

What is a categorical model for Linear Logic?

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The aim of these notes is to give an outline of the categorical structure required for a model of linear logic. That means that we only try to motivate the various conditions imposed on such models; this is by no means intended to be a full account. For intuitionistic linear logic a detailed description of the process of defining such a model can be found in Gavin Bierman's thesis [Bie94]. There are a number of exercises in these notes, they either aim to give examples for the various notions defined or they state minor results that are used subsequently and which hopefully are not too difficult to establish. If the category theory in these notes is too advanced you may want to try [BS] which does not assume any knowledge in that area at all, but proceeds fairly rapidly to the notions we use here.

Originally these notes were written for my PhD students, but since I made the first version available on my webpage a number of people have told me they found them useful. As a result I felt obliged to add details which were rather sketchy in the original account. In particular the account now contains more proofs and so I am hopeful that the various formulations of the notion of linear exponential comonad are finally correct. I would like to thank Paola Maneggia, Peter Selinger, Robin Houston and Nicola Gambino for useful feedback on these notes.

1 Intuitionistic linear logic

When modelling any logic there has to be a valuation telling us how to interpret any variables or literals. In the case of categorical logic, each variable is so interpreted as an object in the modelling category. We take valuations for granted from now on.

When people talk about intuitionistic linear logic they often mean the calculus with the connectives \otimes , \multimap , \Box and $!$, and this is the system we will be referring to when using that name. Since there is no negation, the De Morgan duals of these can not be defined within this system. Further the system only allows *one-sided* judgements to be made. For a full discussion of the rules see Bierman's thesis [Bie94], for a summary of the categorical structure of models for linear logic, [Bie95].

MILL—the multiplicative fragment

The rules for this fragment are given in Table 1, with the exception of the *exchange rule* which allows the exchange of any two neighbouring formulae in a sequent Γ .

Exercise 1 Compare this system to $\mathbf{N}\{\multimap, \Box\}$, a natural deduction system for implication and conjunction. What are the differences, what the similarities? Can you deduce some categorical structure suitable for modelling the above from this?

Identity	Multiplicative connectives		Cut	Constants
	left	right		
$\frac{}{A \vdash A}$	$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}$	$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}$	$\frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}$	$\frac{}{\vdash \mathbf{I}}$
	$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}$	$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}$		

Table 1: Rules for MILL.

Given a formula A we have to define an object of the category, which is usually denoted by $\llbracket A \rrbracket$. The only way we have so far of building new formulae is to apply \otimes , so we want our category to support an operation which we also refer to as \otimes so that we can recursively define

$$\llbracket A_1 \otimes \cdots \otimes A_m \rrbracket = \llbracket A_1 \rrbracket \otimes \cdots \otimes \llbracket A_m \rrbracket.$$

We use the left rule for \otimes is to justify modelling a proof of a judgement

$$A_1, \dots, A_m \vdash B$$

as a morphism

$$\llbracket A_1 \rrbracket \otimes \cdots \otimes \llbracket A_m \rrbracket \longrightarrow \llbracket B \rrbracket \quad \text{in some category } \mathbf{C} \text{ with a suitable } \otimes\text{-operation.}$$

We use $\llbracket A_1, \dots, A_m \rrbracket = A_1 \otimes \cdots \otimes A_m$.

Using semantic brackets becomes somewhat tedious over time, and so we frequently assume that the formula A is interpreted by an object we also refer to as A .

We want to recursively define an assignment from proofs in the logic to morphisms in the category. We do this by saying how to treat each of the proof rules in the system. Some proofs are identified when they are mapped into the category, and how much identification goes on provides some measure of how good a model we might have.

Not all identifications are bad, however: There are some proofs that are ‘similar enough’ for us to want them to be identified. We use this idea to motivate some of the equations we expect to hold in any categorical model. For now the central question is what properties the operation \otimes on the category \mathbf{C} should have.

We start by looking at the rule

$$\frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}.$$

It means that given morphisms

$$\llbracket \Gamma \rrbracket \longrightarrow A \quad \text{and} \quad \llbracket \Delta \rrbracket \longrightarrow B$$

we have to be able to define a morphism

$$\llbracket \Gamma, \Delta \rrbracket = \llbracket \Gamma \rrbracket \otimes \llbracket \Delta \rrbracket \longrightarrow A \otimes B$$

and if \otimes is a bifunctor on \mathbf{C} , that is a functor $\mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$, then this is taken care of for us automatically: We merely take the tensor product of the given morphism $[\Gamma] \longrightarrow A$ with that of the given morphism $[\Delta] \longrightarrow B$.

Functoriality of this assignment means the following: The leaf rule $A \vdash A$ is interpreted by the identity on (the interpretation of) A . So preservation of identities by \otimes means that the identity on $A \otimes B$ is used to interpret

$$A, B \vdash A \otimes B,$$

which itself is a consequence of $A \vdash A$ and $B \vdash B$. Since from there we can, in fact, prove that $A \otimes B \vdash A \otimes B$ this seems to make sense.¹ These are some of the identifications we want to be able to make.

This leaves us with the question of why \otimes should also preserve composition. Composition in the category is the interpretation of the Cut rule:

$$\frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B}.$$

From arrows

$$f: [\Gamma] \longrightarrow A \quad \text{and} \quad g: A \otimes [\Delta] \longrightarrow B$$

we have to define an arrow

$$[\Gamma] \otimes [\Delta] \longrightarrow B,$$

and fortunately there's something to fill the gap:

$$[\Gamma] \otimes [\Delta] \xrightarrow{f \otimes \text{id}_{[\Delta]}} A \otimes [\Delta] \xrightarrow{g} B.$$

Exercise 2 Find a justification for demanding that \otimes preserve composition.

Because lists of formulae are *associative* by definition, we expect \otimes to be associative. Further, because of the exchange rule we expect it to be *symmetric*. And finally we need an ‘empty’ version of the tensor product in order to interpret judgements which have an empty context. We refer to this object as \mathbf{I} , and it is clear that it should behave like a unit for \otimes .

Definition 1 A **(symmetric) monoidal category** is a category \mathbf{C} together with a bifunctor \otimes on \mathbf{C} that is associative, (symmetric) and has a unit \mathbf{I} . By that we mean² that there exist natural isomorphisms α , $(\sigma,)$ ρ and λ such that the diagrams given below commute. These isomorphisms ‘mediate’ between objects and so provide us with associativity, symmetry and unit laws. Their naturality is expressed in the following set of diagrams.

$$\begin{array}{ccc} A \otimes (B \otimes C) & \xrightarrow{\alpha_{A,B,C}} & (A \otimes B) \otimes C \\ \downarrow f \otimes (g \otimes h) & & \downarrow (f \otimes g) \otimes h \\ A' \otimes (B' \otimes C') & \xrightarrow{\alpha_{A',B',C'}} & (A' \otimes B') \otimes C' \end{array}$$

¹If we only allow axioms to introduce literals (say p, q) then tensor preserving the identity amounts to the tensor of the identities on p and q interpreting $p \otimes q \vdash p \otimes q$, which can be proven from $p \vdash p$ and $q \vdash q$.

²Strictly speaking, the isomorphisms are part of the structure that form the symmetric monoidal category, but in practice people are often quite lax about this—it is common to say that a category ‘is symmetric monoidal’ without exhibiting the corresponding tensor product.

$$\begin{array}{ccccc}
A \otimes B & \xrightarrow{\sigma_{A,B}} & B \otimes A & & \mathbf{I} \otimes A & \xrightarrow{\lambda_A} & A & & A \otimes \mathbf{I} & \xrightarrow{\rho_A} & A \\
\downarrow f \otimes g & & \downarrow g \otimes f & & \downarrow \text{id}_{\mathbf{I}} \otimes f & & \downarrow f & & \downarrow f \otimes \text{id}_{\mathbf{I}} & & \downarrow f \\
A' \otimes B' & \xrightarrow{\sigma_{A',B'}} & B' \otimes A' & & \mathbf{I} \otimes A' & \xrightarrow{\lambda_{A'}} & A' & & A' \otimes \mathbf{I} & \xrightarrow{\rho_{A'}} & A'
\end{array}$$

These natural isomorphisms interact with each other as given by the diagrams below.

$$\begin{array}{ccc}
A \otimes (B \otimes (C \otimes D)) & \xrightarrow{\alpha_{A,B,C \otimes D}} & (A \otimes B) \otimes (C \otimes D) \\
\downarrow \text{id}_A \otimes \alpha_{B,C,D} & & \downarrow \alpha_{A \otimes B,C,D} \\
A \otimes ((B \otimes C) \otimes D) & \xrightarrow{\alpha_{A,B \otimes C,D}} & (A \otimes (B \otimes C)) \otimes D \xrightarrow{\alpha_{A,B,C} \text{id}_D} & ((A \otimes B) \otimes C) \otimes D
\end{array}$$

$$\begin{array}{ccccc}
A \otimes (B \otimes C) & \xrightarrow{\alpha_{A,B,C}} & (A \otimes B) \otimes C & \xrightarrow{\sigma_{A \otimes B,C}} & C \otimes (A \otimes B) \\
\downarrow \text{id}_A \otimes \sigma_{B,C} & & \downarrow \alpha_{C,A,B} & & \downarrow \alpha_{C,A,B} \\
A \otimes (C \otimes B) & \xrightarrow{\alpha_{A,C,B}} & (A \otimes C) \otimes B & \xrightarrow{\sigma_{A,C} \text{id}_B} & (C \otimes A) \otimes B
\end{array}$$

$$\begin{array}{ccc}
A \otimes (\mathbf{I} \otimes C) & \xrightarrow{\alpha_{A,\mathbf{I},C}} & (A \otimes \mathbf{I}) \otimes C \\
\searrow \text{id}_A \otimes \lambda_C & & \downarrow \rho_{A \otimes \mathbf{I}} \\
& & A \otimes C
\end{array}
\qquad
\begin{array}{ccc}
A \otimes B & \xrightarrow{\sigma_{A,B}} & B \otimes A \\
\searrow \text{id}_A \otimes \sigma_{B,A} & & \downarrow \sigma_{B,A} \\
& & A \otimes B
\end{array}$$

Remark. There are further equations one might be tempted to impose on (symmetric) monoidal categories. Two of those are the following which commute in every (symmetric) monoidal category.

$$\begin{array}{ccc}
\mathbf{I} \otimes \mathbf{I} & & \mathbf{I} \otimes A \xrightarrow{\sigma_{\mathbf{I},A}} A \otimes \mathbf{I} \\
\downarrow \rho_{\mathbf{I}} & = & \downarrow \rho_A \\
\mathbf{I} & & A
\end{array}$$

The proof is not totally obvious. The coherence theorem tells us that all ‘sensible’ diagrams we may build from the natural isomorphisms do, in fact commute.

Exercise 3 Show that the category of sets and relations **Rel** is symmetric monoidal. Hint: The tensor product we are looking for (because it will return in subsequent exercises) is given by cartesian product. You might want to convince yourself that this is *not* the product in this category.

Exercise 4 Convince yourself that a category with finite products is symmetric monoidal. Do the same for categories with finite coproducts.

So in order to be able to interpret \otimes we require our category \mathbf{C} to be symmetric monoidal. The question of what a functor of symmetric monoidal categories should be can be answered in several ways. Maybe the most straightforward would be to demand that the symmetric monoidal structure is preserved (at least up to isomorphism), but that turns out to be too strong for many examples. Instead, we choose the following notion, which is a *lax* one (in the technical sense). Note that we do not distinguish between the tensor products in different categories in our notation, similarly for the unit \mathbf{I} . Also note that we are sloppy in our notation and write $- \otimes -$ for the functor $\mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{C}$ although nothing in the notation suggests that we assume it to have two (possibly different) arguments rather than just one.

Definition 2 Let \mathbf{C} and \mathbf{D} be symmetric monoidal categories.

(a) A functor $F: \mathbf{C} \longrightarrow \mathbf{D}$ is said to be **symmetric monoidal** if there exist a morphism $m_{\mathbf{I}}: \mathbf{I} \longrightarrow F(\mathbf{I})$ and a natural transformation $m: F(-) \otimes F(-) \longrightarrow F(- \otimes -)$ respecting the symmetric, associative, and unit structures.

(b) Let F and G be two symmetric monoidal functors $\mathbf{C} \longrightarrow \mathbf{D}$ with natural transformations m and n respectively. We say that a natural transformation $\tau: F \longrightarrow G$ is **monoidal** if the following diagrams commute.

$$\begin{array}{ccc}
 \mathbf{I} & \xrightarrow{m_{\mathbf{I}}} & F(\mathbf{I}) \\
 & \searrow \tau_{\mathbf{I}} & \downarrow \tau_{\mathbf{I}} \\
 & & G(\mathbf{I})
 \end{array}
 \qquad
 \begin{array}{ccc}
 F(A) \otimes F(B) & \xrightarrow{m_{A,B}} & F(A \otimes B) \\
 \downarrow \tau_A \otimes \tau_B & & \downarrow \tau_{A \otimes B} \\
 G(A) \otimes G(B) & \xrightarrow{n_{A,B}} & G(A \otimes B)
 \end{array}$$

A priori there is no good reason to demand the natural transformation to go in the given direction, it might as well go the other way—however, some justification for choosing this direction is provided by Exercise 9.

Exercise 5 (a) If you are unsure about what it meant by ‘respecting the ... structures’ try drawing some diagrams.

(b) Show that if F is a symmetric monoidal endofunctor on a symmetric monoidal category \mathbf{C} then so is $F \circ F$. Further show that the identity functor on a symmetric monoidal category is symmetric monoidal.

(c) Show that the following endofunctors are monoidal for every symmetric monoidal category \mathbf{C} :

- The ‘constant \mathbf{I} functor’ $\mathbf{C}_{\mathbf{I}}$ which maps every object to \mathbf{I} and every morphism to the identity on \mathbf{I} .
- The ‘diagonal functor for tensor’ Δ which maps an object A of \mathbf{C} to $A \otimes A$ and a morphism $f: A \longrightarrow B$ to $f \otimes f: A \otimes A \longrightarrow B \otimes B$.

This brings us to the next connective, *linear implication* \multimap . This connective is very tightly linked to the morphisms of our category \mathbf{C} for the following reason. Any judgement $A \vdash B$ is

interpreted by a morphism $A \longrightarrow B$. This judgement allows us to deduce $\vdash A \multimap B$, which is interpreted by a morphism $\mathbf{I} \longrightarrow A \multimap B$. Hence there is a connection between morphism $A \longrightarrow B$ and those $\mathbf{I} \longrightarrow A \multimap B$ which is tightened by the following derivation. Assume now that we have a derivation of $\vdash A \multimap B$.

$$\frac{\vdots \quad \frac{\overline{A \vdash A} \quad \overline{B \vdash B}}{A, A \multimap B \vdash B}}{\vdash A \multimap B} \quad \frac{}{A \vdash B} \text{Cut}$$

For our category that means that we must be able to go the other way, too. We expect the following one-to-one correspondence:

$$\frac{A \longrightarrow B}{\mathbf{I} \longrightarrow A \multimap B}$$

or, more generally,

$$\frac{C \otimes A \longrightarrow B}{C \longrightarrow A \multimap B}$$

Assuming we make the usual naturality assumptions this means that the functor $- \otimes A$ is left-adjoint to $A \multimap -$. The better known notion of a *cartesian closed category* is a special case of this: the monoidal structure, that is the tensor product, is given by a product.

Definition 3 *A symmetric monoidal closed category is a symmetric monoidal category in which each functor $- \otimes A$ has a right adjoint.*

We use \multimap to refer to these right adjoints. Note that the adjunction does impose more structure on the linear function space construct than one might think at first.

Exercise 6 (a) Show that in every symmetric monoidal closed category $\mathbf{I} \multimap A \cong A$ naturally.

(b) Show that in every symmetric monoidal closed category $(A \otimes B) \multimap C$ is naturally isomorphic to $A \multimap (B \multimap C)$.

(c) Show that in every symmetric monoidal closed category \mathbf{C} , $\mathbf{C}(A \otimes B, C)$ is isomorphic to $\mathbf{C}(A, B \multimap C)$ and that this isomorphism is natural in all three of A , B and C . (This is because we have an example of a *parametrized adjunction*, see for example [Mac71], Theorem VI.7.3.)

Symmetric monoidal closed categories have all the categorical structure required to model the multiplicative fragment of intuitionistic linear logic.

Definition 4 *A model for MILL is any symmetric monoidal closed category.*

Exercise 7 Show that \mathbf{Rel} is a symmetric monoidal closed category. Hint: the underlying set for $A \multimap B$ is again the cartesian product of A and B .

Exercise 8 For every rule in Table 1 find the categorical counterpart, that is given the morphisms interpreting the hypothesis, find the morphism interpreting the conclusion. Do the same *vice versa*, that is for every equation imposed on a symmetric monoidal closed category, find a proof justifying it. (For this it is useful to recall how adjunctions can be described using equations only.)

One might assume that something additional is required for a functor between symmetric monoidal closed (rather than merely symmetric monoidal) categories, but this is not the case.

Definition 5 *A morphism of symmetric monoidal closed categories is a symmetric monoidal functor.*

This definition is justified by the following result.

Exercise 9 Assume that \mathbf{C} and \mathbf{D} are symmetric monoidal closed categories. Show that if $F: \mathbf{C} \rightarrow \mathbf{D}$ is a symmetric monoidal functor then there exist natural transformations $F(A \multimap -) \rightarrow F(A) \multimap F(-)$ and $F(- \multimap B) \rightarrow F(-) \multimap F(B)$, and so a natural transformation $F(- \multimap -) \rightarrow F(-) \multimap F(-)$ which is natural in both arguments.

MALL—the multiplicative additive fragment

We add one of the additive connectives to the multiplicative fragment MILL, namely the conjunction \sqcap . For that we add the following rules to those given in Table 1, and call the result MALL.

Additive connectives		Constants
left	right	
$\frac{\Gamma, A \vdash C}{\Gamma, A \sqcap B \vdash C}$	$\frac{\Gamma, B \vdash C}{\Gamma, A \sqcap B \vdash C}$	$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \sqcap B}$
		$\frac{}{\Gamma \vdash \mathbf{1}}$

Table 2: The rules for \sqcap

The categorical interpretation for the new connective \sqcap certainly should be something similar to a tensor product. But looking at the left introduction rules we find that we need something stronger here. In particular given $C \vdash A$ and $C \vdash B$ we obtain $C \vdash A \sqcap B$, so given morphisms $C \rightarrow A$ and $C \rightarrow B$ we have to be able to construct a morphism $C \rightarrow A \sqcap B$. This looks just like the requirement we demand for products, at least the part about being able to construct such morphisms. So how about the second requirement, the projections? From $A \vdash C$ we have to be able to conclude that $A \sqcap B \vdash C$. So given a morphism $A \rightarrow C$ we have to be able to construct one $A \sqcap B \rightarrow C$. If we do have a projection $A \sqcap B \rightarrow A$ then we can simply precompose that with the given morphism to obtain the desired result. Similar for the symmetric version. Since the usual categorical symbol for a product is \times we often write that rather than \sqcap for the interpretation of that connective in the category. The interpretation of $\mathbf{1}$ is similarly determined. Clearly there will have to be a morphism $A \rightarrow \mathbf{1}$ for every object A . We demand that there be precisely one of those for each A , making $\mathbf{1}$ the terminal object or ‘empty product’.

Definition 6 *A model for MALL is given by a symmetric monoidal closed category with finite products.*

Exercise 10 Show that **Rel** is a model for MALL. Hint: the categorical product is given by the disjoint union; projections are the ‘reversed’ injections.

Exercise 11 Justify the equations defining a categorical product from the rules for MALL.

Exercise 12 Show that if \mathbf{C} and \mathbf{D} are categories with products then for every functor $F: \mathbf{C} \longrightarrow \mathbf{D}$, $F^{\text{op}}: \mathbf{C}^{\text{op}} \longrightarrow \mathbf{D}^{\text{op}}$ is monoidal.

Since we are happy with this much connection between the structure of the source and the target category there is no need to define new morphisms of models of MALL.

ILL—the intuitionistic fragment

In order to obtain the intuitionistic fragment of linear logic we have to add one other connective, the unary $!$. This is our way of marking formulae which we allow to be subject to the weakening and contraction rules. This fragment is obtained by adding the rules in Table 3 to MALL.

The linear exponential !	
$\frac{\Gamma \vdash B}{\Gamma, !A \vdash B}$	$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}$
$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}$	$\frac{! \Gamma \vdash A}{! \Gamma \vdash !A}$

Table 3: The rules for the linear exponential !

We wish to interpret $!$ in the categorical model \mathbf{C} by a functor $\mathbf{C} \longrightarrow \mathbf{C}$ which we also refer to as $!$. This functor will have to come with additional structure so that we can define morphisms corresponding to various rules. We postpone the argument for why $!$ should be defined on morphisms too.

The judgement $A \vdash B$ is interpreted by a morphism $A \longrightarrow B$, and from that we have to be able to obtain a morphism $!A \longrightarrow B$. This can be done most easily if we demand that there exist a morphism $\epsilon_A: !A \longrightarrow A$ for each object A , then the composite

$$!A \xrightarrow{\epsilon_A} A \longrightarrow B$$

can be used to interpret the resulting proof of $!A \vdash B$.

Further from $!A \vdash B$ we get $!A \vdash !B$ and thus for a morphism $!A \longrightarrow B$ we want to be able to define a morphism $!A \longrightarrow !B$, that is we want a map $(-)^*: \mathbf{C}(!A, B) \longrightarrow \mathbf{C}(!A, !B)$. If we want to enforce that certain proofs create the same morphisms in our model then we have to impose some conditions on these.

Consider the proof

$$\frac{\frac{\frac{}{A \vdash A}}{!A \vdash A}}{!A \vdash !A}$$

We might just as well have started with the axiom $!A \vdash !A$ and therefore we demand that $(\epsilon_A)^* = \text{id}_{!A}$. Next consider the proof

$$\frac{\frac{\vdots}{!A \vdash B} \quad \frac{}{B \vdash B}}{!A \vdash !B} \quad \frac{}{!B \vdash B} \quad \text{Cut}}{!A \vdash B}$$

So there must be morphisms $m_{A,B}: !A \otimes !B \longrightarrow !(A \otimes B)$ which we expect to be natural in A and B , and there must be a morphism $m_{\mathbf{I}}: \mathbf{I} \longrightarrow !\mathbf{I}$. In other words we expect that $!$ is a monoidal functor. We further demand that ϵ and δ are monoidal natural transformations.

Definition 8 We say that a **comonad** $(!, \epsilon, \delta, m, m_{\mathbf{I}})$ is **monoidal** if $!$ is a monoidal functor (witnessed by m and $m_{\mathbf{I}}$) and ϵ and δ are monoidal transformations.

Exercise 15 Show that the multiset comonad from Exercise 14 is monoidal.

Exercise 16 Can you find proof-theoretic reasons for demanding that ϵ and δ be monoidal? How about the naturality of m ?

We finally turn to the left rules for $!$. Given a morphism $\mathbf{I} \longrightarrow B$ which is the interpretation of (some proof of) $\vdash B$, we have to be able to define a morphism $!A \longrightarrow B$ which serves as the interpretation of $!A \vdash B$. Again we use a composite, namely

$$!A \xrightarrow{e_A} \mathbf{I} \longrightarrow B,$$

demand that for every object A there be a morphism $e_A: !A \longrightarrow \mathbf{I}$.

Similarly we demand the existence of a morphism $d_A: !A \longrightarrow !A \otimes !A$ to interpret the contraction rule. As is to be expected, the mere existence of these morphisms is not sufficient for our purposes.

First of all we expect symmetry, transitivity and unit laws to hold. Clearly when contracting three occurrences of, say, $!A$, it should not matter in which order this happens. Exchanging two such occurrences and then contracting them should be the same as just contracting them. And first introducing $!A$ by weakening and then contracting it with an existing copy should be the same as doing nothing at all. Mathematically, what we are talking about is a *commutative comonoid*, and this is something that can be formulated quite generally in any symmetric monoidal category.

Definition 9 A **commutative comonoid** in a symmetric monoidal category \mathbf{C} is an object A with two morphisms $e: A \longrightarrow \mathbf{I}$ and $d: A \longrightarrow A \otimes A$ such that the following diagrams commute.

$$\begin{array}{ccc}
 A & \xrightarrow{d} & A \otimes A \\
 & \searrow e & \downarrow \sigma_{A,A} \\
 & & A \otimes A
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{d} & A \otimes A \\
 & \searrow \neg_A & \downarrow e \otimes \text{id}_A \\
 & & \mathbf{I} \otimes A
 \end{array}$$

$$\begin{array}{ccccc}
 A & \xrightarrow{d} & A \otimes A & \xrightarrow{d \otimes \text{id}_A} & (A \otimes A) \otimes A \\
 \downarrow d & & \downarrow d & & \downarrow \alpha_{A,A,A} \\
 A & \xrightarrow{d} & A \otimes A & \xrightarrow{\text{id}_A \otimes d} & A \otimes (A \otimes A)
 \end{array}$$

A **morphism of comonoids** from (A, e, d) to (A', e', d') is a morphism $f: A \longrightarrow A'$ respecting the comonoid structure, that is the following diagrams commute.

$$\begin{array}{ccc}
 A & & A \xrightarrow{d} A \otimes A \\
 \downarrow f & \searrow e & \downarrow f \otimes f \\
 A' & \xrightarrow{e'} & \mathbf{I} \\
 & & A' \xrightarrow{d'} A' \otimes A'
 \end{array}$$

Exercise 17 (a) Show that the multiset over a set A is a commutative comonoid in the category **Rel**.

(b) Show that if \mathbf{C} is a category with finite products then every object carries a comonoid structure, and every morphism preserves it.

We want every triple $(!A, e_A, d_A)$ to be a commutative comonoid. Clearly objects of the form $!A$ play a particular role—they are the interpretations of formulae on which we may perform the structural rules such as weakening and contraction. In the first published definition of a model for linear logic [Laf88] it was demanded that $(!A, e_A, d_A)$ be the *free* commutative comonoid over A . However, that turns out to be a definition which leaves out many perfectly good models of linear logic, for example that of coherence spaces with the powerset comonad. Instead the following definition of Bierman’s [Bie95] (building on work by Seely [See89]) has been generally accepted. The following essentially the original definition of a ‘*linear category*’ as given by Bierman [Bie95]. It can be shown that a Lafont-type model is a special case of one of these.

Definition 10 A symmetric monoidal category has a **linear exponential comonad** if it has a monoidal comonad $(!, \epsilon, \delta, m, m_{\mathbf{I}})$ such that

- all objects of \mathbf{C} which are of the form $!A$ are equipped with a commutative comonoid structure (A, e_A, d_A) ;
- the comonoid structure gives rise to monoidal natural transformations $e: ! \longrightarrow \mathbf{C}_{\mathbf{I}}$ and $d: ! \longrightarrow \Delta \circ !$ for the endofunctors $!, \mathbf{C}_{\mathbf{I}}$ and $\Delta \circ !$ on \mathbf{C} ;
- the natural transformations e and d satisfy the following diagrams;

$$\begin{array}{ccc}
 !A & \xrightarrow{\delta_A} & !!A \\
 \downarrow d_A & & \downarrow !d_A \\
 !A \otimes !A & \xrightarrow{\delta_A \otimes \delta_A} & !!A \otimes !!A \xrightarrow{m_{!A, !A}} & !(A \otimes A)
 \end{array}
 \qquad
 \begin{array}{ccc}
 !A & \xrightarrow{\delta_A} & !!A \\
 \downarrow e_A & & \downarrow !e_A \\
 \mathbf{I} & \xrightarrow{m_{\mathbf{I}}} & \mathbf{I}
 \end{array}$$

- every morphism of the form δ_A is a morphism of comonoids.

This is not the only way of defining the concept, and one of the equivalent formulations, see Proposition 4, is a lot shorter, and maybe the easiest to understand.

We introduce various categories that are connected with comonads which allows us to reformulate the desired structure. The first of these concentrates on ‘objects of the form $!A$ ’. We have to be careful about how we talk about them because we cannot afford to forget

the underlying object A . Hence given a category \mathbf{C} with a comonad $(!, \epsilon, \delta)$ we define a new category whose objects are labelled by the objects of \mathbf{C} , but which we secretly think of as objects of the form $!A$. This becomes noticeable in the notion of morphism for such objects.

Definition 11 *The category $\mathbf{C}_!$ of free coalgebras for a comonad $(!, \epsilon, \delta)$ on a category \mathbf{C} has objects A , where A is an object of \mathbf{C} . Morphisms $A \longrightarrow B$ are morphisms $f: !A \longrightarrow !B$ in \mathbf{C} such that the following diagram commutes.*

$$\begin{array}{ccc} !A & \xrightarrow{f} & !B \\ \delta_A \downarrow & & \downarrow \delta_B \\ !!A & \xrightarrow{!f} & !!B \end{array}$$

Exercise 18 Show that every morphism in $\mathbf{C}_!$ can be written as a composite of morphisms of the form δ_A and $!f$.

There is a generalized form of ‘coalgebra’ for a comonad. We have to refer to this because in general there is no monoidal structure on the category of free coalgebras in our situation. (One might want to use tensor and its unit from \mathbf{C} , and indeed there are appropriate symmetry, associativity and unit isomorphisms on $\mathbf{C}_!$; symmetry between $A \otimes B$ and $B \otimes A$, for example, is given by $!\sigma_{A,B}$. However, it is impossible to make \otimes into a bifunctor—there is just nothing sensible one can do on morphisms. Fortunately the category of free coalgebras embeds into a larger category where we can define a suitable monoidal structure.

Definition 12 *The category $\mathbf{C}^!$ of Eilenberg-Moore coalgebras for a comonad $(!, \epsilon, \delta)$ on a category \mathbf{C} has objects \mathbf{C} -morphisms $\xi: A \longrightarrow !A$ such that the following diagrams commute.*

$$\begin{array}{ccc} A & \xrightarrow{\xi} & !A \\ & \searrow \text{id}_A & \downarrow \epsilon_A \\ & & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\xi} & !A \\ \xi \downarrow & & \downarrow \delta_A \\ !A & \xrightarrow{!\xi} & !!A \end{array}$$

We also refer to these objects as **coalgebras (for the comonad)**. Morphisms between two such objects $\xi: A \longrightarrow !A$ and $\zeta: B \longrightarrow !B$ are given by morphisms $f: A \longrightarrow B$ such that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \xi \downarrow & & \downarrow \zeta \\ !A & \xrightarrow{!f} & !B \end{array}$$

Exercise 19 (a) Show that $\mathbf{C}_!$ embeds into $\mathbf{C}^!$ by mapping an object A to the object given by $\delta_A: !A \longrightarrow !!A$, and a morphism $f: A \longrightarrow B$ in $\mathbf{C}_!$ to the underlying $f: !A \longrightarrow !B$ (from \mathbf{C}), viewed now as a morphism in $\mathbf{C}^!$. Note that this embedding is full.

(b) Show that if $\xi: A \longrightarrow !A$ is a coalgebra then ξ is a morphism in $\mathbf{C}^!$ from $\xi: A \longrightarrow !A$ to $\delta_A: !A \longrightarrow !!A$ (which you have just shown to be another coalgebra).

(c) Show that the category of free coalgebras $\mathbf{C}_!$ is isomorphic to the **Kleisli category** of a comonad. This category has objects A where A is an object in \mathbf{C} and morphisms $A \longrightarrow B$ are given by morphisms $!A \longrightarrow B$ in \mathbf{C} . The identity on an object A is ϵ_A and the composite of $f: A \longrightarrow B$ and $g: B \longrightarrow C$ is given by $g \circ !f \circ \delta_A: A \longrightarrow C$. If you haven't seen this construction before you might want to prove that it does indeed give a category.

We define the monoidal structure on $\mathbf{C}^!$ which we are interested in.

Proposition 1 *Let $(!, \epsilon, \delta, m, m_{\mathbf{I}})$ be a monoidal comonad on a symmetric monoidal category \mathbf{C} . Then $\mathbf{C}^!$ is symmetric monoidal. The tensor product of two objects*

$$\xi: A \longrightarrow !A \quad \text{and} \quad \zeta: B \longrightarrow !B$$

is given by

$$A \otimes B \xrightarrow{\xi \otimes \zeta} !A \otimes !B \xrightarrow{m_{A,B}} !(A \otimes B).$$

The unit for the tensor product is given by $m_{\mathbf{I}}: \mathbf{I} \longrightarrow !\mathbf{I}$.

Exercise 20 (a) Prove Proposition 1.

(b) Show that $m_{\mathbf{I}}$ and $m_{A,B}$ are morphisms in $\mathbf{C}^!$ between the appropriate coalgebras.

Whenever we refer to a tensor product on $\mathbf{C}^!$ then this is the one we mean. We shall, however, make this explicit in the results we state. Note that now the tensor product of two objects of the category of free coalgebras A and B , which we think of as $\delta_A: !A \longrightarrow !!A$ and $\delta_B: !B \longrightarrow !!B$ in the larger category, now has $!A \otimes !B$ as its ‘underlying object’ as desired. Let $\mathbf{C}_!^{\otimes}$ be the closure of $\mathbf{C}_!$ under finite tensor products in $\mathbf{C}^!$.

Exercise 21 (a) Show that there is an adjunction between \mathbf{C} and $\mathbf{C}^!$. Hint: The functor $\mathbf{C} \longrightarrow \mathbf{C}^!$ maps an object A to $\delta_A: !A \longrightarrow !!A$; the functor $\mathbf{C}^! \longrightarrow \mathbf{C}$ is the identity on objects. Show that this adjunction is monoidal, that is both functors are monoidal and so are the natural transformations connecting them.

(b) Do the same for \mathbf{C} and $\mathbf{C}_!$. Hint: $\mathbf{C} \longrightarrow \mathbf{C}_!$ maps an object A to A , and a morphism $f: A \longrightarrow B$ of \mathbf{C} to the morphism of $\mathbf{C}_!$ given by $!f: !A \longrightarrow !B$. The functor $\mathbf{C}_! \longrightarrow \mathbf{C}$ maps an object A to $!A$ and maps a morphism $f: A \longrightarrow B$ in $\mathbf{C}_!$ to the underlying morphism $f: !A \longrightarrow !B$ from \mathbf{C} .

(c) Show that the first adjunction factors through the second one by making use of the embedding from Exercise 19(a). Use this to establish an adjunction between \mathbf{C} and $\mathbf{C}_!^{\otimes}$ and show that this adjunction is monoidal.

A linear exponential comonad is, in fact, equivalent to a stronger sounding structure. We began by concentrating on the category of coalgebras.

Proposition 2 *A symmetric monoidal category \mathbf{C} has a linear exponential comonad if and only if it has a monoidal comonad such that the category of Eilenberg-Moore coalgebras, with the induced tensor product (see Proposition 1), is naturally a category of commutative comonoids. By this we mean that*

- every object of the Eilenberg Moore category has a comonoid structure such that

- the comonoid structure yields natural transformations $\text{id}_{\mathbf{C}^!} \longrightarrow \mathbf{C}_{\mathbf{I}}$ and $\text{id}_{\mathbf{C}^!} \longrightarrow \Delta$ and these are monoidal.

Before proving this result it is worth to expand the conditions which appear in it. Exercise 5(b) and (c) establishes that it makes sense to talk about monoidality in this context. We use e and d to refer to the natural transformations given by counit and comultiplication respectively. Note that from here on we typically refrain from bracketing tensored expressions and leave out any mentioning of the associativity isomorphism. This results in better readability, and interested readers can reestablish the results with that isomorphism made explicitly if they are so inclined.

First of all, the components of e and d must be morphisms of $\mathbf{C}^!$. That means that for a coalgebra $\xi: A \longrightarrow !A$ we have the following, where we abbreviate coalgebras by merely naming the underlying morphism.

$$\begin{array}{ccc}
 A & \xrightarrow{e_\xi} & \mathbf{I} \\
 \xi \downarrow & & \downarrow m_{\mathbf{I}} \\
 !A & \xrightarrow{!e_\xi} & !\mathbf{I}
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{d_\xi} & A \otimes A \\
 \xi \downarrow & & \downarrow \xi \otimes \xi \\
 & & !A \otimes !A \\
 & & \downarrow m_{A,A} \\
 !A & \xrightarrow{!d_\xi} & !(A \otimes A)
 \end{array}$$

Secondly, they must be natural. Assume that we have a second coalgebra, namely $\zeta: B \longrightarrow !B$ and a morphism f from ξ to ζ .

$$\begin{array}{ccc}
 A & & \\
 f \downarrow & \searrow e_\xi & \\
 B & \xrightarrow{e_\zeta} & \mathbf{I}
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{d_\xi} & A \otimes A \\
 f \downarrow & & \downarrow f \otimes f \\
 B & \xrightarrow{d_\zeta} & B \otimes B
 \end{array}$$

Remark. Naturality of the comonoid structure is equivalent to saying that every morphism of coalgebras is a morphism of comonoids.

Lastly there are the monoidality conditions.

$$\begin{array}{ccc}
 \mathbf{I} & \xrightarrow{\text{id}_{\mathbf{I}}} & \mathbf{I} \\
 \text{id}_{\mathbf{I}} \searrow & & \downarrow e_{m_{\mathbf{I}}} \\
 & & \mathbf{I}
 \end{array}
 \qquad
 \begin{array}{ccc}
 A \otimes B & \xrightarrow{\text{id}_{A \otimes B}} & A \otimes B \\
 e_\xi \otimes e_\zeta \downarrow & & \downarrow e_\xi \otimes \zeta \\
 \mathbf{I} \otimes \mathbf{I} & \xrightarrow{\lambda_{\mathbf{I}} = \rho_{\mathbf{I}}} & \mathbf{I}
 \end{array}$$

$$\begin{array}{ccc}
\mathbf{I} & \xrightarrow{\text{id}_{\mathbf{I}}} & \mathbf{I} \\
\searrow \lambda_{\mathbf{I}}^{-1} = \rho_{\mathbf{I}}^{-1} & & \downarrow d_{m_{\mathbf{I}}} \\
& & \mathbf{I} \otimes \mathbf{I}
\end{array}
\qquad
\begin{array}{ccc}
A \otimes B & \xrightarrow{\text{id}_{A \otimes B}} & A \otimes B \\
\downarrow d_{\xi \otimes d_{\zeta}} & & \downarrow d_{\xi \otimes \zeta} \\
A \otimes A \otimes B \otimes B & \xrightarrow{\text{id}_{A \otimes \sigma_{A,B} \otimes \text{id}_B}} & A \otimes B \otimes A \otimes B
\end{array}$$

Remark. The monoidality of the natural transformations given by the comonoid structure implies that the comonoid structure is well-behaved with respect to the monoidal structure on $\mathbf{C}^!$.

By this we mean that

- for the tensor unit $m_{\mathbf{I}}: \mathbf{I} \longrightarrow !\mathbf{I}$, the structure is given by

$$\begin{array}{ll}
\text{co-unit} & \text{id}_{\mathbf{I}}: \mathbf{I} \longrightarrow \mathbf{I} \\
\text{comultiplication} & \lambda_{\mathbf{I}}^{-1} = \rho_{\mathbf{I}}^{-1}: \mathbf{I} \longrightarrow \mathbf{I} \otimes \mathbf{I};
\end{array}$$

- the comonoid structure on the tensor product of $\xi: A \longrightarrow !A$ (with comonoid structure (e, d)) and $\xi': A' \longrightarrow !A'$ (with comonoid structure (e', d')) is given by

$$\begin{array}{ll}
\text{co-unit} & A \otimes A' \xrightarrow{e \otimes e'} \mathbf{I} \otimes \mathbf{I} \xrightarrow{\lambda_{\mathbf{I}} = \rho_{\mathbf{I}}} \mathbf{I} \\
\text{comultiplication} & A \otimes A' \xrightarrow{d \otimes d'} A \otimes A \otimes A' \otimes A' \xrightarrow{\text{id}_{A \otimes \sigma_{A,A'} \otimes \text{id}_{A'}}} A \otimes A' \otimes A \otimes A'.
\end{array}$$

Proof. We begin with the ‘if’ part, so let’s assume we have a structure on $\mathbf{C}^!$ as given in the statement of the proposition. For $!A$, pick the comonoid structure for the coalgebra $\delta_A: !A \longrightarrow !!A$ given by the assumption, that is, set $d_A = d_{\delta_A}$, where reusing the letter d hopefully is not too confusing.

Naturality is easy to establish. It follows from the fact that morphisms of the form $!f$ are morphisms of coalgebras and so by the above remark morphisms of comonoids.

We establish monoidality for the natural transformations e and d . Note that $m_{\mathbf{I}}$ is a morphism in $\mathbf{C}^!$ from $m_{\mathbf{I}}: \mathbf{I} \longrightarrow !\mathbf{I}$ to $\delta_{\mathbf{I}}: !\mathbf{I} \longrightarrow !!\mathbf{I}$ by Exercise 20(b) and so a morphism of comonoids by assumption. For similar reasons $m_{A,B}$ is a morphism of coalgebras from

$$\begin{array}{ll}
\text{co-unit:} & !A \otimes !B \xrightarrow{m_{!A,!B} \circ (\delta_A \otimes \delta_B)} !(A \otimes B) \quad \text{to} \quad !(A \otimes B) \xrightarrow{\delta_{A \otimes B}} !!(A \otimes B) \\
& \lambda_{\mathbf{I}} \circ (e_A \otimes e_B) & e_{A \otimes B} \\
\text{comultiplication:} & (\text{id}_{!A} \otimes \sigma_{!A,!B} \otimes \text{id}_{!B}) \circ (d_A \otimes d_B) & d_{A \otimes B}
\end{array}$$

and so a morphism of comonoids.

Since $e_{m_{\mathbf{I}}} = \text{id}_{\mathbf{I}}$ by monoidality of e we get the left hand diagram below from the fact that $m_{\mathbf{I}}$ preserves the co-unit, and the right hand diagram from the fact that $m_{A,B}$ preserves the co-unit as well.

$$\begin{array}{ccc}
\mathbf{I} & \xrightarrow{m_{\mathbf{I}}} & !\mathbf{I} \\
\searrow \text{id}_{\mathbf{I}} & & \downarrow e_{\mathbf{I}} \\
& & \mathbf{I}
\end{array}
\qquad
\begin{array}{ccc}
!A \otimes !B & \xrightarrow{m_{A,B}} & !(A \otimes B) \\
\downarrow e_A \otimes e_B & & \downarrow e_{A \otimes B} \\
\mathbf{I} \otimes \mathbf{I} & \xrightarrow{\lambda_{\mathbf{I}} = \rho_{\mathbf{I}}} & \mathbf{I}
\end{array}$$

Monoidality of the natural transformation d is very similar. We know that $m_{\mathbf{I}}$ must also preserve the comultiplication, and for the algebra $m_{\mathbf{I}}: \mathbf{I} \longrightarrow !\mathbf{I}$ that is given by $\lambda_{\mathbf{I}}^{-1} = \rho_{\mathbf{I}}^{-1}$, which gives the left diagram. For the right diagram we once again use that $m_{A,B}$ is a morphism of comonoids as indicated above and so must preserve the comultiplication.

$$\begin{array}{ccc}
\mathbf{I} & \xrightarrow{m_{\mathbf{I}}} & !\mathbf{I} \\
\downarrow \lambda_{\mathbf{I}}^{-1} = \rho_{\mathbf{I}}^{-1} & & \downarrow d_{\mathbf{I}} \\
\mathbf{I} \otimes \mathbf{I} & \xrightarrow{m_{\mathbf{I}} \otimes m_{\mathbf{I}}} & !\mathbf{I} \otimes !\mathbf{I}
\end{array}
\qquad
\begin{array}{ccc}
!A \otimes !B & \xrightarrow{m_{A,B}} & !(A \otimes B) \\
\downarrow d_A \otimes d_B & & \downarrow d_{A \otimes B} \\
!A \otimes !A \otimes !B \otimes !B & \xrightarrow{\text{id}_{!A} \otimes \sigma_{!A, !B} \otimes \text{id}_{!B}} & !A \otimes !B \otimes !A \otimes !B
\end{array}$$

The two diagrams given explicitly in the definition of linear exponential comonad say that every component of the natural transformations is a morphism of the (appropriate) coalgebras, which is part of the assumptions. Lastly, every morphism of the form δ_A is a morphism of coalgebras as follows

$$\begin{array}{ccc}
!A & \xrightarrow{\delta_A} & !!A \\
\downarrow \delta_A & & \downarrow \delta_{!A} \\
!!A & \xrightarrow{! \delta_A} & !!!A
\end{array}$$

and so covered by the remark that naturality of the comonoid operations is equivalent to every morphism of coalgebras being a morphism of comonoids.

Now for the ‘only if’ part. Assume that we have a linear exponential comonad. For a coalgebra $\xi: A \longrightarrow !A$ we use the fact that this makes A (in \mathbf{C}) a special kind of retract of $!A$. We define the following morphisms which we show give it a comonoid structure:

$$A \xrightarrow{\xi} !A \xrightarrow{d_A} !A \otimes !A \xrightarrow{\epsilon_A \otimes \epsilon_A} A \otimes A$$

and

$$A \xrightarrow{\xi} !A \xrightarrow{e_A} \mathbf{I}.$$

We note in passing that for the coalgebra $\delta_A: !A \longrightarrow !!A$ the new comonoid structure coincides with the old since δ_A is a morphism of comonoids.

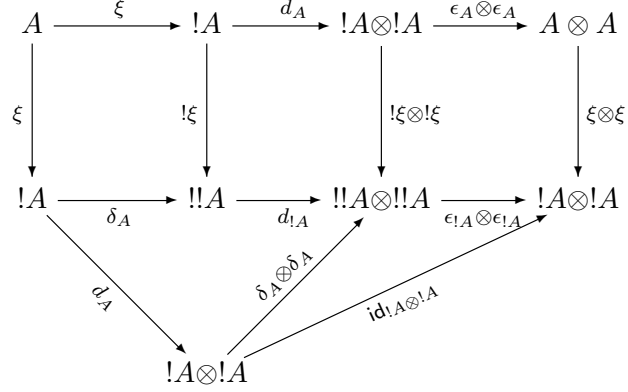
$$\begin{array}{ccc}
!A & \xrightarrow{d_A} & !A \otimes !A \\
\downarrow \delta_A & & \downarrow \delta_A \otimes \delta_A \\
!!A & \xrightarrow{d_{!A}} & !!A \otimes !!A
\end{array}
\qquad
\begin{array}{ccc}
!A & & \mathbf{I} \\
\downarrow \delta_A & \searrow e_A & \\
!!A & \xrightarrow{e_{!A}} & \mathbf{I}
\end{array}$$

We begin by establishing symmetry.

$$\begin{array}{ccccc}
A & \xrightarrow{\xi} & !A & \xrightarrow{d_A} & !A \otimes !A & \xrightarrow{\epsilon_A \otimes \epsilon_A} & A \otimes A \\
& & \searrow d_A & & \downarrow \sigma_{!A, !A} & & \downarrow \sigma_{A, A} \\
& & & & !A \otimes !A & \xrightarrow{\epsilon_A \otimes \epsilon_A} & A \otimes A
\end{array}$$

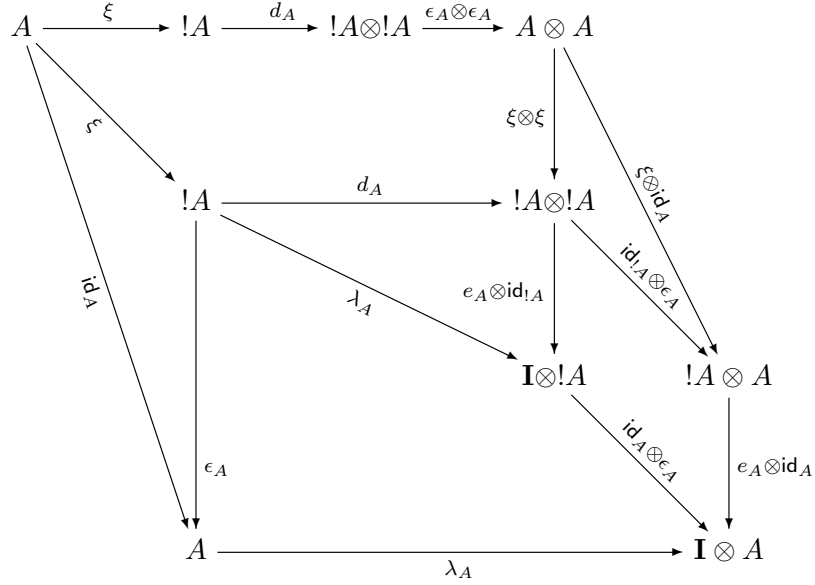
Next we find it useful to establish that

$$(\xi \otimes \xi) \circ (\epsilon_A \otimes \epsilon_A) \circ d_A \circ \xi = d_A \circ \xi. \quad (*)$$



We use in the above diagram that ξ is a coalgebra, naturality of d and ϵ as well as the fact that δ is a morphism of comonoids and one of the comonad equations.

We show that the unit law holds.



In the above diagram we use the just established equation $(*)$ at the top, that $\epsilon_A \circ \xi = id_A$, and various naturality conditions.

Finally we turn to associativity, where we leave out bracketing and the associativity isomorphism for readability.

$$\begin{array}{ccccccc}
A & \xrightarrow{\xi} & !A & \xrightarrow{d_A} & !A \otimes !A & \xrightarrow{\epsilon_A \otimes \epsilon_A} & A \otimes A \\
\downarrow \xi & \searrow \mathfrak{f} & & & \downarrow \xi \otimes \xi & & \downarrow id_A \otimes \epsilon_A \\
!A & & !A & \xrightarrow{d_A} & !A \otimes !A & & A \otimes !A \otimes !A \\
\downarrow d_A & & \downarrow d_A & & \downarrow id_{!A} \otimes d_A & & \downarrow id_A \otimes d_A \\
!A \otimes !A & \xrightarrow{\epsilon_A \otimes \epsilon_A} & A \otimes A & \xrightarrow{\xi \otimes \xi} & !A \otimes !A & \xrightarrow{d_A \otimes id_{!A}} & !A \otimes !A \otimes !A \\
& & \downarrow \xi \otimes id_A & & \downarrow \epsilon_A \otimes \epsilon_A \otimes \epsilon_A & & \downarrow id_A \otimes \epsilon_A \otimes \epsilon_A \\
& & !A \otimes A & \xrightarrow{d_A \otimes id_A} & !A \otimes !A \otimes A & \xrightarrow{\epsilon_A \otimes \epsilon_A \otimes id_A} & A \otimes A \otimes A
\end{array}$$

Again we have used the equation (*) and the same coalgebra equation.

Hence we have indeed defined a comonoid structure. It remains to establish the conditions stated in the bullet points. First we establish that the comultiplication and co-unit we have just defined are morphisms in $\mathbf{C}^!$. Again we use the equation (*), the equation for d stated as a diagram in the definition of linear exponential comonad, and various coalgebra, comonad and naturality equations.

$$\begin{array}{ccccccc}
A & \xrightarrow{\xi} & !A & \xrightarrow{d_A} & !A \otimes !A & \xrightarrow{\epsilon_A \otimes \epsilon_A} & A \otimes A \\
\downarrow \xi & \searrow \mathfrak{f} & & & \downarrow \delta_A \otimes \delta_A & & \downarrow \xi \otimes \xi \\
!A & & !A & \xrightarrow{d_A} & !A \otimes !A & \xrightarrow{id_{!A} \otimes !A} & !A \otimes !A \\
\downarrow \delta_A & & \downarrow \delta_A & & \downarrow m_{!A, !A} & & \downarrow m_{A, A} \\
!A & \xrightarrow{! \xi} & !!A & \xrightarrow{! d_A} & !(A \otimes A) & \xrightarrow{!(\epsilon_A \otimes \epsilon_A)} & !(A \otimes A)
\end{array}$$

For the second diagram we merely require a coalgebra equation and the equation for e stated in the assumptions.

$$\begin{array}{ccccc}
A & \xrightarrow{\xi} & !A & \xrightarrow{e_A} & \mathbf{I} \\
\downarrow \xi & & \downarrow \delta_A & & \downarrow m_{\mathbf{I}} \\
!A & \xrightarrow{! \xi} & !!A & \xrightarrow{! e_A} & !\mathbf{I}
\end{array}$$

Let f be a morphism from $\xi: !A \longrightarrow A$ to $\zeta: !B \longrightarrow B$ in $\mathbf{C}^!$. To establish naturality of the new comonoid operations we provide the following diagrams, where we use naturality of e and d .

$$\begin{array}{ccccc}
 A & \xrightarrow{\xi} & !A & \xrightarrow{d_A} & !A \otimes !A & \xrightarrow{\epsilon_A \otimes \epsilon_A} & A \otimes A \\
 \downarrow f & & \downarrow !f & & \downarrow !f \otimes !f & & \downarrow f \\
 B & \xrightarrow{\zeta} & !B & \xrightarrow{d_B} & !B \otimes !B & \xrightarrow{\epsilon_B \otimes \epsilon_B} & B \otimes B
 \end{array}
 \qquad
 \begin{array}{ccccc}
 A & \xrightarrow{\xi} & !A & & \\
 \downarrow f & & \downarrow !f & \searrow e_A & \\
 B & \xrightarrow{\zeta} & !B & \xrightarrow{e_B} & \mathbf{I}
 \end{array}$$

It remains to establish that these natural transformations are monoidal, compare the diagrams on page 14. For the co-unit this is a straightforward consequence of the monoidality of e —it is harder to convince oneself that the diagrams below do indeed establish what's desired! For that recall that in $\mathbf{C}^!$, the tensor product of $\xi: A \longrightarrow !A$ and $\zeta: B \longrightarrow !B$ is given by

$$A \otimes B \xrightarrow{\xi \otimes \zeta} !A \otimes !B \xrightarrow{m_{A,B}} !(A \otimes B).$$

$$\begin{array}{ccc}
 \mathbf{I} & \xrightarrow{m_{\mathbf{I}}} & !\mathbf{I} \\
 \searrow \text{id}_{\mathbf{I}} & & \downarrow e_{\mathbf{I}} \\
 & & \mathbf{I}
 \end{array}
 \qquad
 \begin{array}{ccccc}
 A \otimes B & \xrightarrow{\xi \otimes \zeta} & !A \otimes !B & \xrightarrow{m_{A,B}} & !(A \otimes B) \\
 \downarrow e_A \otimes e_B & & \downarrow e_{A \otimes B} & & \downarrow e_{A \otimes B} \\
 I \otimes I & \xrightarrow{\lambda_{\mathbf{I}} = \rho_{\mathbf{I}}} & \mathbf{I} & & \mathbf{I}
 \end{array}$$

For the comultiplication, which involves the co-unit of the comonad ϵ , we also require the monoidality of that natural transformation.

$$\begin{array}{ccc}
 \mathbf{I} & \xrightarrow{m_{\mathbf{I}}} & !\mathbf{I} \\
 \downarrow \lambda_{\mathbf{I}}^{-1} = \rho_{\mathbf{I}}^{-1} & & \downarrow d_{\mathbf{I}} \\
 \mathbf{I} \otimes \mathbf{I} & \xrightarrow{m_{\mathbf{I}} \otimes m_{\mathbf{I}}} & !\mathbf{I} \otimes !\mathbf{I} \\
 \searrow \text{id}_{\mathbf{I}} \otimes \text{id}_{\mathbf{I}} & & \downarrow \epsilon_{\mathbf{I}} \otimes \epsilon_{\mathbf{I}} \\
 & & \mathbf{I} \otimes \mathbf{I}
 \end{array}$$

$$\begin{array}{ccccc}
A \otimes B & & & & \\
\downarrow \xi \otimes \zeta & & & & \\
!A \otimes !B & \xrightarrow{m_{A,B}} & & & !(A \otimes B) \\
\downarrow d_A \otimes d_B & & & & \downarrow d_{A \otimes B} \\
!A \otimes !A \otimes !B \otimes !B & \xrightarrow{\text{id}_{!A} \otimes \sigma_{!A,!B} \otimes \text{id}_{!B}} & !A \otimes !B \otimes !A \otimes !B & \xrightarrow{m_{A,B} \otimes m_{A,B}} & !(A \otimes B) \otimes !(A \otimes B) \\
\searrow \epsilon_A \otimes \epsilon_A \otimes \epsilon_B \otimes \epsilon_B & & \searrow \epsilon_A \otimes \epsilon_B \otimes \epsilon_A \otimes \epsilon_B & & \searrow \epsilon_{A \otimes B} \otimes \epsilon_{A \otimes B} \\
A \otimes A \otimes B \otimes B & \xrightarrow{\text{id}_A \otimes \sigma_{A,B} \otimes \text{id}_B} & A \otimes B \otimes A \otimes B & & \\
\end{array}$$

That completes the proof. \square

The following proposition talks in terms of the Kleisli category as far as possible—we have to go to $\mathbf{C}_!^\otimes$ to get a symmetric monoidal category which we need in order to talk about comonoids of the desired type (but see Proposition 5). It is easiest here if we view $\mathbf{C}_!^\otimes$ as a subcategory of $\mathbf{C}^!$. Recall from Exercise 21(a) that there is a functor (which by abuse of notation we are going to call $!$ again

$$!: \mathbf{C} \longrightarrow \mathbf{C}^!$$

which co-restricts to $\mathbf{C}_!^\otimes$ and which maps an object A of \mathbf{C} to the free coalgebra $\delta_A : !A \longrightarrow !!A$ and a morphism f to $!f$. There is also a monoidal functor from \mathbf{C} to $\mathbf{C}_!^\otimes$ which maps everything to the unit for tensor in the target. It is merely a slight generation of the constant I endofunctor from Exercise 5(c). Again we will abuse notation and refer to that functor as $\mathbf{C}_!$.

Proposition 3 *A symmetric monoidal category \mathbf{C} has a linear exponential comonad if and only if it has a monoidal comonad $(!, \epsilon, \delta, m, m_{\mathbf{I}})$ such that the closure under tensor of $\mathbf{C}_!$ in $\mathbf{C}^!$ (under the tensor product from Proposition 1), $\mathbf{C}_!^\otimes$ has the following structure:*

- Every object of the category has a comonoid structure such that
- the comonoid structure yields natural transformations $! \longrightarrow \mathbf{C}_!$ and $! \longrightarrow \Delta \circ !$, where $!, \mathbf{C}_! : \mathbf{C} \longrightarrow \mathbf{C}_!^\otimes$, these are monoidal.

Proof. All the hard work was done in the proof of Proposition 2. For the ‘if’ part we use the given structure on $\mathbf{C}_!^\otimes$ to define the desired structure on \mathbf{C} in precisely the same way as we do in the proof of that proposition.

For the ‘only if’ part, we invoke Proposition 2, and assume that the linear exponential comonad is given as stated there. Then all we have to do here is to restrict that structure to $\mathbf{C}_!^\otimes$. It is easy to see that the resulting structure has the desired properties. \square

As far as I know the various equivalent ways of describing a linear exponential comonad given above have not appeared in print. They have certainly been used because the original definition is rather lengthy, but without a full justification being given

The following result is probably the most surprising restatement of what it means for a category to have a linear exponential comonad, and it certainly is the briefest, and the least cluttered. It appears in Paola Maneggia's thesis [Man04].

Proposition 4 *A symmetric monoidal category \mathbf{C} has a linear exponential comonad if and only if the tensor product on $\mathbf{C}^!$ from Proposition 1 is in fact a product.*

Proof. The 'if' part is easy to establish. We take the comonoid structure from Exercise 17: The co-unit is given by the unique morphism to the unit for product, that is the terminal object. The comultiplication arises from the diagonal arrow $\langle \text{id}, \text{id} \rangle$. We establish that we have a structure as in Proposition 2.

Obviously the unique morphisms to the terminal objects are natural as desired, and similarly for the diagonal arrows. It only remains to cover monoidality.

Since morphisms into the terminal object are unique there is nothing to be done for the co-unit (compare the diagrams on page 14. For the comultiplication note that in a category where tensor is given by product, $\lambda_{\mathbf{I}}^{-1} = \rho_{\mathbf{I}}^{-1} = \langle \text{id}_{\mathbf{I}}, \mathbf{I} \rangle$, and obviously

$$\begin{array}{ccc}
 A \times B & \xrightarrow{\text{id}_{A \times B}} & A \times B \\
 \downarrow \langle \text{id}_A, \text{id}_A \rangle \times \langle \text{id}_B, \text{id}_B \rangle & & \downarrow \langle \text{id}_{A \times B}, \text{id}_{A \times B} \rangle \\
 A \times A \times B \times B & \xrightarrow{\text{id}_A \times \langle \pi_2, \pi_1 \rangle \times \text{id}_B} & A \times B \times A \times B \\
 & & \downarrow \langle \text{id}_A \times \text{id}_B \rangle \times \langle \text{id}_A \times \text{id}_B \rangle
 \end{array}$$

Now for the 'only if' part. Since Proposition 2 already talks about structure on the category of coalgebras, that's the definition of linear exponential comonad we choose to work with for this proof. Assume we have the two natural transformations e and d as stated there.

Assume that $\xi: A \longrightarrow !A$ and $\zeta: B \longrightarrow !B$ are two coalgebras. We have to exhibit projections, for which we propose the arrows

$$A \otimes B \xrightarrow{\text{id}_\xi \otimes e_\zeta} A \otimes \mathbf{I} \xrightarrow{\rho_A} A \quad \text{and} \quad (1)$$

$$A \otimes B \xrightarrow{e_\xi \otimes \text{id}_\zeta} \mathbf{I} \otimes B \xrightarrow{\lambda_B} B. \quad (2)$$

Being composites of morphisms of coalgebras (for ρ_A and λ_B this was established in Exercise 1), these are morphisms of coalgebras.

We also have to give a pairing operation. Assume we have a third coalgebra, $\chi: C \longrightarrow !C$, and morphisms f from χ to ξ and g from χ to ζ . Consider the following morphism from χ to $\xi \otimes \zeta$.

$$C \xrightarrow{d_\chi} C \otimes C \xrightarrow{f \otimes g} A \otimes B$$

We begin by establishing that this composes with the proposed projections as appropriate. Clearly it is sufficient to consider just one case.

$$\begin{array}{ccccc}
C & \xrightarrow{d_\chi} & C \otimes C & \xrightarrow{f \otimes g} & A \otimes B \\
& & \downarrow \text{id}_C \otimes e_C & & \downarrow \text{id}_A \otimes e_C \\
& & C \otimes \mathbf{I} & \xrightarrow{f \otimes \text{id}_\mathbf{I}} & A \otimes \mathbf{I} \\
& & \downarrow \rho_C & & \downarrow \rho_A \\
& & C & \xrightarrow{f} & A
\end{array}$$

$\swarrow \text{id}_C$

Here we use that g is a morphism of comonoids and therefore preserves the co-unit, as well as the unit law for the comonoid structure and naturality of ρ .

It remains to establish the uniqueness of the proposed paired morphism. So assume there is a morphism $h: C \longrightarrow A \otimes B$ which composes with the proposed projections to give f and g as appropriate.

$$\begin{array}{ccccccc}
C & \xrightarrow{d_\chi} & C \otimes C & & & & \\
\downarrow h & & \downarrow h \otimes h & & & & \downarrow f \otimes g \\
A \otimes B & \xrightarrow{d_\xi \otimes d_\zeta} & A \otimes A \otimes B \otimes B & \xrightarrow{\text{id}_A \otimes \sigma_{A,B} \otimes \text{id}_B} & A \otimes B \otimes A \otimes B & & \\
& & \downarrow \text{id}_A \otimes e_\xi \otimes e_C \otimes \text{id}_B & & \downarrow \text{id}_A \otimes e_C \otimes e_\xi \otimes \text{id}_B & & \\
& & A \otimes \mathbf{I} \otimes \mathbf{I} \otimes B & \xrightarrow{\text{id}_A \otimes \sigma_{A,B} \otimes \text{id}_B} & A \otimes \mathbf{I} \otimes \mathbf{I} \otimes B & & \\
& & \downarrow \rho_A \otimes \lambda_B & & \downarrow \rho_A \otimes \lambda_B & & \\
& & A \otimes B & & A \otimes B & & \\
& \searrow \text{id}_{A \otimes B} & & & & & \swarrow \text{id}_{A \otimes B}
\end{array}$$

That completes the proof. □

For completeness' sake we also introduce a notion of morphism between categories with a linear exponential structure. But note that sometimes a stricter version is appropriate, see [Man04].

Definition 13 *Let \mathbf{C} and \mathbf{D} be symmetric monoidal categories with linear exponential comonads. A functor $F: \mathbf{C} \longrightarrow \mathbf{D}$ is **linearly distributive** if and only if F is monoidal (with structure $m_{\mathbf{I}}, m$) and is equipped with a distributive law $\kappa: !F \longrightarrow F!$ of a functor over a*

comonad, that is

$$\begin{array}{ccccc}
 & & !FA & \xrightarrow{\delta_{FA}} & !!FA \\
 & & \downarrow \kappa_A & & \downarrow !\kappa_A \\
 !FA & \searrow \epsilon_{FA} & & & \\
 \downarrow \kappa_A & & & & \\
 F!A & \xrightarrow{F\epsilon_A} & FA & & \\
 & & \downarrow \kappa_A & & \downarrow \kappa_{!A} \\
 & & F!A & \xrightarrow{F\delta_A} & F!!A
 \end{array}$$

respecting the comonoid structure, in the sense that

$$\begin{array}{ccccc}
 \mathbf{I} & \xleftarrow{e_{F(A)}} & !FA & \xrightarrow{d_{F(A)}} & !FA \otimes !FA \\
 \downarrow m_{\mathbf{I}} & & \downarrow \kappa_A & & \downarrow m_{!A, !A} \cdot (\kappa_A \otimes \kappa_A) \\
 F(\mathbf{I}) & \xleftarrow{F(e_A)} & F!A & \xrightarrow{F(d_A)} & F(!A \otimes !A)
 \end{array}$$

Definition 14 We say that a category \mathbf{C} is a **model for intuitionistic linear logic** if

- it is symmetric monoidal closed;
- it has finite products;
- it has a linear exponential comonad.

Exercise 22 Show that **Rel** is a model for intuitionistic linear logic.

Morphisms of models for intuitionistic linear logic are given by linearly distributive functors.

Before we move on to situations with more structure it makes sense to give an overview over other attempts at modelling intuitionistic linear logic. Instead of working in terms of a monoidal comonad one can work with a monoidal adjunction between a symmetric monoidal closed and a cartesian closed category [Ben95]. For comparisons of various definitions found in the literature, see [Mel]. It turns out that if one devises term calculi for linear logic (or looks at the internal languages of these different styles of categorical logic) then there are subtle differences [MMdPR].

2 Cartesian closed categories

Models for intuitionistic linear logic are not just interesting for their own sake. They give rise to a cartesian closed category. Cartesian closed categories are the standard model for the $\{\square, \rightarrow\}$ fragment of classical logic, and this is the right universe to give semantics for various programming languages. We already know from Proposition 4 that if \mathbf{C} has a linear exponential comonad then $\mathbf{C}^!$ has products, and they are given by the tensor product from Proposition 1. But if the underlying category \mathbf{C} also has products (on top of the symmetric monoidal and linear exponential comonad structure) then the Kleisli category is particularly well-behaved.

Proposition 5 *Let \mathbf{C} be a model for intuitionistic linear logic. Then*

$$!A \otimes !B \cong !(A \times B).$$

In particular the Kleisli category $\mathbf{C}_!$ for the linear exponential comonad is closed under the tensor product from $\mathbf{C}^!$. Hence the Kleisli category $\mathbf{C}_!$ has finite products.

Proof. The claim follows from Proposition 4 and Exercise 21(b) by the fact that right adjoints preserve products. \square

This isomorphism is part of Seely's [See89] definition of a model for linear logic, only that he was interested in the classical case which we consider in the last section. The following result also appears in [See89], if once again in the classical setting (which does not affect the proof).

Theorem 6 *Let \mathbf{C} be a model for intuitionistic linear logic. Then its Kleisli category is cartesian closed.*

Proof. By the previous proposition the Kleisli category $\mathbf{C}_!$ has finite products, and the product of A and B is $A \times B$. It remains to find a right adjoint for this functor. This is given by $A \rightarrow B = !A \multimap B$ as is shown by the following series of equations.

$$\begin{aligned} \mathbf{C}_!(A \times B, C) &\cong \mathbf{C}(!A \times !B, C) \\ &\cong \mathbf{C}(!A \otimes !B, C) \\ &\cong \mathbf{C}(!A, !B \multimap C) \\ &\cong \mathbf{C}_!(A, !B \multimap C) \\ &\cong \mathbf{C}_!(A, B \rightarrow C) \end{aligned}$$

\square

3 Classical linear logic

Classical linear logic can be understood to be obtained by adding negation to the intuitionistic fragment—at least if one uses the De Morgan rules to define all the missing connectives (while this does not do the logic justice it is how the model is built). We will not go into details here regarding the full system—see for example [Gir87, Gir95]. Variables in this system come in dual pairs, and then dualization (or negation) can be defined for all formulae using De Morgan-style rules. The system is typically two-sided, that is its judgements are of the form

$$A_1, \dots, A_m \vdash B_1, \dots, B_n.$$

Proofs of such judgements are interpreted by arrows

$$A_1 \otimes \dots \otimes A_m \longrightarrow B_1 \wp \dots \wp B_n,$$

where \wp is defined as the De Morgan dual of \otimes .

Formulae can be moved to the other side of \vdash by dualizing them. In particular, we can freely move between $A \vdash B$ and $B^\perp \vdash A^\perp$. Hence in our category we will have to be able to freely move between morphisms $A \longrightarrow B$ and morphisms $B^\perp \longrightarrow A^\perp$. This suggests having a contravariant functor $\mathbf{C} \longrightarrow \mathbf{C}$ as the interpretation of dualization, and one that is its own inverse.

Definition 15 We say that a category \mathbf{C} has an **involutive duality** if there is a functor $(-)^{\perp}: \mathbf{C} \longrightarrow \mathbf{C}^{\text{op}}$ which is adjoint to its partner $\mathbf{C}^{\text{op}} \longrightarrow \mathbf{C}$ and for which $(-)^{\perp\perp}$ is equivalent to the identity on \mathbf{C} .

Exercise 23 Argue that dualization (as the interpretation of negation) should preserve composition and identities.

Clearly we would like the categorical interpretation of dualization to satisfy the De Morgan equations connecting the various connectives. For \sqcup (the De Morgan dual of \sqcap), \wp , and $?$ (the De Morgan dual of $!$) we may indeed use the De Morgan equations to define interpretations, for example the semantics of $A \wp B$ will be $(A^{\perp} \otimes B^{\perp})^{\perp}$. But the connection we wish to achieve between \otimes and \multimap has to be built into the definition of what a model for (classical) linear logic should be.

Definition 16 A symmetric monoidal closed category \mathbf{C} is ***-autonomous** if has an object \perp such that the canonical arrows $A \longrightarrow (A \multimap \perp) \multimap \perp$ (the transposes of the evaluation maps) are isomorphisms.

Exercise 24 (a) Show that for a *-autonomous category we have an involutive duality given by $- \multimap \perp: \mathbf{C} \longrightarrow \mathbf{C}^{\text{op}}$. Further show that for that duality, for every object A , the functor $- \otimes A: \mathbf{C} \longrightarrow \mathbf{C}$ is left adjoint to $(A \otimes (-)^{\perp})^{\perp}: \mathbf{C} \longrightarrow \mathbf{C}$.

(b) Find conditions to add to the above which are required to ensure that adding an involutive duality to a symmetric monoidal category gives rise to a *-autonomous one.

If we have a *-autonomous category with products then in particular we have a symmetric monoidal closed category with finite products, so we can interpret MALL. But because we now can interpret negation, we can also interpret \wp using the De Morgan rules, so

$$A \wp B := (A^{\perp} \otimes B^{\perp})^{\perp}.$$

Exercise 25 Show that \mathbf{Rel} is *-autonomous. Hint: the dual of a set is the set itself; for morphisms the dual is defined by taking the reverse of the relation in question.

In fact, \mathbf{Rel} is more than just *-autonomous, it is what is called *compact closed*. A *-autonomous category is **compact closed** if for all objects A and B , $(A \otimes B)^{\perp} \cong A^{\perp} \otimes B^{\perp}$ or, equivalently, $A \multimap B \cong A^{\perp} \otimes B$. Compact closed categories are degenerate models of linear logic in the sense that the interpretation of \wp is the same as that of \otimes .

Exercise 26 (a) Show that a category with an involutive duality which has finite products also has finite coproducts.

(b) Define a linear exponential monad (to interpret $?$) and convince yourself that a category with an involutive duality which has a linear exponential comonad also has one of those.

This exercise justifies the following definition.

Definition 17 We say that a category \mathbf{C} is a **model for (classical) linear logic** if and only if it is

- *-autonomous;
- has finite products (and thus finite coproducts);

- has a linear exponential comonad (and thus a linear exponential monad).

In other words, all that is added to a model of intuitionistic linear logic is the self-dualization, which has to behave well when it comes to connecting the tensor and the linear function space. The rest of the structure comes for free, using De Morgan dualization.

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