$\vee \qquad \vee$$

$$I_{00} \models p \leftrightarrow q \qquad I_{01} \models p \leftrightarrow q \qquad I_{10} \models p \leftrightarrow q \qquad I_{11} \models p \leftrightarrow q$$

Truth and satisfiability

Lemma. Let for all free variables p of A we have $I_1(p) = I_2(p)$. Then $I_1 \models A$ if and only if $I_2 \models A$.

Lemma. For every interpretation I and closed formula A the following propositions are equivalent: (i) $I \models A$; (ii) A is satisfiable; and (iii) A is valid.

Lemma. Let A be a formula with free variables $p_1, \ldots, p_n$. Then A is satisfiable (respectively, valid) if and only if the formula $\exists p_1 \ldots \exists p_n A$ (respectively, $\forall p_1 \ldots \forall p_n A$) is satisfiable.

Monotonic and equivalent replacement

Theorem. Let A_1 be a subformula of a formula B_1 and $A_1 \to A_2$ be valid. Let B_2 be obtained from B_1 by replacement of one or more positive (respectively, negative) occurrences of A_1 by A_2. Then $B_1 \to B_2$ (respectively $B_2 \to B_1$).

Theorem. Let A_1 be a subformula of a formula B_1 and $A_1 \equiv A_2$. Let B_2 be obtained from B_1 by replacement of one or more occurrences of A_1 by A_2. Then $B_1 \equiv B_2$.

Renaming bound variables

Definition. We say that a formula A' is obtained from a formula A by renaming bound variables if A' can be obtained from A by a sequence of the following steps. Let $\exists\!\!\!\forall p B_1$ be a subformula of A and q be a variable not occurring in A. Define B_2 as the formula obtained from B_1 by replacement of all free occurrences of p by q. Replace $\exists\!\!\!\forall p B_1$ in A by $\exists\!\!\!\forall q B_2$.

Definition. A formula A is called rectified if (i) no variable appears both free and bound in A, and (ii) for every variable p, the formula A contains at most one occurrence of quantifiers $\forall p$ and $\exists p$ binding p.

Prenex form, CNF

Definition. Prenex formula has the form $\exists\!\!\!/\forall p_1 \ldots \exists\!\!\!/\forall p_n B$, where B is a propositional formula. A formula A is a prenex form of a formula B if A is prenex and $A \equiv B$.

Definition. A quantified boolean formula A is in CNF, if it is either $\bot$, or $\top$, or has the form $\exists\!\!\!/\forall_1 p_1 \ldots \exists\!\!\!/\forall_n p_n (C_1 \wedge \ldots \wedge C_m)$, where $C_1, \ldots, C_m$ are clauses.