The Uses of SAT Solvers in Vampire

Giles Reger and Martin Suda

School of Computer Science, University of Manchester

The 2nd Vampire Workshop
Introduction

In this talk we will:

- Talk about the different use of SAT solvers in Vampire
  1. Finite Model Building
  2. AVATAR
  3. Instance Generation
  4. Global Subsumption
- Talk about how they could be better!
Overview

1. Finite Model Building
2. AVATAR
3. Instance Generation
4. Global Subsumption
5. Other Ideas
Finite Model Building

- Newly added to Vampire this year
- Just implements existing ideas
- Useful for establishing non-theorems i.e. satisfiability checking

Idea: For a domain size $n$ create a ground problem that is satisfiable if the original problem has a finite model of size $n$.

- The ground literals can be (consistently) named/translated into SAT variables and the ground problem decided by a SAT solver
- We can just check for bigger and bigger values of $n$
Preparing the Problem

- **Definition Introduction.** This reduces the size of clauses produced by flattening. A clause $p(f(a, b), g(f(a, b)))$ becomes $p(t_1, t_2)$ and we introduce the definition clauses $t_1 = f(a, b)$ and $t_2 = g(t_1)$.

- **Flattening.** This is necessary for the technique in general. A clause $p(f(a, b), g(f(a, b)))$ becomes

  $$p(x_1, x_2) \lor x_1 \neq f(x_3, x_4) \lor x_2 \neq g(x_1) \lor x_3 \neq a \lor x_4 \neq b$$

- **Splitting.** This can reduce the number of variables in clauses (important later). The clause $p(x, y) \lor q(y, z)$ is transformed to the two clauses $p(x, y) \lor s(y)$ and $\neg s(y) \lor q(y, z)$. 


The Constraints

- **Groundings.** For each (flattened) clause $C[x]$ and each vector of domain constants $d$ translate and add $C[d]

- **Functionality.** For each function symbol $f$ with arity $a$, vector of domain constants $d$ of length $a$ and distinct domain constants $d_1$ and $d_2$ translate and add $f(d) \neq d_1 \lor f(d) \neq d_2

- **Totality.** For each function symbol $f$ with arity $a$ and vector of domain constants $d$ of length $a$ translate and add $f(d) = d_1 \lor \ldots \lor f(d) = d_n$ for (all) the domain constants $d_i

- Note the exponential nature of these constraint sets
Symmetry Breaking and Sort Inference

- **Symmetry Breaking.**
  - Any model will be symmetrical in ordering of domain constants
  - So the SAT solver will be checking the same model multiple times
  - We can (partly) break these symmetries by ordering ground terms
  - Pick and order \( n \) ground terms (include all constants at the front)
  - For term \( t_i \) and domain size \( n \) add the clauses

\[
t_i \neq d_m \lor t_1 = d_{m-1} \lor \ldots \lor t_{i-1} = d_{m-1}
\]

for \( m \leq n \) and if \( i \leq n \) add

\[
t_i = d_1 \lor \ldots \lor t_i = d_i
\]

- **Sort Inference.**
  - Separate constants and function positions into different distinct sorts
  - Under certain conditions we can detect a maximum size for a sort
  - This information can render certain constraints redundant
Importance of the SAT Solver

- The majority of time is spent inside the SAT solver

- Therefore, making the SAT solver faster can improve this method.

- **Variable Elimination.** As implemented in e.g. MiniSAT. Idea is to apply all resolutions on a variable to eliminate it. Only do this if it will reduce the size. Removes pure variables.
  - Can help a lot
  - Can make things worse
Anything Else?

- **Deciding Non-Non-Theorems**
  - This is a decision procedure for EPR i.e. we stop at $n$ where $n$ is the number of constants in the problem.
  - The input can restrict the size of the domain, then we can detect the absence of a model i.e. $X = Y \lor X = Z$ means $n \leq 2$.

- **Incrementality?**
  - Idea (from Paradox): use and update single SAT solver.
  - Requires us to retract totality constraints.
  - Pros: we only have to generate new stuff, we get learned clauses.
  - Cons: we lose variable elimination.
Overview

1 Finite Model Building

2 AVATAR

3 Instance Generation

4 Global Subsumption

5 Other Ideas
A general architecture for proof search based on the idea of splitting

Still relatively new, very exciting, and you will hear about it a lot

Helps Vampire solve a lot of new problems

Allows for exciting new extensions for theory reasoning
  ▶ Combine with decision procedures i.e. use a SMT solver
  ▶ See VampireZ3 in CASC as a proof of idea
Motivation: Reasoning with heavy/long clauses is expensive

The set of clauses $S \cup (C_1 \lor \ldots \lor C_n)$ where $C_i$ are minimal pairwise variable-disjoint components is satisfiable if all of $S \cup C_i$ are

We call $C_i$ a component and say $C$ is splittable if $i > 1$

In general, $C_i$ is nicer than $C_1 \lor \ldots \lor C_n$

Therefore, it suffices to explore each of $S \cup C_i$ separately

To do this we need to

1. Decide which $C_i$ to assert/explore next
2. Backtrack our decision if that branch is unsatisfiable

In AVATAR we use a SAT solver to do this
AVATAR by Example

Input:
\[ p(a), \ q(b), \ \neg p(x) \lor \neg q(y) \]

Repeat
- FO: Process new clauses
  - split clauses into components
- SAT: Construct model
- FO: Use model (do splitting)
  - In FO use clauses with assertions
- FO: Do FO proving
  - Assertions must be preserved in inferences
- Process refutation

Components

FO

SAT
AVATAR by Example

- **Input:**
  
  \[ p(a), \quad q(b), \quad \neg p(x) \lor \neg q(y) \]

- **Repeat**

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**FO**

\[ p(a) \mid {} \]

**SAT**

Components
AVATAR by Example

- **Input:**
  \[ p(a), \ q(b), \ \neg p(x) \lor \neg q(y) \]

- **Repeat**
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  - **SAT: Construct model**
  - **FO: Use model (do splitting)**
    - In FO use clauses with assertions
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\[
\begin{align*}
\text{FO} & \quad \text{SAT} \\
p(a) & \mid \{\} \\
q(b) & \mid \{}
\end{align*}
\]
Input:

\[ p(a), \ q(b), \ \neg p(x) \lor \neg q(y) \]

Repeat

- **FO**: Process new clauses
  - split clauses into components
- **SAT**: Construct model
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  - In FO use clauses with assertions
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- Process refutation

Components:

\[ 1 \leftrightarrow \neg p(x) \]
\[ 2 \leftrightarrow \neg q(y) \]
AVATAR by Example

Input:
\[ p(a), \ q(b), \ \neg p(x) \lor \neg q(y) \]

Repeat

- **FO**: Process new clauses
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<table>
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<td>1 \iff \neg p(x)</td>
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<td>2 \iff \neg q(y)</td>
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\[ FO \]

\[ SAT \]

\[ p(a) \mid \{\} \]
\[ q(b) \mid \{\} \]

\[ 1 \lor 2 \]
AVATAR by Example

- **Input:**
  \[ p(a), \ q(b), \ \neg p(x) \lor \neg q(y) \]

- **Repeat**
  - **FO:** Process new clauses
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<td>( p(a) \mid { } )</td>
<td>( 1 \lor 2 )</td>
</tr>
<tr>
<td>( q(b) \mid { } )</td>
<td></td>
</tr>
<tr>
<td>( \neg p(x) \mid {1} )</td>
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**Components**

1 \( \mapsto \neg p(x) \)
2 \( \mapsto \neg q(y) \)
AVATAR by Example

- **Input:**
  \[ p(a), \ q(b), \neg p(x) \lor \neg q(y) \]

- **Repeat**
  - FO: Process new clauses
    * split clauses into components
  - SAT: Construct model
  - FO: Use model (do splitting)
    * In FO use clauses with assertions
  - FO: Do FO proving
    * Assertions must be preserved in inferences
  - Process refutation

```
Components
1 \mapsto \neg p(x)
2 \mapsto \neg q(y)
```

```
FO
\begin{align*}
p(a) & | \{\} \\
q(b) & | \{\} \\
\neg p(x) & | \{1\} \\
\bot & | \{1\}
\end{align*}
```

```
SAT
1 \lor 2
```
AVATAR by Example

- Input:
  \[ p(a), \; q(b), \; \neg p(x) \lor \neg q(y) \]

- Repeat
  - FO: Process new clauses
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  - SAT: Construct model
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\[
\begin{array}{c|c}
\text{FO} & \text{SAT} \\
\hline
p(a) & 1 \\
q(b) & 2 \\
\neg p(x) & \neg 1 \\
\bot & 1 \\
\end{array}
\]

Components

\[
\begin{align*}
1 & \leftrightarrow \neg p(x) \\
2 & \leftrightarrow \neg q(y)
\end{align*}
\]
AVATAR by Example

- **Input:**
  \[ p(a), \; q(b), \; \neg p(x) \lor \neg q(y) \]

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**FO**

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</tr>
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**SAT**

\[ 1 \lor 2 \]
\[ \neg 1 \]

**Components**

\[ 1 \mapsto \neg p(x) \]
\[ 2 \mapsto \neg q(y) \]
AVATAR by Example

- Input:
  
  \[ p(a), \neg p(x) \lor \neg q(y) \]

- Repeat
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AVATAR by Example

- **Input:**
  \[ p(a), \quad q(b), \quad \neg p(x) \lor \neg q(y) \]

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**FO**

\[
\begin{align*}
p(a) & \mid \{\} \\
q(b) & \mid \{\} \\
\neg p(x) & \mid \{1\} \\
\bot & \mid \{1\} \\
\neg q(y) & \mid \{2\} \\
\bot & \mid \{2\}
\end{align*}
\]

**SAT**

\[ 1 \lor 2 \quad \neg 1 \]

**Components**

\[
\begin{align*}
1 & \mapsto \neg p(x) \\
2 & \mapsto \neg q(y)
\end{align*}
\]
**AVATAR by Example**

- **Input:**
  \[ p(a), \ q(b), \neg p(x) \lor \neg q(y) \]

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Components

1 \[\mapsto\] \[\neg p(x)\]
2 \[\mapsto\] \[\neg q(y)\]
AVATAR by Example

- **Input:**
  \[ p(a), \ q(b), \neg p(x) \lor \neg q(y) \]

- **Repeat**
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    - split clauses into components
  - SAT: Construct model
  - FO: Use model (do splitting)
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- **FO**
  - \( p(a) \mid \{\} \)
  - \( q(b) \mid \{\} \)
  - \( \neg p(x) \mid \{1\} \)
  - \( \perp \mid \{1\} \)
  - \( \neg q(y) \mid \{2\} \)
  - \( \perp \mid \{2\} \)

- **SAT**
  - \( 1 \lor 2 \)
  - \( \neg 1 \)
  - \( \neg 2 \)

- **Components**
  - \( 1 \leftrightarrow \neg p(x) \)
  - \( 2 \leftrightarrow \neg q(y) \)
AVATAR by Example

- **Input:**
  
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- **Repeat**
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  - SAT: Construct model
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- **Refutation**
  - From the SAT solver
Varying the Architecture

- **Component Selection.**
  - What to do with ground literals?
  - What to do with unsplittable clauses?

- **What SAT solver to use, and how?**
  - Our own, MiniSAT, Lingeling
  - Setting various options

- **Minimizing the model.**
  - Do we need the whole model?
  - How does a partial model interact with splitting theory?
SAT Solver Effects

What is clear:
- The model produced by the SAT solver matters
- Faster SAT solving can help
- Incremental SAT solving can help

What is unclear:
- A lot...
- How important the model is, what a nice model is
- How important partial models are, what kind of partialness
- How much information we should give the SAT solver

Martin will say more today and on Thursday :)}
Overview

1. Finite Model Building
2. AVATAR
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Instance Generation

- **Observation**: By Hebrand Theorem, if a set of first-order clauses is unsatisfiable then there is a set of unsatisfiable ground instances that is also unsatisfiable.

- The idea of Instance Generation is then as follows:
  1. Given a set of first-order clauses $S$
  2. Produce ground abstraction $S_{\perp}$ by mapping vars to fresh constant $\perp$
  3. If $S_{\perp}$ is unsatisfiable then $S$ is unsatisfiable
  4. Otherwise, attempt to refine the abstraction by adding clauses to $S$
  5. Goto 2

- Checking satisfiability of $S_{\perp}$ can be done by a SAT solver.
Refine the Abstraction?

- How can the abstraction be too general?

- Consider $S = \{p(f(x, a)), \neg p(f(b, y))\}$
- This gives $S \perp = \{p(f(\perp, a)), \neg p(f(b, \perp))\}$
- Which is SAT but $S$ is unsatisfiable

- To refine the abstraction we add $p(f(b, a))$ and $\neg p(f(b, a))$

- Note that in the SAT solver $p(f(\perp, a))$ and $p(f(b, \perp))$ are just distinct variables
The InstGen rule

- This refinement is carried out by the InstGen rule:

\[
\frac{C \lor L}{(C \lor L)\sigma} \quad \frac{D \lor \overline{K}}{(D \lor \overline{K})\sigma}
\]

where \(\sigma = \text{mgu}(L, K)\) and \(\sigma\) is a proper instantiator of \(L\) or \(K\) and both \(L\) and \(\overline{K}\) are selected.

- A literal is selected if it is appears in the model of the SAT solver.
- This is based on the observation that the conflict that needs to be resolved by refinement is always between such literals.
In Practice

- Instance Generation is applied as a saturation algorithm.
- This means that we saturate (up to redundancy) the set of clauses with respect to the InstGen rule.
- We can use a prolific constant from the problem in groundings.
- We carry out restarts to reset the model periodically.
- We use dismatching constraints to remove some redundant inferences.
- We can combine with superposition by performing superposition proof search alongside this proof search and importing groundings of (unconditional) generated clauses into the SAT solver.
Combination with AVATAR?

- One possible extension to this setup is to share the SAT solver.

- Note that SAT variables are components in AVATAR and ground literals in Instance Generation but all ground literals are components.

- Only get overlap if we use a constant from the problem for grounding.

- Further idea, for component $C$ in AVATAR add $[C] \rightarrow [C\gamma]$.
- This connects non-ground parts of the AVATAR model with the Instance Generation model.
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Global Subsumption: the Ground Case

- This is a very effective simplification technique
- Let us consider the ground case first...

- Assume a set of first order clauses $S$
- Let $S_{gr}$ be a set of ground clauses implied by $S$
  i.e. instances of clauses in $S$
- The ground clause $D \lor D'$ can be replaced by $D$ in $S$ if $S_{gr} \models D$
- This is sound as $D$ follows from $S$ and subsumes $D \lor D'$
- If $D$ is empty then $S_{gr}$ is unsatisfiable and so is $S$
Global Subsumption: the Non-Ground Case

- We can lift this to give the non-ground global subsumption rule:

\[
\frac{C \lor C'}{C}
\]

where \( S_{gr} \models C\gamma \) for non-empty \( C' \) and injective substitution \( \gamma \) from variables in \( C \) to fresh constants.

- For every generated clause \( C \) we
  1. Let \( \gamma = [x_1 \mapsto c_1, \ldots x_n \mapsto c_n] \) for \( x_i \) in \( C \) and fresh \( c_i \)
  2. Add \( C\gamma \) to \( S_{gr} \)
  3. Search for a minimal \( C' \subset C \) such that \( S_{gr} \models C' \)

- We do not add more groundings to \( S_{gr} \) as we want this to be cheap.
Example

- Take the following case:
  - $C = p(x, y) \lor r(x)$
  - $S = \{p(x, y) \lor r(x), p(x, x)\}$

- $C$ cannot be reduced. Injectivity is important
  - If we do things wrong we can get $S_{gr} = \{p(a, b) \lor r(a), p(a, a)\}$
  - We check $\{p(a, a) \lor r(a), p(a, a), \neg p(a, a)\}$
  - We have $S_{gr} \models p(a, a)$ but $p(x, y)$ does not follow from $S$

- If we add $p(x, y)$ to $S$ then $C$ can be reduced
  - The correct grounding of $S$ is $S_{gr} = \{p(a, b) \lor r(a), p(a, a), p(a, b)\}$
  - We check $\{p(a, b) \lor r(a), p(a, a), p(a, b), \neg p(a, b)\}$
  - $C$ can be replaced by $p(x, y)$
SAT Solver Requirements

- As this a simplification technique we want it to be very quick
- Therefore, we only perform propagation in the SAT solver

- This means that we do not need the full power of the SAT solver
- One improvement would be to produce a restricted procedure that performs propagation only
Extending to combine with AVATAR?

- Currently only reason with unconditional clauses

- To reason with conditional clause $C \mid A$ we need to encode $A$ in the SAT solver i.e. translate $A \rightarrow C \gamma$

- Then, when attempting to reduce $C \mid A$ we
  - Assert $A$ for unconditional reduction
  - Assert AVATAR model for conditional reduction
    - Might need to extend $A$ in reduced clause

- Further idea: use this method to attempt to reduce $A$

- Finally, we could share the SAT solver with AVATAR (or Instance Generation) but as noted above, we may want a restricted solver for Global Subsumption
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Why the SAT Solver matters... and can we use this?

- In AVATAR and Instance Generation the model \textit{controls} proof search

- \textit{Idea: use Literal Selection to control the model generated}

- This requires a concept of \textit{nice model} for each technique:
  - For AVATAR this might be about \textit{minimal change} or \textit{minimality}
  - For Instance Generation this might be about \textit{minimising} the number of possible inferences or, conversely, to select \textit{more general} inferences first i.e. those that make others redundant
Conclusions

- SAT solvers can provide powerful mechanisms for implementing effective techniques inside a first-order saturation prover.

- But the way we use SAT solvers is not necessarily the same as the typical SAT usage.

- Therefore, as well as improving the techniques themselves we can consider altering the SAT solver to improve performance.