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The ground-negative fragment of first-order logic is Π_2^p-complete

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Abstract

We prove that for a natural class of first-order formulas the validity problem is Π_2^p-complete.

Section 1

Introduction

Traditionally, the decision problem for first-order logic is studied for classes of formulas characterized by the quantifier prefix and/or signature restrictions (see e.g. the recent book of Börger Grädel and Gurevich [1]). Kozen [4] considered the decision problem for so-called positive first-order formulas. Recently, in connection with model building, some decidability results have been obtained for classes of formulas characterized in terms of the clause form of the formula (e.g. Fermüller and Leitsch [2, 3]).

In this paper we study the complexity of the validity problem for the class of *ground-negative* formulas which is a natural generalization of a class of formulas considered by Kozen [4]. We show that this problem is Π_2^p-complete both for logics with and without equality, which is a quite unusual complexity class for natural fragments of first-order logic.

For technical convenience, we introduce two nullary logical connectives $\top$ and $\bot$, interpreted respectively as a valid formula and as a formula false in all models. A formula φ is called *reduced* if φ is $\top$, or $\bot$, or φ contains neither $\top$ nor $\bot$. We denote the equality predicate by $\simeq$. The symbol $\rightleftharpoons$ means "equal by definition".

We call atomic formulas *atoms.* A *literal* is either an atom or its negation. A literal or term is *ground* if it has no variables. A *clause* is any disjunction of literals. A *sentence* is a formula without free occurrences of variables.

Definition 1.1 (positive and ground-negative formulas)
A formula φ is *positive* if and only if every atomic subformula of φ is positive. A formula φ is called *ground-negative* if every negative atomic subformula of φ is ground.

Obviously, any positive formula is also ground-negative.

We call a $(\wedge, \vee, \exists)$*-formula* any formula constructed from atomic formulas using $(\wedge, \vee, \exists)$. We call a $(L, \wedge, \vee, \exists)$*-formula* any formula constructed from literals using $(\wedge, \vee, \exists)$. We shall use the same notion for other collections of logical connectives, for example, we can speak about $(\wedge, \vee)$-formulas. By using standard transformations into negayion normal form and Skolemization we can prove the following statement.

Lemma 1.2 Given any ground-negative formula φ, one can construct in polynomial time a ground-negative $(L, \wedge, \vee, \exists)$-formula φ' in Skolem negation normal form such that φ is valid if and only if φ' is valid.

The problem dual to the validity problem of ground-negative formulas is the satisfiability problem for ground-positive formulas:

Definition 1.3 (ground-positive formula)
A formula φ is called *ground-positive* if every positive atomic subformula of φ is ground.

In this paper we shall also consider the satisfiability problem for ground-positive clauses.
We shall use several statements proven by Kozen [4].

Lemma 1.4 (Kozen [4], theorem 9)
The following problem is **NP**-complete for logics with equality. Given a conjunction $\mathcal{L}$ of ground literals and a positive formula φ, is the formula $\mathcal{L} \supset \varphi$ valid?

It is easy to prove that this problem is also **NP**-complete for logic without equality.

Section 2

Inclusion in Π_2^p

Let $\mathcal{L}$ be a conjunction of ground literals and φ be a $(L, \wedge, \vee, \exists)$-formula. We define the *set of transformations* $R_{\mathcal{L}}$ as the set consisting of transformations (2.1)–(2.14) given below:

$$
\begin{array}{rcll}
\neg\top & \Rightarrow & \bot & (2.1)\\
\neg\bot & \Rightarrow & \top & (2.2)\\
\top \wedge \varphi & \Rightarrow & \varphi & (2.3)\\
\varphi \wedge \top & \Rightarrow & \varphi & (2.4)\\
\bot \wedge \varphi & \Rightarrow & \bot & (2.5)\\
\varphi \wedge \bot & \Rightarrow & \bot & (2.6)\\
\top \vee \varphi & \Rightarrow & \top & (2.7)
\end{array}
\qquad
\begin{array}{rcll}
\varphi \vee \top & \Rightarrow & \top & (2.8)\\
\bot \vee \varphi & \Rightarrow & \varphi & (2.9)\\
\varphi \vee \bot & \Rightarrow & \varphi & (2.10)\\
\exists x \top & \Rightarrow & \top & (2.11)\\
\exists x \bot & \Rightarrow & \bot & (2.12)\\
A & \Rightarrow & \top, \quad \text{if } A \text{ is a positive atom in } \mathcal{L} & (2.13)\\
A & \Rightarrow & \bot, \quad \text{if } A \text{ is a negative atom in } \mathcal{L} & (2.14)
\end{array}
$$

These transformations replace any subformula of φ to the left of $\Rightarrow$ by the corresponding subformula to the right. We call a *reduced form of* φ *w.r.t.* $\mathcal{L}$, denoted $\varphi\,|_{\mathcal{L}}$, any formula φ' obtained from φ by a sequence of transformations such that no transformation is applicable to φ'.

It is easy to see that the following properties hold for $\varphi\,|_{\mathcal{L}}$:

Lemma 2.1 Let φ be a ground-negative $(L, \wedge, \vee, \exists)$-formula and the set of literals $\mathcal{L}$ be such that for every negative atom A occurring in φ either A or $\neg A$ belong to $\mathcal{L}$. Then (i) $\varphi\,|_{\mathcal{L}}$ is a positive reduced formula; and (ii) $\mathcal{L} \supset \varphi$ is equivalent to $\mathcal{L} \supset (\varphi\,|_{\mathcal{L}})$.

Let φ be any ground-negative $(L, \wedge, \vee, \exists)$-formula and $A_1, \ldots, A_n$ be all negative atomic subformulas in φ in the order of their occurrence in φ. Denote by Π the set of mappings from $\{1, \ldots, n\}$ to $\{0, 1\}$. For any $\pi \in \Pi$ introduce the conjunction of ground literals $\mathcal{L}_\pi$ in the following way:

$$\mathcal{L}_\pi \rightleftharpoons \Big(\bigwedge_{i \in \{i \mid \pi(i)=0\}} A_i\Big) \wedge \Big(\bigwedge_{i \in \{i \mid \pi(i)=1\}} \neg A_i\Big)$$

Note that the formula $\bigvee_{\pi \in \Pi} \mathcal{L}_\pi$ is valid.

Lemma 2.2 The formula φ is equivalent to $\bigwedge_{\pi \in \Pi}(\mathcal{L}_\pi \supset \varphi)$.

Proof. It is easy to check that $\bigwedge_{\pi \in \Pi}(\mathcal{L}_\pi \supset \varphi)$ is equivalent to $(\bigvee_{\pi \in \Pi} \mathcal{L}_\pi) \supset \varphi$. Then use the fact that $\bigvee_{\pi \in \Pi} \mathcal{L}_\pi$ is valid. □

Lemma 2.3 The validity problem for ground-negative formulas with equality is in Π_2^p.

Proof. Let φ be a ground-negative formula. By Lemma 1.2, we can assume that φ is a $(L, \wedge, \vee, \exists)$-formula. Let $\mathcal{L}$, Π and $\mathcal{L}_\pi$ be defined as above. Note the following.

1. φ is valid if and only if for every $\pi \in \Pi$ the formula $\mathcal{L}_\pi \supset \varphi$ is valid (Lemma 2.2).
2. The following problem is in **NP**: given φ and π, is $\mathcal{L}_\pi \supset \varphi$ valid? Indeed, by Lemma 2.1 $\mathcal{L}_\pi \supset \varphi$ is valid if and only if $\mathcal{L}_\pi \supset (\varphi\,|_{\mathcal{L}})$ is valid. The formula $\varphi\,|_{\mathcal{L}}$ is positive (Lemma 2.1). Hence, the validity problem for formulas $\mathcal{L}_\pi \supset (\varphi\,|_{\mathcal{L}})$ is in **NP** (Lemma 1.4).
3. The size of π (e.g. written as a sequence of 0s and 1s of length n) is bounded by a polynomial in the size of φ.

According to a standard characterization of Π_2^p (see e.g. Papadimitriou [5]) in terms of polynomial-size witnesses we have that the validity problem for ground-negative formulas is in Π_2^p. $\square$

Of course, this lemma also holds for formulas without equality.

Section 3

Π_2^p-hardness

In this section we prove that the validity problem for ground-negative formulas is Π_2^p-hard. We shall use a reduction from the QBF_2 problem.

We call a *quantified boolean formula* any formula made of propositional variables, boolean connectives $\neg$, $\wedge$ and quantifiers $\forall x$ and $\exists x$, where x is a propositional variable. The *truth assignment* for a set of variables X is any mapping from X to the set of *boolean values* $\{\mathfrak{t}, \mathfrak{f}\}$.

Definition 3.1 (problem QBF_2)
The QBF_2 is the following problem. Given a boolean formula G with variables divided in two sets X, Y, is there for every truth assignment for X a truth assignment for Y such that G is satisfied by the overall truth assignment?

If $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$, we denote the corresponding instance of QBF_2 by the quantified boolean formula $\forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m G$. If this instance is true we say that this quantified boolean formula is valid.

The following fact is well-known (see e.g. Papadimitriou [5]).

Lemma 3.2 QBF_2 is Π_2^p-complete.

Let $\mathcal{A}$ be a conjunction of ground atoms without equality in a signature σ having at least one constant. The *least Herbrand model of $\mathcal{A}$ in σ*, denoted $\mathfrak{M}_{\mathcal{A}}$, is defined as follows. The domain of $\mathfrak{M}$ is the set of ground terms of σ. The interpretation of terms is defined in such a way that any term r is interpreted by r. The only atoms true in $\mathfrak{M}$ are atoms in $\mathcal{A}$.

Lemma 3.3 (Kozen [4], lemma 4)
For any $(\wedge, \vee, \exists)$-sentence φ the sentence $\mathcal{A} \supset \varphi$ is valid if and only if $\mathfrak{M}_{\mathcal{A}} \models \varphi$.[1]

Let σ be the signature consisting of three predicates *and*, not_1 and not_2 and a countably infinite number of constants $\mathfrak{t}, \mathfrak{f}, c_1, c_2, \ldots$. Note that we denote two constants of this signature in the same way as truth values. We shall define a reduction of QBF_2 to the validity problem for ground-negative formulas without equality in the signature σ.

Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be two countably infinite sequences of variables. For any quantified boolean formula $G = \forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m G'$, where G' is a boolean formula, and any term s,

[1] Kozen proves this statement for arbitrary positive sentences φ, but he uses signatures having function symbols not occurring in $\mathcal{A}$ or φ. His proof remains valid for our formulation.

we define a ground-negative formula G^s called *the s-translation of G*. For technical reasons, we assume that every occurrence of each propositional variable $z \in \{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ in G is in a subformula $\neg z$ (this can be done by replacing z by $\neg\neg z$). The translation is defined as follows:

$$
\begin{aligned}
(\forall x_i F)^s &\rightleftharpoons not_2(c_i, \mathfrak{f}) \vee not_2(c_i, \mathfrak{t}) \supset F^s && (3.1)\\
(\exists y_j F)^s &\rightleftharpoons \exists y_j (F^s) && (3.2)\\
(F_1 \wedge F_2)^s &\rightleftharpoons \exists z_1 z_2 (F_1^{z_1} \wedge F_2^{z_2} \wedge and(z_1, z_2, s)) && (3.3)\\
(\neg F)^s &\rightleftharpoons \exists z (F^z \wedge not_1(z, s)), \quad \text{where } F \text{ is not a variable} && (3.4)\\
(\neg x_i)^s &\rightleftharpoons not_2(c_i, s) && (3.5)\\
(\neg y_i)^s &\rightleftharpoons not_1(y_i, s) && (3.6)
\end{aligned}
$$

Here z, z_1, z_2 are new variables different from all x_i and y_j. It is easy to see that G^s is a ground-negative formula.

Let τ be any truth assignment for $x_1, \ldots, x_n$. Define the conjunction of atoms $\mathcal{A}_\tau$ as follows:

$$
\begin{aligned}
\mathcal{A}_\tau \rightleftharpoons\ & \textstyle\bigwedge_{i \in \{i \mid \tau(x_i) = \mathfrak{t}\}} not_2(c_i, \mathfrak{f}) \wedge \bigwedge_{i \in \{i \mid \tau(x_i) = \mathfrak{f}\}} not_2(c_i, \mathfrak{t}) \wedge \\
& and(\mathfrak{t}, \mathfrak{t}, \mathfrak{t}) \wedge and(\mathfrak{t}, \mathfrak{f}, \mathfrak{f}) \wedge and(\mathfrak{f}, \mathfrak{t}, \mathfrak{f}) \wedge and(\mathfrak{f}, \mathfrak{f}, \mathfrak{f}) \wedge not_1(\mathfrak{t}, \mathfrak{f}) \wedge not_1(\mathfrak{f}, \mathfrak{t}).
\end{aligned}
$$

and let $\mathfrak{M}_\tau$ be the least Herbrand model of $\mathcal{A}_\tau$. As one can see, the formula $\mathcal{A}_\tau$ consists of two parts. The first part "encodes" the truth assignment τ, while the second part "encodes" the truth tables for $\wedge$ and $\neg$.

Lemma 3.4 Let $G(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be a boolean formula with variables $x_1, \ldots, x_n, y_1, \ldots, y_m$ and $\varphi(z, y_1, \ldots, y_m)$ is G^z. Let τ be a truth assignment for $x_1, \ldots, x_n$ and $s, s_1, \ldots, s_m$ be a sequence of truth values. The following conditions are equivalent:

1. $G(\tau(x_1), \ldots, \tau(x_n), s_1, \ldots, s_m)$ evaluates to s;
2. $\mathfrak{M}_\tau \models \varphi(s, s_1, \ldots, s_m)$.

In addition, if for some ground terms $r, r_1, \ldots, r_m$ we have $\mathfrak{M}_\tau \models \varphi(r, r_1, \ldots, r_m)$, then r and each r_i are either $\mathfrak{t}$ or $\mathfrak{f}$.

Proof. The proof uses induction on G.

1. $G = \neg x_i$. In this case G^s is $not_2(c_i, s)$. By the definitions we have $\mathfrak{M}_\tau \models not_2(c_i, \mathfrak{t})$ if and only if $\tau(x_i) = \mathfrak{f}$. Similarly, we have $\mathfrak{M}_\tau \models not_2(c_i, \mathfrak{f})$ if and only if $\tau(x_i) = \mathfrak{t}$.

   Obviously, if $\mathfrak{M}_\tau \models not_2(c_i, r)$, then r is either $\mathfrak{t}$ or $\mathfrak{f}$.

2. $G = \neg y_i$. In this case G^s is $not_1(y_i, s)$. We consider the case $s = \mathfrak{t}$, the other case is similar. By the definition of $\mathfrak{M}_\tau$ we have $\mathfrak{M}_\tau \models not_1(s_i, \mathfrak{t})$ if and only if $s_i = \mathfrak{f}$. Obviously, this is equivalent to the statement that $\neg s_i$ evaluates to $\mathfrak{t}$.

   Obviously, if $\mathfrak{M}_\tau \models not_2(r_i, r)$, then r and r_i are either $\mathfrak{t}$ or $\mathfrak{f}$.

3. $G = \neg G'$, where G' is not a propositional variable. We consider the case $s = \mathfrak{t}$, the other case is similar. Denote $(G')^z$ by $\varphi'(z, y_1, \ldots, y_m)$. Then $G^{\mathfrak{t}} = \exists z (\varphi'(z, y_1, \ldots, y_m) \wedge not_1(z, \mathfrak{t}))$.

$1 \Rightarrow 2$ Suppose that $G(\tau(x_1),\ldots,\tau(x_n),s_1,\ldots,s_m)$ evaluates to $\mathfrak{t}$. Then the boolean formula $G'(\tau(x_1),\ldots,\tau(x_n),s_1,\ldots,s_m)$ evaluates to $\mathfrak{f}$. By the induction hypothesis we have $\mathfrak{M}_\tau \models \varphi'(\mathfrak{f},s_1,\ldots,s_m)$. By the definition of $\mathfrak{M}_\tau$ we have $\mathfrak{M}_\tau \models not_1(\mathfrak{f},\mathfrak{t})$. Hence, we have $\mathfrak{M}_\tau \models \exists z(\varphi'(z,s_1,\ldots,s_m) \wedge not_1(z,\mathfrak{t}))$, i.e. $\mathfrak{M}_\tau \models G^{\mathfrak{t}}$.

$2 \Rightarrow 1$ Suppose that $\mathfrak{M} \models G^{\mathfrak{t}}$, i.e. $\mathfrak{M}_\tau \models \exists z(\varphi'(z,s_1,\ldots,s_m) \wedge not_1(z,\mathfrak{t}))$. Hence, there is a ground term r such that $\mathfrak{M}_\tau \models \varphi'(r,s_1,\ldots,s_m) \wedge not_1(r,\mathfrak{t})$. Since $\mathfrak{M}_\tau \models not_1(r,\mathfrak{t})$, we have $r=\mathfrak{f}$. Then $\mathfrak{M}_\tau \models \varphi'(\mathfrak{f},s_1,\ldots,s_m)$. By the induction hypothesis, the boolean formula $G'(\tau(x_1),\ldots,\tau(x_n),s_1,\ldots,s_m)$ evaluates to $\mathfrak{f}$. Then $G(\tau(x_1),\ldots,\tau(x_n),s_1,\ldots,s_m)$ evaluates to $\mathfrak{t}$.

Suppose $\mathfrak{M} \models G^r$, i.e. $\mathfrak{M}_\tau \models \exists z(\varphi'(z,r_1,\ldots,r_m) \wedge not_1(z,r))$. Then for some ground term r' we have $\mathfrak{M}_\tau \models \varphi'(r',r_1,\ldots,r_m) \wedge not_1(r',r)$. By the induction hypothesis, each r_i is either $\mathfrak{t}$ or $\mathfrak{f}$. By the definition of $\mathfrak{M}_\tau$ we get that r is either $\mathfrak{t}$ or $\mathfrak{f}$.

4. $G = G_1 \wedge G_2$. This case is similar to the previous one, but using properties of *and* on $\mathfrak{M}_\tau$ instead of not_1. □

Lemma 3.5 Let $G(x_1,\ldots,x_n,y_1,\ldots,y_m)$ be a boolean formula with variables $x_1,\ldots,x_n,y_1,\ldots,y_m$ and $\varphi(z,y_1,\ldots,y_m) \rightleftharpoons G^z$. Let τ be a truth assignment for $x_1,\ldots,x_n$. The following conditions are equivalent:

1. there is a truth assignment ρ for $y_1,\ldots,y_m$ so that $G(x_1,\ldots,x_n,y_1,\ldots,y_m)$ evaluates to $\mathfrak{t}$ under the truth assignment combining τ and ρ.
2. $\mathcal{A}_\tau \supset \exists y_1 \ldots \exists y_m \varphi(\mathfrak{t},y_1,\ldots,y_m)$ is valid.

Proof. By Lemma 3.3, the formula $\mathcal{A}_\tau \supset \exists y_1 \ldots \exists y_m \varphi(\mathfrak{t},y_1,\ldots,y_m)$ is valid if and only if $\mathfrak{M}_\tau \models \exists y_1 \ldots \exists y_m \varphi(\mathfrak{t},y_1,\ldots,y_m)$. So we shall consider truth of $\exists y_1 \ldots \exists y_m \varphi(\mathfrak{t},y_1,\ldots,y_m)$ in $\mathfrak{M}_\tau$ instead of validity of $\mathcal{A}_\tau \supset \exists y_1 \ldots \exists y_m \varphi(\mathfrak{t},y_1,\ldots,y_m)$.

$\Rightarrow$ Suppose that there is a truth assignment ρ for $y_1,\ldots,y_m$ so that $G(x_1,\ldots,x_n,y_1,\ldots,y_m)$ evaluates to $\mathfrak{t}$ under the truth assignment combining τ and ρ. By Lemma 3.4, we have $\mathfrak{M}_\tau \models \varphi(\mathfrak{t},\rho(y_1),\ldots,\rho(y_m))$. Hence $\mathfrak{M}_\tau \models \exists y_1 \ldots \exists y_m \varphi(\mathfrak{t},y_1,\ldots,y_m)$.

$\Leftarrow$ Suppose $\mathfrak{M}_\tau \models \exists y_1 \ldots \exists y_m \varphi(\mathfrak{t},y_1,\ldots,y_m)$. Then there exist ground terms $s_1,\ldots,s_m$ such that $\mathfrak{M}_\tau \models \varphi(\mathfrak{t},s_1,\ldots,s_m)$. By Lemma 3.4, each s_i is either $\mathfrak{t}$ or $\mathfrak{f}$. Define the evaluation ρ for variables $y_1,\ldots,y_m$ by $\rho(y_i)=s_i$. By Lemma 3.4, $G(x_1,\ldots,x_n,y_1,\ldots,y_m)$ evaluates to $\mathfrak{t}$ under the truth assignment combining τ and ρ. □

Lemma 3.6 Let $G = \forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m G'$ be an instance of QBF_2. Then G is valid if and only if $G^{\mathfrak{t}}$ is valid.

Proof. The formula $G^{\mathfrak{t}}$ is

$$\bigwedge_{i \in \{1,\ldots,n\}} (not_2(c_i,\mathfrak{f}) \vee not_2(c_i,\mathfrak{t})) \supset \exists y_1 \ldots \exists y_m \varphi(\mathfrak{t},y_1,\ldots,y_m).$$

Denote by $\mathcal{T}$ the set of all truth assignments for $x_1, \ldots, x_n$. It is easy to see that the above formula is equivalent to

$$(\bigvee_{\tau \in \mathcal{T}} \mathcal{A}_\tau) \supset \exists y_1 \ldots \exists y_m \varphi(\mathfrak{t}, y_1, \ldots, y_m).$$

In turn, this formula is equivalent to

$$\bigwedge_{\tau \in \mathcal{T}} (\mathcal{A}_\tau \supset \exists y_1 \ldots \exists y_m \varphi(\mathfrak{t}, y_1, \ldots, y_m)).$$

This formula is valid if and only if for every $\tau \in \mathcal{T}$ the formula $\mathcal{A}_\tau \supset \exists y_1 \ldots \exists y_m \varphi(\mathfrak{t}, y_1, \ldots, y_m)$ is valid. By Lemma 3.5, it is the same as that for every $\tau \in \mathcal{T}$ there exists a truth assignment ρ for $y_1, \ldots, y_m$ such that G evaluates to $\mathfrak{t}$ under the truth assignment combining τ and ρ. $\square$

Lemma 3.7 The validity problem for ground-negative formulas, with or without equality, is Π_2^p-hard.

Proof. It is enough to note that the function $G \mapsto G^{\mathfrak{t}}$ is polynomial-time computable and apply Lemmas 3.2 and 3.6. $\square$

Note that our reduction can be modified to prove a result that is similar to Lemma 1.4 (theorem 9 of Kozen [4]) but for logic without equality:

Theorem 1 The following problem is **NP**-complete for logic without equality. Given a conjunction $\mathcal{L}$ of ground literals and a positive formula φ, is the formula $\mathcal{L} \supset \varphi$ valid?

The same is true even when φ uses only $\wedge, \vee, \exists$ and has no function symbols.

Proof. The complete proof is left to the reader. It follows the proof of Lemma 3.5 with the following modification. We define

$$(\exists y_j F)^s \rightleftharpoons \exists y_j((not_1(y_1, \mathfrak{t}) \vee not_1(y_1, \mathfrak{f})) \wedge F^s).$$

$\square$

Section 4

Conclusion

Lemmas 2.3 and 3.7 give us the main result of this paper.

Theorem 2 The validity problem for ground-negative formulas, with or without equality and with or without function symbols is Π_2^p-complete.

The proof of Theorem 2 also establishes a statement similar to Lemma 1.4 (theorem 9 of Kozen [4]):

Theorem 3 The following problem is Π_2^p-complete, both for logics with and without equality and with or without function symbols. Given a $(\wedge, \vee)$-sentence ψ and a positive formula φ, is the formula $\psi \supset \varphi$ valid?

"A positive formula φ" in this theorem can be replaced by "a $(\wedge, \vee, \exists)$-formula". It is interesting that the complexity of the validity problem for ground-negative formulas does not change when equality, function symbols and universal quantifier are not allowed in formulas.

The decidability of ground-negative formulas is also established by Fermüller and Leitsch [3]. They prove that this class of formulas (and even a more general class) can be decided by hyperresolution with ordered paramodulation. Their proof does not give our complexity bounds since proofs by hyperresolution and ordered paramodulation require exponential space.

Our main theorem implies the following result about the dual satisfiability problem.

Theorem 4 The satisfiability problem for ground-positive formulas, with or without equality, is Σ_2^p-complete.

We also have a similar result for clauses.

Theorem 5 The satisfiability problem for ground-positive clauses, with or without equality, and with or without function symbols is Σ_2^p-complete.

Proof. In view of Theorem 4, it is enough to prove Σ_2^p-hardness for the case without equality and without function symbols. The proof literally follows the proof of Lemma 3.7. We note that the clause form of the formula G^s used in the reduction of that lemma consists of clauses of the forms

$$
\begin{array}{l}
not_2(c_i, \mathfrak{f}) \vee not_2(c_i, \mathfrak{t}) \\
and(\mathfrak{t}, \mathfrak{t}, \mathfrak{t}) \\
and(\mathfrak{t}, \mathfrak{f}, \mathfrak{f}) \\
and(\mathfrak{f}, \mathfrak{t}, \mathfrak{f}) \\
and(\mathfrak{f}, \mathfrak{f}, \mathfrak{f}) \\
not_1(\mathfrak{t}, \mathfrak{f}) \\
not_1(\mathfrak{f}, \mathfrak{t})
\end{array}
$$

plus one negative clause (representing the part of G^s following $\supset$). All these clauses are ground-positive and have neither equality nor function symbols. This clause form can be constructed in time polynomial in the size of G. □

Of course for predicates different from equality there is no difference between negative and positive occurrences. Thus all theorems about ground-negative formulas can be stated for formulas in which all negative occurrences of equality atoms are ground and for every predicate symbol different from equality, either all positive occurrences or all negative occurrences of the corresponding atoms are ground.

Our complexity results cannot be generalized to ground-positive formulas with equality, because the validity problem for formulas of the form $\forall(s_1 \simeq t_1 \wedge \ldots s_n \simeq t_n) \supset s \simeq t$, where s, t are ground, is undecidable.

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