Term-Modal Logics

Melvin Fitting (fitting@alpha.lehman.cuny.edu)
City University of New York

Lars Thalmann (lars.thalmann@csd.uu.se) *
Uppsala University

Andrei Voronkov (voronkov@cs.man.ac.uk) *
University of Manchester

Abstract. Many powerful logics exist today for reasoning about multi-agent systems, but in most of these it is hard to reason about an infinite or indeterminate number of agents. Also the naming schemes used in the logics often lack expressiveness to name agents in an intuitive way.

To obtain a more expressive language for multi-agent reasoning and a better naming scheme for agents, we introduce a family of logics called *term-modal logics*. A main feature of our logics is the use of modal operators indexed by the terms of the logics. Thus, one can *quantify over variables occurring in modal operators*. In term-modal logics agents can be represented by terms, and knowledge of agents is expressed with formulas within the scope of modal operators.

This gives us a flexible and uniform language for reasoning about the agents themselves and their knowledge. This article gives examples of the expressiveness of the languages and provides sequent-style and tableau-based proof systems for the logics. Furthermore we give proofs of soundness and completeness with respect to the possible world semantics.

Keywords: Modal logic, multi-agents, epistemic logic, completeness, tableaux

1. Introduction

In this article, we describe a new family of modal logics, namely the first-order term-modal logics, where we by *term-modal* mean that any term can be used as a modality. The specific logics we discuss are the term-modal versions of the modal logics $\mathsf{K}, \mathsf{D}, \mathsf{T}, \mathsf{K4}, \mathsf{D4}$, and $\mathsf{S4}$. Sequent-style and tableau-style proof systems for the logics are given, and their soundness and completeness are shown.

An earlier version of this article, where the soundness and completeness proofs were omitted, and which contained less discussion about related work, appeared as (Fitting et al., 2000).

* Andrei Voronkov and Lars Thalmann are partially supported by TFR grants. Andrei Voronkov is partially supported by an EPSRC grant and a grant from the Faculty of Science and Engineering, University of Manchester.

1.1. Motivation

Many researchers have been interested in the use of multi-modal logics for knowledge representation see e.g. (Halpern, 1993; Fagin et al., 1995; Meyer and van der Hoek, 1995), although most of them have investigated the use of a finite set of modalities, indexed by the first n natural numbers, usually denoted either $[1], [2], \ldots, [n]$ or $\mathcal{K}_1, \mathcal{K}_2, \ldots, \mathcal{K}_n$. Each number is here naming some agent. By agent we mean any system, e.g. a human or a computer program, to which we can ascribe knowledge. When we instead use an infinite set of modalities we can reason about a dynamic society of agents, where some agents might vanish and new agents may appear.

In the family of multi-modal logics presented in this article, any term can denote an agent. This makes naming of agents easy and the logics expressive. The use of complex names for agents, possibly involving variables, makes it easy to model a society of agents, and give names to new agents by their relationship to already existing agents. For example, to express that the agent $mother(x)$ thinks (or knows, or believes) that the agent x is good, we can write $[mother(x)]good(x)$.

The standard multi-modal logics allow us to reason about beliefs of particular agents, but provide very limited facilities to reason about beliefs of groups of agents or agents themselves. In our language, we can distinguish a group of agents by specifying their properties. For example, to express that every Christian believes in the existence of God, we can write $\forall x(christian(x) \supset [x]\exists y\, God(y))$.

An example of a society of agents is the collection of computer processes on some system. Here the logic with its infinite complex naming mechanism can be used to specify requirements of the system as a whole and the proof system can be used to check that these requirements are satisfied.

When the computer processes spawn new processes, the society of agents (i.e. the number of processes) grows, and the naming mechanism can be used to refer to the newly created processes. As the number of processes spawned by the program may not be known beforehand, it is convenient to have an unlimited set of names for these new agents.

1.2. Background

Fitting (1983) proves soundness and completeness of (single-)modal logics. In this article we introduce some new definitions, and extend the proofs for term-modal logics.

Fagin et al. (1995), van der Hoek and Meyer (1997) all use modal logics to describe multi-agent systems. Their approaches are based on a finite set of agents, and they also discuss the use of common and distributed knowledge. By using the logic presented in this article, their work might be extended to handle dynamic agent societies with an easy naming mechanism, where quantification over agents is possible.

1.3. Overview

The rest of this article is structured as follows. Section 2 defines the syntax of term-modal logics, and Section 3 their semantics. In Section 4 we introduce sequent calculi and in Section 5 tableau calculi for these logics. In Section 6 we establish soundness of the sequent calculi. The completeness proof is rather lengthy and split between two sections. In Section 7 we define the main technical tool used for the completeness proof, the so-called *consistency property*, and prove a Model Existence Theorem for consistency properties. Using this theorem, we establish completeness of sequent calculi in Section 8. Soundness and completeness of sequent calculi implies soundness and completeness of tableau calculi for term-modal logics. As a step toward automated reasoning in term-modal logic in Section 9 we introduce free-variable versions of tableau calculi for term-modal logics. In Section 10 we give an example refutation in such a calculus, in order to illustrate some distinctive features of free-variable calculi for term-modal logics.

2. Syntax

The term-modal logics are obtained from the standard predicate modal logics by adding modal operators indexed by terms. In this section we give a formal definition of the syntax of term-modal logics.

We assume a *signature* Σ consisting of three disjoint sets of constants, function symbols and relation symbols. Usually, the signature is assumed to be fixed, but in some situations we will vary it for technical convenience. In addition to the symbols of Σ, we will use various infinite *sets P of parameters* disjoint from the symbols in Σ. In some situations parameters will behave as new constants, in others as elements of a domain on which formulas are evaluated. The signature is not necessarily finite or countable. For every signature Σ, we denote by Σ^- the signature obtained from Σ by omitting all constants and function symbols.

DEFINITION 1 (Term). *Suppose P is a set of parameters and V a set of* variables *disjoint from the set of parameters. The set of* terms of the signature Σ with parameters in P *and variables in V, denoted $\mathcal{T}(\Sigma \cup P, V)$ is defined inductively as follows.*

1. *Each constant in* Σ *is a term.*

2. *Each variable in* V *is a term.*

3. *Each parameter in* P *is a term.*

4. *If* $t_1, \dots, t_n$ *are terms and* f *is an* n*-ary function symbol, then* $f(t_1, \dots, t_n)$ *is a term.*

In this article, we can restrict ourselves to a fixed set of variables, however the set of parameters (and sometimes the signature) will vary. So we will use a simpler notation $\mathcal{T}(\Sigma \cup P)$. A term is called *ground* if it has no occurrences of variables.

DEFINITION 2 (Formula). *Let* P *be a set of parameters. The set of* formulas of the signature Σ with parameters in P, *denoted* $\mathcal{F}(\Sigma \cup P)$, *is defined inductively as follows.*

1. *If* R *is a relation symbol of arity* n *and* $t_1, \dots, t_n$ *are terms in* $\mathcal{T}(\Sigma \cup P)$, *then* $R(t_1, \dots, t_n)$ *is an* atomic formula. *Any atomic formula is a formula.*

2. *If* A *and* B *are formulas, then so are* $(A \wedge B)$, $(A \vee B)$, $(A \supset B)$ *and* $\neg A$.

3. *If* A *is a formula and* t *is a term in* $\mathcal{T}(\Sigma \cup P)$, *then* $[t]A$ *and* $\langle t \rangle A$ *are formulas.*

4. *If* A *is a formula and* x *is a variable, then* $\forall x A$ *and* $\exists x A$ *are formulas.*

The notions of *free and bound occurrences of variables* are defined as usual, with the exception of the following item:

– The free occurrences of variables in $[t]A$ and $\langle t \rangle A$ are all occurrences of variables in t plus all free occurrences of variables in A.

A formula is called *closed*, or a *sentence* if it has no free occurrences of variables (but note that it may contain parameters). An $\exists$-*formula* is any formula $\exists x A$. A *literal* is either an atomic formula A or its negation $\neg A$. Literals A and $\neg A$ are called *complementary* to each other.

Intuitively, when interpreting the formulas in a multi-agent context, the meaning of the formula $[t]A$ is that the agent denoted by the term t knows (believes etc.) the information represented by the formula A. The formula $\langle t \rangle A$ intuitively means that the agent denoted by t considers it possible that A holds, i.e., it is not the case that the agent knows the contrary (which can also be expressed by $\neg[t]\neg A$).

In the proof systems introduced later we make no assumptions about the kind of knowledge expressed. The knowledge could in fact be just beliefs, i.e., an agent might believe something which is false.

It is easy to add axioms of knowledge, if one is interested in describing a specific kind of knowledge. An example of this is the knowledge axiom, $[x]A \supset A$, which intuitively means that if an agent x knows something, then it is true. More about different epistemic interpretations of modal logic can be found in, e.g., the books by Hintikka (1962) or Lenzen (1978).

NOTATION 3. *For the rest of this article, we will denote*

- *variables by* x, y, z, u, v;
- *parameters by* p;
- *terms by* s, t;
- *sets of parameters by* P;
- *logics* $\mathsf{K}, \mathsf{D}, \mathsf{T}, \mathsf{K4}, \mathsf{D4}, \mathsf{S4}$ *by the generic symbol* $\mathcal{L}$.
- *formulas by* A, B, C;
- *sets of formulas by* S, Ψ;
- *literals by* L;
- *domain elements by* d *(i.e. elements in the set* $\mathbf{D}$*, which is defined later);*

We write $A(x)$ to denote a formula A with (zero or more) free occurrences of the variable x and write $A(t)$ to denote the replacement of all free occurrences of x by a term t. Before the replacement, we rename in $A(x)$ all bound occurrences of variables that have free occurrences in t.

2.1. Related work

The current trend in modal and description logics is to define expressive but still decidable logics. Our logics are undecidable since they contain first-order classical logic. Moreover, the expressiveness of our logics is, in a way, higher than that of the standard first-order modal logics, since first-order modal logics can be interpreted in our logics by using a single constant in modal operators (at least for cumulative domains, but our results can be extended to the constant domain versions of the logics as well). Logics with modalities indexed by terms were studied by Grove and Halpern (1991), (1995). These logics are more expressive in some aspects and less expressive in other aspects than ours. Namely, these logics can handle equality and agents with special properties. However, there are restrictions on how formulas can be built in these logics, so some well-formed formulas of our logics cannot be used as formulas in (Grove and Halpern, 1991; Grove, 1995). For example, the formula $[x]P(x)$ is not a valid formula in (Grove, 1995), since in Grove's framework x in $[x]$ must be of the agent sort, but the formula $P(x)$ in the scope of $[x]$ must not have free variables of the agent sort.

Another related framework are the modal logics with names, see (Passy and Tinchev, 1985; Passay and Tinchev, 1991; Gargov and Goranko, 1993; Blackburn, 1993). In these logics a second sort of atomic formula is introduced (these are called names or nominals) and it is stipulated that such a formula is satisfied at a specific world of the model. Intuitively, a nominal names the world in which it is satisfied. Hybrid logics take this a step further. In these logics, nominals are treated as variables open to binding. See e.g. the work by Blackburn and Seligman (1993), Blackburn and Tzakova (1998) and Areces et al. (2000). A tableau calculi for hybrid logics was presented by Tzakova (1999). The main difference between the logics of our paper and hybrids logics is that in hybrid logic one quantifies over state variables naming worlds, while in this article, we quantify over variables naming accessibility relations.

Propositional dynamic logic (or propositional modal logic of programs, as it was called then) was introduced in (Fischer and Ladner, 1977; Fischer and Ladner, 1979), following ideas of (Pratt, 1976). First-order dynamic logic appear in e.g. (Harel, 1979) and (Kozen and Tiuryn, 1989). Dynamic logic is a logic with complex modalities just as the term-modal logics. The main difference between term-modal and dynamic logic is the structure of the indexes. In dynamic logic modal operators are indexed by programs, either atomic or composed of subprograms or subformulas joined by modal connectives $;, \cup, *$, or ?. In term-modal logic the modal operators are indexed with terms. There is a close syntactic connection between the logics. Term-modal logic can be translated into dynamic logic, by translating every modal operator $[t]$ into $[x := t]$ where x is a dummy programming variable. The semantics of the logics are different though. In dynamic logic, the program $x := t$ is usually deterministic, while in our framework, we are interested in having several worlds reachable by term t. The simplified semantics of the term-modal logic also let us develop the proof theory further. In this article, we provide a free-variable tableau calculus for the term-modal logics.

Modal action logics, see e.g. (Ryan et al., 1991), are another example of application of our logic. In (Ryan et al., 1991) a logic for actions is given, but no proof procedure is given.

In general, the approach in this article contrasts to other approaches, by considering a simple logic with terms in modal operators and develop a complete free-variable proof system, while other authors either limit the expressibility or fail to provide a complete proof procedure.

3. Semantics

In this section, we describe a possible world semantics for the logics. The semantics is defined through the notions of *frames* and *structures*. It differs from the standard semantics of first-order modal logics by the treatment of the reachability relation on worlds: the reachability relation is indexed by elements of the domain. We assume all definitions be given w.r.t. a nonempty set $\mathbf{D}$, called the *domain*.

3.1. Frames

DEFINITION 4 (Frame). *A* frame *over* $\mathbf{D}$ *is a triple* $\langle \mathcal{W}, \mathcal{D}, \longrightarrow \rangle$, *where*

1. $\mathcal{W}$ *is a non-empty set, called the* set of possible worlds.
2. $\mathcal{D}$ *is a mapping from* $\mathcal{W}$ *to the set of subsets of* $\mathbf{D}$. *The set* $\mathcal{D}(w)$ *is denoted by* $\mathcal{D}_w$ *and called the* domain of w.
3. $\longrightarrow$ *is a relation on* $\mathcal{W} \times \mathbf{D} \times \mathcal{W}$, *called the* accessibility relation. *If* $\longrightarrow (w_1, d, w_2)$, *then we say that* w_2 *is* d-reachable *from* w_1 *and write* $w_1 \xrightarrow{d} w_2$.

We require the monotonicity condition *to be satisfied in all frames:*[1]

$$\textit{If } w_1 \xrightarrow{d} w_2 \textit{ then } \mathcal{D}_{w_1} \subseteq \mathcal{D}_{w_2}.$$

The monotonicity condition corresponds to *cumulative domains* e.g. (Wallen, 1990) and *nested domains* e.g. (Garson, 1984).

DEFINITION 5 ($\mathcal{L}$-frame). *We specialize the concept of frames to six different classes:*

K. *All frames are* K-frames.

[1] The monotonicity condition can be replaced by a weaker condition:

$$\text{If } w_1 \xrightarrow{d} w_2 \text{ and } d \in w_1 \text{ then } \mathcal{D}_{w_1} \subseteq \mathcal{D}_{w_2}.$$

The reason is that the first-order language cannot express properties of worlds d-reachable from w, when $d \notin \mathcal{D}_w$.

D. *If for all* d *and* w *there exists* w' *such that* $w \xrightarrow{d} w'$ *(i.e. the accessibility relation is* serial *in its 1st and 3rd arguments) then the frame is a* D-frame.

T. *If* $w \xrightarrow{d} w$ *holds for all* w *and* d *(i.e. the accessibility relation is* reflexive *in its 1st and 3rd arguments), then the frame is a* T-frame.

K4. *If, from* $w \xrightarrow{d} w'$ *and* $w' \xrightarrow{d} w''$ *it follows that* $w \xrightarrow{d} w''$ *for all* d *and* $w, w', w'' \in \mathcal{W}$*, (i.e. the accessibility relation is* transitive *in its 1st and 3rd arguments) then it is a* K4-frame.

D4. *If the accessibility relation is both serial and transitive in its 1st and 3rd arguments, then the frame is a* D4-frame.

S4. *If the accessibility relation is both reflexive and transitive in its 1st and 3rd arguments, then the frame is an* S4-frame.

3.2. First-order modal structures

The first-order modal (Kripke) structures are introduced in the standard way, except for the case of modal operators.

DEFINITION 6 (Structure for $\mathcal{L}$). *Let* $\mathcal{L}$ *be one of* K, D, T, K4, D4, S4. *A* first-order modal structure *for* $\mathcal{L}$*, or simply* $\mathcal{L}$-structure *over a domain* $\mathbf{D}$ *is a tuple* $\mathfrak{S} = \langle \mathcal{W}, \mathcal{D}, \longrightarrow, I, \Vdash \rangle$*, where*

1. $\langle \mathcal{W}, \mathcal{D}, \longrightarrow \rangle$ *is a* $\mathcal{L}$*-frame over* $\mathbf{D}$.

2. $\Vdash$ *is a binary relation between worlds and atomic sentences in* $\mathcal{F}(\Sigma^- \cup \mathbf{D})$. *(Note that elements of* $\mathbf{D}$ *are treated as parameters in* $\mathcal{F}(\Sigma^- \cup \mathbf{D})$.*)*

3. I*, called the* interpretation function*, is a mapping that maps every constant* c *of* Σ *to an element of* $\mathbf{D}$ *and every function symbol* f *of* Σ *of arity* n *to an* n*-place function on* $\mathbf{D}$. *The corresponding element of* $\mathbf{D}$ *and function on* $\mathbf{D}$ *are called the* interpretations *of* c *and* f *respectively. We require the interpretation of any constant and function symbol to be totally defined in every world: this means that* $I(c)$ *belongs to* $\mathcal{D}_w$ *for every world* $w \in \mathcal{W}$ *and for every* $d_1, \ldots, d_n \in \mathcal{D}_w$ *we have* $I(f)(d_1, \ldots, d_n) \in \mathcal{D}_w$.

Note that $\Vdash$ is only defined on formulas without function symbols or constants, but with parameters in $\mathbf{D}$.

We call a *valuation* V *in a structure* $\mathfrak{S}$ any mapping $V : P \to \mathbf{D}$ from a set of parameters to the domain $\mathbf{D}$ of $\mathfrak{S}$. Any valuation V can be extended to the set of all ground terms by defining

$$\begin{aligned} V(c) &= I(c); \\ V(f(t_1, \ldots, t_n)) &= I(f)(V(t_1), \ldots, V(t_n)). \end{aligned}$$

Now we can give the central notion of satisfiability of formulas in structures.

Given a first-order modal structure $\langle \mathcal{W}, \mathcal{D}, \longrightarrow, I, \Vdash \rangle$, we change the relation $\Vdash$ into a ternary relation between worlds in $\mathcal{W}$, valuations, and sentences in $\mathcal{F}(\Sigma \cup \mathbf{D})$ as given below. We write $\mathfrak{S}, w, V \Vdash A$ when this relation holds on w, V, A and denote by $\nVdash$ the complement of $\Vdash$. When we use this notation, we can omit one or both of $\mathfrak{S}, V$, when they are clear from the context.

DEFINITION 7 (Relation $\Vdash$). *Given $\mathfrak{S}$ and V, we define the relation $\Vdash$ as follows.*

1. $w, V \Vdash R(t_1, \ldots, t_n)$ *if* $w \Vdash R(V(t_1), \ldots, V(t_n))$.
2. $w, V \Vdash A \wedge B$ *if* $w, V \Vdash A$ *and* $w, V \Vdash B$.
3. $w, V \Vdash A \vee B$ *if* $w, V \Vdash A$ *or* $w, V \Vdash B$.
4. $w, V \Vdash A \supset B$ *if* $w, V \nVdash A$ *or* $w, V \Vdash B$.
5. $w, V \Vdash \neg A$ *if* $w, V \nVdash A$.
6. $w, V \Vdash [t]A$ *if for all* w' *such that* $w \xrightarrow{V(t)} w'$ *we have* $w', V \Vdash A$.
7. $w, V \Vdash \langle t \rangle A$ *if there exists* w' *such that* $w \xrightarrow{V(t)} w'$ *and* $w', V \Vdash A$.
8. $w, V \Vdash \forall x A(x)$ *if* $w, V \Vdash A(d)$, *for all* $d \in \mathcal{D}_w$.
9. $w, V \Vdash \exists x A(x)$ *if* $w, V \Vdash A(d)$, *for some* $d \in \mathcal{D}_w$.

DEFINITION 8 (Truth, satisfiability). *Let $\mathfrak{S} = \langle \mathcal{W}, \mathcal{D}, \longrightarrow, I, \Vdash \rangle$ be a structure. We say a formula A is* true, *or* holds, *or is* locally satisfied *in $\mathfrak{S}$ at a world $w \in \mathcal{W}$* under a valuation V *if $\mathfrak{S}, w, V \Vdash A$. A formula A is* globally satisfied *in a structure $\mathfrak{S}$ under a valuation V if it is locally satisfied at every world of $\mathfrak{S}$ under V. A formula A is called* locally (respectively, globally) satisfiable in $\mathfrak{S}$ *if it is locally (respectively, globally) satisfied in $\mathfrak{S}$ under some valuation. If A is locally satisfiable in $\mathfrak{S}$ we also say that $\mathfrak{S}$ is a* model *of A.*

Note that the truth of a formula A under a valuation V only depends on the value of V on the parameters occurring in A. Thus, if A is a sentence in $\mathcal{F}(\Sigma)$, its truth does not depend on the valuation at all.

DEFINITION 9 (Model, validity). *Let $\mathcal{L}$ be one of* K, D, T, $\mathsf{K4}$, $\mathsf{D4}$, $\mathsf{S4}$. *We call a model $\langle \mathcal{W}, \mathcal{D}, \longrightarrow, I, \Vdash \rangle$ of a formula A an* $\mathcal{L}$-model *if its frame $\langle \mathcal{W}, \mathcal{D}, \longrightarrow \rangle$ is an $\mathcal{L}$-frame. A formula A is called* $\mathcal{L}$-satisfiable *if it has an $\mathcal{L}$-model. A formula A is called* $\mathcal{L}$-valid *if it is true in every world of every $\mathcal{L}$-structure under every valuation.*

It is not hard to argue that satisfiability and validity are dual notions in the following sense: a formula A is unsatisfiable if and only if $\neg A$ is valid. In view of this duality we will formulate our results in terms of (un)satisfiability only.

When we speak of a *logic* $\mathcal{L}$ in this article, we understand the set of $\mathcal{L}$-valid formulas. So, we will speak of *logics* K, D, T, $\mathsf{K4}$, $\mathsf{D4}$, $\mathsf{S4}$. Another standard way of introducing a logic is to define a suitable *calculus* deriving valid formulas in this logic. In the next section we introduce such calculi for all these logics.

Formulas A and B are called $\mathcal{L}$-*equivalent* if the formulas $A \supset B$ and $B \supset A$ are $\mathcal{L}$-valid. It is evident that in any context when we speak about worlds, structures, valuations, and satisfiability, we can replace formulas by equivalent ones. We will now introduce the negation normal form of formulas, which will simplify our proofs considerably.

DEFINITION 10 (Negation normal form). *A formula A is said to be in* negation normal form *if it is constructed from literals using $\wedge$, $\vee$, $\forall$, $\exists$, $[t]$ and $\langle t \rangle$. A formula B is called a* negation normal form of a formula A, *if B is in negation normal form and B is equivalent to A.*

LEMMA 11. *Every formula A has negation normal form.*

PROOF. It is not hard to argue that one can reduce A to its negation normal form by means of the transformations shown in Figure 1. These transformations replace, in any order, subformulas of A on the left of $\Rightarrow$ by the corresponding subformulas on the right, until no transformation is applicable. $\Box$

4. Sequent calculi

In this section we define sequent calculi for the family of term-modal logics. There are several essentially equivalent notions of sequent giving

$$
\begin{array}{rclcrcl}
B \supset C & \Rightarrow & \neg B \vee C & \qquad & \neg \forall x B & \Rightarrow & \exists x \neg B \\
\neg\neg B & \Rightarrow & B & & \neg \exists x B & \Rightarrow & \forall x \neg B \\
\neg(B \vee C) & \Rightarrow & \neg B \wedge \neg C & & \neg [t] B & \Rightarrow & \langle t \rangle \neg B \\
\neg(B \wedge C) & \Rightarrow & \neg B \vee \neg C & & \neg \langle t \rangle B & \Rightarrow & [t] \neg B
\end{array}
$$

Figure 1. Negation normal form transformation

rise to different calculi. The original definition of Gentzen (1934) defines sequents as expressions $A_1, \ldots, A_n \rightarrow B_1, \ldots, B_m$, where $A_1, \ldots, A_n$, $B_1, \ldots, B_m$ are formulas. Smullyan (1963) represents such a sequent as a collection of formulas $A_1, \ldots, A_n, \neg B_1, \ldots, \neg B_m$ or a collection $T\ A_1, \ldots, T\ A_n, F\ B_1, \ldots,\ F\ B_m$ of *signed formulas* with the intended meaning that all formulas A_i are true and all formulas B_j are false and introduces a *uniform notation* to group together inference rules with similar behavior. We will use the approach due to Schütte (1960). Instead of using arbitrary formulas we will use only formulas in *negation normal form.* Every rule in the uniform notation corresponds to an inference rule introducing a particular connective on formulas in negation normal form. Then we do not need the unifying notation anymore, since we can label inference rules by the corresponding connectives. We will use parameters instead of free variables, and therefore only deal with sentences.

DEFINITION 12 (Sequent). *A* sequent *is a set of sentences. Let S be a sequent and $\mathfrak{S}$ be a structure. We say that a sequent S is* locally satisfied *in $\mathfrak{S}$ at a world $w \in \mathcal{W}$* under a valuation *V if $\mathfrak{S}, w, V \Vdash A$ for all $A \in S$. A sequent S is* globally satisfied *in a structure $\mathfrak{S}$ under a valuation V if S is locally satisfied at every world of $\mathfrak{S}$ under V. A sequent S is called* locally (respectively, globally) satisfiable in $\mathfrak{S}$ *if it is locally (respectively, globally) satisfied in $\mathfrak{S}$ under some valuation. If S is locally satisfiable in $\mathfrak{S}$ we also say that $\mathfrak{S}$ is a* model *of S.*

Thus, a sequent is understood as a (possibly infinite) conjunction of its members.

For a formula A and a set of formulas S we use A, S or S, A to denote the set $S \cup \{A\}$. Likewise, we write S_1, S_2 to denote the union of two sequents $S_1 \cup S_2$.

Sequent calculi for logics K, D, T, $\mathsf{K4}$, $\mathsf{D4}$, $\mathsf{S4}$ are shown in Figure 2. $S^{[t]}$ is a generalization of the notation used in (Fitting, 1983). Semantically, $S^{[t]}$ denotes the set of formulas which must hold in every world that is $V(t)$-reachable from the world in which S holds. Depending on the logic, semantic restrictions on the frame make the definition of $S^{[t]}$ vary between logics.

For all logics $(\mathsf{K}, \mathsf{D}, \mathsf{T}, \mathsf{K4}, \mathsf{D4}, \mathsf{S4})$:

$$\frac{}{S, A, \neg A}\ (\text{ax})$$

$$\frac{S, A \quad S, B}{S, A \vee B}\ (\vee) \qquad \frac{S, A, B}{S, A \wedge B}\ (\wedge)$$

$$\frac{S, A(p)}{S, \exists x A(x)}\ (\exists)^* \qquad \frac{S, \forall x A(x), A(t)}{S, \forall x A(x)}\ (\forall)$$

$$\frac{S^{[t]}, A}{S, \langle t\rangle A}\ (\langle t\rangle)$$

For serial logics $(\mathsf{D}, \mathsf{D4})$:

$$\frac{S^{[t]}}{S}\ ([t])$$

For reflexive logics $(\mathsf{T}, \mathsf{S4})$:

$$\frac{S, A}{S, [t]A}\ ([t])$$

Logic $\mathcal{L}$	**Definition of $S^{[t]}$**
K, D, T	$S^{[t]} = \{A \mid [t]A \in S\}$
K4, D4	$S^{[t]} = \{A \mid [t]A \in S\} \cup \{[t]A \mid [t]A \in S\}$
S4	$S^{[t]} = \{[t]A \mid [t]A \in S\}$

* The rule $(\exists)$ satisfies the *parameter condition*: p is a parameter having no occurrences in the conclusion of the rule.

Figure 2. Sequent calculi

In the rule (ax), A is atomic. We can generalize the calculus for non-atomic axioms in the standard way, but the calculus is complete with atomic axioms.

DEFINITION 13 (Inference, derivation, refutation). *The* inference rules *of the sequent calculi are shown in Figure 2. We call an* inference *any particular instance of an inference rule. The* premises *of any inference or inference rule are the sequents above the bar, its* conclusion *is the sequent below the bar. An* axiom *is any conclusion of* (ax). *A* derivation *of a sequent S is a tree made of inferences and having S as the root. A derivation is called a* refutation *if all leaves in it are axioms.*

We use the term *refutation* instead of a proof because the sequent calculi used in this article establish unsatisfiability rather than validity.

EXAMPLE 14. *Suppose that we wish to establish* K*-validity of the sentence*

$$\forall z([z]\forall x R(x) \supset \forall y[z]R(y)).$$

We turn this formula into its negation

$$\neg\forall z([z]\forall x R(x) \supset \forall y[z]R(y))$$

and establish the unsatisfiability of the latter. To this end, we transform this formula into its negation normal form

$$\exists z([z]\forall x R(x) \wedge \exists y\langle z\rangle\neg R(y))$$

and try to find a refutation in the sequent calculus for K*. An example refutation is as follows.*

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\forall x R(x), R(q), \neg R(q)}\,(\mathrm{ax})}{\forall x R(x), \neg R(q)}\,(\forall)}{[p]\forall x R(x), \langle p\rangle\neg R(q)}\,(\langle p\rangle)}{[p]\forall x R(x), \exists y\langle p\rangle\neg R(y)}\,(\exists)}{[p]\forall x R(x) \wedge \exists y\langle p\rangle\neg R(y)}\,(\wedge)}{\exists z([z]\forall x R(x) \wedge \exists y\langle z\rangle\neg R(y))}\,(\exists)$$

This refutation is also a valid refutation in D *and* T*.*

To obtain a refutation in $\mathsf{K4}$ *and* $\mathsf{D4}$*, we have to modify the top part of this refutation because of the difference in the definition of* $S^{[t]}$*:*

$$\cfrac{\cfrac{\cfrac{}{[p]\forall x R(x), \forall x R(x), R(q), \neg R(q)}\,(\mathrm{ax})}{[p]\forall x R(x), \forall x R(x), \neg R(q)}\,(\forall)}{[p]\forall x R(x), \langle p\rangle\neg R(q)}\,(\langle p\rangle)$$

A refutation in $\mathsf{S4}$ *follows a different strategy because of the difference in the* $([t])$ *rule:*

$$\cfrac{\cfrac{\cfrac{\cfrac{}{\forall x R(x), R(q), \neg R(q)}\,(\mathrm{ax})}{\forall x R(x), \neg R(q)}\,(\forall)}{[p]\forall x R(x), \neg R(q)}\,([p])}{[p]\forall x R(x), \langle p\rangle\neg R(q)}\,(\langle p\rangle)$$

Since every formula has a negation normal form, we can restrict ourselves to negation normal forms. Moreover, using the fact that a formula and its negation normal form obtained by the transformation of Figure 1 have, in a sense, *similar structure*, it is not hard to change sequent calculi introduced below for formulas in negation normal forms, into calculi for arbitrary formulas or signed formulas.

We will augment the logics defined above with so-called *global assumptions*. Let Ψ be a set of sentences and $\mathcal{L}$ be one of the logics defined above. We call a *sequent calculus for $\mathcal{L}$ with global assumptions Ψ* the calculus obtained from $\mathcal{L}$ by adding the rule

$$\frac{S, A}{S}\ (\Psi), \quad \text{where } A \in \Psi.$$

5. Tableau systems

Tableau systems formalize proof-search in sequent calculi. Tableaux are often introduced as trees of formulas, with inference rules on tableaux formulated in terms of *branches*. To simplify the presentation, we introduce tableaux as multisets of branches.

DEFINITION 15 (Tableau, branch, empty branch). *A* tableau *is a finite multiset $S_1, \ldots, S_n$ of sequents, denoted $S_1 \mid \cdots \mid S_n$. The* empty tableau *is denoted by $\#$. Every sequent S_i is called a* branch *of this tableau.*

The tableau calculus for each logic studied in this paper can be obtained by a simple transformation of the corresponding sequent calculus. For every inference rule

$$\frac{S_1 \quad \cdots \quad S_n}{S}$$

of the sequent calculus, the corresponding tableau rule has the form

$$\frac{S \mid \mathcal{T}}{S_1 \mid \cdots \mid S_n \mid \mathcal{T}}$$

where $\mathcal{T}$ is any tableau. Note the reverse order of the sequents. The tableau calculus rules have the following intuitive meaning: suppose that we search for a refutation of S and all sequents in $\mathcal{T}$. Then, since there is a sequent calculus rule reducing S to the sequents $S_1, \ldots, S_n$, it is enough to find a refutation of $S_1, \ldots, S_n$ and all sequents in $\mathcal{T}$. To find a refutation for a formula A, we begin with a tableau consisting of one branch A and try to apply the tableau rules until no (unrefuted) branches remain.

Formally, the *tableau calculi* for $\mathcal{L}$ are shown in Figure 3.

For all logics ($\mathsf{K}, \mathsf{D}, \mathsf{T}, \mathsf{K4}, \mathsf{D4}, \mathsf{S4}$):

$$\frac{S, A, \neg A \mid \mathcal{T}}{\mathcal{T}}\ |\mathrm{ax}|$$

$$\frac{S, A \vee B \mid \mathcal{T}}{S, A \mid S, B \mid \mathcal{T}}\ |\vee| \qquad \frac{S, A \wedge B \mid \mathcal{T}}{S, A, B \mid \mathcal{T}}\ |\wedge|$$

$$\frac{S, \exists x A(x) \mid \mathcal{T}}{S, A(p) \mid \mathcal{T}}\ |\exists|^* \qquad \frac{S, \forall x A(x) \mid \mathcal{T}}{S, \forall x A(x), A(t) \mid \mathcal{T}}\ |\forall|$$

$$\frac{S, \langle t \rangle A \mid \mathcal{T}}{S^{[t]}, A \mid \mathcal{T}}\ |\langle t \rangle|$$

For serial logics (D, $\mathsf{D4}$): **For reflexive logics (T, $\mathsf{S4}$):**

$$\frac{S \mid \mathcal{T}}{S^{[t]} \mid \mathcal{T}}\ |[t]| \qquad \frac{S, [t]A \mid \mathcal{T}}{S, A \mid \mathcal{T}}\ |[t]|$$

For systems with global assumptions Ψ:

$$\frac{S \mid \mathcal{T}}{S, A \mid \mathcal{T}}\ |\Psi|^*$$

Here $S^{[t]}$ is defined in the same way as for the sequent calculi.

* The rule $|\exists|$ satisfies the *parameter condition*: p is a parameter having no occurrences in the premise of the rule. In the rule $|\Psi|$, $A \in \Psi$.

Figure 3. Tableau calculi

THEOREM 16 (Equivalence of tableau calculi and sequent calculi). *A sequent S has a refutation in the sequent calculus for $\mathcal{L}$ (with global assumptions Ψ) if and only if there exists a derivation of # from S in the tableau calculus for $\mathcal{L}$ (with the global assumptions Ψ).*

6. Soundness

The aim of this section is to prove soundness of the introduced sequent calculi. Soundness of the tableau calculi will immediately follow by Theorem 16.

THEOREM 17 (Soundness of sequent calculi). *If a sequent has a refutation in the sequent calculus for $\mathcal{L}$, then it is $\mathcal{L}$-unsatisfiable.*

PROOF. The proof is by induction on the number of inferences in the refutation. The smallest refutations are simply the axioms $S, A, \neg A$. Evidently, this sequent has no model. Take any longer refutation and consider the bottom inference of this refutation

$$\frac{S_1 \quad \cdots \quad S_n}{S} \,. \tag{1}$$

If we prove that any $\mathcal{L}$-model of S is also a $\mathcal{L}$-model for some S_i, then we are done, since all S_i have shorter refutations than S and by the induction hypothesis cannot have $\mathcal{L}$-models.

So we now assume that $\mathfrak{S}$ is a $\mathcal{L}$-model of S and prove that it is also a model of some S_i. The proof is by the case analysis on the inference rule used in inference (1). The proof is standard for most cases, so we only consider two rules ($[t]$) and ($\langle t \rangle$). In the proof, let $[t]S$ denote the set $\{[t]A \mid A \in S\}$.

CASE : rule ($\langle t \rangle$) for logics K, D and T. The rule has the form

$$\frac{S_2, A}{S_1, [t]S_2, \langle t \rangle A} \ (\langle t \rangle)$$

for some S_1, S_2. We assume that there exist a structure $\mathfrak{S}$ and valuation V under which for some world w we have $w \Vdash S_1, [t]S_2, \langle t \rangle A$ and show that the sequent S_2, A is satisfiable in $\mathfrak{S}$. Since $w \Vdash \langle t \rangle A$, there exists a world w' such that $w \xrightarrow{V(t)} w'$ and $w' \Vdash A$. By $w \Vdash [t]S_2$ and $w \xrightarrow{V(t)} w'$ we also have $w' \Vdash S_2$, and therefore $w' \Vdash S_2, A$.

CASE : rule ($[t]$) for logic $\mathsf{D4}$. The rule has the form

$$\frac{S_2, [t]S_2}{S_1, [t]S_2} \ (\langle t \rangle)$$

We assume that there exist a $\mathsf{D4}$-structure $\mathfrak{S}$, world w and valuation V such that under $\mathfrak{S}$ and V we have $w \Vdash S_1, [t]S_2$ and show that the sequent $S_2, [t]S_2$ is satisfiable in $\mathfrak{S}$.

Since $\mathfrak{S}$ is a $\mathsf{D4}$-structure, there exists a world w' such that $w \xrightarrow{V(t)} w'$. By $w \Vdash [t]S_2$ and $w \xrightarrow{V(t)} w'$ we have $w' \Vdash S_2$.

Consider any world w'' such that $w' \xrightarrow{V(t)} w''$. Since $\mathfrak{S}$ is a $\mathsf{D4}$-structure, $w \xrightarrow{V(t)} w''$. This together with $w \Vdash [t]S_2$ gives us $w'' \Vdash S_2$. Since w'' was arbitrary world satisfying $w' \xrightarrow{V(t)} w''$, $w' \Vdash [t]S_2$. Thus, $w' \Vdash S_2, [t]S_2$.

□

7. Model existence

Now our aim is to prove completeness of the sequent calculi.

THEOREM 18 (Completeness of sequent calculi). *Let S be a set of sentences. If S has no refutation in the sequent calculus for $\mathcal{L}$ with the global assumptions Ψ, then there exists an $\mathcal{L}$-structure $\mathfrak{S}$ and a valuation V under which S is locally satisfied and all formulas in Ψ are globally satisfied.*

We will build a $\mathcal{L}$-model for a sequent with no refutation using the construction of Fitting (1983). The construction is roughly as follows. First we define an abstract property capturing the syntactic counterpart of satisfiability, called the *consistency property*. Then we show that the family of non-refutable sets of formulas is such a consistency property. The completeness is finally proved by showing that each set of formulas in this consistency property must be satisfiable. This means that every non-refutable sequent has a model. We will also establish a stronger form of completeness for calculi with global assumptions Ψ: in the constructed $\mathcal{L}$-model all formulas in Ψ will be satisfied *globally*, i.e. in *every* world. Our construction differs from that of Fitting (1983) in several respects. First, the new modal operators require a special treatment. Second, our logic has function symbols which were not treated in (Fitting, 1983). Third, we simplified the construction of Fitting (1983) in several respects.

We will use the fact that valuations do not change in different worlds, and prove the existence of a special kind of model: a structure in which (i) the domain consists of all ground terms in $\mathcal{T}(\Sigma \cup P)$, where P is a set of parameters, and (ii) each term is evaluated to itself. We call such structures *Herbrand structures.*

DEFINITION 19 (Herbrand structure). *Let $\mathfrak{S} = \langle \mathcal{W}, \mathcal{D}, \longrightarrow, I, \Vdash \rangle$ be a structure over a domain $\mathbf{D}$ of a signature Σ. Then $\mathfrak{S}$ is called a* Herbrand structure *if (i) $\mathbf{D}$ is the set of ground terms of $\mathcal{T}(\Sigma \cup P)$ for some set of parameters P and (ii) for the interpretation function I the following holds*

1. *for every constant c, $I(c) = c$;*
2. *for every function symbol f and terms $t_1, \ldots, t_n$, $I(f)(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$;*

Note the following two properties of Herbrand structures.

1. The requirement on functions to be totally defined in a world has a consequence on the structure of the worlds for Herbrand structures: if the domain $\mathcal{D}_w$ of a world w contains parameters $p_1, \ldots, p_n$, it also contains all terms built using the function symbols of Σ and parameters $p_1, \ldots, p_n$.

2. If a valuation V in a Herbrand structure is the identity function on the set of parameters, i.e. $V(p) = p$ for all p, then also $V(t) = t$ for every ground term $t \in \mathcal{T}(\Sigma \cup P)$.

In the proof of the Model Existence Theorem below we will construct a Herbrand structure.

In the proofs below we will assume that we have a set of parameters P, which has the same cardinality as the set of closed formulas of Σ. If, for instance, the number of constants, function symbols and relation symbols are all countable, then we also assume the set P to be countable.

DEFINITION 20 (Consistency property). *A set of sequents* $\mathbf{C}$ *is called a (first-order)* $\mathcal{L}$-consistency property *if, for each* $S \in \mathbf{C}$,

(A) *S contains no atomic formula A and its negation $\neg A$.*

($\wedge$) *If $A \wedge B \in S$, then $S \cup \{A, B\} \in \mathbf{C}$.*

($\vee$) *If $A \vee B \in S$, then $S \cup \{A\} \in \mathbf{C}$ or $S \cup \{B\} \in \mathbf{C}$.*

($\langle t \rangle$) *If $\langle t \rangle A \in S$, then $S^{[t]} \cup \{A\} \in \mathbf{C}$.*

($[t]$) *1. For logics* K *and* $\mathsf{K4}$ *no other conditions.*

2. For T *and* $\mathsf{S4}$, *if $[t]A \in S$, then $S \cup \{A\} \in \mathbf{C}$.*

3. For D *and* $\mathsf{D4}$, *if $S \in \mathbf{C}$, then $S^{[t]} \in \mathbf{C}$.*

($\forall$) *If $\forall x A(x) \in S$, then $S \cup \{A(t)\} \in \mathbf{C}$ for* every *ground term t.*

($\exists$) *If $\exists x A(x) \in S$, then $S \cup \{A(t)\} \in \mathbf{C}$ for* some *ground term t.*

Let Ψ be a set of sentences of the signature Σ. A consistency property $\mathbf{C}$ is called Ψ-compatible *if*

(Ψ) *For every $S \in \mathbf{C}$ and $A \in \Psi$ we have $S \cup \{A\} \in \mathbf{C}$.*

We will simply say *consistency property* instead of $\mathcal{L}$-consistency property, when it causes no ambiguity. Note that we only speak of Ψ-compatible consistency properties when formulas in Ψ use no parameters. Also note that the notion of consistency property depends on the signature and parameters used in formulas, because the $(\forall)$-condition requires a property to be satisfied for *every ground term*. So if a set of sequents is a consistency property in a language with parameters P, it may violate the $(\forall)$-condition considered in a language with more parameters.

The main theorem of this section is the following.

THEOREM 21 (Model existence). *Let $\mathbf{C}$ be a Ψ-compatible $\mathcal{L}$-consistency property and $S \in \mathbf{C}$ be a set of sentences in the signature Σ. Then there exist a Herbrand structure $\mathfrak{S}$ and a valuation V in $\mathfrak{S}$ under which S is locally and Ψ is globally satisfied.*

The proof will be given after a series of lemmas.

The first three lemmas (22, 25 and 27) are applied to the consistency property to close it under subsets, add all parameter variants to it, and add all sets constructed from finite subsets already in it. The three lemmas are summarized as Proposition 28.

The first step in our attempt to build a model is to close the consistency property under subsets.

LEMMA 22 (Subsets closure). *Let $\mathbf{C}$ be a Ψ-compatible $\mathcal{L}$-consistency property and $\mathbf{C}'$ consist of the subsets of all $S \in \mathbf{C}$. Then $\mathbf{C}'$ is also a Ψ-compatible $\mathcal{L}$-consistency property and $\mathbf{C}'$ is closed under subsets.*

PROOF. We consider only one case, the other cases are similar.

$(\wedge)$ Suppose $S' \in \mathbf{C}'$ and $A \wedge B \in S'$. We have to show $S' \cup \{A, B\} \in \mathbf{C}'$. Since $\mathbf{C}'$ consists of the subsets of sets in $\mathbf{C}$, for some $S \in \mathbf{C}$ we have $S' \subseteq S$. Then $A \wedge B \in S$, and by the $(\wedge)$-condition on consistency properties $S \cup \{A, B\} \in \mathbf{C}$. Evidently, $S' \cup \{A, B\} \subseteq S \cup \{A, B\}$, hence $S' \cup \{A, B\} \in \mathbf{C}'$.

□

Now, the conditions on consistency properties reflect both the definition of truth of formulas in structures and the rules of the sequent calculi, but with one exception. The $(\exists)$-condition is in the spirit of the definition of truth (if $\exists x A(x)$ is true, then $A(p)$ is true for some p). However, the corresponding sequent calculus rule is

$$\frac{S, A(p)}{S, \exists x A(x)}\ (\exists),$$

where p is a new parameter. We want to make the notion of consistency property reflect this rule, so we will change the $(\exists)$-condition of consistency properties.

DEFINITION 23 (Alternate $\mathcal{L}$-consistency property). *Let* $\mathbf{C}$ *be a set of sequents. We say that* $\mathbf{C}$ *meets the* new parameter condition *if for each* $S \in \mathbf{C}$*, if* $\exists x A(x) \in S$*, then* $S \cup \{A(p)\} \in \mathbf{C}$ *for* every parameter p *that does not occur in* S*. If* $\mathbf{C}$ *satisfies all conditions for a (*Ψ*-compatible) consistency property except that the (*$\exists$*)-condition is replaced by the new parameter condition, then* $\mathbf{C}$ *is called an* alternate $\mathcal{L}$-consistency property*.*

The condition that $A(p) \in S$ for *every* parameter p that does not occur in S is not restrictive. Since p does not occur in S (and, being a parameter, does not occur in Ψ either), there is from the viewpoint of S no difference between p and any other new parameter.

An alternate consistency property is not necessarily a consistency property. S may already contain all parameters. We will overcome this problem by an iterative construction of consistency properties, throwing more parameters into the language at every iteration step.

DEFINITION 24 (Parameter substitution, parameter variant). *Any function* $\sigma : P \to \mathcal{T}(\Sigma \cup P)$ *is called a* parameter substitution*. For a sentence* A*,* $\sigma(A)$ *denotes the result of replacing every parameter in* A *by its image under* σ*. Similarly,* σ *is extended to sets of sentences. The formula* $\sigma(A)$ *and the set* $\sigma(S)$ *are called the* parameter variants *of* A *and* S*, respectively.*

LEMMA 25 (Parameter variants extension). *Suppose that* $\mathbf{C}'$ *is a* Ψ*-compatible* $\mathcal{L}$*-consistency property closed under subsets. Define* $\mathbf{C}''$ *by:* $S \in \mathbf{C}''$ *if* $\sigma(S) \in \mathbf{C}'$ *for some parameter substitution* σ*. Then* $\mathbf{C}''$ *extends* $\mathbf{C}'$ *and is a* Ψ*-compatible* alternate $\mathcal{L}$*-consistency property closed under subsets.*

PROOF. We will only verify that $\mathbf{C}''$ satisfies the new parameter condition, all other conditions are not difficult to prove.

Suppose $S \in \mathbf{C}''$, $\exists x A(x) \in S$ and p is a parameter that does not occur in S. We have to show that $S \cup \{A(p)\} \in \mathbf{C}''$.

Since $S \in \mathbf{C}''$, there is a parameter substitution σ such that $\sigma(S) \in \mathbf{C}'$. Note that $\sigma(\exists x A(x)) \in \sigma(S)$. Denote $\sigma(A(x))$ by $B(x)$. Since $\mathbf{C}'$ is a $\mathcal{L}$-consistency property and $\exists x B(x) \in \sigma(S)$, there exists a term t such that $\sigma(S) \cup \{B(t)\} \in \mathbf{C}'$. Define σ' to behave exactly as σ except that $\sigma'(p) = t$. Using the fact that p does not occur in S, it is not hard to argue that $\sigma'(S \cup \{A(p)\}) = \sigma(S) \cup \{B(t)\}$. Thus $S \cup \{A(p)\}$ is a parameter variant of a sequent in $\mathbf{C}'$, and hence it is a member of $\mathbf{C}''$. $\square$

Next, we would like the consistency property to satisfy the finite character property defined below.

DEFINITION 26 (Finite character). *A collection* $\mathbf{C}$ *of sets is said to be of* finite character *if for every set* S, S *belongs to* $\mathbf{C}$ *if and only if each finite subset of* S *belongs to* $\mathbf{C}$.

LEMMA 27 (Finite character extension). *Suppose* $\mathbf{C}''$ *is a* Ψ*-compatible alternate* $\mathcal{L}$*-consistency property closed under subsets. Let* $\mathbf{C}'''$ *consist of those sequents* S *all whose finite subsets are in* $\mathbf{C}''$*. Then* $\mathbf{C}'''$ *is again a* Ψ*-compatible alternate* $\mathcal{L}$*-consistency property, which extends* $\mathbf{C}''$ *and is of finite character.*

PROOF. As usual, we will only check some conditions on alternate consistency properties.

($\vee$) Let $S \in \mathbf{C}'''$ and $A \vee B \in S$. We have to prove that either $S \cup \{A\} \in \mathbf{C}'''$ or $S \cup \{B\} \in \mathbf{C}'''$. Suppose, by contradiction, $S \cup \{A\} \notin \mathbf{C}'''$ and $S \cup \{B\} \notin \mathbf{C}'''$. By the definition of $\mathbf{C}'''$, there are finite sets $F_1 \subseteq S \cup \{A\}$ and $F_2 \subseteq S \cup \{B\}$ such that $F_1, F_2 \notin \mathbf{C}''$. Consider the finite set $F = (F_1 - \{A\}) \cup (F_2 - \{B\}) \cup \{A \vee B\}$. Then F is a finite subset of S, hence $F \in \mathbf{C}''$. By the condition ($\vee$) on $\mathbf{C}''$, either $F \cup \{A\} \in \mathbf{C}''$ or $F \cup \{B\} \in \mathbf{C}''$. We show that in either case we obtain a contradiction. Suppose $F \cup \{A\} \in \mathbf{C}''$ (the second case is similar). It is not hard to argue that $F_1 \subseteq F \cup \{A\}$. Since $\mathbf{C}''$ is closed under subsets, $F_1 \in \mathbf{C}''$. Contradiction.

($\exists$) Let $S \in \mathbf{C}'''$ and $\exists x A(x) \in S$. We have to prove that for each parameter p not occurring in S we have $S \cup \{A(p)\} \in \mathbf{C}'''$. Suppose, by contradiction $S \cup \{A(p)\} \notin \mathbf{C}'''$. By the definition of $\mathbf{C}'''$, there is a finite set $F \subseteq S \cup \{A(p)\}$ such that $F \notin \mathbf{C}''$. Consider the finite set $F' = (F - \{A(p)\}) \cup \{\exists x A(x)\}$. Then F' is a finite subset of S, so $F' \in \mathbf{C}''$. Note that p does not occur in F', then by the condition ($\exists$) on $\mathbf{C}''$ we have $F' \cup \{A(p)\} \in \mathbf{C}''$. Evidently, $F \subseteq F' \cup \{A(p)\}$. Since $\mathbf{C}''$ is closed under subsets, $F \in \mathbf{C}''$. Contradiction.

It is easy to see that $\mathbf{C}'''$ is of finite character. □

Let us summarize the results obtained so far:

PROPOSITION 28. *Let* $\mathbf{C}$ *be a* Ψ*-compatible* $\mathcal{L}$*-consistency property. Then* $\mathbf{C}$ *can be extended to a set* $\mathbf{C}^*$ *that is a* Ψ*-compatible alternate* $\mathcal{L}$*-consistency property of finite character.*

Let us now state two lemmas about alternate consistency properties of finite character. Lemma 30 says that a restriction of an alternate consistency property of finite character to certain sublanguages gives us an alternate consistency property of finite character. It will be helpful when we need to throw in new parameters in order to satisfy the $(\exists)$-condition on consistency properties. Lemma 33 asserts the existence of maximal elements in sets of finite character, which will be used as possible worlds in the model construction.

DEFINITION 29 (Section). *Let P be a set of parameters and $\mathbf{C}$ be a set of sequents. By the P-*section *of $\mathbf{C}$, denoted $\mathbf{C}{\restriction_P}$, we mean*

$$\{S \in \mathbf{C} \mid \textit{each parameter occurring in } S \textit{ is a member of } P\}.$$

The following lemma is straightforward.

LEMMA 30 (Section restriction). *Suppose $\mathbf{C}$ is a Ψ-compatible alternate $\mathcal{L}$-consistency property of finite character in the language with parameters P and $P_0 \subseteq P$. Then the P_0-section of $\mathbf{C}$ is a Ψ-compatible alternate $\mathcal{L}$-consistency property of finite character in the language with parameters P_0.*

Note that this lemma does not hold when alternate consistency properties are replaced by consistency properties. Consider, e.g., that while $\mathbf{C} = \{\{\exists x A(x)\}, \{\exists x A(x), A(p)\}\}$ is a consistency property, $\mathbf{C}{\restriction_\emptyset}$ is not.

The following lemma has a straightforward proof by transfinite induction on ordinals.

LEMMA 31. *Let $\mathbf{C}$ be a collection of sets of finite character. Then*

1. *Each member of $\mathbf{C}$ is contained in a maximal member;*
2. *The union of any chain of members is a member.*

Now we give a definition closely related to witness formulas $A(p)$ for $\forall x A(x)$, and then proceed to the proof of Model Existence Theorem 21.

DEFINITION 32 (Downward saturated set). *Suppose $\mathbf{C}^*$ is a Ψ-compatible alternate $\mathcal{L}$-consistency property of finite character and P is a set of parameters. Let S be a set of sentences in $\mathcal{F}(\Sigma \cup P)$. We say S is* downward saturated in $\mathbf{C}^*{\restriction_P}$ *if*

1. *S is maximal in the alternate $\mathcal{L}$-consistency property $\mathbf{C}^*{\restriction_P}$.*
2. *If $\exists x A(x) \in S$, then $A(p) \in S$ for some $p \in P$.*

LEMMA 33 ($\exists$-completion). *Suppose* $\mathbf{C}^*$ *is a* Ψ*-compatible alternate* $\mathcal{L}$*-consistency property of finite character in the language with parameters* P_0*, where* P_0 *has the same cardinality as* $\mathcal{F}(\Sigma)$*. Suppose also that* $P, Q \subseteq P_0$ *are disjoint sets of parameters of the same cardinality as* $\mathcal{F}(\Sigma)$*. If* $S \in \mathbf{C}^*\restriction_P$*, then* S *may be extended to a set that is downward saturated in* $\mathbf{C}^*\restriction_{P\cup Q}$*.*

PROOF. Since Q is infinite, it can be partitioned into countably many pairwise disjoint sets $Q_1, Q_2, \ldots$, all of the same cardinality as Q itself. Note that the sets of $\exists$-sentences in the sets $\mathcal{F}(\Sigma)$, $\mathcal{F}(\Sigma \cup P)$, $\mathcal{F}(\Sigma \cup P \cup Q)$, and $\mathcal{F}(\Sigma \cup P \cup Q_1 \cup Q_2 \cup \cdots \cup Q_n)$ are all of the same cardinality as P.

Now, suppose $S \in \mathbf{C}^*\restriction_P$. Then $S \in \mathbf{C}^*\restriction_{P\cup Q_1}$. Well-order the members of Q_1 as $q_0, q_1, \ldots, q_\alpha, \ldots$ and the $\exists$-sentences of S: $\exists x A_0(x)$, $\exists x A_1(x)$, ..., $\exists x A_\alpha(x)$, ... in such a way that for every ordinal α, the set of parameters occurring in $A_\alpha(x)$ is a subset of $\{q_\beta \mid \beta < \alpha\}$.

Consider the set $S \cup \{A_\beta(q_\beta) \mid \beta = 0, 1, \ldots, \alpha, \ldots\}$. We claim it is a member of $\mathbf{C}^*\restriction_{P\cup Q_1}$. Suppose the contrary, then there is the smallest ordinal α such that $S \cup \{A_\beta(q_\beta) \mid \beta < \alpha\} \in \mathbf{C}^*\restriction_{P\cup Q_1}$ but $S \cup \{A_\beta(q_\beta) \mid \beta \leq \alpha\} \notin \mathbf{C}^*\restriction_{P\cup Q_1}$. Note that q_α does not occur in any member of $S \cup \{A_\beta(q_\beta) \mid \beta < \alpha\} \in \mathbf{C}^*\restriction_{P\cup Q_1}$. By Lemma 30, $\mathbf{C}^*\restriction_{P\cup Q_1}$ is an alternate $\mathcal{L}$-consistency property, so by the ($\exists$)-condition of alternate consistency properties

$$S \cup \{A_\beta(q_\beta) \mid \beta < \alpha\} \cup \{A_\alpha(q_\alpha)\} \in \mathbf{C}^*\restriction_{P\cup Q_1}.$$

But this set is exactly $S \cup \{A_\beta(q_\beta) \mid \beta \leq \alpha\}$, so we have a contradiction.

We have proved that $S \cup \{A_\beta(q_\beta) \mid \beta = 0, 1, \ldots, \alpha, \ldots\}$ is a member of $\mathbf{C}^*\restriction_{P\cup Q_1}$. By Lemma 31, it can be extended to a maximal member S_1. Now S_1 has a "witness parameter" q for every sentence $\exists x A(x)$ that uses only parameters in P, i.e. such that $A(q) \in S_1$. However, our construction does not imply that S_1 contains a witness for $\exists$-formulas containing parameters in Q_1.

To overcome this problem, we iterate the construction again, this time using S_1 instead of S and Q_2 instead of Q_1. Further iterations gives us a sequence of sets $S \subseteq S_1 \subseteq S_2 \subseteq \cdots$, such that

(2) each S_n is maximal in $\mathbf{C}^*\restriction_{P\cup Q_1\cup\cdots\cup Q_n}$;

(3) if $\exists x A(x) \in S_n$, then for some $q \in Q_{n+1}$ we have $A(q) \in S_{n+1}$.

Define $S^* = \bigcup_n S_n$. By Lemma 31, $\mathbf{C}^*$ is closed under chains, so $S^* \in \mathbf{C}^*$. Note that all parameters occurring in S^* are in Q, thus $S^* \in \mathbf{C}^*\restriction_{P\cup Q}$. We claim that S^* is a maximal member of $\mathbf{C}^*\restriction_{P\cup Q}$. This amounts to showing that for every sentence $B \in \mathcal{F}(\Sigma \cup P \cup Q)$, if $S^* \cup \{B\} \in \mathbf{C}^*\restriction_{P\cup Q}$, we have $B \in S^*$. Since B can contain only a finite number of parameters, then $B \in \mathcal{F}(\Sigma \cup P \cup Q_1 \cup \cdots \cup Q_n)$ for some n. Since $S \cup \{B\} \in \mathbf{C}^*\restriction_{P\cup Q}$, which is of finite character, hence closed under subsets, $S_n \cup \{B\} \in \mathbf{C}^*\restriction_{P\cup Q}$. It follows that $S_n \cup \{B\} \in \mathbf{C}^*\restriction_{P\cup Q_1\cup\cdots\cup Q_n}$, hence by maximality of S_n we have $B \in S_n$, and since $S_n \subseteq S^*$ also $B \in S^*$.

Using (3), one may show that if $\exists x A(x) \in S^*$, then $A(q) \in S^*$ for some $q \in Q$. Thus S^* is downward saturated in $\mathbf{C}^*\restriction_{P\cup Q}$. □

Now we are ready to prove the Model Existence Theorem.

PROOF (of Theorem 21). Recall that we, given a Ψ-compatible $\mathcal{L}$-consistency property $\mathbf{C}$ and a set of sentences $S \in \mathbf{C}$ in the signature Σ, are going to build a Herbrand structure $\mathfrak{S}$ and a valuation V in $\mathfrak{S}$ under which S is locally and Ψ is globally satisfied.

Using Proposition 28 extend $\mathbf{C}$ to an alternate $\mathcal{L}$-consistency property $\mathbf{C}^*$ that is also Ψ-compatible. We take a set P of parameters of the same cardinality as $\mathcal{F}(\Sigma)$ and split it in countably many mutually disjoint sets $P_1, P_2, \ldots$, each of them of the same cardinality as P itself. We will define our structure $\mathfrak{S} = \langle \mathcal{W}, \mathcal{D}, \longrightarrow, I, \Vdash \rangle$ over a domain $\mathbf{D}$ as follows.

The domain $\mathbf{D}$ is the set of all ground terms of Σ with parameters in P. We put a set S in $\mathcal{W}$ if S is downward saturated in $\mathbf{C}^*\restriction_{P_1\cup\cdots\cup P_n}$ for some n. The domain of any such world S is the subset of $\mathbf{D}$ consisting of terms with parameters in $P_1 \cup \cdots \cup P_n$. Since we want $\mathfrak{S}$ to be a Herbrand structure, we define the interpretation function I as in Definition 19. Now we define the valuation V as the identity mapping. By our remarks after the definition of Herbrand structures we have $V(t) = t$ for every ground term t of the signature $\Sigma \cup P$.

Define the reachability relation $\longrightarrow$ on $\mathcal{W}$ as follows: $S \xrightarrow{t} S'$ if $S^{[t]} \subseteq S'$ and $\mathcal{D}_S \subseteq \mathcal{D}_{S'}$.

Now we define the relation $\Vdash$ on $\mathfrak{S}$ as follows. For any atomic sentence A we let $S \Vdash A$ if $A \in S$. Thus, the structure $\mathfrak{S}$ and valuation V are defined. We prove the following:

(4) $\quad \mathfrak{S}$ is a $\mathcal{L}$-structure.

First, we note the following useful facts:

(5) $\quad$ For any choice of logic $\mathcal{L}$, if $S_1 \subseteq S_2$, then ${S_1}^{[t]} \subseteq {S_2}^{[t]}$.

(6) $\quad$ For $\mathsf{S4}$ we have $S^{[t]} = {S^{[t]}}^{[t]}$ and $S^{[t]} \subseteq S$.

(7) $\quad$ For $\mathsf{K4}$ and $\mathsf{D4}$ we have $S^{[t]} \subseteq {S^{[t]}}^{[t]}$.

Now we verify (4) for each particular logic $\mathcal{L}$.

Case : $\mathcal{L} = \mathsf{K}$. There is nothing to verify since every structure is a K-structure.

Case : $\mathcal{L} = \mathsf{T}$. We have to prove $S \xrightarrow{t} S$, i.e. $S^{[t]} \subseteq S$. Take any $A \in S^{[t]}$, then $[t]A \in S$. Since $\mathbf{C}^*$ is an alternate T-consistency property, $S \cup \{A\} \in \mathbf{C}^*$. But S is maximal in a section of $\mathbf{C}^*$ containing all parameters in A, hence $A \in S$. Since A was arbitrary, $S^{[t]} \subseteq S$.

The condition on the domains $\mathcal{D}_S \subseteq \mathcal{D}_S$ is obvious. This condition will be obvious for all other cases, so we do not verify it anymore.

Case : $\mathcal{L} = \mathsf{D}$. Suppose $S \in \mathcal{W}$, we have to find $S' \in \mathcal{W}$ such that $S \xrightarrow{t} S'$, i.e. $S^{[t]} \subseteq S'$. Since $\mathbf{C}^*$ is an alternate D-consistency property, $S^{[t]} \in \mathbf{C}^*$. Take S' to be a maximal member extending $S^{[t]}$ in any language containing all parameters in S, then $S^{[t]} \subseteq S'$.

Case : $\mathcal{L} = \mathsf{K4}$. Suppose ${S_1}^{[t]} \subseteq S_2$ and ${S_2}^{[t]} \subseteq S_3$. We have to prove ${S_1}^{[t]} \subseteq S_3$. By (5) above, ${{S_1}^{[t]}}^{[t]} \subseteq {S_2}^{[t]}$. Then by (7) above, ${S_1}^{[t]} \subseteq {S_2}^{[t]}$, this and ${S_2}^{[t]} \subseteq S_3$ implies ${S_1}^{[t]} \subseteq S_3$.

Case : $\mathcal{L} = \mathsf{D4}$. Seriality is proved by the same argument as for D above and transitivity by the same argument as for K4.

Case : $\mathcal{L} = \mathsf{S4}$. Showing reflexivity reduces to $S^{[t]} \subseteq S$, which follows from (6) above. Transitivity is proved by the same argument as for K4.

We proved (4), i.e. that $\mathfrak{S}$ is a $\mathcal{L}$-structure. Next, we prove the following.

(8) Let A be a sentence and $S \in \mathbf{C}^*$. If $A \in S$, then $S \Vdash A$.

The proof is by induction on the structure of A. We take any $S \in \mathbf{C}^*$. Suppose S is downward saturated in $\mathbf{C}^*\restriction_{P_1 \cup \cdots \cup P_n}$ and $A \in S$.

Case : A is atomic. Then $S \Vdash A$ by the definition of $\Vdash$ in $\mathfrak{S}$.

Case : A is a negative literal $\neg B$. By definition of consistency property we have $B \notin S$. Then by the definition of $\Vdash$ in $\mathfrak{S}$ we have $S \nVdash B$, and hence $S \Vdash A$.

Case : A is $A_1 \wedge A_2$. By the definition of consistency property we have $S \cup \{A_1, A_2\} \in \mathbf{C}^*$. Since S is maximal, this implies $A_1, A_2 \in S$. By the induction hypothesis, $S \Vdash A_1$ and $S \Vdash A_2$, hence $S \Vdash A_1 \wedge A_2$.

Case : A is $A_1 \vee A_2$. By the definition of consistency property, either $S \cup \{A_1\} \in \mathbf{C}^*$ or $S \cup \{A_2\} \in \mathbf{C}^*$. Since S is maximal, this implies that either $A_1 \in S$ or $A_2 \in S$. By the induction hypothesis, either $S \Vdash A_1$ or $S \Vdash A_2$, hence $S \Vdash A_1 \vee A_2$.

CASE : A is $\forall x B(x)$. By Lemma 30, $\mathbf{C}^*\restriction_{P_1\cup\cdots\cup P_n}$ is an alternate consistency property in the signature $\Sigma \cup P_1 \cup \cdots \cup P_n$. Then for every term $t \in \mathcal{T}(\Sigma \cup P_1 \cup \cdots \cup P_n)$ we have $S \cup \{B(t)\} \in \mathbf{C}^*\restriction_{P_1\cup\cdots\cup P_n}$. By maximality, S contains all formulas $B(t)$. By the induction hypothesis $S \Vdash B(t)$ for all such t. But $\mathcal{T}(\Sigma \cup P_1 \cup \cdots \cup P_n)$ is the domain of S, hence $S \Vdash \forall x B(x)$.

CASE : A is $\exists x B(x)$. Since S is downward saturated, S contains $B(t)$ for some $t \in \mathcal{T}(\Sigma \cup P_1 \cup \cdots \cup P_n)$, therefore $S \Vdash B(t)$. But t belongs to the domain of S, hence $S \Vdash \exists x B(x)$.

CASE : A is $\langle t\rangle B$. For every choice of logic $\mathcal{L}$ we have $S^{[t]} \cup \{B\} \in \mathbf{C}^*$. Take any downward saturated S' in a signature containing $\Sigma \cup P_1 \cup \cdots \cup P_n$ such that $S^{[t]} \cup \{B\}$. By our construction we have $S \xrightarrow{t} S'$. By the induction hypothesis $S' \Vdash B$, hence $S \Vdash \langle t\rangle B$.

CASE : A is $[t]B$. We take any S' such that $S^{[t]} \subseteq S'$ and S' contains all parameters in S and claim $B \in S'$, then by the induction hypothesis $S' \Vdash B$. Consider two cases.

SUBCASE : $\mathcal{L} = \mathsf{S4}$. By the definition of $S^{[t]}$ for $\mathsf{S4}$, if $[t]B \in S$, then $[t]B \in S^{[t]}$, hence $[t]B \in S'$. By the definition of consistency property for $\mathsf{S4}$, since $[t]B \in S'$, then $S' \cup \{B\} \in \mathbf{C}^*$. Since S' is maximal, $B \in S'$.

SUBCASE : $\mathcal{L}$ is any other logic. By the definition of $S^{[t]}$ for $\mathcal{L}$, if $[t]B \in S$, then $B \in S^{[t]}$, hence $B \in S'$.

So (8) is proved. We return to the proof of the Model Existence Theorem. Recall that we intend to prove that under $\mathfrak{S}$ and V all formulas occurring in any member of $\mathbf{C}$ are locally satisfied and all formulas in Ψ are globally satisfied.

1. Take any $A \in S \in \mathbf{C}$. Extend S to a downward saturated $S' \in \mathbf{C}^*$ (Proposition 28 and Lemma 33). Then S' is a world in $\mathfrak{S}$ and $A \in S'$. By (8), A is satisfied in this world.

2. Take any $A \in \Psi$ and $S \in \mathcal{W}$. Let S be downward saturated in $\mathbf{C}^*\restriction_{P_1\cup\cdots\cup P_n}$. By Lemma 30, $\mathbf{C}^*\restriction_{P_1\cup\cdots\cup P_n}$ is a Ψ-compatible alternate $\mathcal{L}$-consistency property, hence $S \cup \{A\} \in \mathbf{C}^*\restriction_{P_1\cup\cdots\cup P_n}$. Since S is maximal, $A \in S$. By (8), A is satisfied in the world S. Since S was arbitrary, A is satisfied in every world of $\mathfrak{S}$.

The proof of the Model Existence Theorem is completed. $\square$

8. Completeness

The Completeness Theorem 18 can now be proved using the Model Existence Theorem 21 in a rather straightforward way.

Proof (of Theorem 18). Take an infinite set of parameters P and consider the following set $\mathbf{C}$ of sequents of the signature Σ: we put S in $\mathbf{C}$ if S uses only a finite number of parameters and S has no refutation in $\mathcal{L}$ with global assumptions Ψ. We claim

(9) $\mathbf{C}$ is a Ψ-compatible $\mathcal{L}$-consistency property.

We consider only some conditions of $\mathcal{L}$-consistency property, others are rather straightforward. Take any $S \in \mathbf{C}$.

(A) *We prove: S contains no atomic formula A and its negation $\neg A$.* Indeed, if S contains A and $\neg A$, it is an axiom of $\mathcal{L}$, hence has a refutation.

($\wedge$) *We prove: if $A \wedge B \in S$, then $S \cup \{A, B\} \in \mathbf{C}$.* Suppose $A \wedge B \in S$. Consider the inference

$$\frac{S, A, B}{S, A \wedge B}\ (\wedge).$$

If $S \cup \{A, B\}$ had a refutation, so would $S \cup \{A \wedge B\} = S$, hence $S \cup \{A, B\}$ has no refutation. By the definition of $\mathbf{C}$, we have $S \cup \{A, B\} \in \mathbf{C}$.

($\langle t \rangle$) *We prove: if $\langle t \rangle A \in S$, then $S^{[t]} \cup \{A\} \in \mathbf{C}$.* Suppose $\langle t \rangle A \in S$. Consider the inference

$$\frac{S^{[t]}, A}{S, \langle t \rangle A}\ (\langle t \rangle).$$

If $S^{[t]} \cup \{A\}$ had a refutation, so would $S \cup \{\langle t \rangle A\} = S$, hence $S^{[t]} \cup \{A\}$ has no refutation. By the definition of $\mathbf{C}$, we have $S^{[t]} \cup \{A\} \in \mathbf{C}$.

($\exists$) *We prove: if $\exists x A(x) \in S$, then $S \cup \{A(t)\} \in \mathbf{C}$ for* some *ground term t.* Suppose $\exists x A(x) \in S$. Consider the inference

$$\frac{S, A(p)}{S, \exists x A(x)}\ (\exists).$$

If $S \cup \{A(p)\}$ had a refutation, so would $S \cup \{\exists x A(x)\} = S$, hence $S \cup \{A(p)\}$ has no refutation. By the definition of $\mathbf{C}$, since we have an infinite number of parameters and S uses only a finite number of them, we can always choose a new parameter p so that $S \cup \{A(p)\} \in \mathbf{C}$.

(Ψ) *We prove: if $S \in \mathbf{C}$ and $A \in \Psi$, then $S \cup \{A\} \in \mathbf{C}$.* Suppose $S \in \mathbf{C}$ and $A \in \Psi$. Consider the inference

$$\frac{S, A}{S} \; (\Psi).$$

If $S \cup \{A\}$ had a refutation, so would S, hence $S \cup \{A\}$ has no refutation. By the definition of $\mathbf{C}$, we have $S \cup \{A\} \in \mathbf{C}$.

Now take S that has no refutation. By our construction, $S \in \mathbf{C}$. By Model Existence Theorem 21 there exists a $\mathcal{L}$-structure and valuation V in it under which S is locally satisfied and every formula $A \in \Psi$ globally satisfied. $\square$

9. Free-variable tableaux

In this section we change the tableau systems introduced in Section 5 into free-variable tableau systems. We will use the definitions introduced so far, except that we now allow free variables to occur in sequents, and hence in tableaux as well.

To avoid problems with the parameter condition in the ($\forall$)-rules, we introduce so-called *occurrence constraints* similar to those used in (Voronkov, 1996). Note that we could use the "dynamic skolemization" technique introduced in (Fitting, 1988) as well.

In this section we assume knowledge of the standard notions of substitutions and (idempotent, most general) unifiers see, e.g., (Eder, 1985). The application of a substitution σ to a term or formula E is denoted $E\sigma$. As usual, we may need to rename bound variables in a formula before we apply a substitution to it. Any idempotent most general unifier of n expressions $E_1, \ldots, E_n$ is denoted by $mgu(E_1, \ldots, E_n)$. The set of free variables of any expression E (e.g. formula or set of formulas) is denoted by $vars(E)$.

DEFINITION 34 (Occurrence constraint). *A* simple occurrence constraint *is either $\perp$ or an expression $p \notin X$, where p is a parameter and X is a finite set of variables. An* occurrence constraint *is a conjunction of zero or more simple occurrence constraints. A conjunction of zero simple occurrence constraints is denoted by $\top$.*

For any substitution σ and simple occurrence constraint $\mathcal{C} = (p \notin X)$, we denote by $\mathcal{C}\sigma$ the following simple occurrence constraint:

$$\mathcal{C}\sigma = \begin{cases} \bot, & \text{if } p \text{ occurs in } X\sigma; \\ p \notin \mathit{vars}(X\sigma), & \text{otherwise.} \end{cases}$$

When $\mathcal{C}$ is a conjunction $\mathcal{C}_1 \wedge \cdots \wedge \mathcal{C}_n$ of simple occurrence constraints, we denote by $\mathcal{C}\sigma$ the following occurrence constraint:

$$\mathcal{C}\sigma = \begin{cases} \bot, & \text{if } \mathcal{C}_i = \bot \text{ for some } i; \\ \mathcal{C}_1\sigma \wedge \cdots \wedge \mathcal{C}_n\sigma, & \text{otherwise.} \end{cases}$$

An occurrence constraint $\mathcal{C}$ is called *satisfiable* if $\mathcal{C}$ is not $\bot$. A *solution* to an occurrence constraint $\mathcal{C}$ is any substitution σ such that $\mathcal{C}\sigma \neq \bot$ and $x\sigma$ is ground for every variable x occurring in $\mathcal{C}$. Evidently, an occurrence constraint $\mathcal{C}$ is satisfiable if and only if it has a solution: indeed, one can take as a solution any substitution mapping all variables of $\mathcal{C}$ into any ground term not containing parameters in $\mathcal{C}$.

We call a *constrained tableau* any pair consisting of a tableau $\mathcal{T}$ and constraint $\mathcal{C}$, denoted $\mathcal{T} \cdot \mathcal{C}$. Let $\mathcal{L}$ be one of the logics $\mathsf{K}, \mathsf{D}, \mathsf{T}, \mathsf{K4}, \mathsf{D4}$ and $\mathsf{S4}$. The *free-variable tableau calculi* for $\mathcal{L}$ are shown in Figure 4.

We claim

THEOREM 35 (Equivalence of free-variable and sequent calculi). *Let S be a set of sentences of the signature Σ. Then S has a refutation in the sequent calculus for $\mathcal{L}$ (with global assumptions Ψ) if and only if there exists a derivation of $\# \cdot \mathcal{C}$ from $S \cdot \top$ in the tableau calculus for $\mathcal{L}$ (with the global assumptions Ψ) such that $\mathcal{C}$ is satisfiable.*

In order to prove this theorem, we will prove two results showing bisimulation between tableau derivations and free-variable tableau derivations.

Let $\mathcal{T} \cdot \mathcal{C}$ be a constrained tableau and σ be a substitution. We call the tableau $\mathcal{T}\sigma$ the *σ-instance* of $\mathcal{T} \cdot \mathcal{C}$ if $\mathcal{C}\sigma$ is satisfiable. A tableau $\mathcal{T}'$ is called an *instance* of $\mathcal{T} \cdot \mathcal{C}$ if it is a σ-instance of $\mathcal{T} \cdot \mathcal{C}$ for some σ.

The following lemma establishes a simulation of free-variable tableau derivations by tableau derivations.

LEMMA 36. *Suppose there exists a derivation of $\mathcal{T}_2 \cdot \mathcal{C}$ from $\mathcal{T}_1 \cdot \top$ in the free-variable tableau calculus for $\mathcal{L}$ with global assumptions Ψ. Then any instance of $\mathcal{T}_2 \cdot \mathcal{C}$ has a derivation from $\mathcal{T}_1$ in the tableau calculus for $\mathcal{L}$ with the global assumptions Ψ.*

PROOF. The proof is by induction on the length of derivations in the free-variable tableau calculus. When the derivation is of length 0, the

For all logics $(\mathsf{K}, \mathsf{D}, \mathsf{T}, \mathsf{K4}, \mathsf{D4}, \mathsf{S4})$**:**

$$\frac{S, A(\bar{s}), \neg A(\bar{t}) \mid \mathcal{T} \cdot \mathcal{C}}{\mathcal{T} mgu(\bar{s},\bar{t}) \cdot \mathcal{C} mgu(\bar{s},\bar{t})} \ |\mathrm{ax}|$$

$$\frac{S, A \vee B \mid \mathcal{T} \cdot \mathcal{C}}{S, A \mid S, B \mid \mathcal{T} \cdot \mathcal{C}} \ |\vee| \qquad \frac{S, A \wedge B \mid \mathcal{T} \cdot \mathcal{C}}{S, A, B \mid \mathcal{T} \cdot \mathcal{C}} \ |\wedge|$$

$$\frac{S, \exists x A(x) \mid \mathcal{T} \cdot \mathcal{C}}{S, A(p) \mid \mathcal{T} \cdot \mathcal{C} \wedge p \notin vars(S, \exists x A(x))} \ |\exists|^* \qquad \frac{S, \forall x A(x) \mid \mathcal{T} \cdot \mathcal{C}}{S, \forall x A(x), A(y) \mid \mathcal{T} \cdot \mathcal{C}} \ |\forall|^*$$

$$\frac{S, \langle t \rangle A \mid \mathcal{T} \cdot \mathcal{C}}{S^{[t_1]}, \ldots, S^{[t_n]}, A \mid \mathcal{T}\sigma \cdot \mathcal{C}\sigma} \ |\langle t, t_1, \ldots, t_n \rangle|^*$$

For serial logics $(\mathsf{D}, \mathsf{D4})$**:** **For reflexive logics** $(\mathsf{T}, \mathsf{S4})$**:**

$$\frac{S \mid \mathcal{T} \cdot \mathcal{C}}{S^{[t_1]}, \ldots, S^{[t_n]} \mid \mathcal{T}\sigma \cdot \mathcal{C}\sigma} \ |[t_1, \ldots, t_n]|^* \qquad \frac{S, [t]A \mid \mathcal{T} \cdot \mathcal{C}}{S, A \mid \mathcal{T} \cdot \mathcal{C}} \ |[t]|$$

For logics with global assumptions Ψ**:**

$$\frac{S \mid \mathcal{T} \cdot \mathcal{C}}{S, A \mid \mathcal{T} \cdot \mathcal{C}} \ |\Psi|^*$$

* In the rule $|\langle t, t_1, \ldots, t_n \rangle|$, $\sigma = mgu(t, t_1, \ldots, t_n)$. In the rule $|[t_1, \ldots, t_n]|$, $\sigma = mgu(t_1, \ldots, t_n)$. In the rule $|\exists|$, p is new parameter, not occurring in the premise. In the rule $|\forall|$, y is a new variable, not occurring in the premise. In the rule $|\Psi|$, $A \in \Psi$.

Figure 4. Free-variable tableau with constraints calculi

claim is obvious, since $\mathcal{T}_1$ is the only instance of $\mathcal{T}_1 \cdot \top$, when $\mathcal{T}_1$ has no free variables. For derivations with at least one inference, consider the last inference of the derivation. We will consider only two cases, other cases are similar.

CASE : the last inference is $|\mathrm{ax}|$.

$$\frac{S, A(\bar{s}), \neg A(\bar{t}) \mid \mathcal{T} \cdot \mathcal{C}}{\mathcal{T}\sigma \cdot \mathcal{C}\sigma} \ |\mathrm{ax}|,$$

where $\sigma = mgu(\bar{s}, \bar{t})$.

Take any instance of $\mathcal{T}\sigma \cdot \mathcal{C}\sigma$, then this instance has the form $\mathcal{T}\sigma\tau$ for some substitution τ such that $\mathcal{C}\sigma\tau$ is satisfiable. We have to prove that $\mathcal{T}\sigma\tau$ is derivable from $\mathcal{T}_1$.

We claim that the following is a valid inference in the tableau calculus:

$$\frac{(S, A(\bar{s}), \neg A(\bar{t}) \mid \mathcal{T})\sigma\tau}{\mathcal{T}\sigma\tau}\ |\mathrm{ax}|. \tag{10}$$

Indeed, since σ is a unifier of $\bar{s}$ and $\bar{t}$, then $A(\bar{s})\sigma = A(\bar{t})\sigma$, hence $A(\bar{s})\sigma\tau = A(\bar{t})\sigma\tau$.

Since $\mathcal{C}\sigma\tau$ is satisfiable, we get that $(S, A(\bar{s}), \neg A(\bar{t}) \mid \mathcal{T})\sigma\tau$ is a $\sigma\tau$-instance of $S, A(\bar{s}), \neg A(\bar{t}) \mid \mathcal{T} \cdot \mathcal{C}$. By the induction hypothesis, this instance has a derivation from $\mathcal{T}_1$. Add to this derivation the inference (10), then we obtain a required derivation of $\mathcal{T}\sigma\tau$.

CASE : the last inference is $|\exists|$.

$$\frac{S, \exists x A(x) \mid \mathcal{T} \cdot \mathcal{C}}{S, A(p) \mid \mathcal{T} \cdot \mathcal{C} \wedge p \notin \mathit{vars}(S, \exists x A(x))}\ |\exists|.$$

Take any instance of $S, A(p) \mid \mathcal{T} \cdot \mathcal{C} \wedge p \notin \mathit{vars}(S, \exists x A(x))$, then this instance has the form $S\tau, A(p)\tau \mid \mathcal{T}\tau$ for some substitution τ such that $(\mathcal{C} \wedge p \notin \mathit{vars}(S, \exists x A(x)))\tau$ is satisfiable. We have to prove that $S\tau, A(p)\tau \mid \mathcal{T}\tau$ is derivable from $\mathcal{T}_1$.

Since $(\mathcal{C} \wedge p \notin \mathit{vars}(S, \exists x A(x)))\tau$ is satisfiable, by definition of constraint satisfiability, $\mathcal{C}\tau$ is satisfiable and p does not occur in $S\tau, \exists x A(x)\tau$. Since p does not occur in $S\tau, \exists x A(x)\tau$, the following is a valid inference in the tableau calculus:

$$\frac{S\tau, \exists x A(x)\tau \mid \mathcal{T}\tau}{S\tau, A(p)\tau \mid \mathcal{T}\tau}\ |\exists|. \tag{11}$$

By the induction hypothesis, every instance of $S, \exists x A(x) \mid \mathcal{T} \cdot \mathcal{C}$ has a derivation from $\mathcal{T}_1$. Since $\mathcal{C}\tau$ is satisfiable, we can take its τ-instance $S\tau, \exists x A(x)\tau \mid \mathcal{T}\tau$, this instance has a derivation from $\mathcal{T}_1$. Add to this derivation inference (11) and we obtain a required derivation of $S\tau, A(p)\tau \mid \mathcal{T}\tau$ from $\mathcal{T}_1$.

□

Now we want to prove a simulation result in the inverse direction. If we defined a sequent as a multiset of formulas, we could use an argument similar to the previous lemma. The use of sets instead of multisets causes some technical problems because the notion of instance does not work properly any more. To avoid these technical problems we give a definition of generalization that is nearly inverse to the notion of instance but takes into account some specific problems in the inverse simulation proof.

Let $\mathcal{T} = S_1 \mid \cdots \mid S_n$ and $\mathcal{T}' = S_1' \mid \cdots \mid S_n'$ be two tableaux. We write $\mathcal{T} \sqsubseteq \mathcal{T}'$ if (i) for every $i = 1 \ldots n$ we have $S_i \subseteq S_i'$ and (ii) each parameter occurring in some $\mathcal{T}'$ also occurs in $\mathcal{T}$. Let $\mathcal{T} \cdot \mathcal{C}$ be a constrained tableau and $\mathcal{T}'$ a tableau. We call $\mathcal{T} \cdot \mathcal{C}$ a *σ-generalization* of $\mathcal{T}'$ if $\mathcal{T}' \sqsubseteq \mathcal{T}\sigma$ and $\mathcal{C}\sigma$ is satisfiable. We call $\mathcal{T} \cdot \mathcal{C}$ a *generalization* of $\mathcal{T}'$ if $\mathcal{T} \cdot \mathcal{C}$ is a σ-generalization of $\mathcal{T}'$ for some σ.

LEMMA 37. *Suppose there exists a derivation of $\mathcal{T}_2$ from $\mathcal{T}_1$ in the tableau calculus for $\mathcal{L}$ with global assumptions Ψ. Then some generalization of $\mathcal{T}_2$ has a derivation from $\mathcal{T}_1 \cdot \top$ in the free-variable tableau calculus for $\mathcal{L}$ with the global assumptions Ψ.*

PROOF. The proof is by induction on the length of derivations in the tableau calculus. When the derivation is of length 0, the claim is obvious, since $\mathcal{T}_1 \cdot \top$ is a generalization of $\mathcal{T}_1$. For derivations with at least one inference, consider the last inference of the derivation. We will consider only two cases, other cases are similar.

CASE : the last inference is $|\mathrm{ax}|$.

$$\frac{S, A, \neg A \mid \mathcal{T}}{\mathcal{T}}\ |\mathrm{ax}|.$$

By the induction hypothesis, some σ-generalization of $S, A, \neg A \mid \mathcal{T}$ is derivable from $\mathcal{T}_1 \cdot \top$. Then this generalization has a form $S', A', \neg B' \mid \mathcal{T}' \cdot \mathcal{C}$ such that $A'\sigma = A$, $B'\sigma = A$, $\mathcal{T} \sqsubseteq \mathcal{T}'\sigma$ and $\mathcal{C}\sigma$ is satisfiable. Then σ is a unifier of A' and B', therefore, there exists a most general unifier τ of A' and B' and a substitution δ such that $\tau\delta = \sigma$. Consider the following inference in the free-variable tableau calculus.

$$\frac{S', A', \neg B' \mid \mathcal{T}' \cdot \mathcal{C}}{\mathcal{T}'\tau \cdot \mathcal{C}\tau}\ |\mathrm{ax}|.$$

We claim that the conclusion of this inference is a generalization of $\mathcal{T}$, this will complete the proof of this case. To prove the claim, we have to find a substitution δ' such that (i) $\mathcal{T} \sqsubseteq \mathcal{T}'\tau\delta'$ and (ii) $\mathcal{C}\tau\delta'$ is satisfiable. Well, take δ' to be δ, then both (i) and (ii) follow from $\tau\delta = \sigma$.

CASE : the last inference is $|\exists|$.

$$\frac{S, \exists x A(x) \mid \mathcal{T}}{S, A(p) \mid \mathcal{T}}\ |\exists| \tag{12}$$

By the induction hypothesis, some σ-generalization of $S, \exists x A(x) \mid \mathcal{T}$ is derivable from $\mathcal{T}_1 \cdot \top$. Then this generalization has form $S', \exists x A'(x) \mid \mathcal{T}' \cdot$

$\mathcal{C}$ such that (i) $S \subseteq S'\sigma$, (ii) every parameter occurring in $(S', \exists x A'(x))\sigma$ also occurs in $S, \exists x A(x) \mid \mathcal{T}$, (iii) $\exists x A'(x)\sigma = \exists x A(x)$, (iv) $\mathcal{T} \sqsubseteq \mathcal{T}'\sigma$, and (v) $\mathcal{C}\sigma$ is satisfiable.

Consider the following inference in the free-variable tableau calculus.

$$\frac{S', \exists x A'(x) \mid \mathcal{T}' \cdot \mathcal{C}}{S', A'(p) \mid \mathcal{T}' \cdot \mathcal{C} \wedge p \notin \mathit{vars}(S', \exists x A'(x))} \ |\exists|.$$

Let us check that the parameter condition is satisfied. Suppose, by contradiction, that p occurs in $S', \exists x A'(x) \mid \mathcal{T}'$, then it also occurs in $(S', \exists x A'(x) \mid \mathcal{T}')\sigma$, hence also in $S, \exists x A(x) \mid \mathcal{T}$. This violates the parameter condition of (12).

We claim that the conclusion of this inference is a generalization of $S, A(p) \mid \mathcal{T}$, this will complete the proof of this case. We actually claim that the conclusion is the σ-generalization of $S, A(p) \mid \mathcal{T}$. All conditions on σ-generalization except for constraint satisfaction immediately follow from (i)–(iv) above. It remains to verify that $(\mathcal{C} \wedge p \notin \mathit{vars}(S', \exists x A'(x)))\sigma$ is satisfiable. $\mathcal{C}\sigma$ is satisfiable by (v) above, so it remains to check that p does not occur in $(S', \exists x A'(x))\sigma$. If p occurred in $(S', \exists x A'(x))\sigma$, then by (ii) above p would also occur in $S, \exists x A(x) \mid \mathcal{T}$, but this is impossible because of the parameter condition in (12).

□

Now we can prove soundness and completeness of the free-variable calculi.

PROOF (of Theorem 35).

1. Suppose S has a refutation in the sequent calculus for $\mathcal{L}$ with global assumptions Ψ. Then by Theorem 16 there exists a derivation of $\#$ from S in the tableau calculus for $\mathcal{L}$ with the global assumptions Ψ. Hence, by Lemma 37 there exists a derivation of some generalization of $\#$ from $S \cdot \top$ in the free-variable tableau calculus for $\mathcal{L}$. But any generalization of $\#$ has the form $\# \cdot \mathcal{C}$ for a satisfiable $\mathcal{C}$.

2. Suppose there exists a derivation of $\# \cdot \mathcal{C}$ from $S \cdot \top$ in the tableau calculus for $\mathcal{L}$ with the global assumptions Ψ such that $\mathcal{C}$ is satisfiable. By Lemma 36 any instance of $\# \cdot \mathcal{C}$ has a derivation from S in the tableau calculus for $\mathcal{L}$ with the global assumptions Ψ. Obviously, $\#$ is such an instance, so it is derivable from S as well. By Theorem 16 S has a refutation in the sequent calculus for $\mathcal{L}$ with the global assumptions Ψ.

□

10. Example refutation

Consider the following formula valid in term-modal K. (For better readability, we will omit parenthesis in terms like $f(x)$ and write fx instead.)

$$\forall x \exists y([y]R(y,y) \wedge [fy](R(fy,fy) \supset R(y,fy)) \supset [fx]R(x,fx)).$$

We will establish the validity of this formula, i.e. unsatisfiability of its negation using the free-variable tableau calculus for K. First, we negate the formula and transform it into negation normal form:

$$\exists x \forall y([y]R(y,y) \wedge [fy](\neg R(fy,fy) \vee R(y,fy)) \wedge \langle fx \rangle \neg R(x,fx)),$$

and then show its refutation. The refutation is given in Figure 5. In the refutation we do not show the constraint, since it always has the form $p \notin \emptyset$ and is satisfiable. For better readability, we denote the inference steps by $\rightarrow$ followed by the name of the inference rule. We also group similar inferences into one. For example, by $|\forall|^*$ we denote a sequence of $|\forall|$ inferences, and by $|\wedge|^*$ a sequence of $|\wedge|$ inferences.

11. Conclusions

A complete sequent calculus was presented for a logic in which it is possible to quantify over modalities. We note that even though we have restricted ourselves to the logics $\mathsf{K}, \mathsf{D}, \mathsf{T}, \mathsf{K4}, \mathsf{D4}$, and $\mathsf{S4}$, other logics can be included among the Term-Modal Logics.

Term-modal logic can be used to reason about epistemic multi-agent systems or to develop action logics. Interesting future work includes joining epistemic term-modal logic with dynamic logic for actions, e.g. knowledge updates.

References

Areces, C., P. Blackburn, and M. Marx: 2000, 'The Computational Complexity of Hybrid Temporal Logics'. *Logic Journal of the IGPL* **8**(5), 653–679.

Blackburn, P.: 1993, 'Nominal Tense Logic'. *Notre Dame Journal of Formal Logic* **34**(1), 56–83.

Blackburn, P. and J. Seligman: 1993, 'Hybrid Languages'. *Journal of Logic, Language, and Information* **4**(3), 251–272.

Blackburn, P. and M. Tzakova: 1998, 'Hybrid Completeness'. *Logic Journal of the IGPL* **6**(4), 625–650.

$$\exists x \forall y([y]R(y,y) \wedge [fy](\neg R(fy,fy) \vee R(y,fy)) \wedge \langle fx \rangle \neg R(x,fx)) \rightarrow \quad |\exists|$$

$$\forall y([y]R(y,y) \wedge [fy](\neg R(fy,fy) \vee R(y,fy)) \wedge \langle fp \rangle \neg R(p,fp)) \rightarrow \quad |\forall|^*$$

$$\begin{array}{l} \forall y([y]R(y,y) \wedge [fy](\neg R(fy,fy) \vee R(y,fy)) \wedge \langle fp \rangle \neg R(p,fp)), \\ [z]R(z,z) \wedge [fz](\neg R(fz,fz) \vee R(z,fz)) \wedge \langle fp \rangle \neg R(p,fp), \\ [u]R(u,u) \wedge [fu](\neg R(fu,fu) \vee R(u,fu)) \wedge \langle fp \rangle \neg R(p,fp) \rightarrow \quad |\wedge|^* \end{array}$$

$$\begin{array}{l} \forall y([y]R(y,y) \wedge [fy](\neg R(fy,fy) \vee R(y,fy)) \wedge \langle fp \rangle \neg R(p,fp)), \\ [z]R(z,z), \\ [fz](\neg R(fz,fz) \vee R(z,fz)), \\ \langle fp \rangle \neg R(p,fp), \\ [u]R(u,u), \\ [fu](\neg R(fu,fu) \vee R(u,fu)) \rightarrow \quad |\langle fp, z, fu \rangle| \end{array}$$

$$\begin{array}{l} R(fp,fp), \\ \neg R(p,fp), \\ \neg R(fp,fp) \vee R(p,fp) \rightarrow \quad |\vee| \end{array}$$

$$\begin{array}{l} R(fp,fp), \neg R(p,fp), \neg R(fp,fp) \quad | \\ R(fp,fp), \neg R(p,fp), R(p,fp) \rightarrow \quad |\mathrm{ax}|^* \end{array}$$

$$\#$$

Figure 5. Example refutation in the free-variable calculus

Eder, E.: 1985, 'Properties of substitutions and unifications'. *Journal of Symbolic Computations* **1**(1), 31–48.

Fagin, R., J. Halpern, Y. Moses, and M. Vardi: 1995, *Reasoning about Knowledge*. Cambridge: The MIT Press.

Fischer, M. J. and R. E. Ladner: 1977, 'Propositional Modal Logic of Programs'. In: *Ninth Annual ACM Symposium on Theory of Computing*. New York, N.Y., pp. 286–294, ACM.

Fischer, M. J. and R. E. Ladner: 1979, 'Propositional Dynamic Logic of Regular Programs'. *Journal of Computer and System Sciences* **18**(2), 194–211.

Fitting, M.: 1983, *Proof methods for modal and intuitionistic logics*, Vol. 169 of *Synthese Library*. Reidel Publ. Comp.

Fitting, M.: 1988, 'First-Order Modal Tableaux'. *Journal of Automated Reasoning* **4**, 191–213.

Fitting, M., L. Thalmann, and A. Voronkov: 2000, 'Term-Modal Logics'. In: R. Dyckhoff (ed.): *Tableaux 2000*, Vol. 1847 of *Lecture Notes in Artificial Intelligence*. Berlin Heidelberg, pp. 220–236, Springer-Verlag.

Gargov, G. and V. Goranko: 1993, 'Modal Logic with Names'. *Journal of Philosophical Logic* **22**(6), 607–636.

Garson, J.: 1984, 'Quantification in Modal Logic'. In: D. Gabbay and F. Guenther (eds.): *Handbook in Philosophical Logic*, Vol. II. D. Reidel Publishing Company, Chapt. II.5, pp. 249–307.

Gentzen, G.: 1934, 'Untersuchungen über das logische Schließen'. *Mathematical Zeitschrift* **39**, 176–210, 405–431. Translated as (Gentzen, 1969).

Gentzen, G.: 1969, 'Investigations into Logical Deduction'. In: M. Szabo (ed.): *The Collected Papers of Gerhard Gentzen*. Amsterdam: North Holland, pp. 68–131. Originally appeared as (Gentzen, 1934).

Grove, A.: 1995, 'Naming and Identity in Epistemic Logics Part II: A first-order logic for naming'. *Artificial Intelligence* **74**, 311–350.

Grove, A. and J. Halpern: 1991, 'Naming and Identity in a Multi-Agent Epistemic Logic'. In: J. Allen, R. Fikes, and E. Sandewall (eds.): *KR'91. Proc. of the 2nd International Conference on Principles of Knowledge Representation and Reasoning*. Cambridge, Massachusets, pp. 301–312, Morgan Kaufmann.

Halpern, J. Y.: 1993, 'Reasoning about knowledge: a survey circa 1991'. In: A. Kent and J. G. Williams (eds.): *Encyclopedia of Computer Science and Technology, Volume 27 (Supplement 12)*. New York: Marcel Dekker.

Harel, D.: 1979, *First-Order Dynamic Logic*, Vol. 68 of *LNCS*. Springer.

Hintikka, J.: 1962, *Knowledge and Belief*. Ithaca, New York: Cornell University Press.

Kozen, D. and J. Tiuryn: 1989, 'Logics of Programs'. In: J. van Leeuwen (ed.): *Handbook of Theoretical Computer Science*. Amsterdam: North Holland.

Lenzen, W.: 1978, *Recent work in epistemic logic*, Vol. 30 of *Acta Philosophica Fennica*. Amsterdam: North-Holland.

Meyer, J.-J. C. and W. van der Hoek: 1995, *Epistemic Logic for AI and Computer Science*, No. 41 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press.

Passay, S. and T. Tinchev: 1991, 'An Essay in Combinatory Dynamic Logic'. *Information and Computation* **93**(2), 263–332.

Passy, S. and T. Tinchev: 1985, 'Quantifiers in combinatory PDL: completeness, definability, incompleteness'. In: L. Budach (ed.): *5th International Conference on Fundamentals of Computation Theory*, Vol. 199 of *Lecture notes in computer science*. Cottbus, GDR, pp. 512–519, Springer-Verlag.

Pratt, V. R.: 1976, 'Semantical Considerations on Floyd-Hoare Logic'. In: *17th Annual Symposium on Foundations of Computer Science*. pp. 109–121.

Ryan, M., J. Fiadeiro, and T. Maibaum: 1991, 'Sharing Actions and Attributes in Modal Action Logic'. In: T. Ito and A. R. Meyer (eds.): *Theoretical Aspects of Computer Software*, Vol. 526 of *Lecture Notes in Computer Science*. pp. 569–593, Springer-Verlag.

Schütte, K.: 1960, *Beweistheorie (in German)*. Springer Verlag.

Smullyan, R.: 1963, 'A unifying principle in quantification theory'. In: *Proc. Nat. Acad. Sci. U.S.A.*, Vol. 49. pp. 828–832.

Tzakova, M.: 1999, 'Tableau Calculi for Hybrid Logics'. In: N. V. Murray (ed.): *Proceedings of the International Conference on Automated Reasoning with Analytic Tableaux and Related Methods (TABLEAUX-99)*, Vol. 1617 of *LNAI*. Berlin, pp. 278–292, Springer.

van der Hoek, W. and J.-J. C. Meyer: 1997, 'A Complete Epistemic Logic for Multiple Agents–Combining Distributed and Common Knowledge'. In: M. Bacharach, L. Gerard-Varet, P. Mongin, and H. Shin (eds.): *Epistemic Logic and the Theory of Games and Decisions*. Dordrecht: Kluwer Academic Publishers, pp. 35–68.

Voronkov, A.: 1996, 'Proof Search in Intuitionistic Logic Based on Constraint Satisfaction'. In: P. Miglioli, U. Moscato, D. Mundici, and M. Ornaghi (eds.): *Theorem Proving with Analytic Tableaux and Related Methods. 5th International Workshop, TABLEAUX '96*, Vol. 1071 of *Lecture Notes in Artificial Intelligence*. Terrasini, Palermo Italy, pp. 312–329.

Wallen, L.: 1990, *Automated Deduction in Nonclassical Logics*. The MIT Press.