Forty-five minutes

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Mathematical Techniques for Computer Science

16/11/14

Time: 13.00

Marking Scheme Included

Do not publish

This is a CLOSED book examination

The use of electronic calculators is not permitted.

1. a) Consider the following function:

$$\begin{aligned} f\colon \mathbb{Z} &\longrightarrow \mathbb{N} \\ x &\longmapsto \begin{cases} 3x & x \geq 0 \\ -3x+1 & \text{else.} \end{cases} \end{aligned}$$

Is this function injective? Is it surjective? Justify your answers. (5 marks)

Model answer and marking scheme for Q1a
This function is injective but not surjective. We can see that its range consists of elements of $\mathbb{N}$ which leave the remainder 0 or 1 when divided by 3. For example the number 2 is not in the range of this function.
To show that this function is injective assume we have m and n in $\mathbb{Z}$. If $fm = fn$ then clearly there are two cases.

- If $fm = fn$ leaves remainder 0 when divided by 3 then both, m and n must be greater than or equal to 0. Now $fm = fn$ implies that $3m = 3n$, and dividing by 3 on both sides we may conclude that $m = n$.

- If $fm = fn$ leaves remainder 1 when divided by 3 then both, m and n must be less than 0. Now $fm = fn$ implies that $3m+1 = 3n+1$, and first subtracting 1 and then dividing by 3 on both sides we may conclude that $m = n$.

One mark each for the two correct answers, one for a counterexample to surjectivity and two for the proof that it is injective.
I expect student answers to be less organized. If the argument they give is largely correct they should get both marks for the proof. If they can give an element of $\mathbb{N}$ that is not in the range of f they should get the mark, even if their argument why it is not in the range is weak.

b) Consider the following binary operation on $\mathbb{C}$: We set

$$z \circledast z' = z \cdot z' + z + z'.$$

Is this operation associative? Is it commutative? Justify your answers.

(5 marks)

Model answer and marking scheme for Q1b
The operation is both, commutative and associative. To show the former, let z and z' be in $\mathbb{C}$. Then

$$\begin{aligned} z \circledast z' &= zz' + z + z' && \text{def } \circledast \\ &= z'z + z' + z && \text{mult and add on } \mathbb{C} \text{ commut} \\ &= z' \circledast z && \text{def } \circledast. \end{aligned}$$

To show the latter, let z, z' and z'' be elements of $\mathbb{C}$. Then

$$\begin{aligned} (z \circledast z') \circledast z'' &= (zz' + z + z') \circledast z'' && \text{def } \circledast \\ &= zz'z'' + zz'' + z'z'' + + zz' + z + z' + z'' && \text{def } \circledast \\ &= zz'z'' + zz' + zz'' + z + z'z'' + z' + z'' && \text{add on } \mathbb{C} \text{ commut and assoc} \\ &= z \circledast (z'z'' + z' + z'') && \text{def } \circledast \\ &= z \circledast (z' \circledast z'') && \text{def } \circledast. \end{aligned}$$

One mark each for the correct answer, one mark for the proof of commutativity, two for that of associativity. *I would like to see some kind of justification for the calculations, but don't deduct too many marks if the proofs are largely there—maybe one mark in total if the proofs are otherwise correct.*

2. a) Show

$$(P \to Q) \equiv (\neg Q \to \neg P)$$

in the Boolean semantics by using truth tables. (3 marks)

Model answer and marking scheme for Q2a
We construct a truth table to compute the interpretations of the formulas

$$P \to Q \qquad \text{and} \qquad \neg Q \to \neg P$$

for the all possible valuations of P and Q.

P	Q	$P \to Q$	$\neg Q$	$\neg P$	$\neg Q \to \neg P$
1	**1**	1	0	0	1
1	**0**	0	1	0	0
0	**1**	1	0	1	1
0	**0**	1	1	1	1

Since the truth values in column 3 and the last column are the same, the two formulas are semantically equivalent.
2 marks for truth table; 1 mark for final answer and explanation.

b) Give a brief explanation of **one** of the following. (2 marks)

i. subformula
ii. Boolean valuation
iii. substitution for a propositional variable

Model answer and marking scheme for Q2b
Answers will obviously vary. Give 1 mark for each underlined aspects if they appear in their answers and what is written makes sense and is not complete non-sense.

i. A <u>part of a formula</u> that is itself a <u>formula</u> is a subformula of the formula.

ii. A Boolean valuation is a <u>truth assignment</u> to a list of <u>propositional variables</u>. (Or something like: it is a function v that maps a <u>propositional variable</u> to either <u>**1** or **0**</u>.)

iii. Substitution for a propositional variable is an operation where we place for <u>every occurrence</u> of the propositional variable in a formula <u>another formula</u>.

c) Consider the following propositional formula. (5 marks)

$$(\neg P \vee Q) \rightarrow (P \rightarrow (P \rightarrow Q))$$

i) Give a conjunctive normal form for the formula. Simplify as far as possible.
ii) Give a disjunctive normal form for the formula.

Model answer and marking scheme for Q2c
4 marks for part (i), 1 mark for part (ii). If students first computed DNF and ended up doing a lot of calculations here, and then computed the CNF reverse the number of marks awarded for each.
Please only give full marks if almost everything is explained and there are no mistakes. Any methods we covered in the course are acceptable.

i. We use the CNF algorithm and fundamental laws for simplification.

$$
\begin{array}{lll}
(\neg P \vee Q) \rightarrow (P \rightarrow (P \rightarrow Q)) & & \\
\quad \equiv & \neg(\neg P \vee Q) \vee (\neg P \vee (\neg P \vee Q)) & \text{Step 1/elim} \rightarrow \\
\quad \equiv & \neg(\neg P \vee Q) \vee \neg P \vee \neg P \vee Q & \text{assoc.} \\
\quad \equiv & (\neg\neg P \wedge \neg Q) \vee \neg P \vee \neg P \vee Q & \text{Step 2/De Morgan} \\
\quad \equiv & (P \wedge \neg Q) \vee \neg P \vee Q & \text{Step 3/double negation, idempotence} \\
\quad \equiv & (P \vee \neg P \vee Q) \wedge (\neg Q \vee \neg P \vee Q) & \text{Step 4/distr.} \\
\quad \equiv & \top \wedge (\neg Q \vee Q \vee \neg P) & \text{excl. middle, } \top \vee A \equiv \top\text{, comm.} \\
\quad \equiv & \top \wedge \top & \text{excl. middle, } \top \vee A \equiv \top \\
\quad \equiv & \top & \text{idempotence}
\end{array}
$$

The simplified CNF is $\top$.

ii. A DNF is $\top$.

END OF EXAMINATION