Some notes on monads

Andrea Schalk
A.Schalk@cs.man.ac.uk
Department of Computer Science
University of Manchester

March 12, 2002

About these notes. These notes try to motivate the notion of 'monad' in category theory and go on to show some basic facts about them. They are directed at an audience which is familiar with the basic notions of category up to at least adjunctions. To avoid cluttering up the presentation unnecessarily we often use the same name for the 'same' morphism in different categories. Similarly we tend not to distinguish between objects which are naturally isomorphic.

1 Why monads?

When mathematicians first see the definition of an adjunction they will typically have seen lots of examples for these. I certainly remember a sense of relief when finally being shown that what seemed so similar every time I met something called a 'universal property' could be made into instances of one unifying definition.

Monads are somewhat different. For one, when being exposed to the idea in category theory, people often don't find that this is an idea that has been lurking in the background of their experience. The whole question of 'why on Earth should I look at something like that?' is often swept under the carpet.

Sometimes we encounter endofunctors on categories which we think are 'free' in some sense, which means that they have some universal property. But it need not be clear *a priori* just what that property might be. To solve this problem it seems natural to look for an appropriate adjunction, that is:

Given a functor $T\colon \mathbf{C} \longrightarrow \mathbf{C}$, we would like to find a category $\mathbf{D}$ as well as functors $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ and $G\colon \mathbf{D} \longrightarrow \mathbf{C}$ such that

$$T = GF \qquad \text{and} \qquad F \dashv G.$$

That does allow us to express T *via* a universal property, and thus we may make precise in which sense there is a 'free' construction here. The problem becomes one of decomposing T into two adjoint functors. Let us start with some examples.

Example 1.1 Let $M\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ be the functor that maps a set A to all the words that can be formed over the alphabet A, and whose action on morphisms is described as follows. For $f\colon A \longrightarrow B$ in $\mathbf{Set}$ let $Mf\colon MA \longrightarrow MB$ be given by

$$Mf(a_1 \cdots a_n) = f(a_1) \cdots f(a_n).$$

This is an example which you will find in most text books when it comes to describing monads, so you probably already know which free construction this is.

Example 1.2 There are two slight variations on Example 1.1. Instead of taking the set of all words over the alphabet A you could take the set of finite multisets over A, or the set of finite subsets of A. These are easily made into endofunctors on **Set**.

Example 1.3 Let **Pos** be the category of posets with order-preserving functions. For a poset P consider the set of finite subsets of P with one of the following pre-orders. For a and b subsets of P set

- $a \leq b$ if and only if $\downarrow a \subseteq \downarrow b$;
- $a \leq b$ if and only if $\uparrow a \supseteq \uparrow b$;
- $a \leq b$ if and only if $\downarrow a \subseteq \downarrow b$ and $\uparrow a \supseteq \uparrow b$.

Making these into posets in the canonical way gives rise to three endofunctors on **Pos** where the equivalence class of a is mapped to the equivalence class of $f[a]$, the image of a under f.

Example 1.4 Let **DCPO** be the category of directed complete partial orders (where we do not demand that such an object has a least element) with functions preserving directed suprema, that is, Scott-continuous functions. Let $(-)_\perp \colon \mathbf{DCPO} \longrightarrow \mathbf{DCPO}$ be the functor that maps a dcpo D to the dcpo with a (new) bottom element adjoint. The action of $(-)_\perp$ on morphisms is described as follows. For $f \colon D \longrightarrow E$ we set

$$f_\perp(a) = \begin{cases} f(a) & \text{if } a \text{ is in (the embedding of) } D \text{ (in } D_\perp) \\ \perp & \text{if } a = \perp \end{cases}$$

Example 1.5 (a) Let **Top** be the category of T_0-spaces with continuous functions. Let PX be the set of all closed subsets of a T_0-space X with the topology whose open sets are generated by all the

$$\Diamond O := \{A \in PX \mid A \cap O \neq \emptyset\},$$

where O is open in X. For $f \colon X \longrightarrow Y$ and $A \in PX$ set $Pf(A) = \overline{f[A]}$, the closure of the image of A under f.

(b) Let **STop** be the category of locally compact sober spaces with continuous functions. Let PX be the set of all compact subsets of X with the topology whose open sets are generated by the

$$\Box O := \{C \in PX \mid C \subseteq O\},$$

where O is open in X. For $f \colon X \longrightarrow Y$ and $C \in PX$ set $Pf(C) = f[C]$, the image of C under f.

Example 1.6 Let **Rel** be the category of sets and relations. Let MA be the set of finite multisets over A. For $f \colon A \longrightarrow\!\!\!\!\!\!\!+\;\; B$, $x \in MA$ and $y \in MB$ set $x\ Mf\ y$ if and only if there exists a subset c of $Mf \subseteq M(A \times B)$ whose first projection is x and whose second projection is y.

2 Monads

In order for an endofunctor T to be decomposable into such an $F \dashv G$, what do we have to ensure?

One of the possible definitions for such an adjunction, apart from functors F and G as above, demands the existence of natural transformations

$$\eta\colon \mathsf{id}_{\mathbf{C}} \longrightarrow GF \qquad \text{and} \qquad \varepsilon\colon FG \longrightarrow \mathsf{id}_{\mathbf{D}}$$

such that for all $A \in \mathbf{C}$ and all $D \in \mathbf{D}$

$$\begin{tikzcd} FA \arrow[r, "F\eta_A"] \arrow[dr, "\mathsf{id}_{FA}"'] & FGFA \arrow[d, "\varepsilon_{FA}"] \\ & FA \end{tikzcd} \qquad \begin{tikzcd} GD \arrow[r, "\eta_{GD}"] \arrow[dr, "\mathsf{id}_{GD}"'] & GFGD \arrow[d, "G\varepsilon_D"] \\ & GD \end{tikzcd}$$

The first of these natural transformations, η, is something we can straight-forwardly translate into our situation, since we assume that, once we have the adjunction, $T = GF$ will hold. We therefore demand the existence of

$$\eta\colon \mathsf{id}_{\mathbf{C}} \longrightarrow T.$$

Clearly it does not make sense to talk about the other natural transformation, ε, that comes with the adjunction, since we can't talk about FG if we only know $T = GF$. However, there is a derivable natural transformation

$$G\varepsilon F\colon GFGF = TT \longrightarrow GF = T,$$

and therefore we demand that there be a natural transformation

$$\mu\colon T^2 \longrightarrow T.$$

Hence we are really talking about a triple (another name for a monad) (T, η, μ) where η and μ are natural transformations as indicated. However this is not quite enough for the definition we are aiming for.

Exercise 1 Find η and μ for (at least some of) the examples from Section 1.

We wish to derive equations for η and μ from the equations for η and ε given above. By applying G to the first we obtain

$$\begin{tikzcd} TA = GFA \arrow[r, "T\eta_A = GF\eta_A"] \arrow[dr, "\mathsf{id}_{GFA}"'] & T^2A = GFGFA \arrow[d, "\mu_A = G\varepsilon_{FA}"] \\ & TA = GF \end{tikzcd}$$

If, on the other hand, we insert $D = FA$ in the second then we obtain

$$\begin{tikzcd} TA = GFA \arrow[r, "\eta_{TA} = \eta_{GFA}"] \arrow[dr, "\mathsf{id}_{GFA}"'] & T^2A = GFGFA \arrow[d, "\mu_A = G\varepsilon_{FA}"] \\ & TA = GFA \end{tikzcd}$$

In other words we demand that

$$\mathsf{id}_{TA} = \mu_A \circ T\eta_A \qquad \text{and} \qquad \mathsf{id}_{TA} = \mu_A \circ \eta_{TA}.$$

Clearly this is the closest to the original equations that we can ask for in our setting.

But that is still not enough for our purposes just yet. We finally observe that the two ways we have to go from T^3A to TA should coincide since

$$\begin{tikzcd} T^3A = GFGFGFA \arrow[r, "T\mu_A = GFG\varepsilon_{FA}"] \arrow[d, "\mu_{TA} = G\varepsilon_{FGFA}"'] & T^2A = GFGFA \arrow[d, "\mu_A = G\varepsilon_{FA}"] \\ T^2A = GFGFA \arrow[r, "\mu_A = G\varepsilon_{FA}"'] & TA = GFA \end{tikzcd}$$

which follows by naturality of ε in the original setting.

Definition 1 *A* **monad** *on a category* $\mathbf{C}$ *consists of an endofunctor* T *on* $\mathbf{C}$ *together with natural transformations*

$$\eta\colon \mathsf{id}_{\mathbf{C}} \longrightarrow T \qquad \textit{and} \qquad \mu\colon T^2 \longrightarrow T$$

such that

$$\mu_A \circ T\eta_A = \mathsf{id}_{TA} = \mu_A \circ \eta_{TA} \qquad \textit{and} \qquad \mu_A \circ T\mu_A = \mu_A \circ \mu_{TA}.$$

We often speak of a monad (T, η, μ).

Exercise 2 Show that the equations hold for your choice of η and μ in the examples.

Example 2.1 A monad on a poset P (viewed as a category) is an order-preserving function $f\colon P \longrightarrow P$ such that for all $p \in P$

- $p \leq f(p)$, that is f is order-increasing and
- $f(f(p)) \leq f(p)$ which, together with the first inequality, means that $f \circ f = f$.

In other words, f is a closure operator.

Example 2.2 Let $\mathbf{End}_{\mathbf{C}}$ be the category of all endo-functors on some category $\mathbf{C}$, with natural transformations between those as the morphisms. Then $\mathbf{End}_{\mathbf{C}}$ is a strict monoidal closed category with composition as the monoidal operator (or tensor product). A monad on $\mathbf{C}$ is nothing but a monoid with respect to this tensor.

Exercise 3 Convince yourself that the claim made in Example 2.2 is true.

Proposition 2.3 *If* $(F, G, \eta, \varepsilon)$ *describes an adjunction where* $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ *then the triple* $(GF, \eta, G\varepsilon F)$ *defines a monad on* $\mathbf{C}$.

Proof. The proof was given above in the motivation for the concept of a monad. □

It turns out that this definition is sufficient to guarantee that the endofunctor under consideration can be decomposed as desired. But before we get there we will first study an alternative description of the same thing.

3 Kleisli triples

Our definition of monad was clearly driven by a particular way of defining an adjunction. If we start with another such, namely the one defining an adjunction in terms of a universal property, then we obtain a different notion.

Let us repeat that definition here. It demands the existence of a functor $G\colon \mathbf{D} \longrightarrow \mathbf{C}$, whereas F merely has to take objects of $\mathbf{C}$ to objects of $\mathbf{D}$. Further there has to exist, for every object A of $\mathbf{C}$, an arrow $\eta_A\colon A \longrightarrow GFA$ such that for all

$$f\colon A \longrightarrow GD,$$

where D is an arbitrary object of $\mathbf{D}$, there exists a unique

$$f^+\colon FA \longrightarrow D$$

such that $f = Gf^+ \circ \eta_A$. We can view $(-)^+$ as mapping the homset $\mathbf{C}(A, GD)$ to the homset $\mathbf{D}(FA, D)$.

We can describe this in a convenient short form.

$$\begin{tikzcd}
\mathbf{C} & & \mathbf{D} \\
A \arrow[r, "\eta_A"] \arrow[dr, "\forall f"'] & GFA \arrow[d, dashed, "Gf^+"] & FA \arrow[d, dashed, "\exists! f^+"] \\
& GD & D
\end{tikzcd}$$

As before we expect to find F and G such that $T = GF$. Since F now does not have to be a functor we start out with T mapping objects of $\mathbf{C}$ to objects of $\mathbf{C}$.

We further demand, for every $A \in \mathbf{C}$, the existence of an arrow $\eta_A\colon A \longrightarrow TA$. Finally we have to translate the $(-)^+$ operator into our setting. Observing that if

$$f\colon A \longrightarrow GFB$$

then

$$Gf^+\colon GFA \longrightarrow GFB,$$

we demand, for all

$$f\colon A \longrightarrow TB,$$

the existence of a

$$f^*\colon TA \longrightarrow TB.$$

In other words, $(-)^*$ maps the homset $\mathbf{C}(A, TB)$ to the homset $\mathbf{C}(TA, TB)$. It remains to describe the equations connecting the ingredients $(T, \eta, (-)^*)$.

Mimicking the universal property we demand first of all that

$$f^* \circ \eta_A = f.$$

If we consider $\eta_A\colon A \longrightarrow GFA$ then the universal property tells us that

$$G\mathsf{id}_{FA} \circ \eta_A = \eta_A = G\eta_A^+ \circ \eta_A,$$

and by uniqueness it must be the case that

$$\eta_A{}^+ = \mathsf{id}_{FA}.$$

Thus motivated we demand for our triple that

$$\eta_A{}^* = \mathsf{id}_{TA}.$$

Finally we see what we get if we apply the universal property to the following situation. Let

$$f\colon A \longrightarrow GFB \qquad \text{and} \qquad g\colon B \longrightarrow GFC.$$

Then

$$\begin{tikzcd}
A \arrow[rr, "\eta_A"] \arrow[dr, "f"'] & & GFA \arrow[dl, "Gf^+"'] \arrow[dd, "G(g^+ \circ f^+)"] \\
& GFB \arrow[dr, "Gg^+"'] & \\
& & GFC
\end{tikzcd}$$

By uniqueness of the fill-in we deduce that

$$(Gg^+ \circ f)^+ = g^+ \circ f^+$$

and so

$$G(Gg^+ \circ f)^+ = Gg^+ \circ Gf^+.$$

Hence we demand for our $(-)^*$ operator that

$$(g^* \circ f)^* = g^* \circ f^*.$$

Definition 2 *A* **Kleisli triple** *on a category* $\mathbf{C}$ *consists of*

- *an operator T mapping objects of $\mathbf{C}$ to objects of $\mathbf{C}$,*

- *for each* $A \in \mathbf{C}$, *an arrow* η_A *and*
- *for all* $A, B \in \mathbf{C}$ *an operator* $(-)^* \colon \mathbf{C}(A, TB) \longrightarrow \mathbf{C}(TA, TB)$

subject to the following equations:

$$f^* \circ \eta_A = f \qquad \eta_A{}^* = \mathsf{id}_{TA} \qquad (g^* \circ f)^* = g^* \circ f^*.$$

We often talk about the Kleisli triple $(T, \eta, (-)^*)$.

Proposition 3.1 *If* $(F, G, \eta, (-)^+)$ *describes an adjunction with* F *mapping objects of* $\mathbf{C}$ *to objects of* $\mathbf{D}$ *then* $(GF, \eta, G(-)^+)$ *defines a Kleisli triple on* $\mathbf{C}$.

Proof. Again the proof has been given as part of the motivation of the notion of a Kleisli triple. □

There is an obvious question regarding the relation between the two ideas we have developed.

Proposition 3.2 *The notions of monad and Kleisli triple are equivalent.*

Proof. Let $(T, \eta, (-)^*)$ be a Kleisli triple on a category $\mathbf{C}$. We first turn T into an endofunctor on $\mathbf{C}$. For $f \colon A \longrightarrow B$ in $\mathbf{C}$ let $Tf \colon TA \longrightarrow TB$ be given by

$$Tf = (\eta_B \circ f)^*.$$

This is functorial since for all objects $A \in \mathbf{C}$ we have that

$$T\mathsf{id}_A = (\eta_A \circ \mathsf{id}_A)^* = \eta_A{}^* = \mathsf{id}_{TA}$$

and for $f \colon A \longrightarrow B$ and $g \colon B \longrightarrow C$ that

$$\begin{aligned} Tg \circ Tf &= (\eta_C \circ g)^* \circ (\eta_B \circ f)^* \\ &= ((\eta_C \circ g)^* \circ \eta_B \circ f)^* \\ &= (\eta_C \circ g \circ f)^* \\ &= T(g \circ f) \end{aligned}$$

We next have to prove that η is a natural transformation $\mathsf{id}_{\mathbf{C}} \longrightarrow T$, that is that for all $f \colon A \longrightarrow B$ in $\mathbf{C}$ we have

$$\eta_B \circ f = Tf \circ \eta_A.$$

We calculate

$$Tf \circ \eta_A = (\eta_B \circ f)^* \circ \eta_A = \eta_B \circ f.$$

We define the final component of a monad, $\mu \colon T^2 \longrightarrow T$ as

$$\mu_A = \mathsf{id}^*_{TA} \colon T^2A \longrightarrow TA.$$

This is a natural transformation since for all $f\colon A \longrightarrow B$ it is the case that

$$\begin{array}{rcl}\mu_B \circ T^2 f & = & \mathsf{id}^*_{TB} \circ (\eta_{TB} \circ Tf)^* \\ & = & (\mathsf{id}^*_{TB} \circ \eta_{TB} \circ Tf)^* \\ & = & (\mathsf{id}_{TB} \circ Tf)^* \\ & = & (Tf)^* \\ & = & (\eta_B \circ f)^{**} \\ & = & ((\eta_B \circ f)^* \circ \mathsf{id}_{TA})^* \\ & = & (\eta_B \circ f)^* \circ \mathsf{id}^*_{TA} \\ & = & Tf \circ \mu_A.\end{array}$$

It remains to show the three equations. We calculate

$$\mu_A \circ \eta_{TA} = \mathsf{id}^*_{TA} \circ \eta_{TA} = \mathsf{id}_{TA}$$

and

$$\mu_A \circ T\eta_A = \mathsf{id}^*_{TA} \circ (\eta_{TA} \circ \eta_A)^* = (\mathsf{id}^*_{TA} \circ \eta_{TA} \circ \eta_A)^* = (\mathsf{id}_{TA} \circ \eta_A)^* = {\eta_A}^* = \mathsf{id}_{TA}.$$

Finally we have

$$\begin{array}{rcl}\mu_A \circ T\mu_A & = & \mathsf{id}^*_{TA} \circ (\eta_{TA} \circ \mathsf{id}^*_{TA})^* \\ & = & (\mathsf{id}^*_{TA} \circ \eta_{TA} \circ \mathsf{id}^*_{TA})^* \\ & = & (\mathsf{id}_{TA} \circ \mathsf{id}^*_{TA})^* \\ & = & \mathsf{id}^{**}_{TA} \\ & = & (\mathsf{id}^*_{TA} \circ \mathsf{id}_{T^2A})^* \\ & = & \mathsf{id}^*_{TA} \circ \mathsf{id}^*_{T^2A} \\ & = & \mu_A \circ \mu_{TA}\end{array}$$

Hence (T, η, μ) as just defined is a monad.

Now let (T, η, μ) be a monad on a category $\mathbf{C}$. Clearly, T and η satisfy the requirements made for the corresponding components of a Kleisli triple. That leaves the $(-)^*$ operator. For $f\colon A \longrightarrow TB$ in $\mathbf{C}$ we set f^* to be

$$TA \xrightarrow{Tf} T^2 B \xrightarrow{\mu_B} TB.$$

It remains to show the three equations. Let $f\colon A \longrightarrow TB$. Then

$$f^* \circ \eta_A = \mu_B \circ Tf \circ \eta_A = \mu_B \circ \eta_{TB} \circ f = \mathsf{id}_{TB} \circ f = f.$$

Further we have that

$$\eta^*_A = \mu_A \circ T\eta_A = \mathsf{id}_{TA}$$

as required. Now let f be as before and let $g\colon B \longrightarrow TC$. Then

$$\begin{array}{rcl}g^* \circ f^* & = & \mu_C \circ Tg \circ \mu_B \circ Tf \\ & = & \mu_C \circ \mu_{TC} \circ T^2 g \circ Tf \\ & = & \mu_C \circ T\mu_C \circ T^2 g \circ Tf \\ & = & \mu_C \circ T(\mu_C \circ Tg \circ f) \\ & = & (g^* \circ f)^*.\end{array}$$

Hence $(T, \eta, (-)^*)$ as just defined is a Kleisli triple.

If we start with a Kleisli triple $(T, \eta, (-)^*)$, obtain the monad (T, η, μ) as just described and then, for $f\colon A \longrightarrow TB$, we calculate

$$\mu_B \circ Tf = \mathsf{id}_{TA}^* \circ (\eta_B \circ f)^* = (\mathsf{id}_{TA}^* \circ \eta_B \circ f)^* = (\mathsf{id}_{TA} \circ f)^* = f^*,$$

so we can recover the original Kleisli triple from that monad.

If on the other hand we start with a monad (T, η, μ) and then obtain the Kleisli triple according to the above instructions then we find that

$$\mathsf{id}_{TA}^* = \mu_{TA} \circ T\mathsf{id}_{TA} = \mu_{TA} \circ \mathsf{id}_{T^2A} = \mu_{TA}$$

and therefore we can also recover the monad from the Kleisli triple. $\square$

Exercise 4 Find the Kleisli-triple presentation for your example monads.

This allows us two approaches to the problem we are trying to solve.

The problem. Given a monad (T, η, μ) find an adjunction $(F, G, \eta, \varepsilon)$ such that $T = GF$ and such that $\mu = G\varepsilon F$.

(Given a Kleisli triple $(T, \eta, (-)^*)$ find an adjunction $(F, G, \eta, (-)^+)$ such that, on objects, $T = GF$ and such that $(-)^* = G((-)^+)$.)

4 Kleisli's solution

We present the first solution to our problem which is due to Kleisli.

It relies on the observation that if we have an adjunction $F \dashv G$ (where $\mathbf{D}$ is the category through which T factors into F and G) which solves our problem then it must be the case that

$$\mathbf{D}(FA, FB) \cong \mathbf{C}(A, GFB) = \mathbf{C}(A, TB).$$

Definition 3 *The* **Kleisli category** $\mathbf{C}_T$ *of a Kleisli triple* $(T, \eta, (-)^*)$ *on a category* $\mathbf{C}$ *has*

objects: *the objects of* $\mathbf{C}$*;*

morphisms: *a morphism* $A \longrightarrow B$ *is given by a morphism* $A \longrightarrow TB$*;*

identities: *the identity on an object* A *is given by* η_A*;*

composition: *the composite of morphisms* $f\colon A \longrightarrow B$ *and* $g\colon B \longrightarrow C$ *in* $\mathbf{C}_T$ *is given by*

$$A \xrightarrow{f} TB \xrightarrow{g^*} TC.$$

To distinguish it from the composite in $\mathbf{C}$ *we write it a*

$$g \bullet f.$$

We should make sure that we have indeed defined a category. For $f\colon A \longrightarrow B$ in $\mathbf{C}_T$ pre-composing with the identity on A gives

$$f^* \circ \eta_A = f.$$

Post-composing with the identity on B results in

$$\eta_B^* \circ f = \mathsf{id}_B \circ f = f.$$

Let f be as before and let $g\colon B \longrightarrow C$ and $h\colon C \longrightarrow D$ in $\mathbf{C}_T$. Then the two composites work out as

$$h \bullet (g \bullet f) = h^* \circ (g^* \circ f) = (h^* \circ g^*) \circ f = (h^* \circ g)^* \circ f = (h \bullet g) \bullet f.$$

We define F_T to map an object C of $\mathbf{C}$ to the same object in $\mathbf{C}_T$. We further define a functor $G_T\colon \mathbf{C}_T \longrightarrow \mathbf{C}$ by mapping $C \in \mathbf{C}_T$ to TC and by mapping a morphism $f\colon A \longrightarrow B$ in $\mathbf{C}_T$ to $f^*\colon TA \longrightarrow TB$ in $\mathbf{C}$. We check that this is functorial by noting that the identity on A in $\mathbf{C}_T$, η_A, is mapped to ${\eta_A}^* = \mathsf{id}_{TA}$. If $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ in $\mathbf{C}_T$ then

$$G_T g \circ G_T f = g^* \circ f^* = (g^* \circ f)^* = G_T(g \bullet f)$$

and thus G_T preserves composition and is indeed a functor.

It is trivial that now $T = G_T F_T$ on objects. We wish to establish a universal property for this construction. So let $f\colon A \longrightarrow G_T B$ in $\mathbf{C}$, where $A \in \mathbf{C}$ and $B \in \mathbf{C}_T$. We need to establish a unique $f^+\colon F_T A \longrightarrow B$ in $\mathbf{C}_T$ such that

$$G_T f^+ \circ \eta_A = f.$$

But our construction has been precisely so that $f\colon A \longrightarrow G_T B = TB$ is a morphism of the desired type in $\mathbf{D}$. So $G_T f^+ = f^*$ and we calculate

$$G_T f^+ \circ \eta_A = f^* \circ \eta_A = f$$

as desired. It remains to establish uniqueness. But if $g\colon F_T A \longrightarrow B$ in $\mathbf{C}_T$, given by a $g\colon A \longrightarrow TB$ in $\mathbf{C}$ then $G_T g = g^*$ and therefore

$$G_T g \circ \eta_A = g^* \circ \eta_A = g,$$

so if g could replace f^+ then $g = f$ as desired. Hence we have found an adjunction with the desired properties.

We note in passing that the functor F_T (which maps a morphism $f\colon A \longrightarrow B$ in $\mathbf{C}$ to $\eta_B \circ f\colon A \longrightarrow TB$) is not very interesting, whereas G_T does all the work.

Exercise 5 Calculate the Kleisli categories for your example monads. Can you find a different presentation for any of them?

5 Eilenberg and Moore's solution

In practice, Kleisli's solution often is not the one which tells us a lot about the construction under consideration. Instead, another solution to the problem due to Eilenberg and Moore is often favoured.

Definition 4 *The* **category of (Eilenberg-Moore) algebras** $\mathbf{C}^T$ *for a monad* (T, η, μ) *on a category* $\mathbf{C}$ *has*

objects: arrows $\xi \colon TA \longrightarrow A$ *in* $\mathbf{C}$*, called* T**-algebras** *such that*

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "\mathrm{id}_A"'] & TA \arrow[d, "\xi"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} T^2A \arrow[r, "\mu_A"] \arrow[d, "T\xi"'] & TA \arrow[d, "\xi"] \\ TA \arrow[r, "\xi"'] & A \end{tikzcd}$$

morphisms: a morphism from $\xi \colon TA \longrightarrow A$ *to* $\zeta \colon TB \longrightarrow B$ *(called a* **morphism of** T**-algebras***) is given by a morphism* $f \colon A \longrightarrow B$ *in* $\mathbf{C}$ *such that the following diagram commutes*

$$\begin{tikzcd} TA \arrow[r, "\xi"] \arrow[d, "Tf"'] & A \arrow[d, "f"] \\ TB \arrow[r, "\zeta"'] & B \end{tikzcd}$$

Identities and composition are inherited from $\mathbf{C}$.

We briefly return to **Example 2.1**. The category of algebras for a closure operator on a poset P is the set of all fixed points of the operator.

For the general situation we note that for every object A in $\mathbf{C}$ we obtain a T-algebra, namely

$$\mu_A \colon T^2A \longrightarrow TA.$$

These T-algebras are called **free T-algebras**. Further for every $f \colon A \longrightarrow B$ in $\mathbf{C}$, Tf is a morphism of T-algebras from $\mu_A \colon T^2A \longrightarrow TA$ to $\mu_B \colon T^2B \longrightarrow TB$ since

$$\begin{tikzcd} T^2A \arrow[r, "\mu_A"] \arrow[d, "T^2f"'] & TA \arrow[d, "Tf"] \\ T^2B \arrow[r, "\mu_B"'] & TB \end{tikzcd}$$

commutes by naturality of μ. We further note that for a T-algebra $\xi\colon TA \longrightarrow A$, ξ is a morphism from the T-algebra $\mu_A\colon T^2A \longrightarrow TA$ to $\xi\colon TA \longrightarrow A$ since

$$\begin{tikzcd} T^2A \arrow[r, "\mu_A"] \arrow[d, "T\xi"'] & TA \arrow[d, "\xi"] \\ TA \arrow[r, "\xi"'] & A \end{tikzcd}$$

commutes by the definition of a T-algebra.

Again we wish to establish an adjunction decomposing T *via* this category. We define a functor $F^T\colon \mathbf{C} \longrightarrow \mathbf{C}^T$ by mapping $A \in \mathbf{C}$ to the free T-algebra $\mu_A\colon T^2A \longrightarrow TA$, and a morphism $f\colon A \longrightarrow B$ in $\mathbf{C}$ to Tf which we have just observed to be a morphism of T-algebras. F^T is a functor since T is.

There is an obvious forgetful functor $G^T\colon \mathbf{C}^T \longrightarrow \mathbf{C}$ which 'forgets' about the algebra structure, that is it maps $\xi\colon TA \longrightarrow A$ to A and maps a morphism of T-algebras f to itself (viewed as a morphism of $\mathbf{C}$). Clearly G^T is a functor.

Obviously $T = G^TF^T$ and therefore η is a natural transformation $\mathsf{id}_{\mathbf{C}} \longrightarrow G^TF^T$. This time it is the case that F^T does all the work and G^T is a true forgetful functor, in contrast with Kleisli's solution. This fits our original motivation much better and usually is the preferred solution.

It remains to define a natural transformation $\varepsilon\colon F^TG^T \longrightarrow \mathsf{id}_{\mathbf{C}^T}$ and to show that the two equations for adjunctions hold.

For an object $\xi\colon TA \longrightarrow A$ of $\mathbf{C}^T$ we set $\varepsilon_\xi = \xi$ which we know to be a morphism of T-algebras from $F^TG^T\xi = \mu_A$ to ξ. Naturality is easy to establish since it coincides precisely with the definition of a morphism of T-algebras.

We calculate, for $A \in \mathbf{C}$,

$$G^T\varepsilon_{F^TA} = G^T\varepsilon_{\mu_A} = G^T\mu_A = \mu_A$$

and deduce that $G^T\varepsilon F^T = \mu$ as required for a solution to our problem.

Example 1.1 ctd. Here η_A maps an element $a \in A$ to the word consisting of just the letter a, and μ_A maps a word over the alphabet MA to the word over the alphabet A which we obtain by just stringing all the letters together, forgetting that they previously formed words over A. An algebra for this monoid is given by a function $\xi\colon MA \longrightarrow A$ such that the word consisting of just the letter a is mapped to a, and such that for a word $(a_1^1 \cdots a_{n_1}^1)\cdots(a_1^m \cdots a_{n_m}^m)$ over the alphabet MA we have that

$$\begin{aligned} \xi(a_1^1 \cdots a_{n_1}^1 \cdots a_1^m \cdots a_{n_m}^m) &= \xi(\mu_A(a_1^1 \cdots a_{n_1}^1)\cdots(a_1^m \cdots a_{n_m}^m)) \\ &= \xi(T\xi(a_1^1 \cdots a_{n_1}^1)\cdots(a_1^m \cdots a_{n_m}^m)) \\ &= \xi(\xi(a_1^1 \cdots a_{n_1}^1)\cdots\xi(a_1^m \cdots a_{n_m}^m)). \end{aligned}$$

This does not look all that informative, and we have to apply some ingenuity to get a better grip on what's going on here.

We note that all objects of the form MA carry some algebraic operations. There is concatenation

$$\cdot\colon MA \times MA \longrightarrow MA$$

which has the empty word ε as the unit (which we can view as a morphism from the one-point set $\mathbf{1}$ to MA), and which is associative. In other words, every MA is a monoid and the following diagrams commute

$$
\begin{tikzcd}
MA \arrow[r, "\langle \mathsf{id}_{MA}, \varepsilon\rangle"] \arrow[d, "\langle \varepsilon, \mathsf{id}_{MA}\rangle"'] \arrow[dr, "\mathsf{id}_{MA}"] & MA \times MA \arrow[d, "\cdot"] \\
MA \times MA \arrow[r, "\cdot"'] & MA
\end{tikzcd}
\qquad
\begin{tikzcd}
MA \times MA \times MA \arrow[r, "\cdot \times \mathsf{id}_{MA}"] \arrow[d, "\mathsf{id}_{MA} \times \cdot"'] & MA \times MA \arrow[d, "\cdot"] \\
MA \times MA \arrow[r, "\cdot"] & MA
\end{tikzcd}
$$

We also note that if we have an algebra $\xi \colon MA \longrightarrow A$ then we can define a generalized 'concatenation' operation on it via

$$
\begin{tikzcd}
A \times A \arrow[r, "\eta_A \times \eta_A"] & MA \times MA \arrow[d, "\cdot"] \\
A & MA \arrow[l, "\xi"']
\end{tikzcd}
$$

Exercise 6 Show that this derived operation is associative and has $\xi \circ \varepsilon$ as its unit.

So every M-algebra is a monoid in **Set**. How about morphisms?

We start with morphisms of the form Mf and μ_A which we have shown to be morphisms of M-algebras above. For a pair of words over A, say $(a_1 \cdots a_n, a'_1 \cdots a'_m)$ we have that

$$
\begin{tikzcd}
(a_1 \cdots a_n, a'_1 \cdots a'_m) \arrow[r, mapsto, "\cdot"] \arrow[d, mapsto, "Mf \times Mf"'] & a_1 \cdots a_n a'_1 \cdots a'_m \arrow[d, mapsto, "Mf"] \\
(f(a_1) \cdots f(a_n), f(a'_1) \cdots f(a'_m)) \arrow[r, mapsto, "\cdot"] & f(a_1) \cdots f(a_n) f(a'_1) \cdots f(a'_m)
\end{tikzcd}
$$

Similarly we can show that for every set A, μ_A is a morphism of comonoids.

If $f \colon A \longrightarrow B$ is a morphism of M-algebras from $\xi \colon MA \longrightarrow A$ to $\zeta \colon MB \longrightarrow B$ then it is also a morphism of monoids (for the derived monoid structure) since

$$
\begin{tikzcd}
A \times A \arrow[r, "\eta_A \times \eta_A"] \arrow[d, "f \times f"'] & MA \times MA \arrow[r, "\star"] \arrow[d, "Mf \times Mf"'] & MA \arrow[r, "\xi"] \arrow[d, "Mf"] & A \arrow[d, "f"] \\
B \times B \arrow[r, "\eta_B \times \eta_B"'] & MB \times MB \arrow[r, "\star"'] & MB \arrow[r, "\zeta"'] & B
\end{tikzcd}
$$

There is, in fact a general principle at work here which we will briefly outline below.

We claim that this algebraic operation is precisely what *characterizes* M-algebras. We have just shown that every M-algebra is a monoid in **Set** (and that every morphism of M-algebras is a monoid morphism), and we now proceed to show that every monoid in **Set** is an M-algebra (and that every monoid morphism is a morphism of M-algebras).

Definition 5 *Let* **C** *be a category with products. A* **monoid** *in* **C** *is an object A together with arrows*

$$\star\colon A \times A \longrightarrow A \text{ ('multiplication')} \qquad\qquad e\colon \mathbf{1} \longrightarrow A \text{ (unit)}$$

such that the following diagrams commute:

$$\begin{tikzcd} A \arrow[r, "\langle \mathsf{id}_A, e\rangle"] \arrow[d, "\langle e, \mathsf{id}_A\rangle"'] \arrow[dr, "\mathsf{id}_A"] & A \times A \arrow[d, "\star"] \\ A \times A \arrow[r, "\star"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \times A \times A \arrow[r, "\star \times \mathsf{id}_A"] \arrow[d, "\mathsf{id}_A \times \star"'] & A \times A \arrow[d, "\star"] \\ A \times A \arrow[r, "\star"] & A \end{tikzcd}$$

It should be clear that this definition allows us to define algebraic structures over any category.[1]

Let S be a monoid in **Set** which has multiplication

$$\star\colon S \times S \longrightarrow S$$

and unit $e\colon \mathbf{1} \longrightarrow S$. We define an M-algebra $\xi\colon MS \longrightarrow S$ from this by setting

$$\xi(s_1 \cdots s_n) = s_1 \star \cdots \star s_n.$$

Clearly this does map the word consisting of a single letter to that letter. Further we have that

$$\begin{tikzcd} (s_1^1 \cdots s_{n_1}) \cdots (s_1^m \cdots s_{n_m}^m) \arrow[r, mapsto, "M\xi"] \arrow[d, mapsto, "\mu_S"'] & (s_1^1 \star \cdots \star s_{n_1}) \cdots (s_1^m \star \cdots \star s_{n_m}^m) \arrow[d, mapsto, "\xi"] \\ s_1^1 \cdots s_{n_1} \cdots s_1^m \cdots s_{n_m}^m \arrow[r, "\xi"'] & s_1^1 \star \cdots \star s_{n_1} \star \cdots \star s_1^m \cdots s_{n_m}^m \end{tikzcd}$$

Hence every monoid comes with an M-algebra structure. it is trivial to show that every morphism of monoids is a morphism of M-algebras.

Proposition 5.1 *The category of M-algebras is equivalent to the category of monoids in* **Set**.

It is worth examining how we achieved this. We started by looking for some algebraic structure on the *free algebras*. For that it was critical that the category of algebras for the monad carries a product structure which is inherited from that on the original category. There is a general result which states that if **C** has finite products then so does any C^T.[2] We then went on to show that we could define a derived such structure for all algebras, and that it was preserved by morphisms. Finally we showed that the category of these algebras in **C** can be embedded in the category of algebras for the monad. This is a general approach, and it

[1] This does extend to algebraic structures with respect to a tensor product.
[2] This general result can be extended to cover monoidal structures provided that the monad is monoidal.

can be shown that most of it has to work due to general principles. Also note that we can now justify the notion of a 'free algebra'—in this example, the free M-algebras are precisely those monoids which are freely generated over some set.

This should serve as an example that by identifying the category of algebras for a monad we get a deep insight into what the corresponding functor is doing, in particular we find out which free construction it is.

Exercise 7 Can you identify the category of algebras for any of the other examples from Section 1?

6 Comparing the solutions

So far we were only interested in finding *some* solution to our problem, but did not give much thought to how many there might be, or how they might relate to each other.

There is an embedding

$$J\colon \mathbf{C}_T \longrightarrow \mathbf{C}^T.$$

For $A \in \mathbf{C}_T$ let JA be μ_A, the free T-algebra over A, and for a morphism $f\colon A \longrightarrow B$ in $\mathbf{C}_T$ let $Jf\colon JA = TA \longrightarrow JB = TB$ be $f^* = \mu_B \circ Tf$. This is a morphism of T-algebras since both, μ_B and Tf are. The assignment is functorial (and the proof is 'the same' as the one which shows that we can define a monad from a Kleisli triple). It is faithful since for $f, g\colon A \longrightarrow B$ in $\mathbf{C}_T$, given by $f, g\colon A \longrightarrow TB$ in $\mathbf{C}$ it is the case that

$$\mu_B \circ Tf = Jf = Jg = \mu_B \circ Tg$$

implies that

$$\begin{aligned} f &= \mu_B \circ \eta_{TB} \circ f \\ &= \mu_B \circ Tf \circ \eta_A \\ &= \mu_B \circ Tg \circ \eta_A \\ &= \mu_B \circ \eta_{TB} \circ g \\ &= g \end{aligned}$$

We can, in fact, identify the subcategory of $\mathbf{C}^T$ we obtain this way.

Definition 6 *The* **category of free T-algebras** *is the full subcategory of* $\mathbf{C}^T$ *whose objects are all* $\mu_A\colon TA \longrightarrow A$*, where* $A \in \mathbf{C}$.

Proposition 6.1 *The image of* $\mathbf{C}_T$ *under* J *is the category of free* T*-algebras.*

Proof. It remains to show that J is full. Let $f\colon \mu_A \longrightarrow \mu_B$ in the category of free T-algebras. Then

$$\begin{aligned} f &= f \circ \mu_A \circ T\eta_A && (\mu_A \circ T\eta_A = \mathsf{id}_{TA}) \\ &= \mu_B \circ Tf \circ T\eta_A && (f \text{ morphism of } T\text{-algebras}) \\ &= \mu_B \circ T(f \circ \eta_A) \\ &= J(f \circ \eta_A). \end{aligned}$$

Therefore f is in the image of J. $\square$

Hence the category of free T-algebras is equivalent to the Kleisli category for the monad.

We can further define a category whose objects are the solutions to our problems. Let $(F, G, \eta, \varepsilon)$ and $(F', G', \eta, \varepsilon')$ be such solutions with $F\colon \mathbf{C} \longrightarrow \mathbf{D}$ and $F'\colon \mathbf{C} \longrightarrow \mathbf{D}'$. A morphism from the first to the second is given by a functor $L\colon \mathbf{D} \longrightarrow \mathbf{D}'$ such that the following diagram commutes

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "F"] \arrow[dr, "F'"'] & \mathbf{D} \arrow[r, "G"] \arrow[d, "L"] & \mathbf{C} \\
& \mathbf{D}' \arrow[ur, "G'"'] &
\end{tikzcd}$$

and such that for all $D \in \mathbf{D}$

$$F'G'LD = F'GD = LFGD \xrightarrow{L\varepsilon_D} LD \qquad = \qquad F'G'LD \xrightarrow{\varepsilon'_{LD}} LD.$$

We can show that we have found particular solutions.

Proposition 6.2 *The Kleisli adjunction is the initial object in the category of adjunctions, and the Eilenberg-Moore adjunction is the final one.*

Exercise 8 Prove Proposition 6.2.

Exercise 9 Show that J is a morphism in the category of adjunctions (the unique morphism from the initial to the terminal object).

Hence we are in a situation where we have

$$\begin{tikzcd}
\mathbf{C}_T \arrow[r, "J"] \arrow[d, shift left, "G_T"] & \mathbf{C}^T \arrow[d, shift left, "G^T"] \\
\mathbf{C} \arrow[u, shift left, "F_T"] \arrow[r, equal] & \mathbf{C} \arrow[u, shift left, "F^T"]
\end{tikzcd}$$

where

$$G_T = G^T \circ J \qquad \text{and} \qquad F^T = J \circ F_T.$$