The ground-negative fragment of first-order logic is Π_2^p-complete

Andrei Voronkov*

November 19, 1997

Abstract

We prove that for a natural class of first-order formulas the validity problem is Π_2^p-complete.

1 Introduction

Traditionally, the decision problem for first-order logic is studied for classes of formulas characterized by the quantifier prefix and/or signature restrictions, see e.g. the recent book [1]. Kozen [5] considered the decision problem for so-called positive first-order formulas. Recently, in connection with model building, some decidability results have been obtained for classes of formulas characterized in terms of the clause form of the formula, e.g. in [2, 3].

In this paper we study the complexity of the validity problem for the class of *ground-negative* formulas which is a natural generalization of a class of formulas considered in [5]. We show that this problem is Π_2^p-complete both for logics with and without equality, which is quite an unusual complexity class for natural fragments of first-order logic.

For technical convenience, we introduce two nullary logical connectives $\top$ and $\bot$, interpreted respectively as a valid formula and as a formula false in all models. A formula φ is called *reduced* if φ is $\top$, or $\bot$, or φ contains

*Computing Science Department, Uppsala University, Box 311, S-751 05 Uppsala, Sweden. Email `voronkov@csd.uu.se`. URL `http://www.csd.uu.se/ voronkov`. Supported by a TFR grant.

neither $\top$ nor $\bot$. We denote the equality predicate by $\simeq$. The meta-symbol $\rightleftharpoons$ means "equal by definition".

We call atomic formulas *atoms.* A *literal* is either an atom or its negation. A literal or term is *ground* if it has no variables. A *clause* is any disjunction of literals. A *sentence* is a formula without free occurrences of variables.

Definition 1.1 (positive and ground-negative formulas) A formula φ is *positive* if every occurrence of an atomic subformula in φ is positive. A formula φ is called *ground-negative* (respectively, *ground-positive*) if every atomic subformula negatively (respectively, positively) occurring in φ is ground.

Obviously, any positive formula is also ground-negative.

In this article, we shall study the validity (or provability) problem for ground-negative formulas. The dual problem is the satisfiability problem for ground-positive formulas. We shall also consider the satisfiability problem for ground-positive clauses.

We call a $(\wedge, \vee, \exists)$*-formula* any formula constructed from atomic formulas using $(\wedge, \vee, \exists)$. We call a $(L, \wedge, \vee, \exists)$*-formula* any formula constructed from literals using $(\wedge, \vee, \exists)$. We shall use the same notion for other collections of logical connectives, for example, we can speak about $(\wedge, \vee)$-formulas. By using standard transformations into negation normal form and Skolemization we can prove the following statement.

Lemma 1.2 Given any ground-negative formula φ, one can construct in polynomial time a ground-negative $(L, \wedge, \vee, \exists)$-formula φ' such that φ is valid if and only if φ' is valid. $\Box$

We shall use several statements proved in [5].

Lemma 1.3 (Kozen [5]) The following problem is **NP**-complete for logic with equality. Given a conjunction $\mathcal{L}$ of ground literals and a positive formula φ, is the formula $\mathcal{L} \supset \varphi$ valid? $\Box$

It is easy to prove that this problem is also **NP**-complete for logic without equality.

2 Inclusion in Π_2^p

Let $\mathcal{L}$ be a consistent conjunction of ground literals and φ be a $(L, \wedge, \vee, \exists)$-formula. We define the *set of transformations* $R_{\mathcal{L}}$ as the set consisting of transformations (1)–(14) given below:

$$
\begin{array}{rcll rcll}
\neg\top & \Rightarrow & \bot & (1) & \varphi \vee \top & \Rightarrow & \top & (8) \\
\neg\bot & \Rightarrow & \top & (2) & \bot \vee \varphi & \Rightarrow & \varphi & (9) \\
\top \wedge \varphi & \Rightarrow & \varphi & (3) & \varphi \vee \bot & \Rightarrow & \varphi & (10) \\
\varphi \wedge \top & \Rightarrow & \varphi & (4) & \exists x \top & \Rightarrow & \top & (11) \\
\bot \wedge \varphi & \Rightarrow & \bot & (5) & \exists x \bot & \Rightarrow & \bot & (12) \\
\varphi \wedge \bot & \Rightarrow & \bot & (6) & A & \Rightarrow & \top, \text{ if } A \text{ is a positive atom in } \mathcal{L} & (13) \\
\top \vee \varphi & \Rightarrow & \top & (7) & A & \Rightarrow & \bot, \text{ if } A \text{ is a negative atom in } \mathcal{L} & (14)
\end{array}
$$

These transformations replace any subformula of φ to the left of $\Rightarrow$ by the corresponding subformula to the right. We call a *reduced form of* φ *w.r.t.* $\mathcal{L}$, denoted $\varphi \mid_{\mathcal{L}}$, any formula φ' obtained from φ by a sequence of transformations such that no transformation is applicable to φ'.

It is easy to see that the following property holds for $\varphi \mid_{\mathcal{L}}$.

Lemma 2.1 Let φ be a $(L, \wedge, \vee, \exists)$-formula and the set of ground literals $\mathcal{L}$ be such that for every atom A negatively occurring in φ either A or $\neg A$ belong to $\mathcal{L}$. Then (i) $\varphi \mid_{\mathcal{L}}$ is a positive reduced formula; and (ii) $\mathcal{L} \supset \varphi$ is equivalent to $\mathcal{L} \supset (\varphi \mid_{\mathcal{L}})$. □

Let φ be any ground-negative $(L, \wedge, \vee, \exists)$-formula and $A_1, \ldots, A_n$ be all negative atomic subformulas in φ in the order of their occurrence in φ. Denote by Π the powerset of $\{1, \ldots, n\}$. For any $\pi \in \Pi$ introduce the conjunction of ground literals $\mathcal{L}_\pi$ in the following way:

$$\mathcal{L}_\pi \rightleftharpoons (\bigwedge_{i \in \pi} A_i) \wedge (\bigwedge_{i \notin \pi} \neg A_i).$$

Note that the formula $\bigvee_{\pi \in \Pi} \mathcal{L}_\pi$ is valid. Since the formula $\bigvee_{\pi \in \Pi} \mathcal{L}_\pi \supset \varphi$ is equivalent to $\bigwedge_{\pi \in \Pi} (\mathcal{L}_\pi \supset \varphi)$, we have the following statement.

Lemma 2.2 The formula φ is equivalent to $\bigwedge_{\pi \in \Pi} (\mathcal{L}_\pi \supset \varphi)$. □

Lemma 2.3 The validity problem for ground-negative formulas with equality is in Π_2^p.

Proof. Let φ be a ground-negative formula. By Lemma 1.2, we can assume that φ is a $(L, \wedge, \vee, \exists)$-formula. Let Π and $\mathcal{L}_\pi$ be defined as above. Note the following.

1. φ is valid if and only if for every $\pi \in \Pi$ the formula $\mathcal{L}_\pi \supset \varphi$ is valid (Lemma 2.2).
2. The following problem is in **NP**: given φ and π, is $\mathcal{L}_\pi \supset \varphi$ valid? Indeed, by Lemma 2.1 $\mathcal{L}_\pi \supset \varphi$ is valid if and only if $\mathcal{L}_\pi \supset (\varphi \mid_{\mathcal{L}})$ is valid. The formula $\varphi \mid_{\mathcal{L}}$ is positive (Lemma 2.1). Hence, the validity problem for formulas $\mathcal{L}_\pi \supset (\varphi \mid_{\mathcal{L}})$ is in **NP** (Lemma 1.3).
3. The size of π (e.g. written as a sequence of 0s and 1s of length n) is bounded by a polynomial in the size of φ.

According to a standard characterization of Π_2^p, see e.g. [6, corollary 1, page 426] in terms of polynomial-size witnesses we have that the validity problem for ground-negative formulas is in Π_2^p. □

Of course, this lemma also holds for formulas without equality.

3 Π_2^p-hardness

In this section we prove that the validity problem for ground-negative formulas is Π_2^p-hard. We shall use a reduction of the QBF_2 problem.

We call a *quantified boolean formula* any formula made of propositional variables, boolean connectives $\neg, \wedge, \vee$ and quantifiers $\forall x, \exists x$, where x is a propositional variable. A *truth assignment* for a set of variables X is any mapping from X to the set of *boolean values* $\{1, 0\}$.

Definition 3.1 (problem QBF_2) QBF_2 is the following problem. Given a boolean formula G with variables divided in two sets X, Y, is there for every truth assignment for X a truth assignment for Y such that G is satisfied by the overall truth assignment?

If $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$, we denote the corresponding instance of QBF_2 by the quantified boolean formula $\forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m G$.

An instance of QBF_2 is true if and only if the corresponding quantified boolean formula is true in the two-element Boolean algebra $\mathbf{2} = \{1, 0\}$.

The following fact is well-known, see e.g. [6, page 428].

Lemma 3.2 QBF_2 is Π_2^p-complete. $\square$

A *3-clause* is a disjunction of three propositional literals. By using the standard polynomial-time translation of propositional formulas into 3-clauses, using addition of new propositional variables, we can also establish the following fact.

Lemma 3.3 QBF_2 restricted to formulas $\forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_m G$, where G is a conjunction of 3-clauses, is Π_2^p-complete. $\square$

Let

$$\forall \bar{x} \exists \bar{y} B(\bar{x}, \bar{y}) \tag{15}$$

be an instance of QBF_2 in which B is a $(L, \wedge, \vee)$-formula.

Consider a signature Σ consisting of a constant c_i for each x_i, constants t and f, and unary relations T and F. Let B' be the first-order formula obtained from B by replacing

- every positive occurrence of y_i with $T(y_i)$;
- every positive occurrence of x_i with $T(c_i)$;
- every negative occurrence of y_i with $F(y_i)$;
- every negative occurrence of x_i with $F(c_i)$.

Consider the first-order sentence

$$\Phi \supset \exists \bar{y} B'(\bar{c}, \bar{y}), \tag{16}$$

where Φ is the ground sentence

$$T(t) \wedge \neg F(t) \wedge F(f) \wedge \neg T(f) \wedge \bigwedge_i (T(c_i) \equiv \neg F(c_i)).$$

Think of $F(c)$ as saying "c is false" and $T(c)$ as saying "c is true". The sentence Φ says that each c_i is either true or false but not both, t is true but not false and f is false but not true. For example, if (15) is

$$\forall x_1 \forall x_2 \exists y_1 \exists y_2 ((x_1 \wedge \neg y_1) \vee (\neg x_2 \wedge y_2)),$$

then (16) would be

$$\Phi \supset \exists y_1 \exists y_2 ((T(c_1) \wedge F(y_1)) \vee (F(c_2) \wedge T(y_2))).$$

Lemma 3.4 Formula (16) is valid if and only if (15) is true in **2**.

Proof. Assume first that (16) is valid. Let $\sigma : \{x_1, \ldots, x_n\} \to \{1, 0\}$ be any truth valuation. We wish to show that σ extends to a valuation of $\{y_1, \ldots, y_m\}$ satisfying $B(\bar{x}, \bar{y})$. Make **2** into a Σ-structure $\mathfrak{B}$ as follows. Let

$$\begin{array}{ll} c_i^{\mathfrak{B}} \rightleftharpoons \sigma(x_i), & T(1) \rightleftharpoons \text{true}, \\ t^{\mathfrak{B}} \rightleftharpoons 1, & T(0) \rightleftharpoons \text{false}, \\ f^{\mathfrak{B}} \rightleftharpoons 0, & F(1) \rightleftharpoons \text{false}, \\ & F(0) \rightleftharpoons \text{true}. \end{array}$$

Then $\mathfrak{B} \models \Phi$, therefore

$$\mathfrak{B} \models \exists \bar{y} B'(\bar{c}, \bar{y}).$$

Let τ be a valuation of $\bar{y}$ such that

$$\mathfrak{B}, \tau \models B'(\bar{c}, \bar{y}).$$

Extend σ by defining $\sigma(y_i) \rightleftharpoons \tau(y_i)$. It is not hard to argue that

$$\mathbf{2}, \sigma \models B(\bar{x}, \bar{y}).$$

Conversely, suppose that (15) is true in **2**. Let $\mathfrak{A}$ be any Σ-structure with a domain A satisfying Φ. We wish to show that

$$\mathfrak{A} \models \exists \bar{y} B'(\bar{c}, \bar{y});$$

i.e. there exists a $\tau : \{y_1, \ldots, y_m\} \to A$ such that

$$\mathfrak{A}, \tau \models B'(\bar{c}, \bar{y}).$$

Define

$$\sigma(x_i) \rightleftharpoons \begin{cases} 1, & \text{if } T^{\mathfrak{A}}(c_i^{\mathfrak{A}}); \\ 0, & \text{if } F^{\mathfrak{A}}(c_i^{\mathfrak{A}}). \end{cases}$$

For each x_i, exactly one of the two alternatives holds, since $\mathfrak{A} \models \Phi$. Because (15) holds in $\mathbf{2}$, we can extend σ to $\{y_1, \ldots, y_m\}$ in such a way that

$$\mathbf{2}, \sigma \models B(\bar{x}, \bar{y}).$$

Define

$$\tau(x_i) \rightleftharpoons \begin{cases} t^{\mathfrak{A}}, & \text{if } \sigma(y_i) = 1; \\ f^{\mathfrak{A}}, & \text{if } \sigma(y_i) = 0. \end{cases}$$

Again, it is not hard to argue that

$$\mathfrak{A}, \tau \models B'(\bar{c}, \bar{y}).$$

$\square$

Using this lemma and the fact that the reduction (15) $\rightarrow$ (16) is polynomial-time, we obtain

Lemma 3.5 The validity problem for ground-negative $(L, \wedge, \vee, \exists)$-formulas without equality and function symbols is Π_2^p-hard. $\square$

Note that our reduction can be modified to prove a result that is similar to Lemma 1.3 of [5, theorem 9] but for logic without equality:

Theorem 1 The following problem is **NP**-complete for logic without equality. Given a conjunction $\mathcal{L}$ of ground literals and a positive formula φ, is the formula $\mathcal{L} \supset \varphi$ valid?

The same is true even when φ uses only $\wedge, \vee, \exists$ and has no function symbols.

Proof. Inclusion in **NP** follows from Lemma 1.3. To prove **NP**-hardness, we use reduction of 3-SAT, the satisfiability problem for conjunctions of 3-clauses known to be NP-complete. We only give the construction, leaving the complete proof to the reader.

Consider a signature Σ consisting of constants t and f and unary relations T and F. Let C be a conjunction of 3-clauses with variables $x_1, \ldots, x_n$. Let C' be the first-order formula obtained from C by replacing

- every positive occurrence of x_i with $T(x_i)$;
- every negative occurence of x_i with $F(x_i)$.

Consider the formula

$$T(t) \wedge F(f) \supset \exists x_1 \dots \exists x_n C'. \tag{17}$$

Similar to the proof of Lemma 3.4 we can prove that C is satisfiable if and only if (17) is valid. Since the reduction $C \to$ (17) is polynomial-time, we obtain the **NP**-hardness. □

4 Conclusion

Lemmas 2.3 and 3.5 give us the main result of this paper.

Theorem 2 The validity problem for ground-negative formulas, with or without equality and with or without function symbols is Π_2^p-complete.

The proof of Theorem 2 also establishes a statement similar to Lemma 1.3, or [5, theorem 9]:

Theorem 3 The following problem is Π_2^p-complete, both for logics with and without equality and with or without function symbols. Given a $(\wedge, \vee)$-sentence ψ and a positive formula φ, is the formula $\psi \supset \varphi$ valid?

"A positive formula φ" in this theorem can be replaced by "a $(\wedge, \vee, \exists)$-formula". It is interesting that the complexity of the validity problem for ground-negative formulas does not change when equality, function symbols and universal quantifier are not allowed in formulas.

The decidability of ground-negative formulas is also established by Fermüller and Leitsch [3]. They prove that this class of formulas (and even a more general class) can be decided by hyperresolution with ordered paramodulation. Their proof does not give our complexity bounds since proofs by hyperresolution and ordered paramodulation require exponential space.

Our main theorem implies the following result about the dual satisfiability problem.

Theorem 4 The satisfiability problem for ground-positive formulas, with or without equality, is Σ_2^p-complete.

We also have a similar result for clauses.

Theorem 5 The satisfiability problem for ground-positive clauses, with or without equality, and with or without function symbols is Σ_2^p-complete.

Proof. In view of Theorem 4, it is enough to prove Σ_2^p-hardness for the case without equality and without function symbols. The proof literally follows the proof of Lemma 3.5. We note that the clause form of the formula $\Phi \supset \exists \bar{y} B'(\bar{c}, \bar{y})$ used in the reduction of that lemma consists of clauses of the forms

$$\begin{array}{l} \neg F(c_i) \vee T(c_i); \\ \neg T(c_i) \vee F(c_i); \\ T(1); \\ \neg F(1); \\ F(0); \\ \neg T(0), \end{array}$$

plus the clauses for $B'(\bar{c}, \bar{y})$. Using Lemma 3.3, we can assume that B' is alredy in the clause form, which makes this reduction polynomial-time. $\square$

Of course for predicates different from equality there is no difference between negative and positive occurrences. Thus all theorems about ground-negative formulas can be stated for formulas in which all negative occurrences of equality atoms are ground and for every predicate symbol different from equality, either all positive occurrences or all negative occurrences of the corresponding atoms are ground.

Our complexity results cannot be generalized to ground-positive formulas with equality, because the validity problem for formulas of the form $\forall (s_1 \simeq t_1 \wedge \ldots s_n \simeq t_n) \supset s \simeq t$, where s, t are ground, is undecidable.

Acknowledgments

I thank Yuri Gurevich and Margus Veanes whose results [4] were the starting point for this paper. Remarks of an anonymous referee helped to improve a proof that was initially considerably longer.

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