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How to optimize proof-search in modal logics: a new way of proving redundancy for sequent calculi

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Abstract

We present a bottom-up decision procedure for propositional modal logic K based on the inverse method. The procedure is based on the "inverted" version of a sequent calculus. To restrict the search space, we prove a number of redundancy criteria for derivations in the sequent calculus. We introduce a *new technique* of proving redundancy criteria, based on the analysis of tableau-based derivations in K. Moreover, another new technique is used to prove completeness of proof-search with a strong notion of subsumption. This technique is based on so-called *traces*. A new formalization of the inverse method in the form of a *path calculus* considerably simplifies all proofs as compared to the previously published presentations of the inverse method. Experimental results demonstrate that our method is competitive with the best state-of-the-art implementations of K.

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Contents

1 Introduction

Nonclassical (propositional) logics play an increasing role in computer science. They are used in model checking, verification, and knowledge representation. Traditional decision procedures for these logics use semantic tableaux [Baader and Hollunder 1991, Catach 1991, Haarslev, Möller and Turhan 1998, Balsiger, Heuerding and Schwendimann 1998], optimized tableaux [Horrocks 1997, Horrocks 1998, Patel-Schneider 1998], SAT-based methods [Giunchiglia and Sebastiani 1996*a*, Giunchiglia and Sebastiani 1996*b*], or highly optimized translations into first-order classical logic [Hustadt and Schmidt 1997, Schmidt 1998].

In this report we present a bottom-up decision procedure for propositional modal logic K based on the inverse method. All our results can be easily modified to its multi-modal variant K(m) with m modalities, we restrict ourselves to K for simplicity. The modal logic K(m) is important for several reasons, mainly because this logic is a syntactic variant of the knowledge representation formalism $\mathcal{ALC}$.

Our experiments demonstrate that our method is competitive with the best state-of-the-art methods for K. We believe that the technique described in our report opens a new interesting research path in the implementation of decision procedures for other important nonclassical logics, such as temporal, description logics, or quantified boolean formulas.

This report is organized as follows. In Sections 2, 3 and 4 we define main notions and introduce several sequent calculi for K, including an original calculus based on the notion of path in the goal formula. In Section 5 we prove several properties of derivations in the path calculus. In Section 6 we introduce the inverse path calculus and use these properties of derivations to prove that some sequents and derivations in the inverse path calculus are redundant. In Section 7 we introduce new redundancy criteria based on two different versions of subsumption. Section 8 describes an implementation of our method in the system KK and experimental results. Section 9 briefly mentions related work on the inverse method and implementations of nonclassical logics. Section 10 discusses new research directions which we find interesting for efficient automation of nonclassical logics.

2 Preliminaries

In this report we consider propositional modal logic K. K is the minimal normal modal logic, i.e. the set of modal formulas valid in all Kripke models. For the discussion of modal logics and Kripke models see any standard references [e.g. Fitting 1983, Wallen 1990]. Our technique is applicable to other modal logics, at least including T, K4, S4, D and D4 and their versions with global assumptions, but the notions of redundancy for these logics are different.

A *literal* is a propositional variable or its negation. We only consider *modal formulas in negation normal form*, i.e. formulas built from literals using $\wedge$, $\vee$, $\Box$ and $\Diamond$. A formula is called *satisfiable* if it is true in at least one Kripke model. In this report, we will speak about (un)satisfiability of formulas instead of the dual notion of *validity*. The satisfiability problem for K is PSPACE-complete. Satisfiability in K can be characterized using a *sequent calculus* introduced below.

We denote propositional variables by lower-case letters, and formulas by upper-case letters. We will use the formula

$$E = \Box a \wedge \Box(\neg a \vee b) \wedge \Diamond\neg b \wedge \Diamond b \tag{1}$$

for nearly all examples of this report. We consider conjunction and disjunction right-associative, so, for example, formula (1) denotes $\Box a \wedge (\Box(\neg a \vee b) \wedge (\Diamond\neg b \wedge \Diamond b))$.

A *sequent* is a finite multiset of formulas. We denote sequents by Greek letters Γ and Δ. We denote the multiset union of two multisets Γ and Δ by Γ, Δ and the addition of an element A to a multiset Γ by Γ, A. The *multiset difference* of multisets Γ and Δ is denoted $\Gamma \dot{-} \Delta$. If an element occurs in Γ k times and in Δ l times and $k > l$, then it occurs in $\Gamma \dot{-} \Delta$ $k - l$ times, otherwise it does not occur in $\Gamma \dot{-} \Delta$ at all. If A is a member of a multiset Γ, we will write $A \dot{\in} \Gamma$. For a sequent $\Gamma = A_1, \dots, A_n$, we denote by $\Diamond\Gamma$ the sequent $\Diamond A_1, \dots, \Diamond A_n$. A sequent $A_1, \dots, A_n$ is called *satisfiable* if there exists a Kripke model in which all formulas $A_1, \dots, A_n$ are true.

A sequent Γ is a called a *subsequent* of a sequent Δ if for every formula A in Γ, if A occurs in Γ k times at the top level (i.e. as not a proper subformula of another formula), then A occurs in Δ at the top level $\geq k$ times. For example, the sequent a is a subsequent of the sequent a, a, b, but the sequent a, a is not a subsequent of the sequent a.

The *sequent calculus* Seq and an example derivation in this calculus are given in Figures 1 and 2. We call axioms: $\Gamma, p, \neg p$ $\quad \dfrac{\Gamma, A, B}{\Gamma, A \wedge B}\ (\wedge)$

$\dfrac{\Gamma, A \quad \Gamma, B}{\Gamma, A \vee B}\ (\vee)$ $\quad \dfrac{\Delta, A}{\Gamma, \Box\Delta, \Diamond A}\ (\Diamond)$

Figure 1: Sequent calculus Seq

$$\dfrac{\dfrac{\dfrac{a, \neg a, \neg b \quad a, b, \neg b}{a, \neg a \vee b, \neg b}\ (\vee)}{\Box a, \Box(\neg a \vee b), \Diamond\neg b, \Diamond b}\ (\Diamond)}{\Box a \wedge \Box(\neg a \vee b) \wedge \Diamond\neg b \wedge \Diamond b}\ (\wedge), (\wedge), (\wedge)$$

A sequence of inferences $(\wedge)$ is shown as one inference.

Figure 2: Derivation of E in Seq

- an *inference* in Seq any instance of an inference rule;
- a *derivation of a sequent* Γ any tree formed by inferences and having Γ as the root;
- a *refutation* Γ any derivation of Γ whose leaves are axioms.

We will use the notions of inference, derivation and refutation for all other calculi introduced in this report.

THEOREM 2.1 (Completeness of Seq) *A sequent S is unsatisfiable if and only if it has a refutation in* Seq*.*

The proof of this theorem may be found in standard textbooks [e.g. Fitting 1983] or in [Goble 1974].

Let us formulate some properties of Seq that easily follow from the analysis of inference rules of Seq or from the completeness theorem. We call an inference rule *invertible* if for every instance of this inference rule, if its conclusion is unsatisfiable, then so are all its premises.

The following properties of Seq are easy to prove:

LEMMA 2.2 *(i)* Seq *has the* subformula property*: every formula occurring in a derivation of a sequent Γ consists of subformulas of formulas of Γ. (ii) If a sequent Γ has a refutation, then any sequent Γ, Δ has a refutation too. (iii) The rules* $(\wedge)$ *and* $(\vee)$ *are invertible.*

3 The inverse calculus

The calculus Seq is well-known. It is straightforward to design satisfiability-checking procedures that apply the rules of Seq in the top-down way, from the goal to axioms (the tableau method). Much less known is the inverse method, based on a bottom-up version of Seq. In the inverse method, the sequent calculus depends on the goal formula G. Such a calculus, denoted by Inv_G and an example derivation in this calculus are shown in Figures 3 and 4. Compare this derivation with that of Figure 2.

The completeness of the inverse method is formulated as follows:

THEOREM 3.1 (Completeness of Inv_G) *A formula G is unsatisfiable if and only if it has a refutation in* Inv_G*.*

The main technique used in the proof of this theorem is the so-called *Subsequent Lemma*.

axioms: $p, \neg p$ $\qquad \dfrac{\Gamma, A, A}{\Gamma, A}\ (C)$

$$\dfrac{\Gamma, A}{\Gamma, A \wedge B}\ (\wedge_l) \qquad \dfrac{\Gamma, B}{\Gamma, A \wedge B}\ (\wedge_r)$$

$$\dfrac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta, A \vee B}\ (\vee)$$

$$\dfrac{\Gamma, A}{\Box\Gamma, \Diamond A}\ (\Diamond) \qquad \dfrac{\Gamma}{\Box\Gamma, \Diamond A}\ (\Diamond^+)$$

In the rules, all formulas occurring in the rules are subformulas of G. The rule (C) is called the *contraction rule*.

Figure 3: Sequent calculus Inv_G

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a, \neg a \quad b, \neg b}{a, \neg a \vee b, \neg b}\ (\vee)}{\Box a, \Box(\neg a \vee b), \Diamond\neg b}\ (\Diamond)}{\Box a,\ \Box(\neg a \vee b) \wedge \Diamond\neg b \wedge \Diamond b,\ \Box(\neg a \vee b) \wedge \Diamond\neg b \wedge \Diamond b}\ (\wedge),(\wedge),(\wedge)}{\Box a, \Box(\neg a \vee b) \wedge \Diamond\neg b \wedge \Diamond b}\ (C)}{\Box a \wedge \Box(\neg a \vee b) \wedge \Diamond\neg b \wedge \Diamond b}\ (\wedge),(\wedge),(C)$$

Some sequences of inferences are shown as one inference.

Figure 4: Derivation in Inv_E

Lemma 3.2 (Subsequent Lemma) *Let G be a formula and Γ be a sequent consisting of the subformulas of G and having refutation in* Seq. *Then there exists a subsequent Δ of Γ that has a refutation in* Inv_G.

Completeness of Inv_G follows from this lemma immediately: indeed, if G is unsatisfiable, then by Completeness Theorem 2.1 for Seq, the sequent G has refutation in Seq, so by Subsequent Lemma, a subsequent of G has a refutation in Inv_G. The sequent Γ has only two subsequents: G itself, and the empty sequent, but the latter has obviously no refutation, so G has a refutation in Inv_G.

The Subsequent Lemma can be proved by a straightforward induction on derivations. We do not give a full proof here, because below in Section 6 we will prove a much stronger form of completeness (Theorem 6.2) in presence of a number of redundancies. However, we will illustrate an interesting point in the proof that gives rise to two rules for handling $\Diamond$ in Inv_G out of one rule in Seq.

Suppose a sequent $\Gamma, \Box\Delta, \Diamond A$ has refutation in Seq and the last inference of this refutation is

$$\dfrac{\Delta, A}{\Gamma, \Box\Delta, \Diamond A}\ (\Diamond).$$

Since the premise Δ, A has a shorter refutation in Seq, we may assume that a subsequent Δ' of Δ, A has a refutation in Inv_G. We have to find a refutation in Inv_G of a subsequent of $\Gamma, \Box\Delta, \Diamond A$. There are two possible cases, depending on whether A occurs in Δ' or not. If it does, then Δ' has the form Δ'', A where Δ'' is a subsequent of Δ, then we use the following inference in Inv_G to obtain a refutation of a subsequent of $\Gamma, \Box\Delta, \Diamond A$:

$$\dfrac{\Delta'', A}{\Box\Delta'', \Diamond A}\ (\Diamond).$$

If A does not occur in Δ', then Δ' is a subsequent of Δ, and we use the following inference in Inv_G to obtain a refutation of a subsequent of $\Gamma, \Box\Delta, \Diamond A$:

$$\dfrac{\Delta'}{\Box\Delta', \Diamond A}\ (\Diamond^+).$$

It is not hard to find an example showing that the rule $(\Diamond^+)$ is necessary for completeness. Consider the formula $G = \Box a \wedge \Box\neg a \wedge \Diamond b$. Its has a refutation in Inv_G:

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{a, \neg a}{\Box a, \Box\neg a, \Diamond b}\ (\Diamond^+)}{\Box a, \Box\neg a, \Box\neg a \wedge \Diamond b}\ (\wedge_r)}{\Box a, \Box\neg a \wedge \Diamond b, \Box\neg a \wedge \Diamond b}\ (\wedge_l)}{\Box a, \Box\neg a \wedge \Diamond b}\ (C)}{\Box a, \Box a \wedge \Box\neg a \wedge \Diamond b}\ (\wedge_r)}{\Box a \wedge \Box\neg a \wedge \Diamond b, \Box a \wedge \Box\neg a \wedge \Diamond b}\ (\wedge_l)}{\Box a \wedge \Box\neg a \wedge \Diamond b}\ (C)
$$

The only axiom of Inv_G is $a, \neg a$, and the only inference applicable to this axioms is the inference $(\Diamond^+)$ used in this refutation. Therefore, Inv_G is incomplete without the $(\Diamond^+)$ rule.

This example also shows necessity in the contraction rule (C). We could avoid the contraction rule by defining sequents as *sets* of formulas rather than multisets, but the formulation using multisets makes proofs more clear and can also be used for building the inverse method calculi for first-order modal logics.

Note that the calculus Inv_G has the following properties:

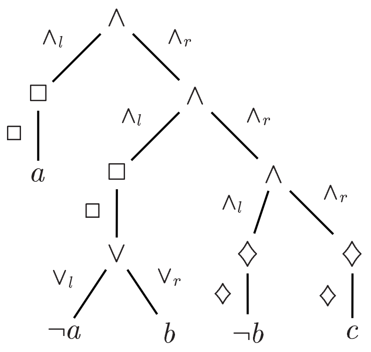
1. Every formula occurring in a derivation is a subformula of G.
2. There exists a finite number of axioms.
3. For every sequent Γ, there exists only a finite number of ways of applying $(\wedge_l)$, $(\wedge_r)$, $(\Diamond)$, $(\Diamond^+)$, or (C) to Γ.
4. For every pair of sequents Γ and Δ, there exists only a finite number of ways of applying $(\vee)$ to Γ and Δ.

A typical proof procedure based on the inverse method uses these properties and works roughly as follows.

1. Let S be a set of sequents, initially $\{p, \neg p \mid p, \neg p \text{ are subformulas of } G\}$.
2. Repeatedly apply all possible inference rules to sequents in S, adding their conclusions to S until no new sequents can be obtained or a refutation of G is found.

However, such naive procedures are hopelessly inefficient, unless they are augmented with powerful *redundancy criteria*. There are two kinds of redundancy criteria, allowing us to get rid of *redundant sequents* and *redundant inferences*. Intuitively, a sequent is redundant if it can be removed from the search space without loosing completeness. An inference is redundant if the calculus obtained by dropping this inference is still complete. Very powerful techniques for proving redundancies have been developed in resolution-based calculi for classical logic, [see e.g. Bachmair and Ganzinger 2000, Niewenhuis and Rubio 2000]. The use of such powerful criteria is one of the reasons for the efficiency of the system **TA** [Hustadt and Schmidt 1997, Schmidt 1998] that translates the goal formula into a fragment of first-order logic and uses a first-order theorem prover SPASS that implements many redundancy criteria.

Redundancy criteria can be incompatible. For example, it is possible that some sequents Γ and Δ are redundant, but every refutation contains at least one of these sequents. Therefore, one usually proves completeness for a collection of redundancy criteria. If we prove several redundancy criteria for a calculus, the proof-search by the inverse method works as follows.



π	$E\|_\pi$
ϵ	$\Box a \wedge \Box(\neg a \vee b) \wedge \Diamond\neg b \wedge \Diamond b$
$\wedge_l$	$\Box a$
$\wedge_l \Box$	a
$\wedge_r$	$\Box(\neg a \vee b) \wedge \Diamond\neg b \wedge \Diamond b$
$\wedge_r\wedge_l$	$\Box(\neg a \vee b)$
$\wedge_r \wedge_l \Box$	$\neg a \vee b$
$\wedge_r \wedge_l \Box\vee_l$	$\neg a$
$\wedge_r \wedge_l \Box\vee_r$	b
$\wedge_r\wedge_r$	$\Diamond\neg b \wedge \Diamond b$
$\wedge_r \wedge_r \wedge_l$	$\Diamond\neg b$
$\wedge_r \wedge_r \wedge_l\Diamond$	$\neg b$
$\wedge_r \wedge_r \wedge_r$	$\Diamond b$
$\wedge_r \wedge_r \wedge_r\Diamond$	c

Figure 5: Paths and subformulas in the formula E

1. Let S be a set of sequents, initially $\{p, \neg p \mid p, \neg p$ are subformulas of $G\}$ *less the redundant sequents of this form.*

2. Repeatedly apply all *nonredundant* inferences to sequents in S, adding to S those conclusions of these inferences *that are nonredundant.*

3. Remove all sequents that *become redundant* due to the addition of these conclusions of inferences.

A number of redundancy criteria for various nonclassical logics are proved in [Voronkov 1992] using a rather general technique (though many proofs are absent). There is a paper on the inverse method for S4 [Mints, Orevkov and Tammet 1996]. This paper is interesting because it discusses many redundancy criteria, but again the proofs of many statements are missing. In particular, [Mints et al. 1996] takes it for granted that some strategies *deleting* clauses from the search space (like subsumption) are complete, though no technique for proving this has been developed.

Here we present a different technique based on proving properties of refutations in Seq and transforming them into redundancy criteria for the inverse method using the Subsequent Lemma. In order to capture redundancies, we give a suitable reformulation of the sequent calculus as a *path calculus.*

4 The path calculus

For a number of reasons, we want now to speak about *occurrences* of subformulas in a goal formula G. Together with each occurrence of a subformula, we also want to keep information about Inv_G-inference rules that can be applied to the subformula corresponding to this occurrence. To achieve both aims, we introduce the notion of path in G.

A *path* is any finite sequence of symbols $\wedge_l$, $\wedge_r$, $\vee_l$, $\vee_r$, $\Box$, $\Diamond$. An example of a path is $\wedge_l\Diamond\Diamond\vee_r$. The empty path is denoted ϵ. Let G be a formula. The notion of *path in G*, or *G-path*, and the *subformula of G on a path π*, denoted $G|_\pi$, are defined inductively as follows.

- ▶ The empty path ϵ is a G-path and $G|_\epsilon$ is G.

$$\text{axiom: } \Gamma, \pi_1, \pi_2 \qquad \frac{\Gamma, \pi\wedge_l, \pi\wedge_r}{\Gamma, \pi}\ (\wedge)$$

$$\frac{\Gamma, \pi\vee_l \quad \Gamma, \pi\vee_r}{\Gamma, \pi}\ (\vee) \qquad \frac{\Pi\Box, \pi\Diamond}{\Gamma, \Pi, \pi}\ (\Diamond)$$

All paths occurring in path sequents are G-paths. In the axioms $G|_{\pi_1} = p$ and $G|_{\pi_2} = \neg p$ for some propositional variable p.

Figure 6: Path sequent calculus Pat_G

$$\cfrac{\cfrac{\cfrac{\begin{array}{c}\wedge_l\Box,\ \wedge_r \wedge_l \Box\vee_l, \\ \wedge_r \wedge_r \wedge_l \Diamond\end{array} \qquad \begin{array}{c}\wedge_l\Box,\ \wedge_r \wedge_l \Box\vee_r, \\ \wedge_r \wedge_r \wedge_l \Diamond\end{array}}{\wedge_l\Box,\ \wedge_r \wedge_l \Box,\ \wedge_r \wedge_r \wedge_l \Diamond}\ (\vee)}{\wedge_l,\ \wedge_r\wedge_l,\ \wedge_r \wedge_r \wedge_l,\ \wedge_r \wedge_r \wedge_r}\ (\Diamond)}{\epsilon}\ (\wedge), (\wedge), (\wedge)$$

A sequence of rules $(\wedge)$ is shown as one inference.

Figure 7: Derivation in Pat_E

- Suppose that π is a G-path and $G|_\pi = F$. Then
  - If F has the form $F_1 \wedge F_2$, then $\pi\wedge_l$ and $\pi\wedge_r$ are G-paths, and $G|_{\pi\wedge_l} = F_1$, $G|_{\pi\wedge_r} = F_2$. In this case we call π a $\wedge$-*path*.
  - If F has the form $F_1 \vee F_2$, then $\pi\vee_l$ and $\pi\vee_r$ are G-paths, and $G|_{\pi\vee_l} = F_1$, $G|_{\pi\vee_r} = F_2$. In this case we call π a $\vee$-*path*.
  - If F has the form $\Box F_1$, then $\pi\Box$ is a G-path and $G|_{\pi\Box} = F_1$. In this case we call π a $\Box$-*path*.
  - If F has the form $\Diamond F_1$, then $\pi\Diamond$ is a G-path and $G|_{\pi\Diamond} = F_1$. In this case we call π a $\Diamond$-*path*.

We call a path π a *literal path* if $G|_\pi$ is a literal. Note that the notions like $\wedge$-path or literal path make sense only when we speak about G-paths for a fixed formula G, and make no sense for paths in general.

The notion of subformula on a path is illustrated in Figure 5.

Evidently, for each subformula F of G, there is a G-path π such that $G|_\pi = F$. The tableau calculus Seq_G can be turned into a calculus where the subformulas of G are replaced by the corresponding paths. We call a *path sequent* any finite multiset of paths. When no ambiguity arises, we will refer to path sequents simply as sequents. For a path sequent $\Pi = \pi_1, \ldots, \pi_n$ we denote by $\Pi\Box$ the path sequent $\pi_1\Box, \ldots, \pi_n\Box$. The *path calculus* Pat_G and a derivation in it are given in Figures 6 and 7. For any inference of Pat_G of Figure 6, the path π is called the *principal path* of this inference. The paths $\pi\wedge_l$, $\pi\wedge_r$, $\pi\vee_l$, $\pi\vee_r$, and $\pi\Diamond$ are called the *side paths* of the respective inference. We will use the notions of principal and side paths for some other calculi in this report.

The following *bisimulation lemma* is the key to proving completeness of the path calculus and various refinements thereof. Let us call the *formula image* of a path sequent or a derivation in Pat_G the figure obtained from this path sequent or derivation by replacing every path π by the formula $G|_\pi$.

LEMMA 4.1 (Bisimulation Lemma for Pat_G) *(i) Let D be a derivation in* Pat_G. *Then the formula image of D is a derivation of G in* Seq. *(ii) Let D' be a derivation of a sequent $A_1, \ldots, A_n$ in* Seq, *and $\pi_1, \ldots, \pi_n$ be positions such that $G|_{\pi_i} = A_i$, for all $i = 1, \ldots, n$. Then there exists a derivation D of $\pi_1, \ldots, \pi_n$ in* Pat_G *such that D' is the formula image of D. (iii) Both (i) and (ii) also hold if "derivation" is replaced by "refutation" everywhere.*

PROOF. (i) is proved by the straightforward analysis of inferences in Pat_G. (iii) easily follows from (i) and (ii). We will only show how to prove (ii). The proof is by induction on the number of inferences in D'. We consider only one case, when the last inference is $(\vee)$, other cases are similar. In this case without loss of generality we assume that $A_n = B \vee C$ and A_n is the principal formula of the last inference. Then the derivation D' has the form

$$\dfrac{\begin{matrix} \vdots\, D'_1 \\ A_1, \dots, A_{n-1}, B \end{matrix} \quad \begin{matrix} \vdots\, D'_2 \\ A_1, \dots, A_{n-1}, C \end{matrix}}{A_1, \dots, A_{n-1}, B \vee C}\ (\vee)$$

Evidently, $\pi_n \wedge_l$ and $\pi_n \wedge_r$ are positions in G. Applying the induction hypothesis to the derivations D'_1 and D'_2, we obtain derivations D_1 and D_2 such that D'_1 and D'_2 are their formula images. Define D as the derivation

$$\dfrac{\begin{matrix} \vdots\, D_1 \\ \pi_1, \dots, \pi_{n-1}, \pi \wedge_l \end{matrix} \quad \begin{matrix} \vdots\, D_2 \\ \pi_1, \dots, \pi_{n-1}, \pi \wedge_r \end{matrix}}{\pi_1, \dots, \pi_{n-1}, \pi}\ (\vee)$$

It is not hard to argue that D' is the formula image of D. □

As a corollary of the Bisimulation Lemma and Completeness Theorem 2.1, we obtain

THEOREM 4.2 (Completeness Theorem for Pat_G) *A formula G is unsatisfiable if and only if ϵ has a refutation in* Pat_G.

PROOF. By the Completeness theorem for Seq, unsatisfiability of G is equivalent to the existence of a refutation of G in Seq.

- $\Rightarrow$ Let ϵ has a refutation Pat_G. By the Bisimulation Lemma there exists a refutation in Seq of the formula $G|_\epsilon$, i.e. of G.
- $\Leftarrow$ Let D' be a refutation of G in Seq. By the Bisimulation Lemma there exists a refutation D in Pat_G whose formula image is D'. Evidently, D is a refutation of ϵ.

□

5 Redundancies

Resolution-based calculi for classical logic become efficient only when they are equipped with powerful redundancy criteria [see e.g. Bachmair and Ganzinger 2000, Niewenhuis and Rubio 2000]. Likewise, a naive implementation of the inverse calculus does not give us an efficient decision procedure. In order to optimize the inverse calculus, we will prove several statements about redundant sequents and derivations. To prove these statements, we will first establish several properties of derivations in the path calculus.

Let us call the *modal length* of a path the number of occurrences of $\Diamond$ and $\Box$ in the path. For example, the modal length of $\Box \wedge_l \Diamond\Diamond\vee_r$ is 3.

LEMMA 5.1 *Let Π be a path sequent occurring in a derivation of ϵ in* Pat_G. *Then all paths in Π have the same modal length.*

PROOF. First, observe the following fact. Let an inference in Pat_G have a conclusion Π satisfying the claim. Then all premises of this inferences satisfy the claim. Second, apply a straightforward induction on the number of inferences. □

A path π' is called a *left subpath*, or simply *subpath* of a path π, denoted $\pi' \sqsubseteq \pi$, if $\pi = \pi'\pi''$ for some path π''. Evidently, if $\pi' \sqsubseteq \pi$ and π is a G-path, then π' is a G-path and $G|_\pi$ is a subformula of $G|_{\pi'}$. A left subpath π' of π is called *proper (left) subpath*, denoted $\pi' \sqsubset \pi$, if $\pi' \neq \pi$.

LEMMA 5.2 *Let Π be a path sequent occurring in a derivation of ϵ in Pat_G and π_1, π_2 be two different occurrences of paths in Π. Then π_1 is not a left subpath of π_2. As a consequence, each path has at most one occurrence in Π.*

PROOF. As in the proof of Lemma 5.1, observe that this property holds for all premises of an inference, whenever it holds for its conclusion. □

We say that two paths π_1 and π_2 are in *conflict* if one of the following conditions is satisfied:

1. $\pi_1 = \pi \vee_l \pi'_1$, $\pi_2 = \pi \vee_r \pi'_2$ for some paths π, π'_1, π'_2; or, symmetrically,
2. $\pi_1 = \pi \vee_r \pi'_1$, $\pi_2 = \pi \vee_l \pi'_2$ for some paths π, π'_1, π'_2; or
3. $\pi_1 = \pi'_1 \Diamond \pi''_1$ and $\pi_2 = \pi'_2 \Diamond \pi''_2$ for some paths $\pi'_1, \pi'_2, \pi''_1, \pi''_2$ such that the paths π'_1 and π'_2 have the same modal length and $\pi'_1 \neq \pi'_2$.

For example, consider the paths of Figure 5. The path $\wedge_r \wedge_l \Box \vee_l$ are in conflict with $\wedge_r \wedge_l \Box \vee_r$ because of the first condition. The paths $\wedge_r \wedge_r \wedge_l \Diamond$ and $\wedge_r \wedge_r \wedge_r \Diamond$ are in conflict because of the last condition.

LEMMA 5.3 *Let Π be a path sequent occurring in a derivation of ϵ in Pat_G. Then every paths π_1, π_2 in Π are not in conflict.*

PROOF. The proof is by contradiction. We assume that there are two paths in conflict π_1, π_2 occurring in Π. Consider all possible cases, depending on the kind of conflict.

1. *Case: $\pi_1 = \pi \vee_l \pi'_1$, $\pi_2 = \pi \vee_r \pi'_2$ for some paths π, π'_1, π'_2.* Consider the branch $\mathcal{B}$ of the derivation on which Π lies and the topmost sequents on $\mathcal{B}$ where $\pi\vee_l$ and $\pi\vee_r$ first appeared. Without loss of generality assume that $\pi\vee_l$ first appeared above $\pi\vee_r$ (the other case is symmetric). Then the derivation contains the subderivation

$$\dfrac{\Gamma_2, \pi\vee_l \qquad \dfrac{\dfrac{\Gamma_1, \pi\vee_l \quad \Gamma_1, \pi\vee_r}{\Gamma_1, \pi}\,(\vee)}{\begin{array}{c}\vdots\\ \Gamma_2, \pi\vee_r\end{array}}}{\Gamma_2, \pi}\,(\vee)$$

Since Γ_1, π is above $\Gamma_2, \pi\vee_r$ in the derivation, Γ_2 must contain a left subpath of π. This contradicts to Lemma 5.2, since the path sequent $\Gamma_2, \pi\vee_r$ now contains a path and its left subpath.

The second case in the definition of conflict is symmetric to this one.

2. *Case:* $\pi_1 = \pi_1'\Diamond\pi_1''$, $\pi_2 = \pi_2'\Diamond\pi_2''$, $\pi_1' \neq \pi_2'$, *and paths* π_1' *and* π_2' *have the same modal length* L. Consider the branch $\mathcal{B}$ of derivation on which Π lies and the topmost sequents on $\mathcal{B}$ where $\pi_1'\Diamond$ and $\pi_2'\Diamond$ first appeared. Without loss of generality assume that $\pi_1'\Diamond$ first appeared above $\pi_2'\Diamond$ (the other case is symmetric). Then the derivation contains a subderivation

$$\begin{array}{c} \dfrac{\Gamma_1\Box, \pi_1'\Diamond}{\Gamma_1, \pi_1', \Delta_1}\ (\Diamond) \\ \vdots \\ \dfrac{\Gamma_2\Box, \pi_2'\Diamond}{\Gamma_2, \pi_2', \Delta_2}\ (\Diamond) \end{array}$$

Since π_2' has the modal length L, it is not hard to argue that all path sequents between $\Gamma_2\Box, \pi_2'\Diamond$ and $\Gamma_1, \pi_1', \Delta_1$ contain a formula with the modal length at least $L+1$. Therefore, the sequent $\Gamma_1, \pi_1', \Delta_1$ contains a path of the modal length at least $L+1$ and the path π_1' with the modal length L. This contradicts to Lemma 5.1, since all paths in a path sequent must have the same modal length.

$\Box$

The lemmas above asserts some properties of all possible derivations in the path calculus. Now we want to restrict search for refutation to only a subset of derivations by introducing an ordering on the set of all G-paths.

In ordered resolution for classical logic, one can order literals in clauses and require that a resolution inference be only applied when the greatest literals in both clauses are resolved. We want to introduce a similar restriction in the modal case, by modifying classical literal orderings to their modal equivalent, called a G*-ordering.* In the classical case, we can use *any* ordering on subformulas of G (or on G-paths) that respects the subpath relation. In the modal case, the situation is more complex, as we will now see: not every ordering on paths preserves completeness. For example, consider the top inference of the derivation of Figure 7:

$$\dfrac{\begin{array}{c}\wedge_l\Box,\ \wedge_r\wedge_l\Box\vee_l, \\ \wedge_r\wedge_r\wedge_l\Diamond\end{array} \qquad \begin{array}{c}\wedge_l\Box,\ \wedge_r\wedge_l\Box\vee_r, \\ \wedge_r\wedge_r\wedge_l\Diamond\end{array}}{\wedge_l\Box,\ \wedge_r\wedge_l\Box,\ \wedge_r\wedge_r\wedge_l\Diamond}\ (\vee)$$

It is essential in that derivation that an $(\vee)$-inference is applied above the rule $(\Diamond)$, because the rule $(\Diamond)$ is not applicable to the top sequents. However, if we impose an ordering on paths in which $\wedge_r\wedge_l\Box\vee_l$ is the smallest path in the first premise, we will rule out the possibility of applying $(\vee)$ first, and no proof will be found. So, the definition of G-ordering is more complex than in the classical case.

We call two paths *adjacent* if one of them has the form $\pi\wedge_l$ and the other $\pi\wedge_r$, or likewise, one of them has the form $\pi\vee_l$ and the other $\pi\vee_r$. For example, the following two paths in our example formula E are adjacent: $\wedge_r\wedge_l\Box\vee_l$ and $\wedge_r\wedge_l\Box\vee_r$, the formulas on this path are the two immediate subformulas of the disjunction $\neg a \vee b$. In general, every conjunction and disjunction in a formula gives rise to a pair of adjacent paths.

We denote by the generic symbol $\mathbin{\wedge\!\!\!\!\vee}$ any of the symbols $\wedge$, $\vee$. Likewise, we denote by $*$ any of the symbols l, r and use notation like $\wedge_*$ or $\vee_*$. Thus $\mathbin{\wedge\!\!\!\!\vee}_*$ denotes any of the symbols $\wedge_l$, $\wedge_r$, $\vee_l$, $\vee_r$. We also denote by the generic symbol $\Diamond\!\!\!\!\Box$ any of the symbols $\Diamond$, $\Box$.

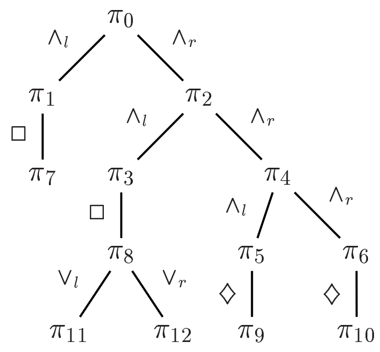


Figure 8: A E-ordering

Let us fix a formula G. We call a *G-ordering* any total order $\succ$ on the set of all G-paths satisfying the following conditions:

1. $\pi_1 \succ \pi_2$, whenever
   (a) the modal depth of π_1 is strictly greater than the modal depth of π_2; or
   (b) π_1, π_2 have the same modal depth, the last symbol of π_1 is $\bowtie_*$, and the last symbol of π_2 is ✡; or
   (c) π_1, π_2 have the same modal depth and $\pi_2 \sqsubset \pi_1$. (Note that this case and the previous one are not exclusive, but they do not contradict each other.)
2. If π_1, π_2 are adjacent, $\pi \succ \pi_1$ and π is not a subpath of π_1, then $\pi \succ \pi_2$.

It will be clear why these conditions are required of G-orderings when we prove Lemma 5.7 below.

To illustrate G-orderings, we will show one possible E-ordering (recall that E is our example formula). The paths in E were illustrated in Figure 5. In Figure 8, we show these paths again, this time each path is labelled as π_i, where i is a natural number. We claim that the ordering given by: $\pi_i \succ \pi_j$ if $i > j$ is a E-ordering. Indeed, the conditions 1a–1c of G-orderings are easy to check. We leave it to the reader to verify that condition 2 holds, too.

We write $\pi_1 \succeq \pi_2$ to express that $\pi_1 \succ \pi_2$ or $\pi_1 = \pi_2$.

In addition to having more complex conditions on orderings for the modal case, another problem will appear in the proof of completeness. For classical logic, one can prove completeness of ordered resolution by purely semantical considerations [see e.g. Bachmair and Ganzinger 2000]. In the modal case, we want to also prove that the strategy of selecting the greatest formula (or path) in a clause is compatible with the redundancies considered so far, so we will follow the same pattern of proof: first prove a property of derivations in the tableau calculus, and then transfer it to the inverse calculus using a variant of the Subsequent Lemma. However, unlike the redundancies considered so far, such a proof of completeness will meet a technical obstacle whose essence can be expressed shortly as follows: *the ordering requirement is formulated in terms of premises of an inference, while the completeness proofs for the tableau calculus expand proofs from conclusions.* This obstacle would lead to a slightly complex definition of the ordering on paths.

Initially, define S_i to be the set of path in G of modal depth i; /* 1 */
<u>for all</u> S_i except the last one **<u>do</u>** /* 2 */
  <u>let</u> $\pi_1, \ldots, \pi_n$ be all path in S_i ending at $\Diamond$; /* 3 */
  split S_i into $S_i - \{\pi_1, \ldots, \pi_n\} \succ \{\pi_1\} \succ \ldots \succ \{\pi_n\}$; /* 4 */
split S_0 into $S_0 - \{\epsilon\} \succ \{\epsilon\}$; /* 5 */
<u>while</u> there exists S_i with more than one member **<u>do</u>** /* 6 */
  <u>let</u> $\pi \bowtie_l, \pi \bowtie_r$ be two adjacent path in S_i such that $\pi \notin S_i$; /* 7 */
  <u>let</u> S_l and S_r be the sets of all proper subpath of $\pi \bowtie_l$ and $\pi \bowtie_r$ in S_i ; /* 8 */
  split S_i into $S_i - (S_l \cup S_r) \succ S_r \succ S_l \succ \{\pi \bowtie_r\} \succ \{\pi \bowtie_l\}$; /* 9 */

Figure 9: An ordering procedure

We will now prove that G-orderings exist for every G. To this end we will describe an *ordering procedure* that orders G-paths and note that the resulting ordering is a G-ordering.

The procedure works with a sequence of sets of paths, this sequence is denoted $S_n \succ S_{n-1} \succ \ldots \succ S_0$. The intuitive meaning of the sequence is that for all $\pi \in S_i$ and $\pi' \in S_{i+1}$ we have $\pi \succ \pi'$. The paths belonging to the same S_i are yet unordered (but will be ordered later). At each step we will pick up one set S_i in the sequence containing one or more members and replace S_i by two or more sets $S_{i1} \succ \ldots \succ S_{ik}$ such that $S_{i1} \cup \ldots \cup S_{ik} = S_i$. The procedure terminates when each sets contains exactly one member. An ordering procedure is shown in Figure 9. Note that some sets obtained by the procedure (for example S_l or S_r) may be empty, in this case we do not include them in the sequence. When the procedure terminates, the sequence consists of singleton sets. We then let $\pi \succ \pi'$ if the sequence has the form $\ldots\{\pi\} \succ \ldots \succ \{\pi'\}\ldots$.

We show how the E-ordering of Figure 8 can be obtained by this procedure. The splitting steps will be denoted by $\rightarrow$ superscribed by the line number in the algorithm. Instead of singleton sets $\{\pi_i\}$ we write π_i.

$$
\begin{array}{l}
\{\pi_7, \pi_8, \pi_9, \pi_{10}, \pi_{11}, \pi_{12}\} \succ \{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6\} \rightarrow^4 \\
\{\pi_{11}, \pi_{12}\} \succ \pi_{10} \succ \pi_9 \succ \pi_8 \succ \pi_7 \succ \{\pi_0, \pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6\} \rightarrow^5 \\
\{\pi_{11}, \pi_{12}\} \succ \pi_{10} \succ \pi_9 \succ \pi_8 \succ \pi_7 \succ \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6\} \succ \pi_0 \rightarrow^9 \\
\pi_{12} \succ \pi_{11} \succ \pi_{10} \succ \pi_9 \succ \pi_8 \succ \pi_7 \succ \{\pi_1, \pi_2, \pi_3, \pi_4, \pi_5, \pi_6\} \succ \pi_0 \rightarrow^9 \\
\pi_{12} \succ \pi_{11} \succ \pi_{10} \succ \pi_9 \succ \pi_8 \succ \pi_7 \succ \{\pi_3, \pi_4, \pi_5, \pi_6\} \succ \pi_2 \succ \pi_1 \succ \pi_0 \rightarrow^9 \\
\pi_{12} \succ \pi_{11} \succ \pi_{10} \succ \pi_9 \succ \pi_8 \succ \pi_7 \succ \{\pi_5, \pi_6\} \succ \pi_4 \succ \pi_3 \succ \pi_2 \succ \pi_1 \succ \pi_0 \rightarrow^9 \\
\pi_{12} \succ \pi_{11} \succ \pi_{10} \succ \pi_9 \succ \pi_8 \succ \pi_7 \succ \pi_6 \succ \pi_5 \succ \pi_4 \succ \pi_3 \succ \pi_2 \succ \pi_1 \succ \pi_0.
\end{array}
$$

LEMMA 5.4 *Every ordering produced by an ordering procedure on G is a G-ordering.*

PROOF. First, we have to ensure that the procedure indeed terminates when all sets are singleton. To this end we have to show that at line 7 we can always choose two adjacent path $\pi \bowtie_l, \pi \bowtie_r \in S_i$ such that $\pi \notin S_i$. This follows from the following observation.

(2) *Every non-singleton set S_i obtained after line 5 is made in the following way: there exists a set of path $\pi'_1, \ldots, \pi'_n$ of the same modal length such that π'_i is not a subpath of π'_j for $i \neq j$ and*

$$S_i = \{\pi'' \mid \textit{for some } n,\ \pi'_n \sqsubset \pi \textit{ and } \pi'', \pi'_n \textit{ have the same modal length}\}.$$

Such a set always contains the required adjacent set of paths, namely those of the form $\pi'_n{\vee\!\!\!\wedge}_*$ for the same n.

Second, we have to show that the ordering satisfies all conditions of the definition of G-orderings. Condition 1a is satisfied at line 1 of the algorithm. Condition 1b is satisfied at lines 2–4.

Let us check condition 1c. It is obviously satisfied at line 1 when π_1, π_2 have different modal length and at line 5 when $\pi_2 = \epsilon$. So we assume π_1, π_2 have the same modal length. Then this condition can only be violated at line 9. We show that it is not the case. Indeed, from (2) it follows that (i) no path in $S_i - (S_l \cup S_r)$ is a subpath of a path in $S_l \cup S_r \cup \{\pi{\vee\!\!\!\wedge}_r, \pi{\vee\!\!\!\wedge}_l\}$; (ii) no path in S_r is a subpath of a path in $S_l \cup \{\pi{\vee\!\!\!\wedge}_r, \pi{\vee\!\!\!\wedge}_l\}$; (iii) no path in S_l is a subpath of either $\pi{\vee\!\!\!\wedge}_r$ or $\pi{\vee\!\!\!\wedge}_l$; (iv) $\pi{\vee\!\!\!\wedge}_r$ is not a subpath of $\pi{\vee\!\!\!\wedge}_l$.

We leave to the reader to check that condition 2 is satisfied. The proof is similar to the one for condition 1c and uses (2). $\square$

Since any run of the ordering procedure results in a G-ordering, we obtain the following corollary.

COROLLARY 5.5 *G-orderings exist.*

The notion of G-ordering is introduced in order to prove the existence of refutations of special form related to these orderings and therefore to restrict the search space to only such refutations.

For a path π and a path sequent Γ we write $\pi \succ \Gamma$ to denote $\pi \succ \pi'$ for every π' in Γ.

A $(\vee)$-inference in Pat_G

$$\frac{\Gamma, \pi\vee_l \quad \Gamma, \pi\vee_r}{\Gamma, \pi}\ (\vee)$$

is said to *respect a G-ordering* $\succ$ if $\pi\vee_l \succ \Gamma$ and $\pi\vee_r \succ \Gamma$. Likewise, a $(\wedge)$-inference in Pat_G

$$\frac{\Gamma, \pi\wedge_l, \pi\wedge_r}{\Gamma, \pi}\ (\wedge)$$

is said to respect $\succ$ if $\pi\wedge_l \succ \Gamma$ and $\pi\wedge_r \succ \Gamma$. A derivation in Pat_G is said to respect $\succ$ if every $(\vee)$ and every $(\wedge)$-inference of this derivation respect $\succ$.

LEMMA 5.6 *Let G be a unsatisfiable formula and $\succ$ be a G-ordering. Then there exists a refutation of ϵ in Pat_G that respects $\succ$.*

It is not immediately obvious how to prove this lemma. If we try to prove this lemma by induction, starting with zero-step derivation of the empty path and trying to extend the derivation from conclusion to premises, we may be in a "deadlock" situation: i.e. obtain a path sequent Γ such that *every* $(\wedge)$- and $(\vee)$-inference having Γ as the conclusion does not respect $\succ$. The conditions on an inference to respect $\succ$ are formulated in terms of the *premises* of the inference, but in such a inductive proof we can only extend the *conclusion* by a new inference. To overcome this problem, we will try to avoid path sequents that bring us into a deadlock, by restricting ourselves to the so-called *$\succ$-compact path sequents* defined below.

Let $\pi_1, \ldots, \pi_n$ be paths in G and $\succ$ be a G-ordering. The path sequent $\Gamma = \pi_1, \ldots, \pi_n$ is called *$\succ$-compact* if for every $i = 1, \ldots, n$,

1. if π_i is a $\wedge$-path, then $\pi_i\wedge_l \succ \Gamma$ and $\pi_i\wedge_r \succ \Gamma$;
2. likewise, if π_i is a $\vee$-path, then $\pi_i\vee_l \succ \Gamma$ and $\pi_i\vee_r \succ \Gamma$.

Using our generic notation, we can reformulate the definition of $\succ$-compactness as "for every $\bowtie$-path π_i in Γ we have $\pi_i\bowtie_* \succ \Gamma$". Note that $\pi_i\bowtie_* \succ \pi_i$ is guaranteed by the conditions on G-orderings, therefore the requirement $\pi_i\bowtie_* \succ \Gamma$ can be replaced by $\pi_i\bowtie_* \succ \Gamma \setminus \{\pi_i\}$.

The next lemma asserts that a compact sequent Γ cannot lead to a deadlock.

Lemma 5.7 *Let $\succ$ be a G-ordering. Then*

1. *the premise of every $(\Diamond)$-inference is $\succ$-compact.*

   Furthermore, let Γ be a $\succ$-compact sequent occurring in a derivation of ϵ. Then

2. *every $(\bowtie)$-inference having Γ as the conclusion, respects $\succ$;*
3. *if Γ contains at least one $\wedge$-path or $\vee$-path, then there exists a $(\bowtie)$-inference all whose premises are $\succ$-compact.*

Proof.

1. We prove: *the premise of every $(\Diamond)$-inference*

   $$\frac{\Pi\Box, \pi\Diamond}{\Gamma, \Pi, \pi}\ (\Diamond)$$

   is $\succ$-compact, i.e. for every π' in the premise, if $\pi\bowtie_*$ is a G-path, then $\pi' \succ \Pi\Box, \pi\Diamond - \{\pi'\}$. Let us prove $\pi\Diamond\bowtie_* \succ \Pi\Box$, the case when $\pi' \in \Pi\Box$ is similar. By Lemma 5.1, all paths in $\Pi\Box, \pi\Diamond$ are of the same modal length, so $\pi\Diamond\bowtie_*$ is of the same modal length as any path in $\Pi\Box$. By condition 1b of the definition of G-ordering we have $\pi\Diamond\bowtie_* \succ \Pi\Box$.

2. We prove: *every $(\bowtie)$-inference having Γ as the conclusion, respects $\succ$.* Consider the case of $(\wedge)$-inferences, the case of $(\vee)$ is similar. Then Γ is of the form Π, π and the inference is

   $$\frac{\Pi, \pi\wedge_l, \pi\wedge_r}{\Pi, \pi}\ (\wedge).$$

   Since Π, π is $\succ$-compact, $\pi\wedge_l \succ \Pi, \pi$ and $\pi\wedge_r \succ \Pi, \pi$, therefore the inference respects $\succ$.

3. We assume that Γ contains at least one $\bowtie$-path and prove: *there exists a $(\bowtie)$-inference all whose premises are $\succ$-compact.* Consider the set of paths

   $$S = \{\pi\bowtie_* \mid \pi \in \Gamma \text{ and } \pi\bowtie_* \text{ is a } G\text{-path}\}.$$

   By the assumption, this set is nonempty. Since $\succ$ is a linear ordering, this set has the least element w.r.t. $\succ$. Without loss of generality we assume that this path is $\pi\wedge_l$. Then Γ has the form Π, π for some path sequent Π. Consider the inference

   $$\frac{\Pi, \pi\wedge_l, \pi\wedge_r}{\Pi, \pi}\ (\wedge).$$

   We claim that the premise of this inference is $\succ$-compact. This amounts to proving that (3a) $\pi \wedge \bowtie_* \succ \Pi$, whenever $\pi \wedge \bowtie_*$ is a G-path; and (3b) for every path $\pi' \in \Pi$ we have $\pi'\bowtie_* \succ \Pi, \pi\wedge$, whenever $\pi'\bowtie_*$ is a G-path. Let us prove this.

(a) Since Π, π is $\succ$-compact, $\pi\wedge \succ \Pi$. Evidently, $\pi\wedge$ is a subpath of $\pi \wedge \wedge\!\!\!\!\vee_*$, thus by the definition of G-orderings (condition 1c) we have $\pi \wedge \wedge\!\!\!\!\vee_* \succ \pi\wedge$. This implies $\pi \wedge \wedge\!\!\!\!\vee_* \succ \Pi$.

(b) Since Π, π is compact, $\pi' \wedge\!\!\!\!\vee_* \succ \Pi$, so it remains to prove $\pi' \wedge\!\!\!\!\vee_* \succ \pi\wedge$, i.e. $\pi' \wedge\!\!\!\!\vee_* \succ \pi\wedge_l$ and $\pi' \wedge\!\!\!\!\vee_* \succ \pi\wedge_r$. Note that $\pi' \wedge\!\!\!\!\vee_* \in S$ and $\pi\wedge_l$ is the least element of S, therefore $\pi' \wedge\!\!\!\!\vee_* \succeq \pi\wedge_l$. By Lemma 5.2 (a sequent in a derivation of ϵ cannot contain a path and its subpath), $\pi' \wedge\!\!\!\!\vee_*$ and $\pi\wedge_l$ cannot coincide, thus $\pi' \wedge\!\!\!\!\vee_* \succ \pi\wedge_l$. Again by this lemma, $\pi' \wedge\!\!\!\!\vee_*$ cannot be a subpath of $\pi\wedge_l$. In addition, we have $\pi' \wedge\!\!\!\!\vee_* \succ \pi\wedge_l$ and $\pi\wedge_l$ is adjacent to $\pi\wedge_r$, then by condition 2 of the definition of G-ordering, $\pi' \wedge\!\!\!\!\vee_* \succ \pi\wedge_r$.

□

This proof explains the conditions used in the definition of G-orderings. The only condition that has not been used is 1a, and indeed this condition can be omitted. This condition ensures that in the multiset ordering on sequents induced by $\succ$ the premise of any rule of Seq is greater than its conclusion.

Now we can prove Lemma 5.6: if G is unsatisfiable, then there exists a refutation of ϵ in Pat_G that respects $\succ$.

Proof (of Lemma 5.6). The lemma is proved by induction. Starting with the path sequent ϵ, we will construct, step by step, a derivation D of ϵ such that (i) D consists only of $\succ$-compact path sequents, (ii) the formula images of all top sequents of D are unsatisfiable; until we obtain a refutation. Lemma 5.7 guarantees that the obtained refutation respects $\succ$. At every step we choose any top path sequent Π of D that is not an axiom and extend the derivation by adding an inference having Π as its conclusion.

1. (Induction base.) *The sequent ϵ is $\succ$-compact.* This means $\wedge\!\!\!\!\vee_* \succ \emptyset$, which is obvious.

2. (Induction step). Denote by Γ the formula image of Π. We have that Γ is unsatisfiable and not an axiom, and Π is $\succ$-compact. We show how to find an inference with the conclusion Π and premises satisfying the same properties.

   If Γ contains any conjunction or disjunction, then by Lemma 5.7 there exists an ($\wedge\!\!\!\!\vee$)-inference having Π as the conclusion and $\succ$-compact premises. Since ($\wedge\!\!\!\!\vee$)-rules are invertible (Lemma 2.2), the formula images of these premises are also unsatisfiable.

   If Γ contains neither conjunction nor disjunction and Γ is unsatisfiable, then there is a ($\Diamond$)-inference in Seq with the conclusion Γ and unsatisfiable premise Γ'. It is not hard to argue that there exists an ($\Diamond$)-inference in Pat_G with a premise Π' such that Γ' is the formula image of Π'. By Lemma 5.7, Π' is $\succ$-compact.

The termination of this construction of derivation can be proved by induction on the length of derivations. □

6 Inverse path calculus

In this section we define the inverse version of the path calculus, obtained by turning Inv_G into a path calculus.

axioms: π_1, π_2 $\qquad \dfrac{\Gamma, \pi, \pi}{\Gamma, \pi}\ (C)$

$$\frac{\Gamma, \pi\wedge_l}{\Gamma, \pi}\ (\wedge_l) \qquad \frac{\Gamma, \pi\wedge_r}{\Gamma, \pi}\ (\wedge_r)$$

$$\frac{\Gamma, \pi\vee_l \quad \Delta, \pi\vee_r}{\Gamma, \Delta, \pi}\ (\vee)$$

$$\frac{\Box\Gamma, \Diamond\pi}{\Gamma, \pi}\ (\Diamond) \qquad \frac{\Box\Gamma}{\Gamma, \pi}\ (\Diamond^+)$$

All paths occurring in path sequents are G-paths. In the rule (ax) we have $G|_{\pi_1} = p$ and $G|_{\pi_2} = \neg p$ for some propositional variable p. In the rule $(\Diamond^+)$, π is an $\Diamond$-path.

Figure 10: Sequent calculus IP_G

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\wedge_l\Box,\ \wedge_r\wedge_l\Box\vee_l \qquad \wedge_r\wedge_l\Box\vee_r,\ \wedge_r\wedge_r\wedge_l\Diamond}{\wedge_l\Box,\ \wedge_r\wedge_l\Box,\ \wedge_r\wedge_r\wedge_l\Diamond}\ (\vee)}{\wedge_l,\ \wedge_r\wedge_l,\ \wedge_r\wedge_r\wedge_l}\ (\Diamond)}{\wedge_l,\ \wedge_r,\ \wedge_r}\ (\wedge),(\wedge),(\wedge)}{\wedge_l,\ \wedge_r}\ (C)}{\epsilon}\ (\wedge),(\wedge),(C)$$

Some sequences of inferences are shown as one inference.

Figure 11: Derivation in IP_E

The *inverse path calculus* IP_G and a derivation in it are given in Figures 10 and 11. For any inference of IP_G of Figure 10, the path π is called the *principal path* of this inference.

The same arguments as in Bisimulation Lemma 4.1 prove the following lemma.

LEMMA 6.1 (Bisimulation Lemma for IP_G) *(i) Let D be a derivation in IP_G. Then the formula image of D is a derivation of G in Inv_G. (ii) Let D' be a derivation of a sequent $A_1, \ldots, A_n$ in Inv_G, and $\pi_1, \ldots, \pi_n$ be positions such that $G|_{\pi_i} = A_i$, for all $i = 1, \ldots, n$. Then there exists a derivation D of $\pi_1, \ldots, \pi_n$ in IP_G such that D' is the formula image of D. (iii) Both (i) and (ii) also hold if "derivation" is replaced by "refutation" everywhere.* $\Box$

The calculus IP_G has different rules for handling $\wedge$, so we change the definition of derivations respecting a G-ordering in the following way.

A derivation in IP_G is said to *respect a G-ordering* $\succ$ if

1. for every $(\vee)$ inference of this derivation
$$\frac{\Gamma, \pi\vee_l \quad \Gamma, \pi\vee_r}{\Gamma, \pi}\ (\vee)$$
we have $\pi\vee_l \succ \Gamma$ and $\pi\vee_r \succ \Gamma$;
2. likewise, for every $(\wedge_l)$ inference (respectively $(\wedge_r)$ inference) of this derivation
$$\frac{\Gamma, \pi\wedge_l}{\Gamma, \pi}\ (\wedge_l) \qquad (\qquad \text{respectively} \qquad \frac{\Gamma, \pi\wedge_r}{\Gamma, \pi}\ (\wedge_r) \qquad)$$
we have $\pi\wedge_l \succ \Gamma$ (respectively $\pi\wedge_r \succ \Gamma$).

The next theorem is the main theorem of this report. It asserts that a formula is unsatisfiable if and only if it has a refutation satisfying several redundancy criteria. Namely, we prove that any sequent containing a conflicting pair is redundant, and that any inference not respecting $\succ$ is redundant as well. Later, we will give an even stronger version of this statement related to the subsumption rule, but we will need to introduce a new notion of derivation first.

THEOREM 6.2 (Completeness of IP_G with Redundancies) *A formula G is unsatisfiable if and only if ϵ has a refutation in the calculus IP_G. Moreover, if ϵ has a refutation in IP_G and $\succ$ is a G-order, then there exists a refutation of ϵ with the following properties:*

1. *For every path sequent Π in the refutation and every paths π_1, π_2 in Π, π_1 and π_2 have the same modal length.*
2. *For every path sequent Π in the refutation and every paths π_1, π_2 in Π, π_1 and π_2 are not in conflict.*
3. *The refutation respects $\succ$.*

We denote by $\mathrm{IP}_{G,\succ}$ the calculus obtained from IP_G by (i) removal of all path sequents containing paths of different modal lengths, (ii) removal of all path sequents containing a conflicting pair of paths, (iii) removal of all inferences that do not respect $\succ$.

The key lemma in the proof of Theorem 6.2 is as follows.

LEMMA 6.3 (Subsequent Lemma for $\mathrm{IP}_{G,\succ}$) *Suppose that D is a refutation of ϵ in Pat_G that respects $\succ$ and I is an inference in D of the form*

$$\frac{\Gamma_1 \quad \cdots \quad \Gamma_n}{\Gamma},$$

and path sequents $\Delta_1, \ldots, \Delta_n$ are such that each Δ_i is a subsequent of Γ_i. Then there exists a derivation D' of Δ in $\mathrm{IP}_{G,\succ}$ from $\Delta_1, \ldots, \Delta_n$ such that Δ is a subsequent of Γ.

PROOF. We will give a proof by case analysis on the inference I, after having noted one general property of $\Delta, \Delta_1, \ldots, \Delta_n$.

By Lemma 5.3 each path sequent among $\Gamma_1, \ldots, \Gamma_n$ contains no conflicting pair of paths. Since $\Delta_1, \ldots, \Delta_n$ are submultisets of $\Gamma_1, \ldots, \Gamma_n$, respectively, we obtain that none of $\Delta_1, \ldots, \Delta_n$ contains a conflicting pair of paths. Likewise, if we find a derivation of Δ from $\Delta_1, \ldots, \Delta_n$, since Δ is a submultiset of Γ, we obtain that Δ contains no conflicting pair of paths. In a similar way we can check that it contains no path sequent with path of different modal length. Therefore, it is enough to check that the intermediate sequents of D' do not violate the condition, which will be obvious in all cases.

Therefore, we will only show how to find the derivation D' and prove that it respects $\succ$. Consider all possible cases of inferences in Pat_G.

1. The inference I is an axiom Γ, π_1, π_2 such that $G|_{\pi_1} = \neg G|_{\pi_2}$. Then π_1, π_2 is a subsequent of Γ, π_1, π_2 and is an axiom of IP_G. Therefore, we can let D' be the derivation consisting of one axiom π_1, π_2. Evidently, it respects $\succ$.
2. The inference I has the form

$$\frac{\Pi, \pi\wedge_l, \pi\wedge_r}{\Pi, \pi}\ (\wedge)$$

Note that $\pi\wedge_l, \pi\wedge_r \succ \Pi$, because D respects $\succ$. We assume that some sequent Δ_1 is a subsequent of $\Pi, \pi\wedge_l, \pi\wedge_r$ and find a derivation in IP_G of a subsequent of Π, π from Δ_1.

Consider the following possible cases:

(a) *Case: Δ_1 contains neither $\pi\wedge_l$, nor $\pi\wedge_r$.* In this case Δ_1 is a subsequent of Π. Then Δ_1 is also a subsequent of Π, π, and the empty derivation of Δ_1 from Δ_1 proves the case.

(b) *Case: Δ_1 contains $\pi\wedge_l$, but not $\pi\wedge_r$.* In this case $\Delta_1 = \Delta', \pi\wedge_l$ for some path sequent Δ' subsuming Π. The following derivation proves the case:

$$\frac{\Delta', \pi\wedge_l}{\Delta', \pi}\ (\wedge_l)$$

Since $\pi\wedge_l, \pi\wedge_r \succ \Pi$ and Δ' is a submultiset of Π, we have $\pi\wedge_l \succ \Delta'$. The ordering condition can be checked in the similar way for all other cases, we do not consider it any more.

(c) *Case: Δ_1 contains $\pi\wedge_r$, but not $\pi\wedge_l$.* This case is symmetric to the previous one.

(d) *Case: Δ_1 contains both $\pi\wedge_l$ and $\pi\wedge_r$.* In this case $\Delta_1 = \Delta', \pi\wedge_l, \pi\wedge_r$ for some path sequent Δ' subsuming Π. Without loss of generality assume $\pi\wedge_l \succ \pi\wedge_r$ (and hence $\pi\wedge_l \succ \Delta', \pi\wedge_r$). The following derivation proves the case:

$$\cfrac{\cfrac{\cfrac{\Delta', \pi\wedge_l, \pi\wedge_r}{\Delta', \pi, \pi\wedge_r}\ (\wedge_l)}{\Delta', \pi, \pi}\ (\wedge_r)}{\Delta', \pi}\ (C)$$

3. The inference I has the form

$$\frac{\Pi, \pi\vee_l \quad \Pi, \pi\vee_r}{\Pi, \pi}\ (\vee)$$

Note that $\pi\vee_l, \pi\vee_r \succ \Pi$, because D respects $\succ$. We assume that some sequents Δ_1, Δ_2 are subsequents of $\Pi, \pi\vee_l$ and $\Pi, \pi\vee_r$, respectively, and find a derivation in IP_G of a sequent subsuming Π, π from Δ_1, Δ_2.

Consider the following possible cases:

(a) *Case: Δ_1 does not contain $\pi\vee_l$.* In this case Δ_1 is a subsequent of Π, and hence a subsequent of Π, π. Then we can take the 0-step derivation of Δ_1 from Δ_1 as D'.

(b) *Case: Δ_2 does not contain $\pi\vee_r$.* This case is symmetric to the previous one.

(c) *Case: Δ_1 contains $\pi\vee_l$ and Δ_2 contains $\pi\vee_r$.* In this case $\Delta_1 = \Delta'_1, \pi\vee_l$ and $\Delta_2 = \Delta'_2, \pi\vee_r$ for some subsequents Δ'_1, Δ'_2 of Π. Evidently, some subsequent Π' of Π can be derived from the sequent Δ_1, Δ_2 by a sequence of inferences by the contraction rule (C). The following derivation proves the case:

$$\begin{array}{c} \dfrac{\Delta'_1, \pi\vee_l \quad \Delta'_2, \pi\vee_r}{\Delta'_1, \Delta'_2, \pi}\ (\vee) \\ \vdots \quad \text{series of contractions} \\ \Pi', \pi \end{array}$$

4. The inference I has the form

$$\frac{\Box\Pi, \Diamond\pi}{\Gamma, \Pi, \pi}\ (\Diamond)$$

We assume that some sequent Δ is a subsequent of $\Box\Pi, \Diamond\pi$ and find a derivation in IP_G of a subsequent of Π, π, and hence also a subsequent of the conclusion Γ, Π, π.

Consider the following possible cases:

(a) *Case:* Δ *does not contain* $\Diamond\pi$. In this case Δ is a subsequent of $\Box\Pi$, so Δ has a form $\Box\Delta'$ for a subsequent Δ' of Π. The following derivation in IP_G proves the case:

$$\frac{\Box\Delta'}{\Delta', \pi}\ (\Diamond^+).$$

(b) *Case:* Δ *contains* $\Diamond\pi$. In this case Δ has the form $\Box\Delta', \Diamond\pi$ for a subsequent Δ' of Π. The following derivation in IP_G proves the case:

$$\frac{\Box\Delta', \Diamond\pi}{\Delta', \pi}\ (\Diamond).$$

$\Box$

By the induction on the length of derivations one can easily generalize Lemma 6.3 from inferences to arbitrary derivations:

Lemma 6.4 *Let D be a refutation of ϵ in Pat_G that respects $\succ$, D'' be a subderivation in D of a sequent Γ from sequents $\Gamma_1, \ldots, \Gamma_n$, and path sequents $\Delta_1, \ldots, \Delta_n$ are subsequents of $\Gamma_1, \ldots, \Gamma_n$, respectively. Then there exists a derivation D' of Δ in $\mathrm{IP}_{G,\succ}$ from $\Delta_1, \ldots, \Delta_n$ such that Δ is a subsequent of Γ.* $\Box$

Now we can prove Theorem 6.2.

Proof (of Theorem 6.2).

($\Rightarrow$) Suppose that a formula G is unsatisfiable. By Completeness Theorem 4.2 for Pat_G, the sequent ϵ has a refutation in Pat_G. By Lemma 6.4, there exists a subsequent of ϵ that has a refutation in $\mathrm{IP}_{G,\succ}$. Evidently, the empty sequent has no refutation in IP_G, therefore ϵ has a refutation in $\mathrm{IP}_{G,\succ}$.

($\Leftarrow$) Suppose that ϵ has a refutation in $\mathrm{IP}_{G,\succ}$. By Bisimulation Lemma 6.1 for IP_G, the sequent G has refutation in Inv_G. It is not hard to argue that Inv_G is sound, i.e. every sequent that has refutation is unsatisfiable. Therefore, G is unsatisfiable.

$\Box$

7 More redundancies: subsumption

We say that a path sequent Π *subsumes* a path sequent Π' if Π is a subsequent of Π'[1]. Saturation-based theorem provers for first-order logic often use subsumption as a redundancy criterion: clauses subsumed by other clauses are removed from the search space.

In order to formalize subsumption as a redundancy criterion, we need a new notion of derivation, since subsumed clauses are not redundant per se, but only redundant with respect to other clauses in the search space. So to prove completeness of subsumption, we need to formalize the notion of search space. In this section we give such a formalization and prove that subsumption can be used to remove clauses from the search space. The formalization here uses two main ideas: inference rules are reformulated as behaving on sets of clauses [Bachmair and Ganzinger 2000], a very general property of derivations on such sets is proved to guarantee completeness in the style of [Voronkov 1992], and then we prove that the system with subsumption satisfies these properties.

As before, we assume to deal with *inferences*, consisting of zero or more premises and one conclusion, we assume that all premises and the conclusion belong to a set of derivable objects, or simply *objects*. Examples of objects used in this report are sequents and path sequents. An *inference rule*, is a collection of inferences. An *inference system* or a *calculus* is a collection of inference rules. Alternatively, we will consider an inference system as consisting of inferences that are instances of inference rules of this system. Suppose that $\mathcal{I}$ is an inference system on objects. We turn it into an inference system $\mathcal{I}^s$ on *sets of objects*, called the *set calculus* as follows. Take any inference

$$\frac{\Pi_1 \cdots \Pi_n}{\Pi} \tag{3}$$

of $\mathcal{I}$. Then for every set S of objects the following is an inference rule of $\mathcal{I}^s$:

$$\frac{S \cup \{\Pi_1, \ldots, \Pi_n\}}{S \cup \{\Pi_1, \ldots, \Pi_n, \Pi\}} \; .$$

The intuitive meaning of $\mathcal{I}^s$ is the following: sets of objects formalize search space, the inference rule above means: if $\Pi_1, \ldots, \Pi_n$ are already proved (belong to the search space), we can add Π to the search space, provided that (3) is an inference of $\mathcal{I}$. Inferences in $\mathcal{I}^s$ will be denoted using the $\triangleright$ sign:

$$S \cup \{\Pi_1, \ldots, \Pi_n\} \triangleright S \cup \{\Pi_1, \ldots, \Pi_n, \Pi\}.$$

Sometimes we will superscribe $\triangleright$ with the name of the inference rule.

We call the *initial set of objects* of $\mathcal{I}^s$ the set of all axioms of $\mathcal{I}$. We also assume that with $\mathcal{I}$ associated is a set of objects, called the *goal object*. For example, in all our path calculi the goal object is the sequent ϵ. We call a *derivation* in the set calculus any sequence of inferences:

$$S_0 \triangleright S_1 \triangleright S_2 \triangleright \ldots \;\; ,$$

possibly infinite, where S_0 is the initial set of objects. We say the derivation *succeeds* if some S_i contains a goal object, and *fails* otherwise.

The subsumption strategy that deletes a clause from the search space can now be formalized as the inference rule in the set calculus

[1] This notion of subsumption should not be mixed with the subsumption of concepts in description logics.

$$S \cup \{\Pi_1, \Pi_2\} \triangleright S \cup \{\Pi_1\},$$

provided that Π_1 subsumes Π_2.

When we have a notion of derivation that deletes objects from the search space, we can have the following problem. Suppose that an inference rule adds an object Π to the search space, and then such a deletion rule removes it. Then we may obtain an infinite failing derivation:

$$S \triangleright S \cup \{\Pi\} \triangleright S \triangleright S \cup \{\Pi\} \triangleright \ldots ,$$

even when a successful derivation exists. Such a derivation is *unfair*: there have been inferences applicable to S but never applied. Let us formalize the notion of fairness.

Consider a derivation $S_0 \triangleright S_1 \triangleright \ldots$ in $\mathcal{I}^s$. The set $\bigcup_i \bigcap_{j \geq i} S_j$ is called the *limit* of this derivation. The derivation is called *fair* if, for every inference of $\mathcal{I}$

$$\frac{\Pi_1 \cdots \Pi_n}{\Pi}$$

such that $\Pi_1, \ldots, \Pi_n$ belong to the limit of the derivation, an inference of $\mathcal{I}^s$

$$S \cup \{\Pi_1, \ldots, \Pi_n\} \triangleright S \cup \{\Pi_1, \ldots, \Pi_n, \Pi\}$$

has been applied in the derivation, i.e., for some i and S, S_i has the form $S \cup \{\Pi_1, \ldots, \Pi_n\}$ and S_{i+1} has the form $S \cup \{\Pi_1, \ldots, \Pi_n, \Pi\}$.

We will now prove a statement about $\mathrm{IP}^s_{G,\succ}$, i.e. the set inference system obtained from $\mathrm{IP}_{G,\succ}$. It will be clear from the proof that the statement is very general and applicable to any inference system without the deletion rule.

THEOREM 7.1 (Completeness of $\mathrm{IP}^s_{G,\succ}$) *Let G be a formula and $\succ$ be a G-ordering. (i) If any derivation in $\mathrm{IP}^s_{G,\succ}$ succeeds, then G is unsatisfiable. (ii) If G is unsatisfiable, then every fair derivation in $\mathrm{IP}^s_{G,\succ}$ succeeds.*

PROOF. (i) Let a derivation $S_0 \triangleright \ldots \triangleright S_n$ succeed. It is easy to see that every $\Pi \in \bigcup_i S_i$ has a refutation in $\mathrm{IP}_{G,\succ}$. Therefore ϵ has a refutation, and by completeness of $\mathrm{IP}_{G,\succ}$ (Theorem 6.2), G is unsatisfiable.

(ii) Let $\mathcal{S} = S_0 \triangleright S_1 \triangleright \ldots$ be a fair derivation. We prove that every path sequent Π that has a refutation in $\mathrm{IP}_{G,\succ}$ occurs in some S_i. The proof is by induction on the length of the refutation of Π in $\mathrm{IP}_{G,\succ}$. If the refutation is of length 0, i.e., Π is an axiom, then Π is already a member of S_0. For refutations of longer lengths, consider the last inference of this refutation:

$$\frac{\Pi_1 \quad \ldots \quad \Pi_n}{\Pi} \, .$$

By the induction hypothesis, every Π_i is a member of some S_{k_i}. Since $\mathrm{IP}^s_{G,\succ}$ does not delete any path sequents, all Π_i belong to the limit of $\mathcal{S}$. Then by fairness of $\mathcal{S}$, Π is a member of some S_m, too. □

Now we are ready to prove that $\mathrm{IP}^s_{G,\succ}$ with subsumption is complete, i.e. every fair derivation in this system contains ϵ, whenever G is unsatisfiable. In fact, we will define an even stronger notion of subsumption first, and then prove that $\mathrm{IP}^s_{G,\succ}$ is complete with this stronger notion.

We say that a path sequent Π *subpath-subsumes*, or *$\sqsubseteq$-subsumes*, a path sequent Π' if for every $\pi \mathrel{\dot{\in}} \Pi \mathbin{\dot{-}} \Pi'$ there exists a $\pi' \mathrel{\dot{\in}} \Pi' \mathbin{\dot{-}} \Pi$ such that $\pi \sqsubset \pi'$.

We call $\mathrm{IP}^s_{G,\succ}$ *with $\sqsubseteq$-subsumption* the inference system obtained from $\mathrm{IP}^s_{G,\succ}$ by adding the *$\sqsubseteq$-subsumption rule*:

$$S \cup \{\Pi, \Pi'\} \triangleright S \cup \{\Pi\},$$

where Π subsumes Π' and $\Pi \neq \Pi'$.

Let us note some obvious properties of $\sqsubseteq$-subsumption:

LEMMA 7.2

1. *Note that Π subsumes Π' if and only if $\Pi \mathbin{\dot{-}} \Pi'$ is empty. Therefore, if Π subsumes Π', then Π also $\sqsubseteq$-subsumes Π'.*
2. *If Π_1 $\sqsubseteq$-subsumes Π_2 and Π_2 $\sqsubseteq$-subsumes Π_1, then $\Pi_1 = \Pi_2$.*
3. *If Π $\sqsubseteq$-subsumes Π' and Π' $\sqsubseteq$-subsumes Π'', then Π $\sqsubseteq$-subsumes Π''.*
4. *If $\pi_1 \sqsubset \pi'$, ..., $\pi_n \sqsubset \pi'$, and Π $\sqsubseteq$-subsumes Π', then $\Pi, \pi_1, \ldots, \pi_n$ $\sqsubseteq$-subsumes Π', π'.*
5. *If $\pi \sqsubseteq \pi'$ and Π $\sqsubseteq$-subsumes Π', then Π, π $\sqsubseteq$-subsumes Π', π'.*
6. *There is no infinite sequence $\Pi_0, \Pi_1, \ldots$ such that Π_{i+1} $\sqsubseteq$-subsumes Π_i and $\Pi_{i+1} \neq \Pi_i$.*

We will note two further properties of $\sqsubseteq$ related to proof-search optimization after proving completeness of $\mathrm{IP}^s_{G,\succ}$ with $\sqsubseteq$-subsumption.

THEOREM 7.3 (Completeness of $\mathrm{IP}^s_{G,\succ}$ with $\sqsubseteq$-subsumption) *(i) If any derivation in the system $\mathrm{IP}^s_{G,\succ}$ with $\sqsubseteq$-subsumption succeeds, then G is unsatisfiable. (ii) If G is unsatisfiable, then every fair derivation $\mathcal{S}$ in $\mathrm{IP}^s_{G,\succ}$ with $\sqsubseteq$-subsumption succeeds.*

PROOF. (i) is proved as in Theorem 7.1 (completeness of $\mathrm{IP}^s_{G,\succ}$). For (ii), we will use Theorem 7.1. Let us show that (ii) immediately follows from

(4) Let $\mathcal{T}$ be a fair derivation in $\mathrm{IP}^s_{G,\succ}$ and a path sequent Π appears in $\mathcal{T}$. Then there exists a path sequent Γ that $\sqsubseteq$-subsumes Π and belongs to the limit of $\mathcal{S}$.

Suppose G is unsatisfiable, then by Theorem 7.1 $\mathcal{T}$ contains the path sequent ϵ. Then by (4), $\mathcal{S}$ must contain a path sequent that $\sqsubseteq$-subsumes ϵ, but there is only one such path sequent: ϵ itself. So, it remains to prove (4).

Suppose $\mathcal{S} = S_0 \triangleright S_1 \triangleright \ldots$ and denote by S_∞ the limit of $\mathcal{S}$. We say that a path sequent Π has been *removed* in $\mathcal{S}$ by Π' if some S_{i+1} is $S_i \setminus \{\Pi\}$, and $\Pi' \in S_{i+1}$ $\sqsubseteq$-subsumes Π. If Π has been removed by Π_1, it is possible that Π_1 is later removed by Π_2 etc., but item 6 of Lemma 7.2 guarantees that this cannot continue infinitely often, so, starting from some j, all S_j will contain the same path sequent Π' that $\sqsubseteq$-subsumes Π. Thus, we proved:

(5) *for every path sequent Π in any S_i there exists a $\Pi' \in S_\infty$ such that Π' $\sqsubseteq$-subsumes Π.*

Moreover, using similar considerations one can strengthen this result to

(6) *for every path sequent* Π *in any* S_i *there exists a* $\Pi' \in S_\infty$ *such that* Π' $\sqsubseteq$*-subsumes* Π *and no* $\Pi'' \in S_\infty$ $\sqsubseteq$*-subsumes* Π'.

For any path sequent Π (not necessarily in some S_i) we call a *trace* of Π in $\mathcal{S}$ any $\Pi' \in S_\infty$ such that Π' $\sqsubseteq$-subsumes Π and no $\Pi'' \in S_\infty$ $\sqsubseteq$-subsumes Π'. Now (6) can be reformulated: every $\Pi \in S_i$ has a trace in $\mathcal{S}$.

Note that (4) obviously follows from a slightly more general statement

(7) *Every path sequent* Π *that appears in a fair derivation* $\mathcal{T}$ *in* $\mathrm{IP}^s_{G,\succ}$ *has a trace in* $\mathcal{S}$.

We will now prove (7) and then we are done. Let $\mathcal{T} = T_0 \triangleright T_1 \triangleright \ldots$. Take the least i such that $\Pi \in T_i$. (7) will be proved by induction on i. If $i = 0$, then Π is an axiom, therefore it belongs to S_0, and by (6) has a trace in $\mathcal{S}$. If $i > 0$, then Π is obtained by an inference I of $\mathrm{IP}_{G,\succ}$ from path sequents in T_{i-1}. By the induction hypothesis, these path sequents have traces in $\mathcal{S}$. We use the case analysis on the inference I, but before we will note some general properties related to redundancies and $\sqsubseteq$-subsumption. Namely, we prove the following 3 statements.

(8) *If every path sequent* Π *in the initial set* S_0 *contains no two paths* $\pi_1, \pi_2 \dot{\in} \Pi$ *with different modal lengths, then so is every path sequent* Π' *occurring in any* S_i.

(9) *Suppose* Π *is a path sequent that contains a pair of conflicting paths* π_1, π_2 *and* Π $\sqsubseteq$*-subsumes* Π'. *Then* Π' *contains a pair of conflicting paths, too.*

(10) *Suppose* $\pi \succ \Pi'$ *and* Π $\sqsubseteq$*-subsumes* Π'. *Then* $\pi \succ \Pi$.

(8) is straightforward. (9) follows from the following observation: if $\pi_1, \pi_2 \dot{\in} \Pi$ and Π $\sqsubseteq$-subsumes Π', then there exist $\pi_1', \pi_2' \dot{\in} \Pi'$ such that $\pi_1 \sqsubseteq \pi_1'$ and $\pi_2 \sqsubseteq \pi_2'$, then π_1 being in conflict with π_2 implies π_1', π_2' are conflicting, too. To prove (10), note the following: Π consists of subpaths of Π', and every path is greater than any its subpath w.r.t. $\succ$, so π is greater than any path sequent in Π.

Let us now return to the proof of (7).

1. The inference I has the form

$$\frac{\Pi, \pi\wedge_l}{\Pi, \pi}\ (\wedge_l)$$

Let Γ be a trace of $\Pi, \pi\wedge_l$ in $\mathcal{S}$. Consider three possible cases:

(a) *Case:* Γ $\sqsubseteq$*-subsumes* Π. Then Γ also $\sqsubseteq$-subsumes Π, π, and is a trace of Π, π.

(b) *Case:* Γ *has the form* $\Gamma', \pi\wedge_l$, *where* Γ' $\sqsubseteq$*-subsumes* Π. Since $\mathcal{S}$ is fair, it contains an inference

$$\frac{\Gamma', \pi\wedge_l}{\Gamma', \pi}\ (\wedge_l)$$

By (6), $\mathcal{S}$ contains a trace of Γ', π, evidently, this trace is also a trace of Π, π. To complete the proof of this case, we have to prove that this inference is not redundant in $\mathrm{IP}_{G,\succ}$. (8) guarantees that Γ', π contains no subpaths of different modal lengths, (9) guaranteeses that Γ', π contains no conflicting pairs of paths, because so is Π, π, and finally, (10) guarantees that $\pi \succ \Gamma'$, because $\pi \succ \Pi$.

Below we will not prove any more that the inferences in $\mathcal{S}$ are nonredundant, since the proof is similar to the one in this case.

(c) *Case:* Γ *has the form* Γ', π', *where* Γ' $\sqsubseteq$*-subsumes* Π *and* $\pi' \sqsubset \pi\wedge_l$. Note that if $\pi' \sqsubset \pi$, then Γ, π' also $\sqsubseteq$-subsumes Π, π. So we only consider the case $\pi' = \pi$. Using the definition of $\sqsubseteq$-subsumption, one can show that either Γ, π $\sqsubseteq$-subsumes Π', in which case we are done, or $\pi \mathrel{\dot\in} \Gamma$. Let us show that $\pi \mathrel{\dot\in} \Gamma$ is impossible. Indeed, $\pi \mathrel{\dot\in} \Gamma$ implies that Γ, π has a form Γ', π, π for some Γ'. Since $\mathcal{S}$ is fair, the contraction rule should be applied in $\mathcal{S}$ to Γ', π, π, obtaining Γ', π which $\sqsubseteq$-subsumes Γ', π, π. By (6) Γ', π has a trace in $\mathcal{S}$, then this trace $\sqsubseteq$-subsumes Γ', π, π. But Γ', π, π is a trace itself, and cannot be $\sqsubseteq$-subsumed by any other path sequent in S_∞. Contradiction.

2. The inference I has the form

$$\frac{\Pi_1, \pi\vee_l \quad \Pi_2, \pi\vee_r}{\Pi_1, \Pi_2, \pi}\ (\vee)$$

Take traces Γ_1 and Γ_2 of $\Pi, \pi\vee_l$ and $\Pi, \pi\vee_r$, respectively, in $\mathcal{S}$. Similar to the previous case, we consider the following subcases:

(a) *Case: Either* Γ_1 $\sqsubseteq$*-subsumes* Π_1 *or* Γ_2 $\sqsubseteq$*-subsumes* Π_2. Then either Γ_1 or Γ_2 $\sqsubseteq$-subsumes Π_1, Π_2, π.

(b) *Case:* Γ_1 *contains* $\pi\vee_l$ *and* Γ_2 *contains* $\pi\vee_r$. Then Γ_1 has the form $\Gamma'_1, \pi\vee_l$ and Γ_2 has the form $\Gamma'_2, \pi\vee_r$ such that Γ_1 and Γ_2 $\sqsubseteq$-subsume Π_1 and Π_2, respectively. Since $\mathcal{S}$ is fair, it contains an inference

$$\frac{\Gamma'_1, \pi\vee_l \quad \Gamma'_2, \pi\vee_r}{\Gamma'_1, \Gamma'_2, \pi}\ (\vee)$$

By (6), $\mathcal{S}$ contains a trace of $\Gamma'_1, \Gamma'_2, \pi$, evidently, this trace is also a trace of Π, π.

(c) *Case:* Γ_1 *has the form* Γ'_1, π', *where* Γ'_1 $\sqsubseteq$*-subsumes* Π_1 *and* $\pi' \sqsubset \pi\vee_l$. This case is similar to the respective case for the conjunction above.

(d) *Case:* Γ_2 *has the form* Γ'_2, π', *where* Γ'_2 $\sqsubseteq$*-subsumes* Π_2 *and* $\pi' \sqsubset \pi\vee_r$. This case is symmetric to the previous one.

3. The inference I has the form

$$\frac{\Box\Pi, \Diamond\pi}{\Pi, \pi}\ (\Diamond)$$

Take a trace Γ of $\Box\Pi, \Diamond\pi$. All paths in $\Box\Pi, \Diamond\pi$ have the same modal length, so every proper subpath of any path in $\Box\Pi, \Diamond\pi$ has the modal length less than that of $\Diamond\pi$. This implies that Γ either only consists of paths in $\Box\Pi, \Diamond\pi$ or subpaths of Π, π. In the latter case, using that Γ is a trace, one can prove as in item 1c above that Γ $\sqsubseteq$-subsumes Π, π. So it remains to consider the case when Γ consists of paths in $\Box\Pi, \Diamond\pi$. Note that Γ cannot contain two copies of $\Diamond\pi$, since Γ is a trace, so it contains either one copy of $\Diamond\pi$, or none at all. So we consider two cases:

(a) *Case:* Γ *does not contain* $\Diamond\pi$. In this case Γ is a subsequent of $\Box\Pi$, so Γ has a form $\Box\Gamma'$ for a subsequent Γ' of Π. Consider the following inference:

$$\frac{\Box\Gamma'}{\Gamma', \pi}\ (\Diamond^+).$$

Since $\mathcal{S}$ is fair, it contains this inference. It is easy to see that Γ', π subsumes Π, π, and thus also $\sqsubseteq$-subsumes Π, π. Take a trace of Γ', π in $\mathcal{S}$, evidently it is also a trace of Π, π.

(b) *Case: Γ contains $\Diamond\pi$.* In this case Γ has the form $\Box\Gamma'\Diamond\pi$ for some subsequent Γ' of Π. Consider the following inference:

$$\frac{\Box\Gamma', \Diamond\pi}{\Gamma', \pi}\ (\Diamond).$$

Since $\mathcal{S}$ is fair, it contains this inference. It is easy to see that Γ', π subsumes Π, π, and thus also $\sqsubseteq$-subsumes Π, π. Take a trace of Γ', π in $\mathcal{S}$, evidently it is also a trace of Π, π.

4. The inference I is $(\Diamond^+)$. This case is nearly identical to the previous.

The proof is complete. □

We did not implement $\sqsubseteq$-subsumption, since we could not yet find efficient data structures that make $\sqsubseteq$-subsumption checks fast, but we plan such an implementation. The current version of our system KЯ [Voronkov 2000] uses the ordinary subsumption. Note that the completeness of $\mathrm{IP}_{G,\succ}$ with subsumption immediately follows from completeness of $\mathrm{IP}_{G,\succ}$ with $\sqsubseteq$-subsumption.

However, the completeness result for $\sqsubseteq$-subsumption has some applications even without implementation. Let us look at two interesting observations that come out of this results. Consider any $(\wedge_*)$-inference:

$$\frac{\Pi, \pi\wedge_*}{\Pi, \pi}\ (\wedge_*).$$

Note that the conclusion of this inference $\sqsubseteq$-subsumes the premise. Thus, the premise can be discarded as soon as the rule is applied. Even more, when $\pi\wedge_*$ is not maximal in $\Pi, \pi\wedge_*$, so the inference is redundant, it makes sense to apply this inference, because the premise can be discarded after. If we apply $(\wedge_*)$-rules in an eager way, as soon as they are applicable, any path $\pi\wedge_*$ becomes "short-lived". This observation has a far-reaching consequence: one can modify the calculus $\mathrm{IP}_{G,\succ}$ so that such paths do not occur in sequents at all. Such an optimization first appeared for classical logic in [Voronkov 1985] published in Russian, and then in a more accessible [Voronkov 1990*b*]. It has also been used in tableau calculi in [Degtyarev and Voronkov 1996] as a *least conjunctive superformula* optimization. It is implemented in our system KЯ [Voronkov 2000].

Another interesting consequence of the use of $\sqsubseteq$-subsumption is the following. Suppose that during search for a refutation we obtained a path sequent $\Delta = \Pi, \pi, \pi'$ such that $\pi \sqsubseteq \pi'$. Consider the path sequent $\Gamma = \Pi, \pi', \pi'$. This sequent is $\sqsubseteq$-subsumed by Δ, so it can be added to the search space without compromising soundness or completeness. But if we add Γ to the search space, by contraction we can derive Π, π' that subsumes Δ. Therefore Π, π, π' can be replaced by a simpler sequent Π, π', as soon as π is a subpath of π'.

K	*branch* p	*branch* n	*d4* p	*d4* n	*dum* p	*dum* n	*grz* p	*grz* n	*lin* p	*lin* n	*path* p	*path* n	*ph* p	*ph* n	*poly* p	*poly* n	*t4p* p	*t4p* n
leanK 2.0	1	0	1	1	0	0	0	>	>	4	2	0	3	1	2	0	0	0
Crack 1.0	2	1	2	3	3	>	1	>	5	2	2	6	2	3	>	>	1	1
□**KE**	13	3	13	3	4	4	3	1	>	2	17	5	4	3	17	0	0	3
LWB 1.0	6	7	8	6	13	19	7	13	11	8	12	10	4	8	8	11	8	7
Kris	3	3	8	6	15	>	13	>	6	9	3	11	4	5	11	>	7	5
KSAT	8	8	8	5	11	>	17	>	>	3	4	8	5	5	13	12	10	18
FaCT 1.2	6	4	>	8	>	>	>	>	>	>	7	6	6	7	>	>	>	>
Kꓘ	6	7	>	16	>	>	>	>	>	>	15	14	2	2	>	>	>	>
TA	9	9	>	18	>	>	>	>	>	>	20	20	6	9	16	17	>	19
DLP 3.1	19	13	>	>	>	>	>	>	>	>	>	>	7	9	>	>	>	>
∗**SAT** 1.2	>	12	>	>	>	>	>	>	>	>	>	>	8	12	>	>	>	>

Table 1: Results for K

8 Experiments

The calculus Inv_G for K with the redundancy criteria mentioned above is implemented in version 1.0 of our system Kꓘ (read "K-inverse K"). A preliminary much slower version of Kꓘ was implemented in ML and described in [Voronkov 1999]. The current version 1.0 of Kꓘ used for benchmarking is implemented in C++ and comprises about 13,000 lines of C++ code, of which 5,000 lines implement a parser generated automatically using the ANTLR parser generator and PCCTS tool.

In this section we consider the results of running Kꓘ on the collection of tests used for the Tableau'98 competition. We give a comparison with other systems for K. The results for other systems are taken from [Giunchiglia, Giunchiglia and Tacchella 1999, Horrocks, Patel-Schneider and Sebastiani 2000]. The results for other systems were obtained on a computer with the Pentium 350MHz processor using the time limit of 100 seconds. We ran our experiments on a Pentium 450MHz processor and set the limit to 77 seconds, in order to compensate the difference in performance. Some of the results for other systems are from the Tableaux'98 comparison, but some are more recent [see for details Horrocks et al. 2000].

The experiments were performed on 9 collections of tests. Each collection consists of 21 provable and 21 unprovable formulas of presumably growing difficulty. The results are given in Table 1. p in every column means provable formulas, while n means unprovable. The table contains the number of problems solved by systems within the time limit of 100 seconds. If all 21 problems can be solved, we put $>$ in the table. Table 2 shows the time spent by some systems for solving the most difficult problems in each category.

One can see that Kꓘ 1.0 is weaker than the two best systems ∗**SAT** and **DLP**. However, we would like to note that our current implementation is completely nonoptimized as to proof-search strategies and nonlogical optimizations. For example, both ∗**SAT** and **DLP** have strategies for intelligent backtracking and **DLP** implements tabulation. In addition, all top systems have powerful preprocessors that simplify the input formula, while the current version of Kꓘ has no preprocessor. Kꓘ is stronger on this collection of benchmarks than the less optimized implementations **FaCT** of the tableau method, **KSAT** of the SAT-based method, and the earlier versions of **TA** of the first-order translation-based methods. This shows the high potential of future, more optimized versions of Kꓘ.

Our system is based on a systematic proof procedure without any ad hoc optimizations.

	∗**SAT** 1.2		**DLP** 3.2		**TA** 1.4		Kꓘ	
Test	Size	Time	Size	Time	Size	Time	Size	Time
branch_p	>	0.21	19	46.06	6	65.76	6	29.08
branch_n	12	94.49	13	53.63	6	96.44	7	65.33
d4_p	>	0.06	>	0.05	15	71.11	>	0.09
d4_n	>	2.87	>	1.12	14	44.06	16	69. 17
dum_p	>	0.04	>	0.02	17	64.99	>	0.01
dum_n	>	0.12	>	0.02	16	65.82	>	0
grz_p	>	0.04	>	0.04	>	0.51	>	0.11
grz_n	>	0.01	>	0.05	>	0.33	>	0.08
lin_p	>	0.01	>	0.03	>	9.24	>	0.03
lin_n	>	47.80	>	0.13	>	80.01	>	0.15
path_p	>	0.72	>	0.32	5	25.03	15	36.43
path_n	>	0.96	>	0.36	4	60.84	14	35.92
ph_p	8	48.54	7	10.23	6	43.16	2	0.08
ph_n	12	0.60	>	2.69	9	55.13	2	0.02
poly_p	>	1.73	>	0.11	5	53.48	>	0.54
poly_n	>	2.25	>	0.18	4	9.09	>	0.49
t4p_p	>	0.29	>	0.06	16	88.66	>	0.01
t4p_n	>	1.28	>	0.13	9	87.72	>	0.04

Table 2: Timing Results for ∗**SAT** (options `-k1 -e -m6`), **DLP**, **TA** and Kꓘ.

We expect a considerable speedup by implementing such optimizations. Our expectations are confirmed by the experience of other provers, whose first versions were very considerably slower than the current versions. Among such future optimizations we consider the following.

1. Implementing a formula preprocessor that makes general simplifications of the goal formula and also some specific simplifications that make the inverse proof search easier.

2. Implementing a greedy search for redundancies, similar to the SPASS theorem prover for first-order classical logic [Weidenbach, Gaede and Rock 1996] used as a subsystem in **TA**. This means that we give priority to those inferences that can make some sequents from the search space redundant. Currently, Kꓘ behaves like a very naive Davis-Putnam procedure for propositional classical logic.

3. Optimizing the system for the cases when propositional reasoning dominates the modal reasoning (cf. the weak performance of Kꓘ on the *ph* benchmarks in which essentially no modal reasoning is involved).

9 Related works

The inverse method for nonclassical logics. Gentzen [1934] was the first to formulate the inverse method: he has shown how to prove the decidability of intuitionistic propositional logic using the bottom-up search in a sequent calculus based on the subformula property. The remark of Gentzen [1934] remained unnoticed for quite some time. The general interest to the inverse method appeared due to a series of Maslov's works, e.g. [Maslov 1971].

Voronkov [1992] sketches a general scheme of the inverse method for nonclassical logics. Some redundancy criteria considered in this report are similar to strategies introduced in [Voronkov 1992]. However, [Voronkov 1992] contains no complete proofs.

Mints [1993] and Tammet [1994] consider the inverse method for linear logic. Mints [1994] and Tammet [1996] discuss the inverse method for intuitionistic logic, however Tammet's [1996] presentation contains several inaccuracies that are difficult to validate in the absence of proofs. For example, his general scheme of constructing inverse method calculi may result in incomplete calculi, and the intuitionistic calculus of Tammet [1996] is indeed incomplete. The ordering strategy of our report is similar to a strategy based on the permutability of rules introduced by [Tammet 1996], with a difference that our ordering is computed in compile-time, while Tammet [1996] has to store some runtime information in derived sequents.

Mints et al. [1996] discuss the inverse method for S4 with several redundancies, but several statements of their paper are difficult to validate in the absence of proofs.

Implementation of the inverse method. Tammet [1996] reported some results on the implementation of the inverse method for intuitionistic predicate logic, however it is difficult to comment on his results since calculi underlining his implementation [Tammet 1996] are incomplete.

Our implementation seems to be the first to show that calculi based on forward reasoning may compete with calculi based on backward reasoning for *propositional* logics.

Implementations of other methods. Implementations of other state-of-the-art systems are described in the following papers. SAT-based decision procedures are considered in [Giunchiglia, Giunchiglia, Sebastiani and Tacchella 1998, Tacchella 1999, Giunchiglia et al. 1999]. Hustadt and Schmidt [1997] describe an implementation based on a translation to classical predicate logic and the use of the classical logic theorem prover SPASS [Weidenbach et al. 1996]. Two best tableau-based systems are discussed in [Horrocks and Patel-Schneider 1998, Horrocks 1998, Patel-Schneider 1998]. A comparison of several approaches and a general discussion of evaluation of decision procedures for nonclassical logics can be found in [Hustadt and Schmidt 1997, Horrocks et al. 2000].

10 Future research

The framework described in this paper is applicable to a number of nonclassical logics. It would be interesting to *extend it to more expressive description logics* [Calvanese, De Giacomo, Lenzerini and Nardi 2000]. An obstacle to such an extension is the way these logics are usually introduced. The technique for designing calculi for various logics developed in the description logic community is essentially oriented to tableau-based procedures. These calculi are formulated in a top-down way and often have explicit annotation for worlds. This simplifies completeness proofs but the resulting calculi do not have obvious proof-theoretical properties to be exploited. As a result, only two kinds of satisfiability checking procedures can be developed: tableau-based ones and semantics-based ones.

To design an inverse-method based calculus one needs to *obtain reformulations of these tableau-oriented calculi in a more traditional way*, where the inference rules are formulated solely in terms of formulas, or otherwise find new techniques for designing bottom-up calculi. Such a logical reformulation is an interesting line of research by itself and can lead to better versions of tableau-based procedures that also exploit proof-theoretical properties of calculi, while the current tableau-based systems are mostly based on various nonlogical optimizations, such as tabulation or intelligent backtracking.

It seems that bottom-up calculi can offer considerable advantages for handling some inference rules. For examples, some description logics have a restricted form of the cut rule:

$$\frac{\Gamma, A \quad \Gamma, \overline{A}}{\Gamma} \ (cut),$$

where A and $\overline{A}$ (equivalent to the negation of A) can be arbitrary subformulas of the goal formula. The problem of using this rule in the direction from the conclusion to the premises is high nondeterminism in selecting the cut formula A. The application in the inverse direction does not meet this problem because Γ, A and $\Gamma, \overline{A}$ must be already derived sequents. Therefore, from the viewpoint of bottom-up calculi there is no essential difference between this rule and, say, the $(\vee)$-rule:

$$\frac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta A \vee B} \ (\vee),$$

in which Γ, A and Δ, B should be already proved sequents.

Another interesting line of research is *combination of top-down and bottom-up proof search*, or *bidirectional search*. In such a combination the inverse method would produce sequents that can be used by the tableau method as additional axioms. This is similar to tabulation, but with an essential difference. The traditional tabulation is used to store *derivable* sequents and checking whether a newly derived sequent is unsatisfiable by verifying whether it is subsumed by a tabulated sequent. If it is, then the newly derived sequent is unsatisfiable, if not, then we have to continue the search for refutation. The inverse method tabulates *all* sequents up to a certain modal depth k, which means that a newly derived sequent of a modal depth $\geq k$ is satisfiable *if and only if* it is not subsumed by any of the tabulated sequent. Therefore, if the subsumption-check fails one should *not* continue search for the proof of this newly derived clause.

A combination of the tableau and the inverse method can also solve the memory problem for the inverse method. On some benchmarks, our implementation of the inverse method uses more than 400M of memory. If one discovers that the memory consumption is too large, one can terminate the inverse proof search and continue the tableau search with the tabulated sequents collected so far.

However, such bidirectional search meets essential difficulties on the theoretical level. The inverse method without redundancies is not efficient at all. The naive tableau proof-search is inefficient as well. When we run bidirectional proof-search, one has to be sure that the optimizations used in the tableau proof-search are compatible with the redundancies used in the inverse proof search. This is a challenging task, since the completeness proofs of the redundancies for the inverse method require a nontrivial technique.

We feel that a new technique can be used to give model-theoretical completeness proofs of the inverse path calculus with redundancies built in directly into the proof system, similar to completeness proofs for resolution with redundancies [Bachmair and Ganzinger 2000].

11 Conclusion

The main contribution of this report is the demonstration that bottom-up procedures can be a viable alternative to existing methods of automated reasoning for modal and description logics. In addition, three novelties related to the inverse method are presented:

1. We introduce a new technique of proving completeness of calculi with redundancies by analyzing tableau-based derivations. The previous completeness proofs were either based on ad-hoc considerations [Tammet 1996, Mints et al. 1996] or on a different technique based on a suitable notion of subsumption [Voronkov 1992][2].

2. We give a new formalization of the inverse method based on a path calculus. This formalization simplifies the definition of redundancies and completeness proofs. An example is the definition of conflicting paths and related proofs.

3. A new technique for proving completeness of subsumption-like redundancies is introduced.

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