Tableau Development for a Bi-Intuitionistic Tense Logic*

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Abstract. The paper introduces a bi-intuitionistic logic with two modal operators and their tense versions. The semantics is defined by Kripke models in which the set of worlds carries a pre-order relation as well as an accessibility relation, and the two relations are linked by a stability condition. A special case of these models arises from graphs in which the worlds are interpreted as nodes and edges of graphs, and formulae represent subgraphs. The pre-order is the incidence structure of the graphs. These examples provide an account of time including both time points and intervals, with the accessibility relation providing the order on the time structure. The logic we present is decidable and has the effective finite model property. We present a tableau calculus for the logic which is sound, complete and terminating. The MetTel system has been used to generate a prover from this tableau calculus.

1 Introduction

We start by reviewing the motivation for developing a theory of 'relations on graphs' which generalizes that of 'relations on sets'. One novel feature of relations on graphs is a pair of adjoint converse operations instead of the involution found with relations on sets. One half of this pair (the 'left converse') is used later in defining a relational semantics for a novel bi-intuitionistic modal logic.

Relations on sets underlie the most fundamental of the operations used in mathematical morphology [Ser82,BHR07]. Using $\mathbb{Z}^2$ to model a grid of pixels, binary (i.e., black and white) images are modelled by subsets of $\mathbb{Z}^2$. One aim of processing images is to accentuate significant features and to lessen the visual impact of the less important aspects. Several basic transformations on images are parameterized by small patterns of pixels called structuring elements. These structuring elements generate relations which transform subsets of $\mathbb{Z}^2$ via the correspondence between relations $R \subseteq \mathbb{Z}^2 \times \mathbb{Z}^2$ and union-preserving operations on the powerset $\mathscr{P}\mathbb{Z}^2$. Several fundamental properties of image processing operations can be derived using only properties of these relations.

There have been various proposals for developing a version of mathematical morphology for graphs, one of the earliest being [HV93]. However, most work in

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this area does not use a relational approach, probably because a theory of relations on graphs may be constructed in several different ways. In one way [Ste12], the set of relations on a graph, or more generally a hypergraph, forms a generalization of a relation algebra where, in particular, the usual involutive converse operation becomes a pair of operations (the left converse and the right converse) forming an adjoint pair. In the present paper, we use these relations to give a semantics for a bi-intuitionistic modal logic in which propositions are interpreted over subgraphs as opposed to subsets of worlds as in standard Kripke semantics.

Accessibility relations with additional structure are already well-known in intuitionistic modal logic [ZWC01]. However, the semantics for the logic we present is distinguished both from this work, and from other related work we discuss in Section 5, by the use of the left-converse operation. This leads to a logic with novel features which include $\Diamond\varphi$ being equivalent to $\lrcorner\Box\neg\varphi$, where $\lrcorner$ and $\neg$ are respectively the co-intuitionistic and the intuitionistic negation.

Connections between mathematical morphology and modal logic, have been developed by Aiello and Ottens [AO07], who implemented a hybrid modal logic for spatial reasoning, and by Aiello and van Benthem [Av02] who pointed out connections with linear logic. Bloch [Blo02], also motivated by applications to spatial reasoning, exploited connections between relational semantics for modal logic and mathematical morphology. These approaches used morphology operations on sets, and one motivation for our own work is to extend these techniques to relations on graphs. This has potential for applications to spatial reasoning about discrete spaces based on graphs.

In this paper we restrict our attention to the logic itself, rather than its applications, and the semantic setting we use is more general than that arising from relations on graphs or hypergraphs.

The main contribution of the paper is a bi-intuitionistic tense logic, called BISKT, for which a Kripke frame consists of a pre-order H interacting with an accessibility relation R via a stability condition. The semantics interprets formulae as H-sets, the downwardly closed sets of the pre-order. A particular case arises when the worlds represent the edges and nodes of a graph and formulae are interpreted as subgraphs. We show that BISKT is decidable and has the effective finite model property, by showing that BISKT can be mapped to the guarded fragment which is known to be decidable and has the effective finite model property [ANvB98,Grä99].

The semantic setting for BISKT is a relational setting in which it is not difficult to develop deduction calculi. Semantic tableau deduction calculi in particular are easy to develop. In this paper we follow the methodology of tableau calculus synthesis and refinement as introduced in [ST11,TS13] to develop a tableau calculus for the logic. We give soundness, completeness and termination results as consequences of results of tableau synthesis and that BISKT has the effective finite model property.

Implementing a prover is normally a time-consuming undertaking but MetTeL is software for automatically generating a tableau prover from a set of tableau rules given by the user [Met,TSK12]. For us using MetTeL turned out to be useful because we could experiment with implementations of different initial versions of the calculus. In combination with the tableau synthesis method it was easy to run tests on a growing collection of problems with different provers for several preliminary versions of formalisations of bi-intuitionistic tense logics before settling on the definition given in this paper. MetTeL has also allowed to us experiment with different refinements of the rules and different forms of blocking. Blocking is a technique for forcing termination of tableau calculi for decidable logics.

The paper is structured as follows. Section 2 presents the basic notions of bi-Heyting algebras and relations on downwardly closed sets as well as graphs. Section 3 defines the logic BISKT as a bi-intuitionistic stable tense logic in which subgraphs are represented as downwardly closed sets. In Section 4 we present a terminating labelled tableau calculus for BISKT. With a MetTeL generated prover we have tested several formulae for validity and invalidity in BISKT; a selection of the validities shown are given in the section. Connections to other work is discussed in Section 5.

The MetTeL specification of the tableau calculus for BISKT and the generated prover can be downloaded from the accompanying website: `http://staff.cs.manchester.ac.uk/~schmidt/publications/biskt13/`. There also our current set of problems and performance graphs can be found.

2 Relations on pre-orders and on graphs

2.1 The bi-Heyting algebra of *H*-sets

Let U be a set with a subset $X \subseteq U$, and let $R \subseteq U \times U$ be a binary relation. We recall the definitions of the key operations used in mathematical morphology.

Definition 1 *The* ***dilation****, $\oplus$, and the* ***erosion****, $\ominus$, are given by:*

$$X \oplus R = \{u \in U : \exists x\, ((x, u) \in R \wedge x \in X)\},$$
$$R \ominus X = \{u \in U : \forall x\, ((u, x) \in R \to x \in X)\}.$$

For a fixed R these operations form an adjunction from the lattice $\mathscr{P}U$ to itself in the following sense, with $\_ \oplus R$ being left adjoint to $R \ominus \_$.

Definition 2 *Let $(X, \leq_X), (Y, \leq_Y)$ be partially ordered sets. An* ***adjunction*** *between X and Y consists of a pair of order-preserving functions $f : X \to Y$ and $g : Y \to X$ such that $x \leq_X g(y)$ iff $f(x) \leq_Y y$ for all $x \in X$ and all $y \in Y$. The function f is said to be* ***left adjoint to*** *g, and g is* ***right adjoint to*** *f.*

Erosion and dilation interact with composition of relations as follows.

Lemma 1 *If R and S are any binary relations on U and $X \subseteq U$, then*

$$(S\,;R) \ominus X = S \ominus (R \ominus X) \qquad \textit{and} \qquad X \oplus (R\,;S) = (X \oplus R) \oplus S.$$

The operations $\oplus$ and $\ominus$ can be applied to subgraphs of a graph when relations on graphs are defined. We see how this works in Section 2.3, but first work in a more general setting of H-sets which we now define. Let U be a set and H a pre-order on U (i.e., a reflexive and transitive relation).

Definition 3 *A subset $X \subseteq U$ is an H-**set** if $X \oplus H \subseteq X$.*

Since H is reflexive, the condition is equivalent to $X \oplus H = X$, and if we were to write H as $\geqslant$ these would be downsets. It follows from the adjunction between dilation and erosion that X satisfies $X = X \oplus H$ iff it satisfies $X = H \ominus X$.

The set of all H-sets forms a lattice which is a bi-Heyting algebra. The H-sets are closed under unions and intersections but not under complements. When A and B are H-sets, we can construct the following H-sets where $-$ denotes the complement of a subset of U.

$$
\begin{array}{rll}
A \to B = H \ominus (-A \cup B) & & \text{relative pseudocomplement} \\
A \succ B = (A \cap -B) \oplus H & & \text{dual relative pseudocomplement} \\
\neg A = H \ominus (-A) & & \text{pseudocomplement} \\
\llcorner A = (-A) \oplus H & & \text{dual pseudocomplement}
\end{array}
$$

2.2 Relations on H-sets

Relations on a set U can be identified with the union-preserving functions on the lattice of subsets. When U carries a pre-order H, the union-preserving functions on the lattice of H-sets correspond to relations on U which are stable:

Definition 4 *A binary relation R on U is **stable** if $H\,;R\,;H \subseteq R$.*

Stable relations are closed under composition, with H as the identity element for this operation, but they are not closed under converse. They do however support an adjoint pair of operations, the left and the right converse, denoted by $\curvearrowright$ and $\curvearrowleft$ respectively. Properties of these include $\curvearrowright\curvearrowleft R \subseteq R \subseteq \curvearrowleft\curvearrowright R$ for any stable relation R.

Definition 5 *The **left converse** of a stable relation R is $\curvearrowright R = H;\breve{R};H$ where $\breve{R}$ is the (ordinary) converse of R.*

The stability of $\curvearrowright R$ is immediate since H is a pre-order, and the left converse can be characterized as the smallest stable relation which contains $\breve{R}$. The right converse is characterized as the largest stable relation contained in $\breve{R}$, but it plays no role in this paper, so we omit an explicit construction (see [Ste12] for details).

The connection between erosion, dilation, complementation and converse in the lemma below is well-known [BHR07]. We need it to prove Theorem 3 below which generalizes the lemma to the case of a stable relation acting on an H-set.

Lemma 2 *For any relation R on U and any $X \subseteq U$, $X \oplus \breve{R} = -(R \ominus (-X))$.*

The following was proved in [Ste12] for the special case of hypergraphs, but we give a direct proof of the general case as it underlies one of the novel features of the logic we consider.

Theorem 3 *For any stable relation R and any H-set A,*

$$A \oplus (\curvearrowright R) = \llcorner(R \ominus (\neg A)) \quad \textit{and} \quad (\curvearrowright R) \ominus A = \neg((\llcorner A) \oplus R).$$

Proof.

$$\begin{aligned} A \oplus (\curvearrowright R) &= A \oplus (H \,;\, \breve{R} \,;\, H) \\ &= ((A \oplus H) \oplus \breve{R}) \oplus H \\ &= (A \oplus \breve{R}) \oplus H \\ &= (-(R \ominus (-A))) \oplus H \\ &= \llcorner(R \ominus (-A))) \\ &= \llcorner((R \,;\, H) \ominus (-A))) \\ &= \llcorner(R \ominus (H \ominus (-A))) \\ &= \llcorner(R \ominus (\neg A)) \end{aligned}$$

$$\begin{aligned} (\curvearrowright R) \ominus A &= (H \,;\, \breve{R} \,;\, H) \ominus A \\ &= H \ominus (\breve{R} \ominus (H \ominus A)) \\ &= H \ominus (\breve{R} \ominus A) \\ &= H \ominus -((-A) \oplus R) \\ &= \neg((-A) \oplus R) \\ &= \neg((-A) \oplus (H \,;\, R)) \\ &= \neg(((-A) \oplus H) \oplus R) \\ &= \neg((\llcorner A) \oplus R) \end{aligned}$$

2.3 Relations on graphs

A special case of the above constructions is when U is the set of all edges and nodes of a graph (that is, an undirected multigraph with multiple loops permitted). For an edge e and a node n we put $(e, n) \in H$ iff e is incident with n, and otherwise $(u, v) \in H$ holds only when $u = v$.

In this setting, the H-sets are exactly the subgraphs, that is sets of nodes and edges which include the incident nodes of every edge in the set. The importance of the bi-Heyting algebra of subgraphs of a directed graph has been highlighted by Lawvere as explained in [RZ96].

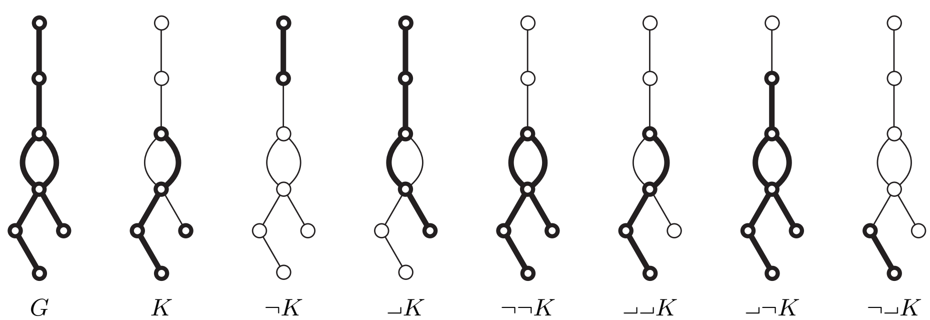

Fig. 1. Graph G with subgraph K and the pseudocomplement operation and its dual.

We give an example for an undirected graph of the operations $\neg$ and $\llcorner$ of Section 2.1 since these motivate the semantics for the two negations in our logic. Figure 1 shows various subgraphs of a graph where the subgraphs are distinguished by depicting the edges and nodes in bold. For a subgraph K, the operations $\neg$ and $\lrcorner$ yield respectively the largest subgraph disjoint from K and the smallest subgraph containing all the edges and nodes not present in K. In Figure 1 it can be seen that neither $\neg\neg K$ nor $\lrcorner\lrcorner K$ is equal to K. The subgraph $\neg\neg K$ consists of K completed by the addition of any edges all of whose incident nodes are in K. The subgraph $\lrcorner\lrcorner K$ is K with the removal of any nodes that have incident edges none of which is present in K. The subgraph $\lrcorner\neg K$ can be interpreted as the expansion of K to include things up to one edge away from K, and $\neg\lrcorner K$ is a kind of contraction, removing any nodes on the boundary of K and any edges incident on them.

The graph includes edges v, w, x, y, z and nodes a, b, c. The relation is the set $\{(z, a), (x, z), (b, z), (v, z), (w, z), (x, a), (b, a), (v, a), (w, a), (y, b), (y, v), (y, c)\}$.

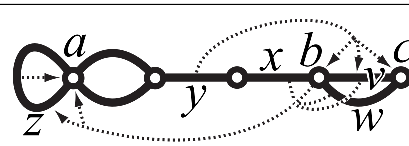

Fig. 2. Relation on a graph drawn as arrows with multiple heads and tails.

The stable relations on a graph can be visualized as in Figure 2. The arrows used may have multiple heads and multiple tails; the meaning is that every node or edge at a tail is related to all the edges and nodes at the various heads. The stability condition implies that if a node, n, is related to something, u say, then every edge incident with n is also related to u. Stablility also implies that if u, which may be an edge or a node, is related to an edge e, then u is also related to every node incident with e.

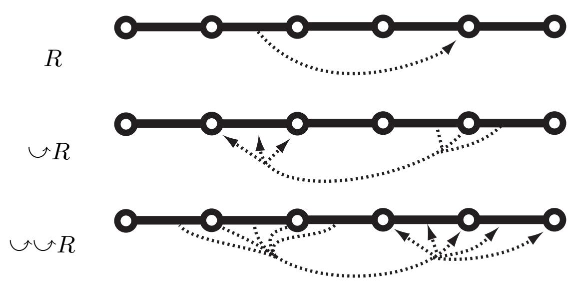

Fig. 3. A relation R, its left converse, and the left converse of the left converse of R

The left converse operation is illustrated in Figure 3 showing that iterating this operation can lead to successively larger relations.

3 Bi-intuitionistic stable tense logic

We now propose a modal logic BISKT for which a Kripke frame is a pre-ordered set (U, H) together with a stable relation. The semantics in this section interprets formulae as H-sets. A particular case is when the worlds are the edges and nodes of a graph and formulae are interpreted as subgraphs.

Definition 6 *The language of* BISKT *consists of a set Vars of propositional variables:* $p, q, \ldots$, *a constant:* $\bot$, *unary connectives:* $\neg$ *and* $\lrcorner$, *binary connectives:* $\wedge$, $\vee$, $\to$, $\succ$, *and unary modal operators:* $\Box$, $\blacklozenge$, $\Diamond$, *and* $\blacksquare$. *The set Form of formulae is defined in the usual way.*

Definition 7 *An H-**frame** $\mathcal{F} = (U, H, R)$ is a pre-order (U, H) together with a stable relation R. A **valuation** on an H-frame $\mathcal{F}$ is a function $\mathcal{V}$: Vars $\to$ H-Set, where H-Set is the set of all H-sets.*

A valuation $\mathcal{V}$ on $\mathcal{F}$ extends to a function $[\![\ ]\!]$: *Form* $\to$ H-Set by putting $[\![v]\!] = \mathcal{V}\, v$ for any propositional variable v, and making the following definitions.

$$
\begin{array}{rlrl}
[\![\bot]\!] &= \varnothing & [\![\top]\!] &= U \\
[\![\alpha \vee \beta]\!] &= [\![\alpha]\!] \cup [\![\beta]\!] & [\![\alpha \wedge \beta]\!] &= [\![\alpha]\!] \cap [\![\beta]\!] \\
[\![\neg\alpha]\!] &= \neg[\![\alpha]\!] & [\![\lrcorner\alpha]\!] &= \lrcorner[\![\alpha]\!] \\
[\![\alpha \to \beta]\!] &= [\![\alpha]\!] \to [\![\beta]\!] & [\![\alpha \succ \beta]\!] &= [\![\alpha]\!] \succ [\![\beta]\!] \\
[\![\Box\alpha]\!] &= R \ominus [\![\alpha]\!] & [\![\Diamond\alpha]\!] &= [\![\alpha]\!] \oplus (\smile R) \\
[\![\blacklozenge\alpha]\!] &= [\![\alpha]\!] \oplus R & [\![\blacksquare\alpha]\!] &= (\smile R) \ominus [\![\alpha]\!]
\end{array}
$$

From Theorem 3 we have $[\![\Diamond\alpha]\!] = [\![\lrcorner\Box\neg\alpha]\!]$ and $[\![\blacksquare\alpha]\!] = [\![\neg\blacklozenge\lrcorner\alpha]\!]$.

Definition 8 *Given a frame $\mathcal{F}$, a valuation $\mathcal{V}$ on $\mathcal{F}$ and a world $w \in U$ we define $\Vdash$ by*

$$\mathcal{F}, \mathcal{V}, w \Vdash \alpha \text{ iff } w \in [\![\alpha]\!].$$

When $\mathcal{F}, \mathcal{V}, w \Vdash \alpha$ holds for all $w \in U$ we write $\mathcal{F}, \mathcal{V} \Vdash \alpha$, and when $\mathcal{F}, \mathcal{V} \Vdash \alpha$ holds for all valuations $\mathcal{V}$ we write $\mathcal{F} \Vdash \alpha$.

In the special case that H is the identity relation on U, stability places no restriction on R and $\smile R$ is just the ordinary converse of R. The semantics is then equivalent to the usual relational semantics for tense logic when time is not assumed to have any specific properties. Figure 4 illustrates how the semantics of $\Diamond$ and $\blacksquare$ can differ from this usual case. In the figure, R is denoted by the broken lines and H is determined by the graph. The H-sets are shown in bold. We can give a temporal interpretation to the example by taking the nodes to be time points, the edges to be open intervals, and R to relate each open interval to all instants that either end the interval or end some later interval. The times when $\Diamond p$ holds are then the open intervals for which p holds at some later instant, together with both endpoints of those intervals. The times when $\blacksquare q$ holds are all the closed intervals where q holds at all times and has always held.

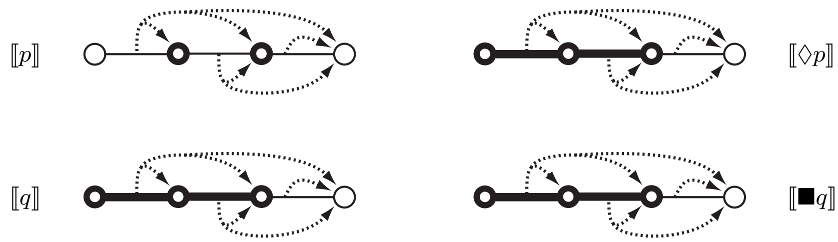

Fig. 4. Example of semantics for $\Diamond$ and $\blacksquare$

Theorem 4 *The logic* BISKT *is decidable and has the effective finite model property.*

The proof involves an embedding of BISKT into a traditional modal logic with forward and backward looking modal operators defined by H and R as accessibility relations. The frame conditions are reflexivity and transitivity of H, and the stability of R with respect to H. Monotonicity also needs to be suitably ensured. This logic can be shown to be decidable and have the effective finite model property by mapping it to the guarded fragment. This can be done using the axiomatic translation principle introduced in [SH07]. As the guarded fragment is decidable and has the effective finite model property [ANvB98,Grä99] these properties are inherited by the modal logic and also BISKT. It further follows that the computational complexity of reasoning in BISKT is no worse than EXPTIME. (Due to space constraints the detailed proof has been omitted.)

4 Tableau calculus for BISKT

Since the accessibility relations in the Kripke models of BISKT involve converse relations it is natural to use a semantic tableau method, which does not place any limitations on how the proof search can be performed. In particular, we use a labelled tableau approach because this ensures proof confluence. This means there is no need for backtracking over the proof search, and there is more flexibility in defining search heuristics in an implementation. These are aspects which make it harder to develop tableau calculi where the rules operate on formulae of the logic and do not include any syntactic entities referring to semantics. An additional advantage of semantic tableau calculi is that they return concrete models for satisfiable formulae.

Since BISKT is based on intuitionistic logic the calculus presented in this section is a labelled *signed* tableau calclulus. Tableau formulae have these forms.

$$\bot \qquad s : S\,\varphi \qquad H(s,t) \qquad R(s,t) \qquad s \approx t \qquad s \not\approx t$$

S denotes a sign (either T or F for true or false), s and t represent worlds in the models constructed by the tableau calculus, and $\approx$ is the standard equality symbol. Technically, s and t denote terms in the term algebra freely generated from a finite set of constants and a finite set of unary function symbols f_θ, $g_{\theta'}$, $g'_{\theta'}$, which are uniquely associated with subformulae of the input set. Specifically, the f_θ are associated with subformulae involving quantification in their semantic definition (i.e., $\neg\varphi$, $\lrcorner\varphi$, $\varphi \to \psi$, $\varphi \succ \psi$, $\Box\varphi$, $\blacklozenge\varphi$, $\Diamond\varphi$, $\blacksquare\varphi$), and for each subformula θ' of the form $\Diamond\varphi$ or $\blacksquare\varphi$ there is a unique function symbol $g_{\theta'}$ and $g'_{\theta'}$. These symbols are Skolem functions and provide a convenient technical device to generate witnesses for formulae of existential extent.

The semantics of tableau formulae is defined by extended Kripke models. An extended Kripke model is a structure $(\mathcal{M}, \iota)$, where $\mathcal{M} = (\mathcal{F}, \mathcal{V})$ is a Kripke model defined as in the previous section ($\mathcal{F}$ denotes a frame and $\mathcal{V}$ a valuation function) and ι is an assignment mapping terms in the tableau language to worlds in U. Satisfiability of tableau formulae is defined by:

$$
\begin{array}{lll@{\qquad}lll}
\mathcal{M}, \iota \not\Vdash \bot & & & & & \\
\mathcal{M}, \iota \Vdash s : T\ \varphi & \text{iff} & \mathcal{M}, \iota(s) \Vdash \varphi & \mathcal{M}, \iota \Vdash s : F\ \varphi & \text{iff} & \mathcal{M}, \iota(s) \not\Vdash \varphi \\
\mathcal{M}, \iota \Vdash H(s,t) & \text{iff} & (\iota(s), \iota(t)) \in H & \mathcal{M}, \iota \Vdash R(s,t) & \text{iff} & (\iota(s), \iota(t)) \in R \\
\mathcal{M}, \iota \Vdash s \approx t & \text{iff} & \iota(s) = \iota(t) & \mathcal{M}, \iota \Vdash s \not\approx t & \text{iff} & \iota(s) \neq \iota(t)
\end{array}
$$

Let Tab_{BISKT} be the calculus consisting of the rules in Figure 5. The rules are to be applied top-down. Starting with set of tableau formulae the rules are used to decompose formulae in a goal-directed way. Since some of the rules are branching the inference process constructs a tree derivation. As soon as a contradiction is derived in a branch (that is, when $\bot$ has been derived) that branch is regarded **closed** and no more rules are applied to it. If a branch is not closed then it is **open**. When in an open branch no more rules are applicable then the derivation can stop because a model for the input set can be read off from the branch.

The way to use the tableau calculus is as follows. Suppose we are interested in the validity of a formula, say φ, in BISKT. Then the input to the tableau derivation is the set $\{a : F\varphi\}$ where a is a constant representing the initial world, and the aim is to find a counter-model for φ. If a counter-model is found then φ is not valid, on the other hand, if a closed tableau is constructed then φ is valid.

The first group of rules in the calculus are the closure rule and the decomposition rules of the operators of bi-intuitionistic logic. The closure rule derives $\bot$ when for a formula φ both $s : T\ \varphi$ and $s : F\ \varphi$ occur on the branch. The branch is then closed. The other rules can be thought of as 'decomposing' labelled formulae and building an ever growing tree derivation. These inference steps basically follow the semantics of the main logical operator of the formula being decomposed. For example, the rule for positive occurrences of implication extends the current branch with $t : F\ \varphi$ and $t : T\ \psi$ thereby creating two branches, if formulae of the form $s : T\ \varphi \to \psi$ and $H(s,t)$ belong to the current branch. The rule for negative occurrences of implication extends the current branch with the three formulae $H(s, f_{\varphi\to\psi}(s))$, $f_{\varphi\to\psi}(s) : T\ \varphi$ and $f_{\varphi\to\psi}(s) : F\ \psi$, if the formula $s : F\ \varphi \to \psi$ occurs on the current branch. The effect is that an H-successor

Rules for operators of bi-intuitionistic logic and the closure rule:

$$\frac{s : T\ \varphi,\ s : F\ \varphi}{\bot}\ \text{pv. 0} \qquad \frac{s : T\ \bot}{\bot}\ \text{pv. 0}$$

$$\frac{s : T\ \varphi \wedge \psi}{s : T\ \varphi,\ s : T\ \psi}\ \text{pv. 1} \qquad \frac{s : F\ \varphi \wedge \psi}{s : F\ \varphi \mid s : F\ \psi}\ \text{pv. 7}$$

$$\frac{s : F\ \varphi \vee \psi}{s : F\ \varphi,\ s : F\ \psi}\ \text{pv. 1} \qquad \frac{s : T\ \varphi \vee \psi}{s : T\ \varphi \mid s : T\ \psi}\ \text{pv. 7}$$

$$\frac{s : T\ \neg\varphi,\ H(s,t)}{t : F\ \varphi}\ \text{pv. 2} \qquad \frac{s : F\ \neg\varphi}{H(s, f_{\neg\varphi}(s)),\ f_{\neg\varphi}(s) : T\ \varphi}\ \text{pv. 10}$$

$$\frac{s : F\ ⌐\varphi,\ H(t,s)}{t : T\ \varphi}\ \text{pv. 2} \qquad \frac{s : T\ ⌐\varphi}{H(f_{⌐\varphi}(s), s),\ f_{⌐\varphi}(s) : F\ \varphi}\ \text{pv. 10}$$

$$\frac{s : T\ \varphi \to \psi,\ H(s,t)}{t : F\ \varphi \mid t : T\ \psi}\ \text{pv. 2} \qquad \frac{s : F\ \varphi \to \psi}{H(s, f_{\varphi\to\psi}(s)),\ f_{\varphi\to\psi}(s) : T\ \varphi,\ f_{\varphi\to\psi}(s) : F\ \psi}\ \text{pv. 10}$$

$$\frac{s : F\ \varphi \succ \psi,\ H(t,s)}{t : F\ \varphi \mid t : T\ \psi}\ \text{pv. 2} \qquad \frac{s : T\ \varphi \succ \psi}{H(f_{\varphi\succ\psi}(s), s),\ f_{\varphi\succ\psi}(s) : T\ \varphi,\ f_{\varphi\succ\psi}(s) : F\ \psi}\ \text{pv. 10}$$

Rules for the tense operators:

$$\frac{s : T\ \Box\varphi,\ R(s,t)}{t : T\ \varphi}\ \text{pv. 2} \qquad \frac{s : F\ \Box\varphi}{R(s, f_{\Box\varphi}(s)),\ f_{\Box\varphi}(s) : F\ \varphi}\ \text{pv. 10}$$

$$\frac{s : F\ \blacklozenge\varphi,\ R(t,s)}{t : F\ \varphi}\ \text{pv. 2} \qquad \frac{s : T\ \blacklozenge\varphi}{R(f_{\blacklozenge\varphi}(s), s),\ f_{\blacklozenge\varphi}(s) : T\ \varphi}\ \text{pv. 10}$$

$$\frac{s : F\ \Diamond\varphi,\ H(t,s),\ R(t,u),\ H(v,u)}{v : F\ \varphi}\ \text{pv. 4}$$

$$\frac{s : T\ \Diamond\varphi}{H(g_{\Diamond\varphi}(s), s),\ R(g_{\Diamond\varphi}(s), g'_{\Diamond\varphi}(s)),\ H(f_{\Diamond\varphi}(s), g'_{\Diamond\varphi}(s)),\ f_{\Diamond\varphi}(s) : T\ \varphi}\ \text{pv. 10}$$

$$\frac{s : T\ \blacksquare\varphi,\ H(s,t),\ R(u,t),\ H(u,v)}{v : T\ \varphi}\ \text{pv. 4}$$

$$\frac{s : F\ \blacksquare\varphi}{H(s, g_{\blacksquare\varphi}(s)),\ R(g'_{\blacksquare\varphi}(s), g_{\blacksquare\varphi}(s)),\ H(g'_{\blacksquare\varphi}(s), f_{\blacksquare\varphi}(s)),\ f_{\blacksquare\varphi}(s) : F\ \varphi}\ \text{pv. 10}$$

Rules for frame and model conditions:

$$(\mathsf{refl})\ \frac{}{H(s,s)}\ \text{pv. 3} \qquad (\mathsf{tr})\ \frac{H(s,t),\ H(t,u)}{H(s,u)}\ \text{pv. 2}$$

$$(\mathsf{mon})\ \frac{s : T\ \varphi,\ H(s,t)}{t : T\ \varphi}\ \text{pv. 2} \qquad (\mathsf{stab})\ \frac{H(s,t),\ R(t,u),\ H(u,v)}{R(s,v)}\ \text{pv. 4}$$

Fig. 5. Tableau calculus Tab_{BISKT}. (pv = priority value. Rules of highest priority have pv 0, rules of lowest priority have pv 10.)

world is created for s and ψ is assigned false in this successor, while φ is assigned true.

The rules for the $\Box$ and $\blacklozenge$ operators are signed versions of the standard rules for tense modalities in semantic tableaux for traditional modal logics. The rules for the $\Diamond$ and $\blacksquare$ operators are more complicated versions, as they refer to the composite relation $H; \breve{R}; H$.

The third group of rules ensures the models constructed have the required properties. For example, the (refl)-rule ensures all terms (representing worlds) are reflexive in a fully expanded branch, and (tr)-rule ensures the H-relation is transitively closed. The rule (mon) accommodates the property that the truth sets form downsets. It is justified since we can show monotonicity not only for atomic formulae but any formulae of the logic. The rule (stab) ensures the relation R will be stable with respect to H in any generated model.

The rules have been systematically derived from the definition of the semantics of BISKT as given in Section 3. We first expressed the semantics in first-order logic and then converted the formulae to inference rules following the tableau synthesis method described in [ST11]. We do not describe the conversion here because it is completely analogous to the conversion for intuitionistic logic considered as a case study in [ST11, Section 9]. The subset of the rules in the calculus Tab_{BISKT} restricted to the operators and frame conditions relevant to intuitionistic logic in fact coincides with the tableau calculus derived there for intuitionistic logic (there are just insignificant variations in notation). We just note for the rule refinement step in the synthesis process atomic rule refinement as introduced in [TS13] is sufficient. Atomic rule refinement is a specialization of a general rule refinement technique described in [ST11] with the distinct advantage that it is automatic because separate proofs do no need to be given.

The values accompanying each rule in Figure 5 are the priority values we used in the implementation using the MetTeL tableau prover generator [Met,TSK12]. Lower values mean higher prority.

Theorem 5 *The tableau calculus Tab_{BISKT} is sound and (constructively) complete with respect to the semantics of* BISKT.

This follows by the results of the tableau synthesis framework and atomic rule refinement [ST11,TS13], because we can show the semantics of BISKT defined in Section 3 is well-defined in the sense of [ST11].

To obtain a terminating tableau calculus, adding the unrestricted blocking mechanism provides an easy way to obtain a terminating tableau calculus for any logic with the finite model property [ST08,ST11,ST13]. The main ingredient of the unrestricted blocking mechanism is the following rule.

Unrestricted blocking rule: $\quad (\mathsf{ub})\ \dfrac{}{s \approx t \mid s \not\approx t}$ pv. 9

Since this involves equality $\approx$, provision needs to be made for equality reasoning. This can be achieved, for example, via the inclusion of these paramodulation rules.

$$\frac{s \not\approx s}{\bot} \qquad \frac{s \approx t}{t \approx s} \qquad \frac{s \approx t,\ G[s]_\lambda}{G[\lambda/t]}$$

Here, G denotes any tableau formula. The notation $G[s]_\lambda$ means that s occurs as a subterm at position λ in G, and $G[\lambda/t]$ denotes the formula obtained by replacing s at position λ with t. In MetTeL equality reasoning is provided in the form of ordered rewriting which is more efficient [TSK12].

Unrestricted blocking systematically merges terms (sets them to be equal) in order to find small models if they exists. The intuition of the (ub)-rule is that merging two terms s and t, either leads to a model, or it does not, in which case s and t cannot be equal. In order that small models are found it is crucial that blocking is performed eagerly before the application of any term creating rules. The term creating rules are the rules expanding formulae with implicit existential quantification. As can be seen in our implementation using MetTeL the (ub)-rule has been given higher priority (a lower priority value 9) than all the rule creating new Skolem terms (priority value 10).

We denote the extension of the calculus Tab_{BISKT} by the unrestricted blocking mechanism, including some form of reasoning with equality $\approx$, by $Tab_{\text{BISKT}}(ub)$.

Using unrestricted blocking, because the logic has the finite model property, as we have shown in Theorem 4, the tableau calculus of Figure 5 extended with unrestricted blocking provides the basis for a decision procedure.

Theorem 6 *The tableau calculus $Tab_{\text{BISKT}}(ub)$ is sound, (constructively) complete and terminating for* BISKT.

Implementing a prover requires lots of specialist knowledge and there are various non-trivial obstacles to overcome, but using the MetTeL tableau prover generator requires just feeding in the rules of the calculus into the tool which then fully automatically generates an implemention in Java. The tableau calculus in Figure 5 is in exactly the form as supported by MetTeL and unrestricted blocking is available in MetTeL. We have therefore implemented the calculi using MetTeL. MetTeL turned out to be useful to experiment with several initial versions of the calculus. Moreover, in combination with the tableau synthesis method it was easy to experiment with different tableau provers for several preliminary versions of formalizations of bi-intuitionistic tense logics before settling on BISKT. MetTeL has also allowed us to experiment with different rule refinements, limiting the monotonicity rule to atomic formulae or not, and alternative forms of blocking. A natural variation of the (ub) rule is given by these two rules.

Predecessor blocking rules: $\quad \frac{H(s,t)}{s \approx t \mid s \not\approx t}$ pv. 9 $\quad \frac{R(s,t)}{s \approx t \mid s \not\approx t}$ pv. 9

Whereas the (ub)-rule blocks any two distinct terms, these rules restrict blocking to terms related via the H and R relations. The rules implement a form of predecessor blocking, which is known to give decision procedures for basic multi-modal logic $K_{(m)}$. We conjecture it also provides a decision procedure for

BISKT. Because Tab_{BISKT} is sound and complete and both rules are sound, basing blocking on these rules instead of (ub) preserves soundness and completeness.

The following are examples of formulae provable using the calculus as have been verified by a MetTeL generated prover.

Lemma 7 *The following hold in* BISKT.

1. (a) $\llbracket \bot \to \varphi \rrbracket = \llbracket \top \rrbracket$
   (b) $\llbracket \varphi \to \neg\bot \rrbracket = \llbracket \top \rrbracket$
   (c) $\llbracket \neg(\varphi \to \psi) \to (\neg\varphi \to \neg\psi) \rrbracket = \llbracket \top \rrbracket$
   (d) $\llbracket ⌐(\varphi \to \psi) \to (⌐\varphi \to ⌐\psi) \rrbracket = \llbracket \top \rrbracket$
   (e) $\llbracket \varphi \wedge \neg\varphi \rrbracket = \llbracket \bot \rrbracket$
   (f) $\llbracket \varphi \vee ⌐\varphi \rrbracket = \llbracket \top \rrbracket$
   (g) $\llbracket \neg\neg\neg\varphi \rrbracket = \llbracket \neg\varphi \rrbracket$
   (h) $\llbracket ⌐⌐⌐\varphi \rrbracket = \llbracket ⌐\varphi \rrbracket$
   (i) $\llbracket \neg(\varphi \vee \psi) \rrbracket = \llbracket \neg\varphi \wedge \neg\psi \rrbracket$
   (j) $\llbracket \neg(\varphi \wedge \psi) \rrbracket = \llbracket \neg\neg(\neg\varphi \vee \neg\psi) \rrbracket$
   (k) $\llbracket ⌐(\varphi \wedge \psi) \rrbracket = \llbracket ⌐\varphi \vee ⌐\psi \rrbracket$
   (l) $\llbracket ⌐(\varphi \vee \psi) \rrbracket = \llbracket ⌐⌐(⌐\varphi \wedge ⌐\psi) \rrbracket$
   (m) $\llbracket \varphi \rrbracket \subseteq \llbracket \neg\neg\varphi \rrbracket$
   (n) $\llbracket ⌐⌐\varphi \rrbracket \subseteq \llbracket \varphi \rrbracket$
   (o) $\llbracket \neg\varphi \vee \neg\psi \rrbracket \subseteq \llbracket \neg(\varphi \wedge \psi) \rrbracket$
   (p) $\llbracket ⌐(\varphi \vee \psi) \rrbracket \subseteq \llbracket ⌐\varphi \wedge ⌐\psi \rrbracket$
   (q) $\llbracket \varphi \wedge \psi \rrbracket \subseteq \llbracket \neg(\neg\varphi \vee \neg\psi) \rrbracket$
   (r) $\llbracket \varphi \vee \psi \rrbracket \subseteq \llbracket \neg(\neg\varphi \wedge \neg\psi) \rrbracket$
   (s) $\llbracket \neg\neg(\varphi \wedge \psi) \rrbracket = \llbracket \neg\neg\varphi \wedge \neg\neg\psi \rrbracket$
   (t) $\llbracket \neg\neg(\varphi \to \psi) \rrbracket = \llbracket \neg\neg\varphi \to \neg\neg\psi \rrbracket$

2. (a) $\llbracket \varphi \rrbracket \subseteq \llbracket ⌐\neg\varphi \rrbracket$
   (b) $\llbracket \neg\varphi \rrbracket \subseteq \llbracket ⌐\varphi \rrbracket$
   (c) $\llbracket \neg⌐\varphi \rrbracket \subseteq \llbracket \varphi \rrbracket$
   (d) $\llbracket ⌐⌐\varphi \rrbracket \subseteq \llbracket ⌐\neg\varphi \rrbracket$
   (e) $\llbracket \neg⌐\varphi \rrbracket \subseteq \llbracket \neg\neg\varphi \rrbracket$
   (f) $\llbracket \neg⌐\varphi \rrbracket \subseteq \llbracket ⌐⌐\varphi \rrbracket$
   (g) $\llbracket ⌐\varphi \rrbracket = \llbracket ⌐\varphi \vee \bot \rrbracket$

3. (a) $\llbracket \Box\neg\bot \rrbracket = \llbracket \top \rrbracket$, $\llbracket \Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi) \rrbracket = \llbracket \top \rrbracket$
   (b) $\llbracket \blacklozenge\bot \rrbracket = \llbracket \bot \rrbracket$, $\llbracket \neg\blacklozenge(\varphi \to \psi) \to (\neg\blacklozenge\varphi \to \neg\blacklozenge\psi) \rrbracket = \llbracket \top \rrbracket$
   (c) $\llbracket \Diamond\bot \rrbracket = \llbracket \bot \rrbracket$, $\llbracket \neg\Diamond(\varphi \to \psi) \to (\neg\Diamond\varphi \to \neg\Diamond\psi) \rrbracket = \llbracket \top \rrbracket$
   (d) $\llbracket \blacksquare\neg\bot \rrbracket = \llbracket \top \rrbracket$, $\llbracket \blacksquare(\varphi \to \psi) \to (\blacksquare\varphi \to \blacksquare\psi) \rrbracket = \llbracket \top \rrbracket$
   These properties mean all box and diamond operators are in fact 'modal'.

4. (a) $\llbracket \Diamond\varphi \rrbracket = \llbracket ⌐\Box\neg\varphi \rrbracket$
   (b) $\llbracket \Diamond\varphi \rrbracket \subseteq \llbracket ⌐\Box\neg⌐\neg\varphi \rrbracket$
   (c) $\llbracket \neg\Box⌐\varphi \rrbracket \subseteq \llbracket \Diamond\varphi \rrbracket$
   (d) $\llbracket \neg\Diamond⌐\varphi \rrbracket \subseteq \llbracket \Box\varphi \rrbracket$
   (e) $\llbracket \blacksquare\varphi \rrbracket = \llbracket \neg\blacklozenge⌐\varphi \rrbracket$

5. (a) $\llbracket ⌐\varphi \rrbracket = \llbracket \neg\bot \succ\!\!- \varphi \rrbracket$
   (b) $\neg(\bot \succ\!\!- \varphi)$
   (c) $\neg(\varphi \succ\!\!- \neg\bot)$
   (d) $\llbracket \Diamond\varphi \rrbracket \subseteq \llbracket \neg\bot \succ\!\!- \Box\neg(\neg\bot \succ\!\!- \neg\varphi) \rrbracket$

Since we can show the following hold in BISKT, the inclusions in Lemma 7 also hold as implications.

Lemma 8 *1.* $\llbracket \varphi \rrbracket \subseteq \llbracket \psi \rrbracket$ *implies* $\Vdash \varphi \to \psi$.
2. $\llbracket \varphi \rrbracket = \llbracket \top \rrbracket$ *implies* $\Vdash \varphi$.
3. $\llbracket \varphi \rrbracket = \llbracket \bot \rrbracket$ *implies* φ *is contradictory.*

5 Connections to other work

Propositional bi-intuitionistic logic was studied by Rauszer [Rau74] who referred to it as H-B logic, standing for Heyting-Brouwer logic. The co-intuitionistic fragment of H-B logic is one of the propositional paraconsistent logics investigated by Wansing [Wan08], but neither of these papers was concerned with bi-intuitionistic modal logic. The stable relations we used are already well known in intuitionistic modal logic [ZWC01, p219] and they provide a special case of the category-theoretic notion of a distributor [Bén00]. However, as far as we are aware, the left converse operation that we use has not featured in either of these contexts.

Reyes and Zolfaghari [RZ96] present modal operators with semantics in bi-Heyting algebras. Graphs are an important example, as in our work, but the modalities are quite different, arising from iterating alternations of $\neg$ and $\lrcorner$.

Goré et al [GPT10] studied a bi-intuitionistic modal logic, BiKt, with the same langauge as BISKT but with a semantics producing no relationship between the box and diamond operators. The four modal operators form two residuated pairs $(\Box, \blacklozenge)$ and $(\blacksquare, \Diamond)$ but without any necessary relationship between $\Box$ and $\Diamond$ or between $\blacksquare$ and $\blacklozenge$. In our case the same pairs are residuated (or adjoint) but we have a different semantics for the $(\blacksquare, \Diamond)$ pair, and consequently we do get relationships between $\Box$ and $\Diamond$ and between $\blacksquare$ and $\blacklozenge$.

We next describe the semantics for BiKt [GPT10, p26], using our notation and terminology to clarify the connection with BISKT.

Definition 9 *A **BiKt Frame**, $\langle U, H, R, S\rangle$, consists of a set U, a relation H on U which is reflexive and transitive, and two relations R and S on U that satisfy $R\,;H \subseteq H\,;R$ and $\breve{H}\,;S \subseteq S\,;\breve{H}$.*

The condition $R\,;H \subseteq H\,;R$ is strictly more general than $R = H\,;R\,;H$. For example, if $U = \{a, b\}$ and $H = \{(a, a), (a, b), (b, b)\}$ then taking R to be $\{(b, a), (b, b)\}$ we find that $R\,;H \subseteq H\,;R$ holds but not $R = H\,;R\,;H$. However, we shall see shortly that this additional generality is not essential. The semantics interprets formulae by H-sets and the modalities are defined as follows.

$$[\![\Diamond\alpha]\!] = [\![\alpha]\!] \oplus \breve{S} \quad [\![\Box\alpha]\!] = (H\,;R) \ominus [\![\alpha]\!] \quad [\![\blacklozenge\alpha]\!] = [\![\alpha]\!] \oplus R \quad [\![\blacksquare\alpha]\!] = (H\,;\breve{S}) \ominus [\![\alpha]\!]$$

Since $[\![\alpha]\!]$ is an H-set $[\![\alpha]\!] \oplus \breve{S} = [\![\alpha]\!] \oplus (H\,;\breve{S})$ and $[\![\alpha]\!] \oplus R = [\![\alpha]\!] \oplus (H\,;R)$. Thus the only accessibility relations needed in the semantics are $R' = H\,;R$ and $S' = H\,;\breve{S}$. The following lemma shows that the constraints on R and S are equivalent to R' and S' being stable with respect to H.

Lemma 9 *Let U be any set, let H be any pre-order on U, and let S be any binary relation on U. Then the following are equivalent.*

1. *$S = H\,;S\,;H$.*
2. *There is some relation $R \subseteq U \times U$ such that $S = H\,;R$ and $R\,;H \subseteq H\,;R$.*

Thus we can rephrase the semantics in [GPT10, p26] as

1. A frame $\langle U, H, R', S' \rangle$, consists of a set U, a pre-order H on U, and two stable relations R' and S' on U. A valuation assigns an H-set to each propositional variable, and the connectives are interpreted as for BISKT.
2. The semantics of the modal operators is:

$$[\![\Diamond\alpha]\!] = [\![\alpha]\!] \oplus S' \quad [\![\Box\alpha]\!] = R' \ominus [\![\alpha]\!] \quad [\![\blacklozenge\alpha]\!] = [\![\alpha]\!] \oplus R' \quad [\![\blacksquare\alpha]\!] = S' \ominus [\![\alpha]\!]$$

There is no relationship between R' and S' whereas the approach in Section 3 above is the special case in which $S' = \circlearrowleft R'$. The significance of our semantics for BISKT is that we are able to define all four modalities from a single accessibility relation. In developing, for example, a bi-intuitionistic modal logic of time it seems reasonable that the forward procession of time should not be completely unrelated to the backwards view looking into the past. It is the left converse operation on stable relations that allows us to state what appears to be the appropriate connection between the two directions.

6 Conclusion

Motivated by the theory of relations on graphs and applications to spatial reasoning, we have presented a bi-intuitionistic logic BISKT with tense operators. The need to interact well with a graph structure, and more generally with a pre-order, meant that our accessibility relations needed to be stable. The stability condition itself is not novel, but stable relations are not closed under the usual converse operation and our work is the first to show that the weaker left converse, which does respect stability, can be used to define semantics for modalities.

In contrast to other intuitionistic and bi-intuitionistic tense logics where all the modal operators are independent of each other, in BISKT the white diamond can be defined in terms of the white box, although not conversely. Dually the black box can be defined from the black diamond, also by using a pairing of intuitionistic negation and dual intuitionistic negation.

We showed BISKT is decidable and has the effective finite model property. The proof is via a reduction to the guarded fragment, which also gives an upper complexity bound of EXPTIME. Future work includes giving a tight complexity result for BISKT.

We have presented a tableau calculus for BISKT, which was shown to be sound, complete and terminating. This was obtained and refined using the tableau synthesis methodology and refinement techniques of [ST11,TS13]. We used a prover generated with the MetTeL tool [Met,TSK12] to analyse the logic and investigate the properties that hold in it, including relationship between different logical operators.

As the reduction to the guarded fragment is via an embedding into a multimodal logic, this provides another route to obtaining a tableau calculus and tableau prover for BISKT. Preliminary experiments on about 120 problems have however shown that the performance of this alternative prover is not better than the prover for the calculus presented in this paper. The reason is that the rules are more fine grained and less tailor-made, which allows fewer rule refinements. However deeper analysis and more experiments are needed.

Other future work includes extending BISKT with modalities based on the right converse operator.

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