Decidability Problems for the Prenex Fragment of Intuitionistic Logic

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Abstract

We develop a constraint-based technique which allows one to prove decidability and complexity results for sequent calculi. Specifically, we study decidability problems for the prenex fragment of intuitionistic logic. We introduce an analogue of Skolemization for intuitionistic logic with equality, prove PSPACE-completeness of two fragments of intuitionistic logic with and without equality and some other results. In the proofs, we use a combination of techniques of constraint satisfaction, loop-free sequent systems of intuitionistic logic and properties of simultaneous rigid E-unification.

1 Introduction

In this paper we develop a constraint-based technique which allows one to prove decidability and complexity results for sequent calculi. Specifically, we study decidability problems for the prenex fragment of intuitionistic logic. We introduce an analogue of Skolemization for intuitionistic logic with equality, prove PSPACE-completeness of two fragments of intuitionistic logic with and without equality and some other results. In the proofs, we use a combination of techniques of constraint satisfaction, loop-free sequent systems of intuitionistic logic and properties of simultaneous rigid E-unification.

Simultaneous rigid E-unification plays a crucial role in automatic proof methods for first-order logic with equality based on sequent calculi, such as semantic tableaux [12], the connection method [3] (also known as the mating method [1]), model elimination [20] and a dozen other procedures. It was proposed by Gallier, Raatz and Snyder [15] as a method for automated theorem proving in classical logic with equality in sequent-based methods, for example the tableau method or the connection method. After the proof of the undecidability of simultaneous rigid E-unification [6, 10] it has become clear that one should look for more refined techniques to deal with equality in matrix-based methods (see Degtyarev and Voronkov [5, 7, 8, 9], Moser, Lynch, and Steinbach [23] and Plaisted [27]).

In [6] we have also noted that simultaneous rigid E-unification is representable in the $\exists^*$-fragment of intuitionistic logic with equality, obtaining the undecidability result of this fragment. This paper studies further relations between intuitionistic logic with equality and simultaneous rigid E-unification. In particular, we study prenex formulas of intuitionistic logic. The results of our paper can be used in automated theorem proving in intuitionistic logic with equality.

In Section 2 we define all formal notions necessary to understand the paper. In Section 3 we introduce a sequent calculus $\mathbf{LJ}^{\approx}$ for intuitionistic logic with equality. We prove an analogue of Skolemization for intuitionistic logic with equality. Section 4 formulates main technical definitions and results used in this paper. The technique we use is essentially a combination of a constraint calculus for intuitionistic logic introduced in [32, 30] and a calculus LG of Hudelmaier [16] which gives a polynomial-space decision procedure for intuitionistic propositional logic. In Section 4 we prove a general statement characterizing the connection between derivability in the constraint calculus $\mathbf{LJ}_c^{\approx}$ and the calculus $\mathbf{LJ}^{\approx}$. Using this result, in Section 5 we prove PSPACE-completeness of the prenex fragment of intuitionistic logic without equality. Section 6 studies the prenex fragment of intuitionistic logic with equality but without function symbols. The decidability of this fragment was proved in Orevkov [26]. We prove that this fragment is PSPACE-complete. In Section 7 we discuss the (still open) decidability problem for the prenex fragment of intuitionis-

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tic logic with function symbols of arity ≤ 1.

The lack of space does not allow us to give all details of proofs. The proofs of some statements of this paper are quite long and technical. However, we tried to explain the main ideas behind the proofs.

2 Preliminaries

Formulas are defined as usual, using atomic formulas, the constant Λ, the connectives $\wedge, \vee, \supset$ and the quantifiers $\forall, \exists$. Negation $\neg\varphi$ can be introduced as $\varphi \supset \Lambda$. A *sequent* is an expression of the form $\Gamma \to \varphi$ where Γ is a finite sequence of formulas and φ is a formula. A formula is *closed* iff it has no free variables. A sequent $\Gamma \to \varphi$ is *closed* iff all formulas in Γ, φ are closed.

The notions of terms, substitutions and equations are standard and may be found e.g. in Dershowitz and Jouannaud [11]. Equations will be denoted by $s \approx t$. For any sequence for formulas Γ, we denote by $\Gamma^{\approx}$ its subsequence consisting of all equations in Γ. For example, if Γ is $f(x)\approx x, P(x) \wedge Q(x), y\approx a, \neg(y\approx b)$, then $\Gamma^{\approx}$ is Γ is $f(x)\approx x, y\approx a$.

An expression is *ground* iff it has no variables. By a *language* we understand a set of function symbols. *Constants* are function symbols of arity 0. A *formula in the language* Σ can use function symbols of Σ and any predicate symbols. The set of all terms of a language Σ with variables in the set X is denoted by $T_\Sigma(X)$, the set of all ground terms of the language Σ by T_Σ. We denote substitutions by $\{t_1/x_1, \ldots, t_n/x_n\}$. The notation $\rightleftharpoons$ stands for "equal by definition". For a substitution σ of the form $\{t_1/x_1, \ldots, t_n/x_n\}$ we define $\mathit{dom}(\sigma) \rightleftharpoons \{x_1, \ldots, x_n\}$ and $\mathit{ran}(\sigma) \rightleftharpoons \{t_1, \ldots, t_n\}$.

Let E be a term, formula, sequence of formulas or sequent. For such a substitution σ, $E\sigma$ denotes the result of *applying σ to E*, i.e. renaming of those bound variables in E that coincide with a variable occurring in any t_i and the replacement of all free occurrences of x_i by t_i. $\mathit{var}(E)$ denotes the set of all (free) variables of E. Let φ be a formula all whose free variables are $x_1, \ldots, x_n$, in the order of their occurrence in φ. By $\forall\varphi$ we denote the formula $\forall x_1 \ldots \forall x_n \varphi$, and similar for $\exists\varphi$.

Definition 2.1 (Rigid equations) *A* rigid equation *is an expression $E \vdash_\forall s\approx t$, where E is a finite set of equations. A* solution to a rigid equation $\{s_1\approx t_1, \ldots, s_n\approx t_n\} \vdash_\forall s\approx t$ *is any substitution σ such that the formula $\forall(s_1\sigma\approx t_1\sigma \wedge \ldots \wedge s_n\sigma\approx t_n\sigma \supset s\sigma\approx t\sigma)$ is true*[1]*. A* system of rigid equations *is a finite set of rigid equations. A* solution to a system of rigid equations S *is any substitution that is a solution to every rigid equation in S. The problem of solvability of rigid equations is known as* rigid E-unification. *The problem of solvability of systems of rigid equations is known as* simultaneous rigid E-unification[2].

We shall sometimes write the left-hand side of a rigid equation as a *sequence* of equations, for example $x\approx a \vdash_\forall g(x)\approx x$ instead of $\{x\approx a\} \vdash_\forall g(x)\approx x$.

If a system of rigid equations is solvable, it has a relevant ground solution:

Lemma 2.1 *Let S be a solvable system of rigid equations in a language Σ with at least one constant. Then there is a solution σ of S such that $\mathit{dom}(\sigma) = \mathit{var}(S)$ and $\mathit{ran}(\sigma) \subseteq T_\Sigma$.*

The following theorem was originally proved in [6]:

Theorem 2.1 *Simultaneous rigid E-unifiability is undecidable.*

3 Intuitionistic logic

In this section we define a formalization of intuitionistic logic with equality in the form of a sequent calculus. We show close connections between simultaneous rigid E-unification and the prenex fragment of intuitionistic logic.

Definition 3.1 (Sequent calculus $\mathbf{LJ}^{\approx}$) *A* sequent *is an expression of the form $\Gamma \to \varphi$, where Γ is a sequence of formulas. The calculus $\mathbf{LJ}^{\approx}$ for intuitionistic logic with equality derives sequents. The inference rules of $\mathbf{LJ}^{\approx}$ are given in Figure 1.*

A formula φ is provable in intuitionistic logic, *denoted by $\vdash_I \varphi$, iff the sequent $\to \varphi$ is derivable in $\mathbf{LJ}^{\approx}$.*

The equivalence of $\mathbf{LJ}^{\approx}$ to other formalizations of intuitionistic logic with equality is proved e.g. in Orevkov [25].

Definition 3.2 (Prenex formulas) *A formula φ is called* prenex *iff it has the form $Q_1x_1 \ldots Q_nx_n\psi$, where $n \geq 0$, any Q_i is either $\forall$ or $\exists$ and ψ is a quantifier-free formula. The* prenex fragment *of intuitionistic logic is the set of prenex formulas provable in intuitionistic logic.*

In classical logic, every formula is (effectively) equivalent to a prenex formula, so that the problem of the provability of prenex formulas is equivalent to the general provability problem and hence, it is undecidable. In intuitionistic logic, the situation is rather different. For example, the prenex fragment of intuitionistic logic without equality is decidable

[1] I.e. this formula is provable in first-order (classical or intuitionistic) logic. Equivalently, we can say that $s\sigma$ and $t\sigma$ lie in the same class of the congruence generated by $\{s_1\sigma\approx t_1\sigma, \ldots, s_n\sigma\approx t_n\sigma\}$.

[2] The term "rigid equation" could be more adequately expressed as "instance of a (non-simultaneous) rigid E-unification problem", but this would be too lengthy.

(Theorem 5.1 below). In intuitionistic logic (with and without equality), formulas close to prenex formulas have been studied in Orevkov [24, 26].

Let S be the system of rigid equations $\{E_i \vdash_\forall s_i \approx t_i \mid i \in \{1,\dots,n\}\}$. By $\mathcal{F}(S)$ we denote the formula

$$\bigwedge_{i=1}^{n} (\bigwedge_{e \in E_i} e \supset s_i \approx t_i)$$

For example, if S consists of three rigid equations $a \approx x \vdash_\forall x \approx f(x,y)$, $\vdash_\forall f(y,y) \approx y$ and $x \approx y, v \approx y \vdash_\forall a \approx u$, then $\mathcal{F}(S)$ is the formula

$$(a \approx x \supset x \approx f(x,y)) \wedge f(y,y) \approx y \\ \wedge (x \approx y \wedge v \approx y \supset a \approx u)$$

By a simple analysis of possible derivations of $\mathcal{F}(S)$ and of $\exists \mathcal{F}(S)$ in $\mathbf{LJ}^{\approx}$, we obtain

Lemma 3.1 *Let S be a system of rigid equations and σ be any substitution. Then*

1. *σ is a solution to S iff $\vdash_I \mathcal{F}(S)\sigma$;*
2. *S has a solution iff $\vdash_I \exists \mathcal{F}(S)$.*

As a consequence, we obtain that simultaneous rigid E-unification is polynomially reducible to the prenex fragment of intuitionistic logic. This fact and Theorem 2.1 immediately yield

Theorem 3.1 *The prenex fragment of $\mathbf{LJ}^{\approx}$ is undecidable.*

We prove one technical property of systems of rigid equations.

Lemma 3.2 *Let F be a finite language containing at least one constant. Then there is a rigid equation R with one variable x such that for every substitution σ, σ is a solution to R if and only if $x\sigma \in T_F$.*

Proof. Select any constant $a \in F$ and define R as $\{f(a,\dots,a) \approx a \mid f \in F\} \vdash_\forall a \approx x$. □

Combining Lemmas 3.1 and 3.2 we obtain

Lemma 3.3 *Let F be a finite language containing at least one constant. There is a quantifier-free formula with equality $\alpha_{F,x}$ of one variable x such that for any substitution σ we have that $\vdash_I \alpha_{F,x}\sigma$ if and only if $x\sigma \in T_F$.*

The following result proves an analogue of Skolemization for intuitionistic logic:

Theorem 3.2 *Let $\varphi = Q_1 x_1 \dots Q_n x_n \psi$ be a closed formula. Let $I_\forall \rightleftharpoons \{i_1,\dots,i_k\}$ be the set of all i such that $Q_i = \forall$ and $I_\exists \rightleftharpoons \{j_1,\dots,j_l\}$ be the set of all j such that $Q_j = \exists$. Let $c_{i_1},\dots,c_{i_k}$ be constants not occurring in φ. Denote by θ the substitution $\{c_{i_1}/x_{i_1},\dots,c_{i_k}/x_{i_k}\}$. There is a quantifier-free formula χ with variables in $x_1,\dots,x_n$ depending only on $Q_1 x_1 \dots Q_n x_n$ but not on ψ such that the following conditions are equivalent:*

1. *$\vdash_I \varphi$;*
2. *$\vdash_I \forall x_{i_1} \dots \forall x_{i_k} \exists x_{j_1} \dots \exists x_{j_l} (\psi \wedge \chi)$;*
3. *$\vdash_I \exists x_{j_1} \dots \exists x_{j_l} (\psi\theta \wedge \chi\theta)$.*

In addition, the only predicate symbol of χ is $\approx$ and function symbols occurring in χ are function symbols occurring in φ and maybe one additional constant.

Proof. Let F be the set of function symbols occurring in φ. If there is no constant occurring in φ, we add to F a constant.

All proofs of $\rightarrow \varphi$ in $\mathbf{LJ}^{\approx}$ have the form

$$\dfrac{\dfrac{\vdots}{\psi\sigma}}{(Q_n x_n \psi)\sigma} \\ \vdots \\ \dfrac{(Q_2 x_2 \dots Q_n x_n \psi)\sigma}{(Q_1 x_1 \dots Q_n x_n \psi)\sigma}$$

where σ is a substitution satisfying the property

(I) For any $i \in I_\forall$ we have that $x_i\sigma$ is a variable not occurring in $x_1\sigma,\dots,x_{i-1}\sigma$.

Moreover, for any substitution σ satisfying (I), if $\vdash_I \psi\sigma$, then $\vdash_I \varphi$. Thus, $\vdash_I \varphi$ if and only if there is a substitution σ with property (I) such that $\vdash_I \psi\sigma$. Using the fact that constants c_i do not occur in ψ, we prove that $\vdash_I \varphi$ if and only if there is a substitution τ such that $\vdash_I \psi\theta\tau$ and

(II) For every $j \in I_\exists$ we have that $x_j\tau$ is a ground term all whose function symbols are in the set $F_j \rightleftharpoons F \cup \{c_i \mid i \in I_\forall \text{ and } i < j\}$.

Define a formula χ' as $\bigwedge_{j \in I_\exists} \alpha_{F_j,x_j}$, where the formulas α_{F_j,x_j} are defined as in Lemma 3.3. Applying Lemma 3.3, we can prove that for any substitution τ, we have $\vdash_I (\psi\theta \wedge \chi')\tau$ if and only if $\vdash_I \psi\theta\tau$ and τ satisfies (II). Define χ as the formula obtained from χ' by the replacement of constants c_i by variables x_i, for all $i \in I_\forall$. Then $\chi\theta = \chi'$. The rest of the proof is straightforward. □

4 Calculus $\mathbf{LJ}_c^{\approx}$

In contrast to Theorem 3.1, the prenex fragment of logic without equality is decidable. In this section, we prove PSPACE-completeness of this fragment. Before proving this and other results, we have to introduce some auxiliary technical definitions. In particular, the proof of some theorems of this paper uses a combination of a propositional calculus LG of Hudelmaier [16] with a special calculus of *constraints* of [32, 30]. We introduce constraints following [32] (with some modifications):

Definition 4.1 (Constraint) Constraints *are defined inductively as follows:*

1. $\top$ *and* $\bot$ *are constraints;*
2. *Any rigid equation* $E \vdash_\forall s{\approx}t$ *is a constraint;*
3. *For any terms* $s, t_1, \ldots, t_n$*, where* $n \geq 0$*,* $s \not\preceq \{t_1, \ldots, t_n\}$ *is a constraint;*
4. *If* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are constraints, then* $\mathcal{C}_1 \wedge \mathcal{C}_2$ *is a constraint;*
5. *If* $\mathcal{C}$ *is a constraint and* x *is variable, then* $\exists x\mathcal{C}$ *is a constraint.*

Constraints describe the domain of substitutions. The semantics of constraints is defined in the following way:

Definition 4.2 (Constraint satisfiability) *The notion "a substitution* σ *satisfies the constraint* $\mathcal{C}$*", denoted* $\sigma \models \mathcal{C}$*, is defined inductively as follows:*

1. $\sigma \models \top$ *for every substitution* σ*;*
2. *There is no substitution* σ *such that* $\sigma \models \bot$*;*
3. $\sigma \models E \vdash_\forall s{\approx}t$ *iff* σ *is a solution of* $E \vdash_\forall s{\approx}t$*;*
4. $\sigma \models s \not\preceq \{t_1, \ldots, t_n\}$ *iff* $s\sigma$ *is a variable not occurring in* $t_1\sigma, \ldots, t_n\sigma$*;*
5. $\sigma \models \mathcal{C}_1 \wedge \mathcal{C}_2$ *iff* $\sigma \models \mathcal{C}_1$ *and* $\sigma \models \mathcal{C}_2$*;*
6. $\sigma \models \exists x\mathcal{C}$ *iff there is a substitution* τ *such that* $\tau \models \mathcal{C}$ *and* $y\tau = y\sigma$ *for every variable* y *different from* x*.*

Constraints $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *are* equivalent *iff for every substitution* σ *we have* $\sigma \models \mathcal{C}_1$ *iff* $\sigma \models \mathcal{C}_2$*. A constraint* $\mathcal{C}$ *is* satisfiable *iff there is a substitution* σ *satisfying* $\mathcal{C}$*.*

We introduce a technical definition.

Definition 4.3 *For any quantifier-free formulas* φ, ψ*, the constraint* $\mathcal{E}_{\varphi=\psi}$ *is defined inductively as follows:*

1. *If* φ *has the form* $\varphi_1 \wedge \varphi_2$ *and* ψ *has the form* $\psi_1 \wedge \psi_2$*, then* $\mathcal{E}_{\varphi=\psi} \rightleftharpoons \mathcal{E}_{\varphi_1=\psi_1} \wedge \mathcal{E}_{\varphi_2=\psi_2}$*;*
2. *If* φ *has the form* $\varphi_1 \vee \varphi_2$ *and* ψ *has the form* $\psi_1 \vee \psi_2$*, then* $\mathcal{E}_{\varphi=\psi} \rightleftharpoons \mathcal{E}_{\varphi_1=\psi_1} \wedge \mathcal{E}_{\varphi_2=\psi_2}$*;*
3. *If* φ *has the form* $\varphi_1 \supset \varphi_2$ *and* ψ *has the form* $\psi_1 \supset \psi_2$*, then* $\mathcal{E}_{\varphi=\psi} \rightleftharpoons \mathcal{E}_{\varphi_1=\psi_1} \wedge \mathcal{E}_{\varphi_2=\psi_2}$*;*
4. *If* φ *has the form* $P(s_1, \ldots, s_n)$ *and* ψ *has the form* $P(t_1, \ldots, t_n)$*, then* $\mathcal{E}_{\varphi=\psi} \rightleftharpoons \vdash_\forall s_1{\approx}t_1 \wedge \ldots \wedge \vdash_\forall s_n{\approx}t_n$*;*
5. *In all other cases* $\mathcal{E}_{\varphi=\psi} \rightleftharpoons \bot$*.*

The following straightforward lemma shows that $\mathcal{E}_{\varphi=\psi}$ expresses all unifiers of φ and ψ:

Lemma 4.1 *Let* φ *and* ψ *be quantifier-free formulas and* θ *be a substitution. Then* $\varphi\theta = \psi\theta$ *iff* $\theta \models \mathcal{E}_{\varphi=\psi}$*.*

The calculus $\mathbf{LJ}_c^{\approx}$ derives *constrained sequents*:

Definition 4.4 ($\mathbf{LJ}_c^{\approx}$) *A* constrained sequent *is an expression* $S \cdot \mathcal{C}$ *where* S *is a sequent and* $\mathcal{C}$ *is a constraint. The* constraint calculus $\mathbf{LJ}_c^{\approx}$ *of constrained sequents is shown in Figure 2.*

The idea of the constraint calculus of [32, 30] is that the existence of a derivation with the given skeleton reduces to constraint satisfiability. The constraint satisfiability problem is, in general, undecidable, but it has decidable subcases giving decidability results for corresponding fragments of intuitionistic logic. The soundness and completeness of the constraint system $\mathbf{LJ}_c^{\approx}$ are expressed by the following lemma:

Lemma 4.2 *Let* $\Gamma \to \varphi$ *be a sequent and* σ *be a substitution. The following conditions are equivalent:*

1. *The sequent* $\Gamma\sigma \to \varphi\sigma$ *is derivable in* $\mathbf{LJ}^{\approx}$*;*
2. *There is a constraint* $\mathcal{C}$ *such that* $\sigma \models \mathcal{C}$ *and* $\Gamma \to \varphi \cdot \mathcal{C}$ *is provable in* $\mathbf{LJ}_c^{\approx}$*;*

Proof. We call a derivation in $\mathbf{LJ}^{\approx}$ *regular* iff the equality rules $(\approx_1)$ and $(\approx_2)$ in this derivation are applied before all other rules except (Ax), (Λ) and $(\approx)$. First we note that any sequent S derivable in $\mathbf{LJ}^{\approx}$ has a regular derivation. Then we apply induction on the number of applications of rules different from $(\approx_1)$ and $(\approx_2)$. For more detail see [31]. $\square$

Let us note that this statement is valid for *arbitrary sequents.* For closed sequents, a trivial consequence of Lemma 4.2 is the following statement:

Lemma 4.3 *Let* S *be a closed sequent. The following conditions are equivalent:*

1. *S is derivable in* $\mathbf{LJ}^{\approx}$*;*

2. *There is a satisfiable constraint $\mathcal{C}$ such that $S \cdot \mathcal{C}$ is derivable in* $\mathbf{LJ}^{\approx}_{c}$.

We introduce one more calculus $\mathbf{LG}^{\approx}_{c}$ which is a combination of $\mathbf{LJ}^{\approx}_{c}$ and the propositional calculus LG of Hudelmaier [16]. The calculus $\mathbf{LG}$ is shown in Figure 3. The idea of $\mathbf{LG}$ is that the backward proofs in LG give us a polynomial space decision procedure for intuitionistic propositional logic. The calculus $\mathbf{LJ}^{\approx}_{c}$ of constrained sequents is shown in Figure 2. Unlike $\mathbf{LJ}^{\approx}_{c}$, the calculus $\mathbf{LG}^{\approx}_{c}$ is not complete in general. However, it is complete for the prenex fragment:

Lemma 4.4 *Let φ be a closed prenex formula. The following conditions are equivalent:*

1. *$\rightarrow \varphi$ is derivable in* $\mathbf{LJ}^{\approx}$*;*

2. *There is a satisfiable constraint $\mathcal{C}$ such that $\rightarrow \varphi \cdot \mathcal{C}$ is derivable in* $\mathbf{LG}^{\approx}_{c}$.

Proof. The proof of this lemma uses the techniques similar to that of Hudelmaier [16] to show that for any constraint sequent $S = (\Gamma \rightarrow \varphi \cdot \mathcal{C})$ such that φ and every formula in Γ are quantifier-free, the derivability of S in $\mathbf{LJ}^{\approx}_{c}$ is equivalent to its derivability in $\mathbf{LG}^{\approx}_{c}$. Then this result is lifted in a straightforward way to arbitrary prenex formulas. Then we use Lemma 4.3. □

5 Logic without equality

Lemma 4.4 implies the decidability result for logic without equality:

Theorem 5.1 *The prenex fragment of intuitionistic logic without equality is PSPACE-complete.*

Proof. Intuitionistic propositional logic is PSPACE-complete (Statman [28]). Hence, the prenex fragment of intuitionistic logic without equality is PSPACE-hard.

The complete proof of PSPACE-completeness requires many technical details. The main ideas can be explained as follows. The *skeleton* of a derivation Π in $\mathbf{LG}^{\approx}_{c}$ is obtained by removing all constraints in Π and adding some information about (Ax) rules and antecedent rules. These information uniquely identifies an $\mathbf{LJ}^{\approx}_{c}$-derivation of a given sequent. Given a skeleton of a derivation of $S \cdot \mathcal{C}$, we can effectively reconstruct $\mathcal{C}$. Let φ be any closed prenex formula. By using the techniques of Hudelmaier [16], we note that all skeletons of $\mathbf{LG}^{\approx}_{c}$-derivations of $\rightarrow \varphi \cdot \mathcal{C}$, where $\mathcal{C}$ is some constraint, can be found by using a polynomial-space algorithm. Any such constraint has the form $\exists(D_1 \wedge \ldots \wedge D_k \wedge R_1 \wedge \ldots \wedge R_m)$, where D_i are $\not\preceq \{\}$-constraints, the size of $D_1 \wedge \ldots \wedge D_k$ is polynomial in the size of the quantifier prefix of φ, and each R_i has the form $\vdash_{\forall} s{\approx}t$ such that there exist a predicate symbol P and a number j such that

1. s is the jth argument of a negative occurrence of P in φ and

2. t is the jth argument of a positive occurrence of P in φ.

The number of different R_i of this form is polynomial in the size of φ. Thus, $\mathcal{C}$ can be effectively replaced by an equivalent constraint $\mathcal{C}'$ whose size is polynomial in the size of φ. Finally, we use a polynomial time constraint satisfiability algorithm for such constraints $\mathcal{C}'$ [29]. □

6 The function-free case

In this section we consider the prenex fragment of intuitionistic logic in languages without function symbols of arity > 0. We denote by Σ_0 a language consisting of a countable number of constants. Any prenex formula in Σ_0 belongs to the class $(\Pi_1, \sigma_\infty)^{\approx}$ introduced in Orevkov [26]. Since the provability problem for the class $(\Pi_1, \sigma_\infty)^{\approx}$ is decidable (Orevkov [26]), the prenex fragment of $\mathbf{LJ}^{\approx}$ in the language Σ_0 is also decidable. The following theorem describes the complexity of this fragment:

Theorem 6.1 *The prenex fragment of* $\mathbf{LJ}^{\approx}$ *in the language Σ_0 is PSPACE-complete.*

Proof. The problem is PSPACE-hard because intuitionistic propositional logic is PSPACE-hard.

By Theorem 3.2 we can restrict ourselves to closed formulas of the form $\varphi = (\exists x_1 \ldots \exists x_n \psi)$, where ψ is quantifier-free. Let $c_1, \ldots, c_m$ be all constants occurring in φ. Evidently, $\vdash_I \varphi$ if and only if there is a substitution σ with *dom*$(\sigma) = \{x_1, \ldots, x_m\}$ and *ran*$(\sigma) \subseteq \{c_1, \ldots, c_m\}$ such that $\vdash_I \psi\sigma$. We can "guess" the values of $x_i\sigma$ in polynomial space. For a given σ, we have to establish the provability of the quantifier-free closed formula $\psi\sigma$. For this we note that the backward proof-search in $\mathbf{LG}^{\approx}_{c}$ can be done in polynomial space, by solving ground rigid equations occurring in constraints immediately when we apply the (Ax) and $(\approx)$ rules. □

7 The monadic case

For the undecidability of the prenex fragment of intuitionistic logic with equality, it is enough to have one function symbol of arity ≥ 2. In this section, we consider the case when all function symbols have arity ≤ 1. Let Σ_n be a language with n unary function symbols and a countable number of constants.

Theorem 7.1 *The prenex fragment of intuitionistic logic with equality in the language Σ_1 is decidable.*

Proof. In [4] it is proved that simultaneous rigid E-unification in Σ_1 is decidable[3]. Let φ be any prenex formula in such a language. By Theorem 3.2 we can assume that all quantifiers in φ are $\exists$. There is only a finite number of proofs of $\to \varphi \cdot \mathcal{C}$ for some $\mathcal{C}$ in $\mathbf{LG}_c^{\approx}$ (up to renaming variables). Consider all such proofs. The constraint $\mathcal{C}$ has the form $\exists x_1 \ldots \exists x_n(R_1 \wedge \ldots \wedge R_m)$, where all R_i are rigid equations. Hence, $\mathcal{C}$ is satisfiable iff the system of rigid equations $\{R_1, \ldots, R_m\}$ has a solution. This implies the claim of the theorem. □

Similar considerations prove

Lemma 7.1 *The following conditions are equivalent:*

1. *Simultaneous rigid E-unification in the language Σ_n is decidable;*
2. *The prenex fragment of intuitionistic logic in the language Σ_n is decidable.*

The same can be stated for other, non-monadic languages, although the complexity of the two problems can be different (provided that PSPACE $\neq$ NP). For example, simultaneous rigid E-unification in the language Σ_0 is NP-complete [4], but the prenex fragment of intuitionistic logic with equality in Σ_0 is PSPACE-complete.

The decidability of the simultaneous rigid E-unification in the language Σ_n for $n > 1$ is an open question. If this problem is decidable, then the proof should be very non-trivial for the following reason: there is a simple polynomial-time reduction of word equation solvability (Makanin [21]) (also known as unification under associativity) to simultaneous rigid E-unification in the language Σ_n [4]. We also note that the decidability of simultaneous rigid E-unification in the language Σ_n for any $n > 1$ is equivalent to the decidability of simultaneous rigid E-unification in the language Σ_2 [4].

Related works

The decidability problems for some other fragments of intuitionistic logic with and without equality have been studied in Orevkov [24, 26], Mints [22] and Statman [28]. Orevkov [24] proves that the $\neg\neg\forall\exists$-fragment of intuitionistic logic with function symbols is undecidable. Lifschitz [17] proves the undecidability of the "pure constructive theory of equality", i.e. intuitionistic logic with equality without function symbols whose only predicate symbol is $\approx$. Mints [22] introduces a system of Skolem functions for formulas of intuitionistic logic restricting the application of quantifier rules, but does not define a Skolemized form of a formula. Orevkov [26] gives a partial classification of decidable pseudo-prenex[4] fragments of intuitionistic logic with equality but without function symbols. Statman [28] proves PSPACE-completeness of intuitionistic propositional logic.

A proof-theoretic technique similar to ours has been introduced by Lincoln and Shankar [18]. They handle quantifier-rules using skolemization and prove some complexity results for fragments of linear logic.

We do not know any previous proof of decidability of the prenex fragment of intuitionistic logic without equality (for the case without function symbols it was proved in Orevkov [26]). The decidability of the prenex fragment of intuitionistic logic with equality and without function symbols follows from any of Mints [22] or Orevkov [26], but they did not prove the complexity of the problem.

All our results on the monadic case are new.

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[3] This decidability proof is rather non-trivial and we have no space to describe its details or idea here. There is hardly a simple proof because the Diophantine problem for addition and divisibility is easily representable in rigid unification (this Diophantine problem has only non-trivial proofs of decidability, e.g. Bel'tykov [2], Lipshitz [19]).

[4] A *pseudo-prenex* is a finite sequence of $\forall x_i$, $\exists x_i$ and $\neg\neg$.

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$$\frac{}{\Gamma, A, \Delta \to A}\ (Ax) \qquad \frac{}{\Gamma, \Lambda, \Delta \to \varphi}\ (\Lambda) \quad \frac{}{\Gamma \to t{\approx}t}\ (\approx)$$

$$\frac{\Gamma\{t/x\}, s{\approx}t, \Delta\{t/x\} \to \chi\{t/x\}}{\Gamma\{s/x\}, s{\approx}t, \Delta\{s/x\} \to \chi\{s/x\}}\ (\approx_1) \qquad \frac{\Gamma\{s/x\}, s{\approx}t, \Delta\{s/x\} \to \chi\{s/x\}}{\Gamma\{t/x\}, s{\approx}t, \Delta\{t/x\} \to \chi\{t/x\}}\ (\approx_2)$$

$$\frac{\Gamma, \varphi, \psi, \Delta \to \chi}{\Gamma, \varphi \wedge \psi, \Delta \to \chi}\ (\wedge \to) \qquad \frac{\Gamma \to \varphi \quad \Gamma \to \psi}{\Gamma \to \varphi \wedge \psi}\ (\to \wedge)$$

$$\frac{\Gamma, \varphi, \Delta \to \chi \quad \Gamma, \psi, \Delta \to \chi}{\Gamma, \varphi \vee \psi, \Delta \to \chi}\ (\vee \to) \qquad \frac{\Gamma \to \varphi}{\Gamma \to \varphi \vee \psi}\ (\to \vee_1) \quad \frac{\Gamma \to \psi}{\Gamma \to \varphi \vee \psi}\ (\to \vee_2)$$

$$\frac{\Gamma, \psi, \Delta \to \chi \quad \Gamma, \varphi \supset \psi, \Delta \to \varphi}{\Gamma, \varphi \supset \psi, \Delta \to \chi}\ (\supset\to) \qquad \frac{\varphi, \Gamma \to \psi}{\Gamma \to \varphi \supset \psi}\ (\to\supset)$$

$$\frac{\Gamma, \varphi\{t/x\}, \forall x\varphi, \Delta \to \chi}{\Gamma, \forall x\varphi, \Delta \to \chi}\ (\forall \to) \qquad \frac{\Gamma \to \varphi\{y/x\}}{\Gamma \to \forall x\varphi}\ (\to \forall)$$

$$\frac{\Gamma, \varphi\{y/x\}, \Delta \to \chi}{\Gamma, \exists x\varphi, \Delta \to \chi}\ (\exists \to) \qquad \frac{\Gamma \to \varphi\{t/x\}}{\Gamma \to \exists x\varphi}\ (\to \exists)$$

In this figure, A ranges over atomic formulas, φ, ψ, χ range over arbitrary formulas and Γ, Δ range over finite sequences of formulas. In the rules $(\exists \to)$ and $(\to \forall)$ the variable y has no free occurrences in the conclusions of the rules. The variable y is called the *eigenvariable* of these rules.

Figure 1. Calculus $\mathbf{LJ}^{\approx}$

$$\frac{}{\Gamma, A(s_1, \ldots, s_n), \Delta \to A(t_1, \ldots, t_n) \cdot \Gamma^{\approx}, \Delta^{\approx} \vdash_\forall s_1{\approx}t_1 \wedge \ldots \wedge \Gamma^{\approx}, \Delta^{\approx} \vdash_\forall s_n{\approx}t_n}\ (Ax)$$

$$\frac{}{\Gamma, \Lambda, \Delta \to \varphi \cdot \top}\ (\Lambda) \qquad \frac{}{\Gamma \to s{\approx}t \cdot \Gamma^{\approx} \vdash_\forall s{\approx}t}\ (\approx)$$

$$\frac{\Gamma, \varphi, \psi, \Delta \to \chi \cdot \mathcal{C}}{\Gamma, \varphi \wedge \psi, \Delta \to \chi \cdot \mathcal{C}}\ (\wedge \to) \qquad \frac{\Gamma \to \varphi \cdot \mathcal{C}_1 \quad \Gamma \to \psi \cdot \mathcal{C}_2}{\Gamma \to \varphi \wedge \psi \cdot \mathcal{C}_1 \wedge \mathcal{C}_2}\ (\to \wedge)$$

$$\frac{\Gamma, \varphi, \Delta \to \chi \cdot \mathcal{C}_1 \quad \Gamma, \psi, \Delta \to \chi \cdot \mathcal{C}_2}{\Gamma, \varphi \vee \psi, \Delta \to \chi \cdot \mathcal{C}_1 \wedge \mathcal{C}_2}\ (\vee \to) \qquad \frac{\Gamma \to \varphi \cdot \mathcal{C}}{\Gamma \to \varphi \vee \psi \cdot \mathcal{C}}\ (\to \vee_1) \quad \frac{\Gamma \to \psi \cdot \mathcal{C}}{\Gamma \to \varphi \vee \psi \cdot \mathcal{C}}\ (\to \vee_2)$$

$$\frac{\Gamma, \psi, \Delta \to \chi \cdot \mathcal{C}_1 \quad \Gamma, \varphi \supset \psi, \Delta \to \varphi \cdot \mathcal{C}_2}{\Gamma, \varphi \supset \psi, \Delta \to \chi \cdot \mathcal{C}_1 \wedge \mathcal{C}_2}\ (\supset\to) \qquad \frac{\varphi, \Gamma \to \psi \cdot \mathcal{C}}{\Gamma \to \varphi \supset \psi \cdot \mathcal{C}}\ (\to\supset)$$

$$\frac{\Gamma, \varphi\{y/x\}, \forall x\varphi, \Delta \to \chi \cdot \mathcal{C}}{\Gamma, \forall x\varphi, \Delta \to \chi \cdot \exists y\mathcal{C}}\ (\forall \to) \qquad \frac{\Gamma \to \varphi\{y/x\} \cdot \mathcal{C}}{\Gamma \to \forall x\varphi \cdot \exists y(y \not\preceq \{v_1, \ldots, v_n\} \wedge \mathcal{C})}\ (\to \forall)$$

$$\frac{\Gamma, \varphi\{y/x\}, \Delta \to \chi \cdot \mathcal{C}}{\Gamma, \exists x\varphi, \Delta \to \chi \cdot \exists y(y \not\preceq \{v_1, \ldots, v_n\} \wedge \mathcal{C})}\ (\exists \to) \qquad \frac{\Gamma \to \varphi\{y/x\} \cdot \mathcal{C}}{\Gamma \to \exists x\varphi \cdot \exists y\mathcal{C}}\ (\to \exists)$$

In the rule (Ax), A is a predicate symbol different from equality. In the rules $(\forall \to)$–$(\to \exists)$ the variable y has no free occurrences in the sequent in the conclusion of the rules. In the rule $(\to \forall)$, $v_1, \ldots, v_n$ are all free variables of $\Gamma, \forall x\varphi$. In the rule $(\exists \to)$, $v_1, \ldots, v_n$ are all free variables of $\Gamma, \exists x\varphi, \Delta, \chi$.

Figure 2. Calculus $\mathbf{LJ}_c^{\approx}$

$$\frac{}{\Gamma, A, \Delta \to A}\ (Ax) \qquad \frac{}{\Gamma, \Lambda, \Delta \to \varphi}\ (\Lambda)$$

$$\frac{\Gamma \to \varphi \quad \Gamma \to \psi}{\Gamma \to \varphi \wedge \psi}\ (GI1\wedge) \qquad \frac{\Gamma, \varphi \supset \chi, \Delta \to \varphi \quad \Gamma, \psi \supset \chi, \Delta \to \psi}{\Gamma, \varphi \wedge \psi \supset \chi, \Delta \to \varphi \wedge \psi}\ (GI2\wedge)$$

$$\frac{\Gamma \to \varphi}{\Gamma \to \varphi \vee \psi}\ (GI1\vee) \qquad \frac{\Gamma \to \psi}{\Gamma \to \varphi \vee \psi}\ (GI1\vee)$$

$$\frac{\Gamma, \varphi \supset P, \psi \supset P, P \supset \chi, \Delta \to \varphi}{\Gamma, (\varphi \vee \psi) \supset \chi, \Delta \to \varphi \vee \psi}\ (GI2\vee) \qquad \frac{\Gamma, \varphi \supset P, \psi \supset P, P \supset \chi, \Delta \to \psi}{\Gamma, (\varphi \vee \psi) \supset \chi, \Delta \to \varphi \vee \psi}\ (GI2\vee)$$

$$\frac{\varphi, \Gamma \to \psi}{\Gamma \to \varphi \supset \psi}\ (GI1 \supset) \qquad \frac{\Gamma, \varphi, \psi \supset \chi, \Delta \to \psi}{\Gamma, (\varphi \supset \psi) \supset \chi, \Delta \to \varphi \supset \psi}\ (GI2 \supset)$$

$$\frac{\Gamma, \varphi, \psi, \Delta \to \chi}{\Gamma, \varphi \wedge \psi, \Delta \to \chi}\ (GE\wedge) \qquad \frac{\Gamma, \varphi, \Delta \to \chi \quad \Gamma, \psi, \Delta \to \chi}{\Gamma, \varphi \vee \psi, \Delta \to \chi}\ (GE\vee)$$

$$\frac{\Gamma, A, \Delta, \psi, \Xi \to \chi}{\Gamma, A, \Delta, A \supset \psi, \Xi \to \chi}\ (GE \supset P) \qquad \frac{\Gamma, \psi, \Delta, A, \Xi \to \chi}{\Gamma, A \supset \psi, \Delta, A, \Xi \to \chi}\ (GE \supset P)$$

$$\frac{\Gamma, \varphi \supset (\psi \supset \chi), \Delta \to \rho}{\Gamma, (\varphi \wedge \psi) \supset \chi, \Delta \to \rho}\ (GE \supset \wedge)$$

$$\frac{\Gamma, \varphi \supset P, \psi \supset P, P \supset \chi, \Delta \to \rho}{\Gamma, (\varphi \vee \psi) \supset \chi, \Delta \to \rho}\ (GE \supset \vee) \qquad \frac{\Gamma, \varphi, \psi \supset \chi, \Delta \to \psi \quad \Gamma, \chi, \Delta \to \rho}{\Gamma, (\varphi \supset \psi) \supset \chi, \Delta \to \rho}\ (GE \supset\supset)$$

The rules satisfy the following restrictions:

1. Conclusions of $(GI1)$-rules must not be of the form $\Gamma', \varphi' \supset \psi', \Delta' \to \varphi'$;
2. In every rule where P occurs, P is a propositional variable not occurring in the conclusion of the rule;
3. In the rule $(GE \supset \wedge)$ we have $\rho \neq \varphi \wedge \psi$, in the rule $(GE \supset \vee)$ we have $\rho \neq \varphi \vee \psi$, in the rule $(GE \supset\supset)$ we have $\rho \neq \varphi \supset \psi$.

Figure 3. Calculus LG

$$\frac{}{\Gamma, A(s_1,\ldots,s_n), \Delta \to A(t_1,\ldots,t_n) \cdot \Gamma^{\approx}, \Delta^{\approx} \vdash_{\forall} s_1 \approx t_1 \wedge \ldots \wedge \Gamma^{\approx}, \Delta^{\approx} \vdash_{\forall} s_n \approx t_n}\ (Ax)$$

$$\frac{}{\Gamma, \Lambda, \Delta \to \varphi \cdot \top}\ (\Lambda) \qquad \frac{}{\Gamma \to s \approx t \cdot \Gamma^{\approx} \vdash_{\forall} s \approx t}\ (\approx)$$

$$\frac{\Gamma, \varphi, \psi, \Delta \to \chi \cdot \mathcal{C}}{\Gamma, \varphi \wedge \psi, \Delta \to \chi \cdot \mathcal{C}}\ (\wedge \to)$$

$$\frac{\Gamma \to \varphi \cdot \mathcal{C}_1 \quad \Gamma \to \psi \cdot \mathcal{C}_2}{\Gamma \to \varphi \wedge \psi \cdot \mathcal{C}_1 \wedge \mathcal{C}_2}\ (\to \wedge) \qquad \frac{\Gamma, \varphi \supset \chi, \Delta \to \varphi \cdot \mathcal{C}_1 \quad \Gamma, \psi \supset \chi, \Delta \to \psi \cdot \mathcal{C}_2}{\Gamma, \varphi \wedge \psi \supset \chi, \Delta \to \varphi' \wedge \psi' \cdot \mathcal{C}_1 \wedge \mathcal{C}_2 \wedge \mathcal{E}_{\varphi=\varphi'} \wedge \mathcal{E}_{\psi=\psi'}}\ (\to \wedge')$$

$$\frac{\Gamma, \varphi, \Delta \to \chi \cdot \mathcal{C}_1 \quad \Gamma, \psi, \Delta \to \chi \cdot \mathcal{C}_2}{\Gamma, \varphi \vee \psi, \Delta \to \chi \cdot \mathcal{C}_1 \wedge \mathcal{C}_2}\ (\vee \to)$$

$$\frac{\Gamma \to \varphi \cdot \mathcal{C}}{\Gamma \to \varphi \vee \psi \cdot \mathcal{C}}\ (\to \vee_1) \qquad \frac{\Gamma, \varphi \supset P, \psi \supset P, P \supset \chi, \Delta \to \varphi \cdot \mathcal{C}}{\Gamma, \varphi \vee \psi \supset \chi, \Delta \to \varphi' \vee \psi' \cdot \mathcal{C} \wedge \mathcal{E}_{\varphi=\varphi'} \wedge \mathcal{E}_{\psi=\psi'}}\ (\to \vee_1')$$

$$\frac{\Gamma \to \psi \cdot \mathcal{C}}{\Gamma \to \varphi \vee \psi \cdot \mathcal{C}}\ (\to \vee_2) \qquad \frac{\Gamma, \varphi \supset P, \psi \supset P, P \supset \chi, \Delta \to \psi \cdot \mathcal{C}}{\Gamma, \varphi \vee \psi \supset \chi, \Delta \to \varphi' \vee \psi' \cdot \mathcal{C} \wedge \mathcal{E}_{\varphi=\varphi'} \wedge \mathcal{E}_{\psi=\psi'}}\ (\to \vee_2')$$

$$\frac{\Gamma, \psi, \Delta \to \chi \cdot \mathcal{C}_1 \quad \Gamma, \varphi \supset \psi, \Delta \to \varphi \cdot \mathcal{C}_2}{\Gamma, \varphi \supset \psi, \Delta \to \chi \cdot \mathcal{C}_1 \wedge \mathcal{C}_2}\ (\supset\to)$$

$$\frac{\varphi, \Gamma \to \psi \cdot \mathcal{C}}{\Gamma \to \varphi \supset \psi \cdot \mathcal{C}}\ (\to\supset) \qquad \frac{\Gamma, \varphi, \psi \supset \chi, \Delta \to \psi \cdot \mathcal{C}}{\Gamma, (\varphi \supset \psi) \supset \chi, \Delta \to \varphi' \supset \psi' \cdot \mathcal{C} \wedge \mathcal{E}_{\varphi=\varphi'} \wedge \mathcal{E}_{\psi=\psi'}}\ (\to\supset')$$

$$\frac{\to \varphi\{y/x\} \cdot \mathcal{C}}{\to \forall x \varphi \cdot \exists y (y \not\precsim \{v_1,\ldots,v_n\} \wedge \mathcal{C})}\ (\to \forall) \qquad \frac{\to \varphi\{y/x\} \cdot \mathcal{C}}{\to \exists x \varphi \cdot \exists y \mathcal{C}}\ (\to \exists)$$

$$\frac{\Gamma, A, \Delta, \varphi, \Xi \to \psi \cdot \mathcal{C}}{\Gamma, A, \Delta, A' \supset \varphi, \Xi \to \psi \cdot \mathcal{C} \wedge \mathcal{E}_{A=A'}}\ (A_1 \to) \qquad \frac{\Gamma, \varphi, \Delta, A, \Xi \to \psi \cdot \mathcal{C}}{\Gamma, A' \supset \varphi, \Delta, A, \Xi \to \psi \cdot \mathcal{C} \wedge \mathcal{E}_{A=A'}}\ (A_2 \to)$$

$$\frac{\Gamma, \varphi \supset (\psi \supset \chi), \Delta \to \gamma \cdot \mathcal{C}}{\Gamma, \varphi \wedge \psi \supset \chi, \Delta \to \gamma \cdot \mathcal{C}}\ (\wedge \supset\to)$$

$$\frac{\Gamma, \varphi \supset P, \psi \supset P, P \supset \chi, \Delta \to \gamma \cdot \mathcal{C}}{\Gamma, \varphi \vee \psi \supset \chi, \Delta \to \gamma \cdot \mathcal{C}}\ (\vee \supset\to)$$

$$\frac{\Gamma, \varphi, \psi \supset \chi, \Delta \to \gamma \cdot \mathcal{C}_1 \quad \Gamma, \chi, \Delta \to \gamma \cdot \mathcal{C}_2}{\Gamma, (\varphi \supset \psi) \supset \chi, \Delta \to \gamma \cdot \mathcal{C}_1 \wedge \mathcal{C}_2}\ (\supset\supset\to)$$

This calculus has the same restrictions as $\mathbf{LJ}_c^{\approx}$ and the following restrictions:

1. In the rules $(\to \vee_1')$, $(\to \vee_2')$ and $(\vee \supset\to)$ the formula P is a propositional variable not occurring in the conclusions of the rules.
2. In the rules $(A_1 \to)$ and $(A_2 \to)$ the formula A is atomic.
3. In the rules $(\to \wedge)$, $(\to \vee_1)$, $(\to \vee_2)$ and $(\to\supset)$, the sequent at the conclusion is not of the form $\Gamma', \chi \supset \gamma, \Delta' \to \chi$.
4. In the rule $(\wedge \supset\to)$ the formula γ is not the formula $\varphi \wedge \psi$, in the rule $(\vee \supset\to)$ the formula γ is not the formula $\varphi \vee \psi$ and in the rule $(\supset\supset\to)$ the formula γ is not the formula $\varphi \supset \psi$.

Figure 4. Calculus $\mathbf{LG}_c^{\approx}$