Chapter 2

Fourier Imaging

To understand the principles of image formation in optical imaging systems it is necessary to have a clear understanding of how the optical field propagates through the optical system. This requires knowledge of scalar diffraction theory and Fourier imaging.

The current chapter introduces the wave nature description of light, and how it can be used to describe the propagation of the optical field between two arbitrary planes. The analysis will begin with a description of Huygens' principle as a solution to the scalar wave equation, and Kirchhoff's formula as an interpretation of Huygens' principle. The analysis continues with the development of an expression describing the far-field diffraction pattern for a propagating optical field and its similarity with the two-dimensional Fourier transform. Finally, the far-field diffraction due to an optical field propagating through a bounded aperture is discussed as is the case of focusing via the thin lens.

2.1 Propagation of the optical field

To understand the diffraction of the optical field it is important to determine the manner in which the optical field propagates from one surface to another. This involves the solution of the scalar wave equation illustrated in eq. (2.1)

$$\nabla^2 V(r,t) - \frac{1}{c^2}\frac{d^2 V(r,t)}{dt^2} = 0 \tag{2.1}$$

where ∇^2 is the Laplacian operator, c is the velocity of the electromagnetic field, r is radial position $r = \{x, y\}$ and V is the optical disturbance [23,35,39]. Any solution to eq. (2.1) represents a possible optical field [39].

A monochromatic optical field can be expressed in the form

$$V(r,t) = a(r)\cos(2pvt + f(r)) \tag{2.2}$$

where $a(r)$ is the amplitude of the optical field, $f(r)$ is the phase of the optical field, v is the frequency of the wave and t is time. Alternatively, the monochromatic optical field may be expressed in the complex form

$$V(r,t) = y(r)\exp\{j2pvt\} \tag{2.3}$$

where $y(r)$ represents the spatial variation of the amplitude and phase of the disturbance. Substituting eq. (2.3) into eq. (2.1) gives

$$\nabla^2 y + k^2 y = 0 \tag{2.4}$$

which is the well-known Helmholtz equation, where the time dependence has been removed and

$$k = \frac{2pv}{c} = \frac{2p}{l} \tag{2.5}$$

is termed the wavenumber [35,39].

The solution to the Helmholtz equation is the mathematical description of the propagation of the optical field. Green's theorem [35] can be used to solve rigorously eq. (2.4). However, an adequate solution is provided by Huygens' principle

" ….every point on a primary wavefront serves as the source of secondary wavelets, such that the primary wavefront at some later time is the envelope of these wavelets… " [4,23,41,42,43]

where the terms wavefront and optical field can be used interchangeably.

Kirchhoff's diffraction formula is the mathematical interpretation of Huygens' principle. Since the Helmholtz equation is linear, then the linear superposition of solutions is permitted. Hence the optical field at any point in space is given by the sum of the corresponding spherical wavelets emanating from a point source, as described by

$$y(r) = \frac{\exp\{\pm jkr\}}{r} \tag{2.6}$$

where r is the radial distance from the source to the point of observation and $\pm$ indicates a diverging $(+)$ and a converging $(-)$ optical field respectively [23,35,39].

If the wave function, $y(x_1, y_1)$, describes the distribution of the field in a plane $\{x_1, y_1\}$, then Kirchhoff's diffraction formula allows the field, $y(x_2, y_2)$, in a plane $\{x_2, y_2\}$ to be expressed in terms of $y(x_1, y_1)$ as

$$y\left(x_2, y_2\right) = \iint_{-\infty}^{\infty} \frac{1}{j l r} y\left(x_1, y_1\right) \exp\left\{- jkr\right\} dx_1 dy_1 \tag{2.7}$$

where the diffraction geometry is illustrated in Fig. 2.1. Hence, an arbitrary optical field can be seen as a collection of spherical point sources and the field at any point is simply given by the sum of propagating spherical waves.

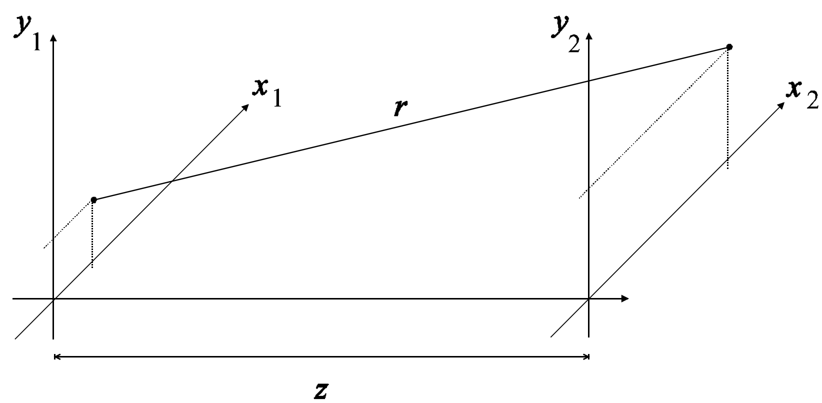

Figure 2.1 : *Diffraction geometry for Kirchhoff's diffraction formula.*

Equation (2.7) assumes that the optical field is slowly varying compared to the wavelength, and that the optical field is appreciable in a region around the optic axis that is small compared with the axial separation of the two planes.

Expanding *r*, using simple geometry, gives

$$r^2 = \left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + z^2 = z\sqrt{\left(1 + \frac{\left(x_2 - x_1\right)^2}{z^2} + \frac{\left(y_2 - y_1\right)^2}{z^2}\right)} \tag{2.8}$$

which by using the Binomial expansion gives

$$r = \left(z + \frac{\left(x_2 - x_1\right)^2}{2z} + \frac{\left(y_2 - y_1\right)^2}{2z}\right) \tag{2.9}$$

where higher terms than the second power have been ignored. Replacing *r* in the denominator of eq. (2.7) with *z*, and replacing *r* in the exponent of eq. (2.7) with eq. (2.9) allows us to express the Fresnel approximation for the diffracted field

$$y\left(x_2, y_2\right) = \frac{\exp\left\{-\,j l z\right\}}{j l z} \iint_{-\infty}^{\infty} y\left(x_1, y_1\right) \exp\left\{-\frac{jk}{2z}\left[\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2\right]\right\} dx_1 dy_1 \tag{2.10}$$

where the symbols have their usual meaning. If *z* is large compared to the maximum of x_1 and y_1 i.e.

$$z >> \left[\frac{k}{2}\left(x_1^2 + y_1^2\right)\right]_{\max} \tag{2.11}$$

then eq. (2.10) can be simplified to give the Fraunhofer or far-field approximation for the diffracted field [4,20,23,38,40,41,42,43], i.e.

$$\begin{aligned} y\left(x_2, y_2\right) = & \frac{\exp\left\{-\,j l z\right\}}{j l z} \exp\left\{\frac{-\,jk}{2z}\left(x_2^2 + y_2^2\right)\right\} \\ & \cdot \iint_{-\infty}^{\infty} y\left(x_1, y_1\right) \exp\left\{\frac{jk}{2z}\left(x_1 x_2 + y_1 y_2\right)\right\} dx_1 dy_1 \end{aligned} \tag{2.12}$$

where the symbols have their usual meaning. Equation (2.12) describes the far-field diffraction pattern, $y\,(x_2, y_2)$, in the plane $\{x_2, y_2\}$ due to an optical field, $y\,(x_1, y_1)$, in the plane $\{x_1, y_1\}$.

2.2 Scalar diffraction theory and the Fourier transform

The two-dimensional Fourier transform may be defined by

$$U(m,n) \;=\; \iint_{-\infty}^{+\infty} u(x,y) \exp\left\{2p\, j\left(mx + ny\right)\right\} dx\, dy \tag{2.13}$$

and its inverse transform defined by

$$u(x,y) \;=\; \iint_{-\infty}^{+\infty} U(m,n) \exp\left\{-\,2p\, j\left(mx + ny\right)\right\} dm\, dn \tag{2.14}$$

where (x,y) are co-ordinates in the spatial domain, (m,n) are co-ordinates in the spatial frequency domain, and $U(m,n)$ and $u(x,y)$ are Fourier transform pairs [20,40,42].

Recasting eq. (2.12) in the form

$$y\left(x_2, y_2\right) = \frac{\exp\{-jlz\}}{jlz} \exp\left\{\frac{-jk}{2z}\left(x_2^2 + y_2^2\right)\right\} \cdot \iint_{-\infty}^{\infty} y\left(x_1, y_1\right) \exp\left\{j2p\left(\frac{x_2}{lz}x_1 + \frac{y_2}{lz}y_1\right)\right\} dx_1 dy_1 \tag{2.15}$$

and comparing with eq. (2.14), it can be seen that the integral term in eq. (2.15) can be expressed simply as the Fourier transform of the field $y(x_1, y_1)$ in the plane $\{x_1, y_1\}$. Therefore eq. (2.15) can be recast in the form

$$y\left(x_2, y_2\right) \exp\left\{\frac{jk}{2z}\left(x_2^2 + y_2^2\right)\right\} = \frac{\exp\{-jlz\}}{jlz} \Psi\left(\frac{x_2}{lz}, \frac{y_2}{lz}\right) \tag{2.16}$$

where

$$\Psi\left(\frac{x_2}{lz}, \frac{y_2}{lz}\right) \Leftrightarrow y\left(x_1, y_1\right) \tag{2.17}$$

are Fourier transform pairs.

Hence, by neglecting phase factors and scaling constants, it can be seen that the far-field diffraction pattern in an arbitrary plane is given by the Fourier transform of the field in some prior plane, i.e.

$$y\left(x_2, y_2\right) = \Psi\left(\frac{x_2}{lz}, \frac{y_2}{lz}\right) \tag{2.18}$$

where z is the distance between the two planes $\{x_1, y_1\}$ and $\{x_2, y_2\}$, as illustrated in Fig. 2.1, and l is the wavelength of the incident illumination.

2.3 Diffraction due to finite sized apertures

It was shown above that the far-field diffraction pattern of an optical field is simply given by the two-dimensional Fourier transform of the field distribution in some prior plane. However, what would be the effect if the optical field was to propagate through some finite sized aperture? What effect would the physical bounds of the aperture have upon the propagation of the optical field?

As the optical field propagates through the aperture, its propagation will be altered by the aperture itself, and hence the far-field diffraction pattern becomes a function of the shape and transmission properties of the aperture. The transmission properties of the aperture are described by the so called ***aperture pupil function,*** $p(x,y)$ [4,38,42]. The optical field immediately after the plane of the aperture is simply given by the multiplication of the incident optical field distribution, $y\,(x_1, y_1)$, and the aperture pupil function, $p(x_1, y_1)$. If the incident optical field distribution is a uniform plane wave then the far-field diffraction pattern at some arbitrary plane $\{x_2, y_2\}$, is given by the two-dimensional Fourier transform of the aperture pupil function, i.e.

$$y\left(x_2, y_2\right) = \iint_{-\infty}^{\infty} p\left(x_1, y_1\right) \exp\left\{ j2p\left(\frac{x_2}{lz} x_1 + \frac{y_2}{lz} y_1\right)\right\} dx_1 dy_1 = FT\left\{p\left(x_1, y_1\right)\right\} \quad (2.19)$$

where the symbols have their usual meaning.

It is instructive to investigate the effects that the shape of the aperture has on the far-field diffraction pattern by considering three simple, and common, aperture pupil shapes: circular, square and annular.

Diffraction due to a circular aperture - the Airy disc

For a circular aperture of radius a, the aperture pupil function, $p(x, y)$, is given by

$$p\left(x,y\right) = \begin{cases} 1, & \sqrt{\left(x^2+y^2\right)} \le a \\ 0, & \sqrt{\left(x^2+y^2\right)} > a \end{cases}. \quad (2.20)$$

Since the circular aperture pupil function is radially symmetrical in the plane $\{x, y\}$ then its two-dimensional Fourier transform is given by the Fourier-Bessel, or Hankel, transform, as defined by eq. (2.21),

$$P(m,\ n) = \frac{2p\, aJ_1(2p\, a\sqrt{m^2+n^2}\,)}{2p\,\sqrt{m^2+n^2}} \quad (2.21)$$

where J_1 is the Bessel function of the first kind of order one [23,40,43] and m and n are the transform co-ordinates. Therefore, substituting into eq. (2.19), the far-field diffraction pattern in the plane $\{x_2, y_2\}$, due to a circular aperture of radius a in the plane $\{x_1, y_1\}$, is given by

$$y\,(x_2, y_2) = \frac{1}{lz}\;\frac{2p\,a^2 J_1\left(\dfrac{2p\,a\sqrt{x_2^2+y_2^2}}{l\,z}\right)}{\dfrac{2p\,a\sqrt{x_2^2+y_2^2}}{l\,z}} \qquad (2.22)$$

where z is the distance between the two planes, all other symbols have their usual meaning and phase factors have been ignored.

Hence, the diffracted irradiance due to the circular aperture is given by the square magnitude of eq. (2.22), i.e.

$$I(x_2, y_2) = \left|y\,(x_2, y_2)\right|^2 = \left(\frac{p\,a^2}{l\,z}\right)^2 \left(\frac{2J_1\left(\dfrac{2p\,a\sqrt{x_2^2+y_2^2}}{l\,z}\right)}{\dfrac{2p\,a\sqrt{x_2^2+y_2^2}}{l\,z}}\right)^2 . \qquad (2.23)$$

Equation (2.23) represents the well known ***Airy disc*** formula, and it describes the irradiance distribution that would be observed on a screen placed a distance z from the plane containing the circular aperture. Figure 2.2 illustrates the irradiance distribution of the Airy disc [23,38,41,42]. The transmission property of the aperture determines what proportion of the incident field will propagate through the aperture.

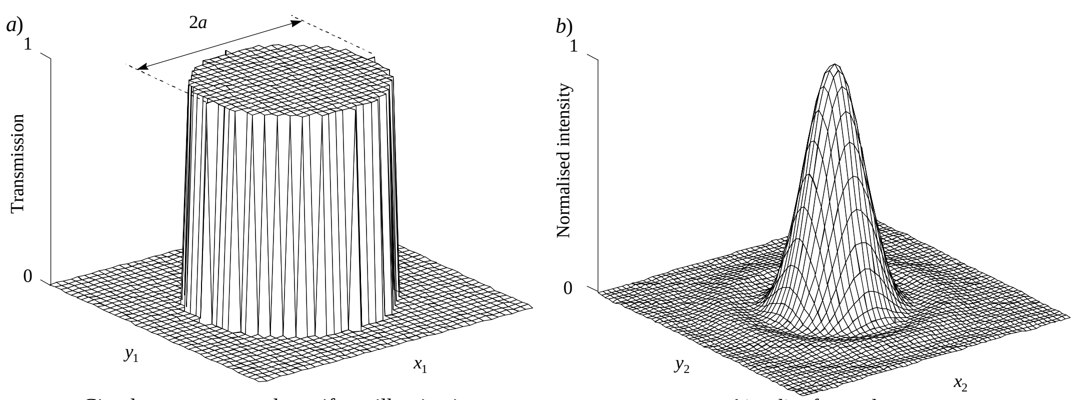

Figure 2.2 : *Diffraction due to a circular aperture, a),of radius a and the resulting Airy disc, b).*

Diffraction due to a rectangular aperture

For a rectangular aperture, in one-dimension, the aperture pupil function is given by the *rect* function, as defined by

$$p(x) = rect(x) = \begin{cases} 1, & |x| < \frac{1}{2} \\ 0, & |x| > \frac{1}{2} \end{cases} \quad . \tag{2.24}$$

The one-dimensional Fourier transform of the *rect* function, eq. (2.24), is given by the *sinc* function,

$$sinc(am) = \frac{\sin(pam)}{(pam)} \quad . \tag{2.25}$$

Hence, for a two-dimensional rectangular aperture pupil function the far-field diffraction pattern, when illuminated by a uniform plane wave, is given by the two-dimensional Fourier transform of the *rect* function, which by using eq. (2.19) gives

$$y\left(x_2, y_2\right) = a\ b\ sinc\left(\frac{ax_2}{lz}\right) sinc\left(\frac{by_2}{lz}\right) \tag{2.26}$$

where the rectangular aperture is of dimensions a and b. Figure 2.3 illustrates the far-field diffraction pattern of a square aperture under uniform illumination.

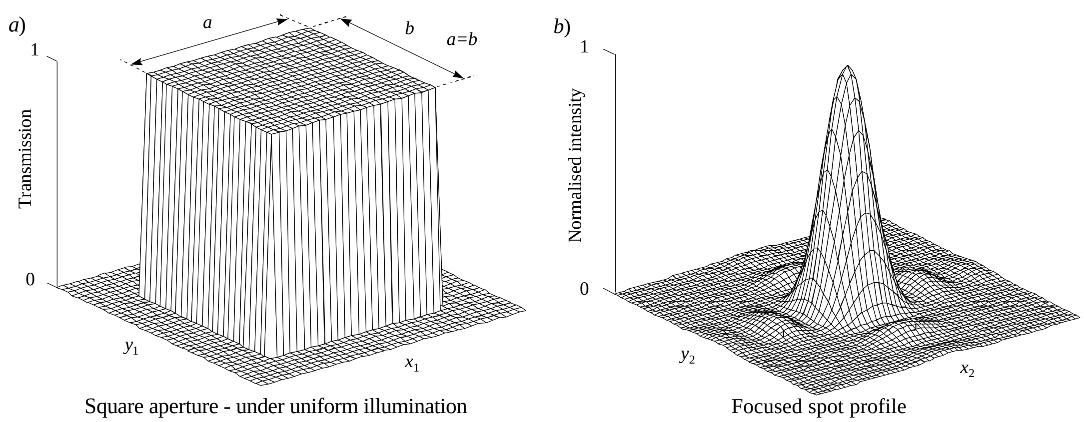

Figure 2.3 : *Diffraction due to a square aperture a), and the resulting diffracted irradiance profile b).*

Diffraction due to an annular aperture

For an annular aperture, with inner radius r_i and outer radius r_o, where $r_i < r_o$, the far-field diffraction pattern can be calculated following a similar procedure to the circular aperture. Therefore, using eq. (2.21), the diffracted field in the plane $\{x_2, y_2\}$ is given by

$$y(x_2, y_2) = \frac{1}{lz}\left\{\left[\frac{2pr_o^2 J_1\left(\frac{2p\, r_o\sqrt{x_2^2+y_2^2}}{l\, z}\right)}{\frac{2p\, r_o\sqrt{x_2^2+y_2^2}}{l\, z}}\right] - \left[\frac{2pr_i^2 J_1\left(\frac{2p\, r_i\sqrt{x_2^2+y_2^2}}{l\, z}\right)}{\frac{2p\, r_i\sqrt{x_2^2+y_2^2}}{l\, z}}\right]\right\} \quad (2.27)$$

and the resulting profile can be seen to be a function of two Airy discs. Figure 2.4 illustrates the resulting far-field diffraction pattern for an annular aperture under uniform illumination, where the ratio of the outer radius to the inner radius is 2:1.

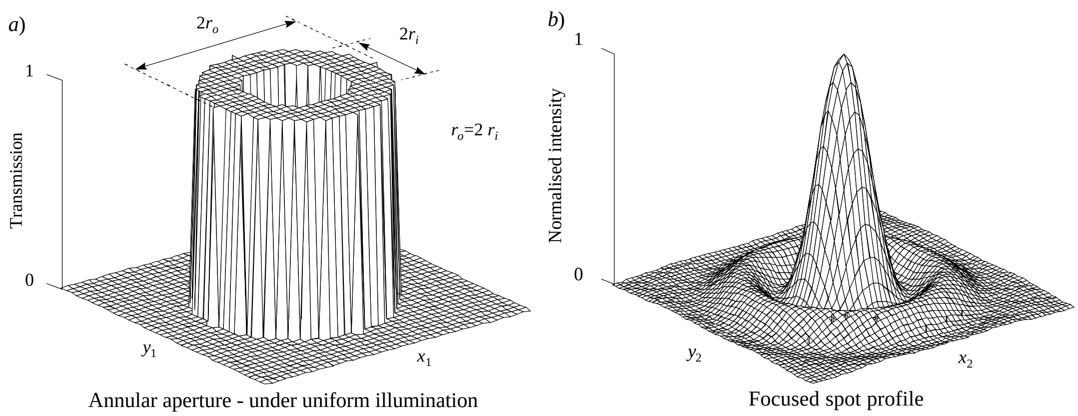

Figure 2.4 : *Diffraction due to an annular aperture, a), and the resulting diffracted irradiance profile, b).*

2.4 The Gaussian field distribution

It has been discussed that under uniform illumination the far-field diffraction pattern due to a finite sized aperture depends upon the shape and the transmission properties of the aperture. In practical laser imaging systems the incident optical field distribution on the aperture is often not uniform, but Gaussian in shape. The Gaussian distribution is given by

$$y(x, y) = y(0,0)\exp\left\{\frac{-\left(x^2+y^2\right)}{w^2}\right\} \quad (2.28)$$

where $y(0,0)$ is the magnitude of the field at the origin, and *w*, or $w_{e^{-2}}$, is termed the width parameter that defines the spatial extent of the Gaussian distribution; i.e. the point where the beam irradiance falls to e^{-2} the maximum value. Figure 2.5 illustrates the Gaussian intensity distribution for a width of $w = 1$.

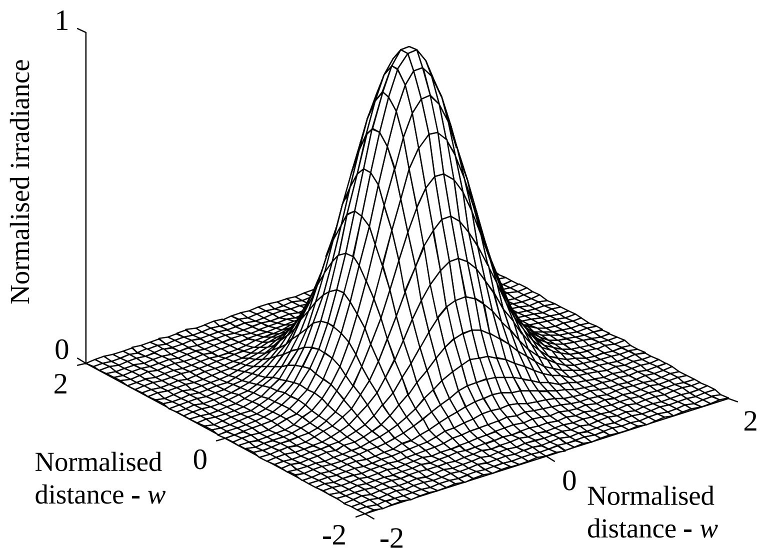

Figure 2.5 : *The Gaussian intensity distribution, of width w=1.*

The effect of the aperture pupil function on the incident Gaussian field distribution is to truncate it to zero outside the bounds of the aperture. There are two limiting situations of interest:

i) The *under-filled* case, where w is small compared to the dimensions of the aperture. In this case the Gaussian field distribution is effectively untruncated since the majority of the Gaussian field distribution passes through aperture.
ii) The *over-filled* case, where w is large compared to the dimensions of the aperture. In this case the Gaussian field distribution is severely truncated, and the field distribution immediately after propagation through the aperture approximates to the case of uniform illumination.

The far-field diffraction pattern under Gaussian illumination is a function of the product of the Gaussian distribution and the transmission properties of the aperture. In the under-filled case, the field distribution immediately after propagation through the aperture remains Gaussian, and the far-field diffraction pattern can be shown to exhibit the Gaussian distribution of eq. (2.29),

$$y\left(x_2, y_2\right) = \frac{pw^2}{lz}\exp\left\{-\left(\frac{kw\sqrt{\left(x_2^2+y_2^2\right)}}{2z}\right)^2\right\} \tag{2.29}$$

where the symbols have their usual meaning.

2.5 The thin lens

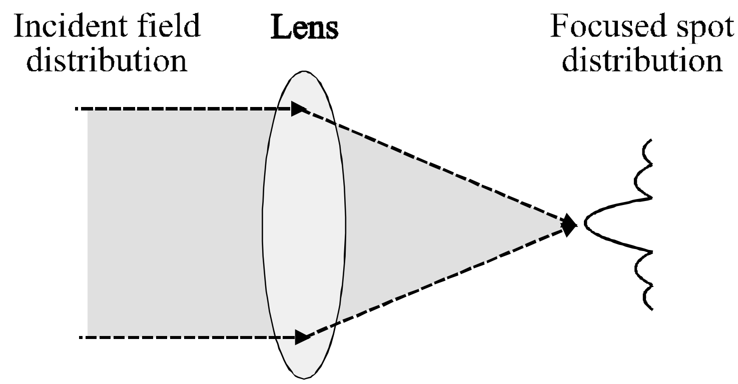

Figure 2.6 : *The thin lens.*

Figure 2.6 illustrates the operation of the thin lens, the purpose of which is to introduce a phase change into the propagating optical field. This phase change can introduce one of two effects,

(i) it collimates a spherical wave diverging from a point at a distance *f* from the lens into a plane wave, or alternatively,

(ii) it focuses a collimated beam to a diffraction limited spot at a distance *f* from the lens.

A real lens possesses a finite areal size, which is taken into account by introducing the aperture pupil function of the lens, $p(x,y)$, that is assumed for simplicity to take the value unity inside the aperture of the lens and zero outside. If the lens is illuminated by an optical field distribution $y\,(x_1, y_1)$, then the distribution of the optical field, $y\,'(x_1, y_1)$, immediately after the lens is given by

$$y\,'(x_1, y_1) = p_1\left(x_1, y_1\right)y\left(x_1, y_1\right)\exp\left\{\frac{jk}{2f}\left(x_1^2 + y_1^2\right)\right\} \tag{2.30}$$

where the exponential term represents the phase factor introduced by the lens. The phase term has the effect of focusing the collimated beam to a diffraction limited spot a distance *f*, the focal length of the lens, from the plane of the lens. Using Fresnel's diffraction formula, given by eq. (2.10), the optical field distribution in the plane, $\{x_2, y_2\}$, some distance after the lens, can be given by

$$y\left(x_2, y_2\right) = \frac{\exp(-jkz)}{j l\ z} \iint_{-\infty}^{\infty} y'\left(x_1, y_1\right) \exp\left\{\frac{-jk}{2z}\left[\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2\right]\right\} dx_1 dy_1 \tag{2.31}$$

which by expanding the exponential, and substituting *f* for *z* gives

$$\begin{aligned} y\left(x_2, y_2\right) = & \frac{\exp(-jkf)}{j l\ f} \exp\left\{\frac{-jk}{2f}\left(x_2^2 + y_2^2\right)\right\} \\ & \cdot \iint_{-\infty}^{\infty} y\left(x_1, y_1\right) p_1\left(x_1, y_1\right) . \exp\left\{\frac{jk}{f}\left(x_1 x_2 + y_1 y_2\right)\right\} dx_1 dy_1 \end{aligned} \tag{2.32}$$

which represents the field distribution in the focal plane, i.e. the distribution of the focused spot. Comparing eq. (2.32) with eq. (2.12), it can be seen that they are identical if the distance *z* is replaced by the distance *f* , the focal length of the lens. This simple but important result tells us that the field at the focal point of a lens obeys the far-field diffraction analysis presented previously, and hence this simple approach can be applied to modelling the behaviour of the optical field as it propagates through a thin lens.

If the lens is illuminated by a uniform plane wave, then the focused spot is given by the two-dimensional Fourier transform of the pupil function of the lens, $p_1(x_1, y_1)$, i.e.

$$y\left(x_2, y_2\right) = \iint_{-\infty}^{\infty} p_1\left(x_2, y_2\right) \exp\left\{\frac{jk}{f}\left(x_1 x_2 + y_1 y_2\right)\right\} dx_1 dy_1 = h_1\left(x_2, y_2\right) \tag{2.33}$$

where $h_1(x_2, y_2)$ is termed the ***amplitude point spread function*** of the lens.

If the lens has a circular aperture pupil function as described by eq. (2.20), then it can be seen that the far-field diffraction pattern at the focal point of the lens is given by eq. (2.22) but with the distance *z* replaced by *f*, the focal length of the lens.

If the incident illumination is uniform in distribution, then the irradiance distribution of the focused spot is given by the well known *Airy disc* formula as described by eq. (2.23), and illustrated in Fig. 2.2 [4,42]. The width of the Airy disc focused spot, w_{Airy} , is usually taken to be the distance between the first two minima, located either side of the main peak, and is a quantitative measure of the width of the focused spot. It can be shown that the width of the Airy disc is given by

$$w_{Airy} = 1.22\frac{fl}{a} \approx 1.22\frac{l}{NA} \tag{2.34}$$

where a is the radius of the circular aperture, f is the focal length of the lens, l is the wavelength of the incident illumination and NA is the numerical aperture of the lens[4,41,42].

An alternative expression for the spot size of the Airy disc is that of the full width at half maximum (FWHM). The FWHM of the Airy disc can be shown to be equal to

$$w_{Airy(FWHM)} = 0.6\frac{l}{NA} \tag{2.35}$$

where the symbols have their usual meaning. Figure 2.7 illustrates the Airy disc focused spot profile, and the measurement of w_{Airy} and $w_{Airy(FWHM)}$.

If the incident field distribution is Gaussian and assuming the field is untruncated by the aperture of the lens, then the diffraction pattern at the focal point of the lens is given by the Gaussian distribution of eq. (2.29). The width of the Gaussian focused spot is governed by the width of the incident Gaussian distribution in relation to the width of the aperture of the lens. For example, when $w = a/2$ (effectively an untruncated Gaussian) where a is the radius of the aperture of the lens, then the e^{-2} width of the Gaussian irradiance distribution in the focal plane is given by

$$w_{e^{-2}} = \frac{4fl}{pa} = \frac{4l}{pNA} \approx 1.22\frac{l}{NA} \tag{2.36}$$

where the symbols have their usual meaning. Figure 2.7 illustrates the Gaussian focused spot profile, and the measurement of $w_{e^{-2}}$.

By comparing the Airy disc and Gaussian spot profiles illustrated in Fig. 2.7 it can be seen that the Airy disc spot is narrower than that of the Gaussian spot, for an untruncated incident Gaussian field distribution. Hence, the highest imaging resolution will be observed for a uniform incident field distribution. However, the sidelobes in the Airy disc focused spot can give rise to undesirable imaging characteristics that will be described in chapter 3.

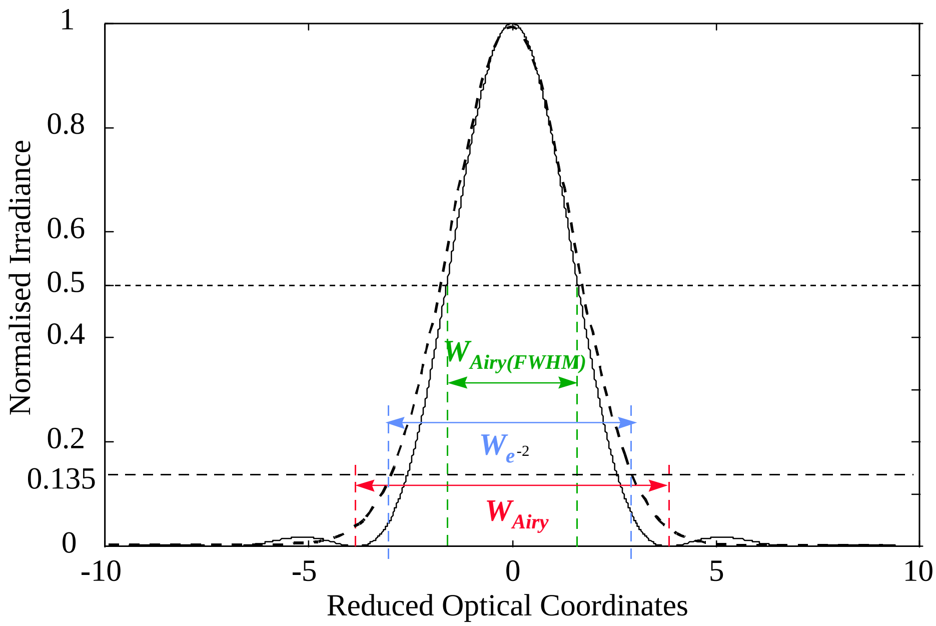

Figure 2.7 : *The irradiance distribution of the Airy disc (solid line) and Gaussian (bold dashed line) focused spot profiles illustrating the measurement of* w_{Airy} *and* $w_{Airy(FWHM)}$ *of the Airy disc profile, and* $w_{e^{-2}}$ *of the Gaussian profile.*

The form and width of the focused spot will vary depending upon the form of the incident illumination, and will change from Airy disc to Gaussian as the form of the incident illumination changes from uniform to untruncated Gaussian.