On Voevodsky's Univalence Axiom

6th July, 2011, Edinburgh

Third European Set Theory Conference .

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Plan of Lecture

I(1): Introduction

II(3): Higher dimensional category theory homotopy theory and homotopy type theory (HoTT)

III(3): The Structure Identity Principle (SIP)

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References:
Use google on Vladimir Voevodsky, Univalence Axiom and Homotopy Type Theory or HoTT

I: Introduction

Voevodsky's Univalence Axiom (UA) is a fundamental axiom, to be added to (intensional dependent) type theory, for a proposed Univalent Foundations of mathematics.

• Vladimir Voevodsky and Steve Awodey were the independent originators, around 2005/06, of the ideas at the basis of UA and Homotopy Type Theory (HoTT), an amalgam of

- Higher dimensional groupoid/category theory
- Homotopy theory
- Type theory

• My talk will focus on an application of UA, pointed out by Thierry Coquand, to a strong version of a

Structure Identity Principle (SIP).

II.1: Higher dimensional category theory

dim 0: Sets have elements/objects.

dim 1: Categories also have arrows between objects

dim 2: 2-categories have in addition arrows between those arrows.

⋮ ⋮

Identity between elements/objects

dim 0: standard equality between elements of a set.

dim 1: isomorphism between objects of a category

dim 2: equivalence between objects of a 2-category

⋮ ⋮

II.2: Groupoids and Homotopy Theory

- A (weak) $n+1$-category need only have identity and associative laws up to an n-equivalence.
- A groupoid is a category in which every arrow is invertible.
- A (weak) $n+1$-groupoid is a (weak) $n+1$-category in which each arrow is invertible up to an n-equivalence.

Homotopy Theory

- A Space has points, paths between points, homotopies (i.e. paths) between paths, etc ...
- Each space X has a set $\Pi_0(X)$ of its path connected, components, its fundamental groupoid $\Pi_1(X)$ and its higher dimensional groupoids $\Pi_n(X)$ for $n > 1$.
- A cts function $f : X \to Y$ is a weak equivalence if it induces isomorphisms $\Pi_n(X) \cong \Pi_n(Y)$ for all $n \geq 0$.

II.3: Homotopy Type Theory (HoTT)

- Interpretation of types as spaces and identity types as path spaces.

- Higher dimensional inductive definitions of the standard spaces.

- Hierarchy of homotopy levels of types.

- Univalence Axiom and the structure identity principle.

- Simplicial sets model of HoTT.

- HoTT in the coq proof development system.

III.1: Structure Identity Principle (SIP)

Isomorphic mathematical structures are structurally identical; i.e. have the same structural properties.

$$\mathcal{A} \cong \mathcal{B} \Rightarrow \mathcal{A} =_{str} \mathcal{B},$$

where, for structures $\mathcal{A}, \mathcal{B}$ of the same signature,

$$\mathcal{A} =_{str} \mathcal{B} \;:=$$

$P(\mathcal{A}) \;\Leftrightarrow\; P(\mathcal{B})$ for all structural properties P of structures of that signature.

• Structures may be higher order, many-sorted (or even dependently-sorted), infinitary, etc ...

III.2: What is a structural property?

• In mathematical practise the notion is usually not precisely defined, but is usually intuitively understood.
• In logic there can be a precise answer.
A structural property has the form P_T where
T is a set of L-sentences of a formal language L for the signature and

$$P_T(\mathcal{A}) := \ \mathcal{A} \text{ is a model of } T.$$

• There can be a variety of possible languages L for a signature, depending on the logic of L, which has to be able to express the ingredients of the signature.
• In category theory, when working with a category of structures, equality between objects is considered not meaningful. So the language being used only allows structural properties.

Homotopy Type Theory (HoTT)

- HoTT is intentional dependent type theory with the Univalence Axiom (UA).

- SIP in HoTT:

Isomorphic structures are identical

i.e. if C is the type of structures of some signature then

$$(A \cong_C B) \;\rightarrow\; Id_C(A, B)$$

where

- $(A \cong_C B)$ is the type of isomorphisms from A to B,
- $Id_C(A, B)$ is the type of witnesses that A, B are identical.

IV: Review of

Intensional Dependent Type Theory

A formal language in which only structural properties can be represented.

IV.1: The Forms of Judgment

A judgment has the form $\Gamma \vdash \mathcal{B}$ where Γ is a context

$$x_1 : A_1, x_2 : A_2[x_1], \ldots, x_n : A_n[x_1, \ldots, x_{n-1}]$$

and $\mathcal{B}$ has one of the forms

$$A[x_1, \ldots, x_n] \text{ type}$$
$$a[x_1, \ldots, x_n] : A[x_1, \ldots, x_n]$$
$$A[x_1, \ldots, x_n] = A'[x_1, \ldots, x_n]$$
$$a[x_1, \ldots, x_n] = a'[x_1, \ldots, x_n]$$

The $x_1, \ldots, x_n$ are distinct variables and each

$$x_i : A_i[x_1, \ldots, x_{i-1}]$$

is a variable declaration.

IV.2: Rules of Inference

Each instance of a rule of inference has the form

$$\frac{J_1 \quad \cdots \quad J_n}{J_0}$$

where each J_i is a possible judgment. Rules are presented schematically using obvious conventions such as the suppression of parametric declarations.

For example the scheme $\dfrac{A, B \text{ type}}{(A \to B) \text{ type}}$

will have instances, for any context Γ,

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash B \text{ type}}{\Gamma \vdash (A \to B) \text{ type}}.$$

IV.3: Some more schemes for $(A \to B)$

$$\frac{x : A \vdash b[x] : B}{(\lambda x : A)b[x] : (A \to B)} \qquad \frac{f : A \to B \quad a : A}{fa : B}$$

$$\frac{x : A \vdash b[x] : B \quad a : A}{((\lambda x : A)b[x])a = b[a] : B}$$

IV.4: Basic forms of type

$\mathbb{0}, \mathbb{1}, \mathbb{B}, \mathbb{N}$:	standard ground types
$A \to B$:	Function type
$A \times B$:	Cartesian Product type
$A + B$:	Disjoint Union type

and when there are dependent types

$(\Pi x : A)B[x]$:	type of functions $fx : B[x]$ for $x : A$
$(\Sigma x : A)B[x]$:	type of pairs (x, y) for $x : A, y : B[x]$

We could define

$$A \to B := \quad (\Pi\_ : A)B$$

$$A \times B := \quad (\Sigma\_ : A)B$$

IV.5: Propositions as Types

The dictionary for representing logic in the Curry-Howard correspondence:

prop	A true	$\bot$	$\top$	$A \to B$	$A \wedge B$	$A \vee B$
type	$- : A$	$\mathbb{0}$	$\mathbb{1}$	$A \to B$	$A \times B$	$A + B$

prop	$(\forall x : A)B[x]$	$(\exists x : A)B[x]$	$a =_A a'$
type	$(\Pi x : A)B[x]$	$(\Sigma x : A)B[x]$	$??$

Per Martin-Löf introduced identity types into type theory:

prop	$a =_A a'$
type	$Id_A(a, a')$

IV.6: Identity Rules

Logical Identity Rules: $x : A \vdash x =_A x$

$$\frac{\begin{cases} x, y : A \vdash \phi[x, y] \text{ prop} \\ x : A \vdash \phi[x, x] \text{ true} \end{cases}}{x, y : A, x =_A y \vdash \phi[x, y] \text{ true}}$$

Type Theory Identity Rules: $x : A \vdash rx : Id_A(x, x)$

$$\frac{\begin{cases} x, y : A, z : Id_A(x, y) \vdash C[x, y, z] \text{ type} \\ x : A \vdash b[x] : C[x, x, rx] \end{cases}}{\begin{cases} x, y : A, z : Id_A(x, y) \vdash J(x, y, z) : C[x, y, z] \\ x : A \vdash J(x, x, rx) = b[x] : C[x, x, rx] \end{cases}}$$

We write $a \sim_A b$ or just $a \sim b$ for $Id_A(a, b)$.

IV.7: Type Universe (à la Russell)

A type universe $\mathbb{U}$ is a type, whose elements are types (the small types). It has the closure properties given by the basic forms of type; i.e.

$$\mathbb{0}, \mathbb{1}, \mathbb{B}, \mathbb{N} : \mathbb{U} \qquad \frac{A, B : \mathbb{U}}{A + B : \mathbb{U}}$$

$$\frac{A : \mathbb{U} \qquad x : A \vdash B[x] : \mathbb{U}}{\begin{cases} (\Pi x : A) B[x] : \mathbb{U} \\ (\Sigma x : A) B[x] : \mathbb{U} \end{cases}} \qquad \frac{A : \mathbb{U}}{x, x' : A \vdash (x \sim_A x') : \mathbb{U}}$$

IV.8: Type Theoretic AC

Let $C := (\Pi x : A)B[x]$, where A is a type and $B[x]$ is a type for $x : A$.

Theorem: If $R[x, y]$ is a type for $x : A, y : B[x]$ then $AC_{C,R}$, where $AC_{C,R}$ is the type

$$(\Pi x : A)(\Sigma y : B[x])R[x, y] \;\rightarrow\; (\Sigma f : C)(\Pi x : A)R[x, fx].$$

IV.9: Function Extensionality Axiom

Let $C := (\Pi x : A)B[x]$, where A is a type and $B[x]$ is a type for $x : A$.

The Axiom:

$$FEA_C := (\Pi f, f' : C)[f \approx f' \;\to\; f \sim f'],$$

where

$$f \approx f' := (\Pi x : A)\; fx \sim f'x.$$

As $(\Pi f : C)[f \approx (\lambda x : A)fx]$, an immediate consequence of FEA_C is the Eta Axiom (EA_C), where EA_C is the type

$$(\Pi f : C)\; f \sim (\lambda x : A)fx.$$

V: The Univalence Axiom

V.1: Type Equivalence

- A type is contractible if $contr(X)$, where $contr(X)$ is the type
$$(\Sigma x : X)(\Pi x' : X)\ x \sim x'.$$
- In PaT $contr(X)$ expresses the proposition that X is a singleton.
- If $f : A \to B$ let $A \overset{f}{\simeq} B \ := \ (\Pi y : B)contr(f^{-1}y)$, where $f^{-1}y := (\Sigma x : A)fx \sim y$ for $y : B$.
- In PaT $A \overset{f}{\simeq} B$ expresses the proposition that $f : A \to B$ is injective and surjective.
- In HoTT it can express that $f : A \to B$ is a weak equivalence.
- Let $A \simeq B := (\Sigma f : A \to B)(A \overset{f}{\simeq} B)$.

Proposition: There is $r^{\simeq}_{A} = (id_A, w_A) : A \simeq A$.

V.2: Type Isomorphism

- Let A, B, C be types. Define $id_A := (\lambda x : A)x : A \to A$ and, if $f : A \to B$ and $g : B \to C$, $g \circ f := (\lambda x : A)g(fx)$.

- If $f : A \to B$ let

$$A \overset{f}{\cong} B := (\Sigma g : B \to A)[(g \circ f \approx id_A) \times (f \circ g \approx id_B)]$$

- In PaT the type $A \overset{f}{\cong} B$ expresses the proposition that $f : A \to B$ is an isomorphism.

- In HoTT the type $A \overset{f}{\cong} B$ can express that $f : A \to B$ is a homotopy equivalence.

- Let $A \cong B \ := \ (\Sigma f : A \to B)\ A \overset{f}{\cong} B$.

Proposition: $A \cong B \ \to \ (A \ \leftrightarrow \ B)$.

Proposition: $(A \simeq B) \ \leftrightarrow \ (A \cong B)$.

V.3: The Univalence Axiom (UA)

Let $\mathbb{U}$ be a type universe. Using the Elimination rule for $\mathbb{U}$ there is

$$EXYZ : (X \simeq Y) \text{ for } X, Y : \mathbb{U}, Z : (X \sim Y)$$

such that $EXX(rX) = r_X^{\simeq} : (X \simeq X)$ for $X : \mathbb{U}$.

So $EXY : (X \sim Y) \to (X \simeq Y)$ for $X, Y : \mathbb{U}$.

The Axiom:

$$UA(\mathbb{U}) \;:=\; (\Pi X, Y : \mathbb{U})[(X \sim Y) \overset{EXY}{\simeq} (X \simeq Y)].$$

Theorem: $UA(\mathbb{U}) \;\to\; [X \cong Y \;\leftrightarrow\; X \sim Y]$ for $X, Y : \mathbb{U}$.

Theorem: Univalence Axiom and Eta Axiom for $\mathbb{U}$ implies Function Extensionality for $\mathbb{U}$.

Theorem (Voevodsky)

In ZFC+ a Grothendiek Universe, intensional dependent type theory with $UA(\mathbb{U})$ has an interpretation in the category of simplicial sets.

I have yet to study and understand the proof!

Conclusion

Assuming $UA(\mathbb{U})$

SIP in HoTT

e.g. for $A, B : \mathbb{U}$

- $A \cong B \;\rightarrow\; A \sim_{\mathbb{U}} B$,
- for $a : A$, $b : B$,

$(A, a) \cong (B, b) \;\rightarrow\; (A, a) \sim_C (B, b)$ where $C := (\Sigma X : \mathbb{U})X$,

- for $G, G' : AbGrp(\mathbb{U})$ where $G = (|G|, 0_G, +_G)$, $G' = (|G'|, 0_{G'}, +_{G'})$

$$G \cong G' \;\rightarrow\; G \sim_{AbGrp(\mathbb{U})} G'.$$