Term Induction and Herbrand's Theorem

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We investigate the computational consequences of the addition of the conservative Rule of term induction

$$\frac{A(0) \quad A(a) \supset A(S(a))}{A(S^n(0))}\ TI$$

(n is arbitrary, side formulas are admitted) to a usual system of first-order logic FL.

FL+TI does not admit bounds on the width of Herbrand disjunctions in terms of proof length (contrary to FL), but FL+ q.f.TI. admits such bounds on the number of equivalence classes in the Herbrand disjunction modulo 0 and S. Consequently, the depth of the function symbols different to 0 and S can be limited. (This result does not holt in case of unrestricted TI).

In case of q.f.TI the Herbrand disjunctions for $\exists x_1 \dots x_r B$ can be written as

$$\bigvee_{i_1} \dots \bigvee_{i_n} A(S^{i_1}(0), \dots, S^{i_n}(0))$$

$$A \equiv B\sigma_1 \vee \dots \vee B\sigma_k$$

where A is determined by $\exists x_1 \dots x_r B$ and the length of the proof. n is bounded by the number of nested applications of TI. The existence of a valid Herbrand disjunction

$$\bigvee_{i_1} \dots \bigvee_{i_n} A(S^{i_1}(0), \dots, S^{i_n}(0))$$

for

$$\lambda x_1 \dots x_n A(x_1, \dots, x_n)$$

is shown to be decidable for $n = 1$, undecidable else. In presence of an additional parameter the general existence of a valid Herbrand disjunction

$$\bigvee_{i} A(S^i(0), S^m(0))$$

for

$$\lambda x, z A(x, z)$$

is decidable iff no atomic subformula contains both x and z. In this case, the inductions can be reconstucted providing a proof of $\exists x A(x, S^n(0))$ uniform for all n. The proof of this Theorem requires methods known from propositional temporal logic. Applications to concrete mathematical proofs and to ATP are discussed.