Explicit Set Existence


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Explicit Set Existence

I  Inexplicity of AC over ZF?
II  Core Mathematics
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I: Inexplicitness of AC over ZF?

- All the set existence axioms and schemes of ZF are explicit; e.g.

  **Pairing:** The class \( \{ a, b \} = \{ x \mid x = a \lor x = b \} \) is a set, for all sets \( a, b \).

- **Replacement:** For all sets \( a, \ldots, \)

\[
\forall x \in a \exists! y \phi(x, y, \ldots) \Rightarrow \{ y \mid \exists x \in a \phi(x, y, \ldots) \} \text{ is a set.}
\]

- AC seems to be essentially inexplicit over ZF; i.e. every explicit theorem of ZFC seems to be ‘equivalent’ to an explicit theorem of ZF.

- Can this idea be made precise?

- ZFC ⊢ ‘\( \{ x \mid \neg AC \} \text{ is a set} \)’, but if ZF is consistent

ZF ⊭ ‘\( \{ x \mid \neg AC \} \text{ is a set} \)’.
Some brands of mathematics

- Classical, with AC
- Classical, without any choice
- Topos
- Constructive, Brouwer style - Intuitionism
- Constructive, Markov style - Recursive
- Constructive, Bishop style
- Constructive, Richman style (= Bishop without any choice)
II: Core Mathematics

- All these, and others, are brands of mathematics.
- They are open conceptual frameworks.
- A lot of constructive mathematics can be derived in all these brands.
- Some mathematical principles are brand-essential.
  - Choice principles: AC, CC, DC, RDC, PA, ... etc
  - Logical: EM, REM, LPO, LLPO, MP, ... etc
  - Impredicative: Powerset, Full Separation, ...
Some criteria for Core Mathematics

- Extensional
- Adequate
- Compatible
- Local
- Explicit

Some problems with $\text{CZF}$ for a core system:
- Strong Collection is inexplicit.
- Subset Collection (Fullness) is inexplicit.
- Set Induction is not local.

Problems with $\text{CZF}_{R,E}^-$:
- Cannot show that $\mathbb{R}_d$ is a set.
- Do not have apparatus to define the class of hereditarily countable sets, etc.
III: The Fullness Axiom, 1

- The Fullness axiom is an inexplicit set existence axiom that can be used instead of the Subset Collection Scheme in axiomatizing CZF.

- In CZF the axiom has been used to prove Myhill’s Exponentiation axiom and also to prove that the class of Dedekind reals, $\mathbb{R}_d$, is a set and several other results.

- Some notation, for classes $A, B, R$:
  - $R : A \succ B$ if $\forall x \in A \exists y \in B (x, y) \in R$.
  - $R : A \rhd B$ if $R : A \succ B$ & $R^{-1} : B \succ A$.
  - $mv(A, B) = \{ r \in Pow(A \times B) | r : A \succ B \}$.
  - $B^A = \{ f \in mv(A, B) | f \text{ is single valued} \}$. 
The Fullness Axiom, 2

- **Exponentiation Axiom:** $\text{Exp}(A, B)$ for all sets $A, B$, where
  \[ \text{Exp}(A, B) \equiv B^A \text{ is a set.} \]

- **Fullness Axiom:** $\text{Full}(A, B)$ for all sets $A, B$, where
  \[ \text{Full}(A, B) \equiv \text{mv}(A, B) \text{ has a full subset,} \]
  where, for a class $C \subseteq X$,
  \[ C \text{ is a full subclass of } X \text{ if } \forall r \in X \exists s \in C \text{ s } s \subseteq r. \]

- **Strong Collection Scheme:** For each class $R$ and every set $A$, if $R : A \gtrless V$ then $R : A \gtrless < B$ for some set $B$.

- **AC(A,B):** $B^A$ is a full subclass of $\text{mv}(A, B)$. 

Fullness and Exponentiation

The axiom system BCST has Extensionality, Pairing, Union, $\Delta_0$-Separation and Replacement.

**Theorem:** In BCST,

1. $Full(A, B) \implies Exp(A, B)$,

2. $AC(A, B) + Exp(A, B) \implies Full(A, B)$. 
IV: The Dedekind Reals, 1: Weak cuts

- **$X \subseteq \mathbb{Q}$ is a weak left cut if**
  1-l: $\exists r (r \in X) \& \exists s (s \notin X)$,
  2-l: $r \in X \iff \exists s \ r < s \in X$.

- **$Y \subseteq \mathbb{Q}$ is a weak right cut if**
  1-r: $\exists r (r \in Y) \& \exists s (s \notin Y)$,
  2-r: $r \in Y \iff \exists s \ r > s \in Y$.

- **$(X, Y)$ is a weak cut if**
  - $X$ is a weak left cut and $Y$ is a weak right cut,
  - $X \cap Y = \emptyset$,
  - $r < s \Rightarrow (r \notin X \Rightarrow s \in Y) \& (s \notin Y \Rightarrow r \in X)$.

- **$(X, Y)$ is located if** $r < s \Rightarrow (r \in X \lor s \in Y)$.

- **$X$ is located if** $r < s \Rightarrow (r \in X \lor s \notin X)$.
The Dedekind Reals,2

A (left) cut is a located weak (left) cut.

Note: Classically every weak (left) cut is located.

Proposition: The following are equivalent:

- $X$ is a left cut,
- $(X, Y)$ is a cut for some $Y$,
- $(X, Y)$ is a cut, where $Y = \{s \in \mathbb{Q} \mid \exists r < s \ r \not\in X\}$.

Definition: The class $\mathbb{R}_d$ of Dedekind reals is the class of all left cuts. Note: $\mathbb{R}_d$ is a $\Delta_0$-class.

Prop: A weak left cut $X$ is located (and so in $\mathbb{R}_d$) iff

$$\forall \epsilon > 0 \ \exists r \in X \ r + \epsilon \not\in X; \quad \text{i.e.} \quad R_X \in \text{mv}(\mathbb{Q}>0, \mathbb{Q}),$$

where $R_X = \{(\epsilon, r) \in \mathbb{Q}>0 \times X \mid r + \epsilon \not\in X\}$. 

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The Dedekind Reals

**Theorem:** Assuming $Full(\mathbb{N}, \mathbb{N})$, the class of Dedekind reals is a set.

**Proof:** Assuming $Full(\mathbb{N}, \mathbb{N})$, as $\mathbb{Q}^>0 \sim \mathbb{N}$ and $\mathbb{Q} \sim \mathbb{N}$, we also have $Full(\mathbb{Q}^>0, \mathbb{Q})$.

So we may choose a full subset $\mathcal{C}$ of $mv(\mathbb{Q}^>0, \mathbb{Q})$

For $R \in mv(\mathbb{Q}^>0)$ let

$$X_R = \{ r \in \mathbb{Q} \mid r < s \text{ for some } (\epsilon, s) \in R \},$$

Now let $\mathcal{C}_X = \{ R \in \mathcal{C} \mid X_R \in \mathbb{R}_d \}$ and

$$\mathbb{R}' = \{ X_R \mid R \in \mathcal{C} \& X_R \in \mathbb{R}_d \} = \{ X_R \mid R \in \mathcal{C}_X \}.$$

Then, by $\Delta_0$-Separation and Replacement $\mathbb{R}'$ is a set.
The Dedekind Reals,

- If $R \in \mathcal{C}$, $X_R = \{ r \in \mathbb{Q} \mid r < s $ for some $(\epsilon, s) \in R \}$. 
- $\mathbb{R}' = \{ X_R \mid R \in \mathcal{C} \& X_R \in \mathbb{R}_d \}$ is a set.
- It suffices to show that $\mathbb{R}_d = \mathbb{R}'$.
- $\mathbb{R}_d \supseteq \mathbb{R}'$ trivially.
- If $X \in \mathbb{R}_d$,
  - $R_X = \{ (\epsilon, r) \in \mathbb{Q}^>0 \times X \mid r + \epsilon \notin X \} \in \text{mv}(\mathbb{Q}^>0, \mathbb{Q})$.
- For $\mathbb{R}_d \subseteq \mathbb{R}'$ it suffices to prove

**Lemma (ECST)**: Let $X \in \mathbb{R}_d$ and $R \in \mathcal{C}$. Then $R \subseteq R_X \Rightarrow X = X_R$.

$X \in \mathbb{R}_d \Rightarrow R_X \in \text{mv}(\mathbb{Q}^>0, \mathbb{Q})$, as $X$ is located
$\Rightarrow R \subseteq R_X$ for some $R \in \mathcal{C}$, as $\mathcal{C}$ is a full subset
$\Rightarrow X = X_R$, by the lemma.
V: Explicit Fullness; the scheme

For classes $F, X, A$ such that $F : X \to V$ and $A \subseteq X$, $F$ is $A$-powerful if, for all $r \in A$ there is $r' \in A$ such that

$$(*) \quad \forall s \in X [s \subseteq r' \Rightarrow s \in A \& Fs = Fr].$$

The Explicit Fullness Scheme (EFS): If $F : \text{mv}(B, C) \to V$ is $A$-powerful, where $B, C$ are sets and $A$ is a $\Delta_0$-subclass of $\text{mv}(B, C)$ then $FA = \{Fr \mid r \in A\}$ is a set.

Note that EFS is an explicit set existence scheme.

Theorem: In BCST, Fullness implies each instance of EFS. In fact Full($B,C$) implies the above instances, EFull($B,C$), of EFS.

Lemma: If $F : X \to V$ is $A$-powerful, $A$ is a $\Delta_0$-subclass of $X$ and $X$ has a full subset $D$ then $FA$ is a set.
V: Explicit Fullness; applications

- **Theorem (BCST+EFS):** Exponentiation
  - **Proof:** Given sets $B, C$, to show that $C^B$ is a set, apply EFS with $A = C^B$ and $Fr = r$ for $r \in mv(B, C)$.

- **Theorem (BCST+EFS):** Let $Q, A$ be sets such that $A \subseteq Q \times Q$. Then $R$ is a set, where $R$ is the class of subsets $X$ of $Q$ such that
  - $X$ is open; i.e. $\forall x \in X \exists y \in X (x, y) \in A$, and
  - $X$ is located; i.e. $\forall (x, y) \in A [x \in X \lor y \notin X]$.

  - **Note:** The proof only uses EFS(A,2).

- **Corollary (ECST+EFS(ℕ, 2)):** $R^e_d$ and $R_d$ are sets.

  - **Here** $R^e_d$ is the class of open, located subsets of $Q$, where $A = \{(r, s) \in Q \times Q \mid r < s\}$.

  - **Note that** $R_d \subseteq R^e_d$. 
VI: Deterministic Inductive Definitions, 1

Let $\Phi$ be a class. A $\Phi$-step, $X/y$, is a pair $(X, y) \in \Phi$. A class $A$ is $\Phi$-closed if

$$X \subseteq A \Rightarrow y \in A, \text{ for all } \Phi\text{-steps } X/y.$$ 

Theorem (CZF-Subset Collection): For each class $\Phi$ there is a smallest $\Phi$-closed class $I(\Phi)$.

The proof makes essential use of Strong Collection and Set Induction.

$\Phi$ is deterministic if

If $X_1/y$ and $X_2/y$ are $\Phi$-steps then $X_1 = X_2$.

ECST is BCST+Strong Infinity.
**Theorem (ECST + Set Induction):** The smallest class $I(\Phi)$ exists for each deterministic class $\Phi$.

**Examples:**

1. For each class $A$, $H(A) = I(\Phi_A)$, where $\Phi_A$ is the class of steps $y/y$ such that $y$ is an image of a set in $A$. So $H(\omega)$ is the class of hereditarily finite sets and $H(\omega \cup \{\omega\})$ is the class of hereditarily countable sets; i.e. hereditarily finite or an image of $\omega$. Here $\omega$ is the smallest inductive set, given by Strong Infinity.

2. If $A, R$ are classes, with $R \subseteq A \times A$ such that $R_y = \{x \in A \mid (x, y) \in R\}$ is a set for each $y \in A$, the class $WF(A, R) = I(\{R_y/y \mid y \in A\})$ is the well-founded part of $R$ in $A$.

3. Also the W-classes are given by deterministic inductive definitions.
CONCLUSION

- A possible useful axiom system for my core mathematics might be ECST+EFS+DIDS, where DIDS is a scheme in an extension of the language so as to obtain a class $I(\Phi)$ from a class $\Phi$.
- The scheme should express that if $\Phi$ is deterministic then $I(\Phi)$ is the smallest $\Phi$-closed class.
- The Replacement scheme and EFS need to be extended to the extended language.
- I conjecture that it has the same logical strength as CZF.