Formulating The Extended Josephus Problem

Armin Shams-Baragh
Ferdowsi University of Mashhad
armin_shams@behineh.com

Abstract

Josephus problem is a historical algorithmic problem, which is used for educational purposes in algorithm analysis books and courses. There's a simple algorithm to solve this problem, however the answer can also be found using a formula. Josephus problem can be extended to form Extended Josephus Problem. No formula had been seen for solving this problem before. This paper proposes and proves both recursive and non-recursive formulation for the extended form of this famous problem.

Keywords: Extended Josephus Problem, Josephus Problem, Recursive Formula, Algorithms

1. Introduction

This old problem was introduced by first century historian, Flavius Josephus. During a war between Romans and Jews, he was among forty-one Jews captured by Romans in a cave. They preferred suicide to that situation and decided to sit around a circle and begin to kill third of every three remaining persons from beginning of the circle, until no one is alive (at last, there were two persons alive who should kill themselves). Josephus that didn't want to suicide calculated that he and his friend should sit where to remain alive (to be those last two persons). The problem known as Josephus problem is something similar to the problem that Josephus solved [1].

Josephus Problem: There are n persons, numbered 1 to n, around a circle. We start from person number 1 and eliminate (kill) second of every two remaining persons until one person remains. Given the n, determine the number of that person. For example, if n=10, elimination is done this way: 2, 4, 6, 8, 10, 3, 7, 1, 9 and finally 5 remains.

Formula for Josephus Problem: The answer is obtained by circular shifting of binary form of n to left [1]. Following formula for the answer, was derived from a more general formula that will be discussed:

$$J(n) = 2(n - 2^{\lceil \lg(n) \rceil}) + 1 = 2n + 1 - 2^{\lceil \lg(n) \rceil + 1}$$

(Please note that lg(0) is assumed 0 here.)

Extended Josephus Problem: There are n persons, numbered 1 to n, around a circle. We eliminate second of every two remaining persons until one person remains. Given the n, determine the number of "x"th person who is eliminated.

Formula for the Extended Josephus Problem: We are going to introduce and prove the following non-recursive formula for the Extended Josephus Problem:

$$J(n,x) = 2n+1-(2n-2x+1)(2^{\lceil \lg \lceil n/(2n-2x+1) \rceil \rceil +1} - \operatorname{sgn}\left\lceil \frac{\lceil n/2 \rceil}{x} \right\rceil)$$

(lg(0) is assumed 0 here.)

2. Introducing and Proving Recursive and Non-recursive Formulas

First, we introduce a recursive formula for solving the Extended Josephus Problem and prove it. Then, a non-recursive formula is suggested and proved using the proved recursive formula and a lemma.

Recursive Formula: We propose the following recursive formula for the Extended Josephus Problem:

$$J(n,x) = \begin{cases} 1 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (n,x>1; J(n-1,x-1)=n-1) \\ J(n-1,x-1)+2 \cdots\cdots\cdots (n,x>1; J(n-1,x-1) \le n-2) \\ 1 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots n=x=1 \\ 2 \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots n>1, x=1 \end{cases}$$

Proof of the Recursive Formula: Correctness of the last two parts of the above recursive formula is obvious. The first two parts complete each other to form a circular computation for J (n,x) that changes J (n,x) = n+1 to J (n,x) = 1 (that is when the second part of the recursive formula wants to say J (n,x)=n+1, the first part corrects it as J (n,x)=1). When J (n-1, x-1) = n-1, according to the second part we have J (n,x) = n+1, but n+1 is not acceptable because we have only n numbers in our circle, so according to circularity of elimination, the next number to be eliminated is 1 (if there is any 1 remained, and we will see that there is). For proving the correctness of first two parts of the formula, we eliminate 2 from the circle of numbers and shift the circle to left, assuming that the circle begins with 3 and assuming that 1 is the last element (after n). Now, we call (rename) 1 as n+1 from now on (it obviously dose not hurt!). This way we have an n-1 element circle that it's "i"th element is i+2. Considering the fact that shifting to left has no effect on circular elimination of numbers and that we should eliminate 4 after 2 (which was eliminated) and the fact that elimination will begin with 4 in our new shifted circle, we can conclude that if the "k"th number to be eliminated in a n-1 member circle of numbers 1 to n-1 (corresponding to our new shifted circle) is i, then the "k"th number to be eliminated in our new shifted circle is i+2. So, we can conclude that the "k+1"th number to be eliminated in our original n member circle is i+2, with the exception that if i=n-1, then i+2 = (n-1)+2 = n+1 that is actually 1 (we called 1 as n+1 before). This exception is shown in form of the first part of the recursive formula. So, we proved that the mentioned recursive formula for the Extended Josephus Problem is correct.

Non-recursive Formula: Now, we suggest the following non-recursive formula and prove it using the proved recursive formula and a lemma:

$$J(n,x) = 2n+1-(2n-2x+1)(2^{\lceil \lg \lceil n/(2n-2x+1) \rceil \rceil + 1} - \operatorname{sgn}\left\lceil \frac{\lceil n/2 \rceil}{x} \right\rceil)$$

(lg(0) is assumed 0 here.)

Answer Triangle: To make the problem and proofs better to be understood, we build and use an answer triangle for the Extended Josephus Problem, based on the recursive formula, in which in its "k"th row, J(k,x), 1≤ x ≤ k, are listed increasingly:

n	J(n,x)																
1	1																
2	2	1															
3	2	1	3														
4	2	4	3	1													
5	2	4	1	5	3												
6	2	4	6	3	1	5											
7	2	4	6	1	5	3	7										
8	2	4	6	8	3	7	5	1									
9	2	4	6	8	1	5	9	7	3								
10	2	4	6	8	10	3	7	1	9	5							
11	2	4	6	8	10	1	5	9	3	11	7						
12	2	4	6	8	10	12	3	7	11	5	1	9					
13	2	4	6	8	10	12	1	5	9	13	7	3	11				
14	2	4	6	8	10	12	14	3	7	11	1	9	5	13			
15	2	4	6	8	10	12	14	1	5	9	13	3	11	7	15		
16	2	4	6	8	10	12	14	16	3	7	11	15	5	13	9	1	
17	2	4	6	8	10	12	14	16	1	5	9	13	17	7	15	11	3

We need to prove the following lemma before proving the formula:

Lemma: Row number of the first previous 1 that exists in diagonal row of the number J(n,x) is:

$$O = (2n-2x+1) \times 2^{\lceil \lg \lceil n/(2n-2x+1) \rceil \rceil}$$

(lg(0) is assumed 0 here.)

For example: Second diagonal row from right is 2 1 3 5 1 3 5 7 9 … . If J (x,n) = J (9,8) = 7, then the row number of the first previous 1 that exists in diagonal row of the number J(n,x) – which is the second diagonal row from right - is 6.

Proof of the Lemma: Based on two first parts of the recursive formula, we can easily see that if J(n,x) = 1 then J (2n, 2n-(n-x)) = 1. J(n,x) and J (2n, 2n-(n-x)) are in a diagonal row of the triangle. So, if we see 1 in a favorite diagonal row, in horizontal row n, then we see next 1 of that diagonal row in horizontal row 2n. In the other hand, the first 1 of each diagonal row is seen in horizontal row 2n-2x+1 (cause that all such 1 numbers are in odd rows, and in odd rows, 1 comes immediately after all even numbers come). Based on this fact and the previous fact, we can conclude that the second 1 occurs in row 2×(2n-2x+!), the third occurs in row 2×2×(2n-2x+!) and so on. So, the nearest 1 before

J(n,x) and in its diagonal row is in horizontal row O = $2^f \times (2n-2x+1)$ such that f is the maximum integer that satisfies $2^f \times (2n-2x+1) \le$ n. That means $2^f \le$ n / $(2n-2x+1)$, 2n-2x+1 > 0, that is $2^f \le$ [n / $(2n-2x+1)$], f∈Z, when f is maximum. *Of course, we could omit brackets and continue the proof, but brackets ensures that we can use integer division to divide n to 2n-2x+1.* We will have: f = $\lfloor \lg\lceil n/(2n-2x+1)\rceil \rfloor$. Substituting f with this value in $2^f \times (2n-2x+1)$ we will have:

$$O = (2n-2x+1)\times 2^{\lceil \lg\lceil n/(2n-2x+1)\rceil\rceil}$$

(lg(0) is assumed 0 here.)

Proof of the Non-recursive Formula: The proof of formula for all x ≤ $\lceil n/2 \rceil$ is obvious, because in this case J(n,x) should be 2x and we can easily see that $\mathrm{sgn}\left\lceil \frac{\lceil n/2 \rceil}{x} \right\rceil$ = 1 and $2^{\lceil \lg\lceil n/(2n-2x+1)\rceil\rceil+1}$ = 2, so J(n,x) = 2x (as it should). For x > $\lceil n/2 \rceil$, based on the lemma and the recursive formula we can easily prove (just take a look at the Answer Triangle):

$$J(n,x) = 2\,(n - O) + 1 = 2n+1-(2n-2x+1)2^{\lceil \lg\lceil n/(2n-2x+1)\rceil\rceil+1}$$

In the other hand, for x > $\lceil n/2 \rceil$ we see that $\mathrm{sgn}\left\lceil \frac{\lceil n/2 \rceil}{x} \right\rceil$ = 0 and can be eliminated from the formula in this case, so the formula is correct for x > $\lceil n/2 \rceil$, too.

So, we proved that:

$$J(n,x) = 2n+1-(2n-2x+1)\left(2^{\lceil \lg\lceil n/(2n-2x+1)\rceil\rceil+1} - \mathrm{sgn}\left\lceil \frac{\lceil n/2 \rceil}{x} \right\rceil\right)$$

(lg(0) is assumed 0 here.)

Prof. Donald E. Knuth's suggested idea for proving the non-recursive formula: "Give everybody a number starting 2n, 2n-1, ..., n+1. Then when you pass over a person, divide his/her number by 2. But when a person dies, you can figure out the original number by multiplying by a power of 2 until reaching the interval [n+1..2n]. This proof leads to a slightly simpler form of the result."

3. Conclusions

The problem can be extended more.

More Extended Josephus Problem: There are n persons, numbered 1 to n, around a circle. We eliminate "k"th of every k remaining persons until k-1 person(s) remain(s). Given the n, determine the number of "x"th person who is eliminated and determine the remaining k-1 persons. We can assume that last k-1 person(s) are eliminated consecutively starting from the person immediately after the last eliminated “k”th of k persons.

Generally, trying to obtain formula or formulation approaches for algorithmic problems is very interesting and may lead to a new branch in theoretical foundations of computer science. The old famous problems of finding “m”th prime number and checking primality of a given number may help a lot in establishing such a branch, because famous open problems have helped us a lot in such establishments. Considering the fact that an algorithm is a large formula itself, we may be able to find compression approaches to convert some groups of these large formulas (algorithms) to little ordinary formulas. Based on the fact that an algorithm is a logical circuit, “Karnaugh Table” approach, widely used in logical circuits, can be assumed the ancestor of such compression approaches.

4. Acknowledgments

The author thanks Professor Donald E. Knuth for his valuable notes and for mentioning the formula in his famous “The Art of Computer Programming” book.

Reference

[1] Sh. Rostami, Recursive Problems, Olympiad Quarterly, Fall 1994 (Translated from: Graham – Knuth – Patashnik, Concrete Mathematics, Addison-Wesley, 1988)