On avoiding dependent choices in Formal Topology

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Why avoid countable/dependent choice?

• “Is it reasonable to do constructive mathematics without the axiom of countable choice? Serious schools of constructive mathematics all assume it one way or another, but the arguments for it are not compelling.” - from “The fundamental theorem of algebra: a constructive development without choice”, by Fred Richman.
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- “The theory should be \textit{compatible} with known theories such as classical set theory and the theory of a generic topos, ... and hence it should be \textit{minimal} with respect to such more expressive existing theories” - from “Toward a minimalist foundation for constructive mathematics”, by Maria Emilia Maietti and Giovanni Sambin.
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- “The theory should be compatible with known theories such as classical set theory and the theory of a generic topos, ... and hence it should be minimal with respect to such more expressive existing theories” - from “Toward a minimalist foundation for constructive mathematics”, by Maria Emilia Maietti and Giovanni Sambin.

- I suggest: A good core (minimal?) theory for constructive set theory should, at least, be preserved by the Heyting valued model construction.
Three results of $\text{CZF} + \text{uREA} + \text{DC}$
Three results of $CZF + uREA + DC$

1: Inductive and Coinductive definitions can be used to generate set-presented, balanced formal topologies. (A formal topology is balanced if it comes with a compatible binary positivity predicate).
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1: Inductive and Coinductive definitions can be used to generate set-presented, balanced formal topologies. (A formal topology is balanced if it comes with a compatible binary positivity predicate).

2: The category of set-presented formal topologies and continuous maps has coequalisers. (Erik Palmgren)
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3: The dcpo of formal points of a set-presented formal topology is set-generated.
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For 1,2 $uREA + DC$ can be weakened to $*REA$.
For 3 something that seems a bit stronger than $*REA$, $*_2REA$, seems to be needed.
Plan of talk

- Review
- Relation Reflection
- Coinductive Definitions
- If time: a slide on characterising the collection of positivity relations
Some axiom systems for Constructive Set Theory

These are formulated in the language of ZF, but use intuitionistic logic.

\[ CZF^o \equiv \text{Extensionality, Pairing, Union, Infinity axioms,} \]
\[ \text{Restricted Separation scheme,} \]
\[ \text{Set Induction and Strong Collection schemes.} \]
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\[ CZF \equiv CZF^o + \text{Subset Collection Axiom.} \]
Some axiom systems for Constructive Set Theory

These are formulated in the language of ZF, but use intuitionistic logic.

- $CZF^o \equiv$ Extensionality, Pairing, Union, Infinity axioms, Restricted Separation scheme, Set Induction and Strong Collection schemes.
- $CZF \equiv CZF^o +$ Subset Collection Axiom.
- $CZF^+ \equiv CZF^o +$ Regular Extension Axiom.
We use class notation in the standard way. Each class $X$ is defined by a formula with parameters $a, \ldots$:

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Class notation and terminology

- We use class notation in the standard way. Each class \( X \) is defined by a formula with parameters \( a, \ldots: \)

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\forall x [x \subseteq X \Rightarrow x \in X] \Rightarrow \forall x [x \in X].
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- Restricted Separation: For each restricted class \( X \), if \( a \) is a set the class \( a \cap X \) is a set.

- A class is **restricted** if it is defined by a restricted formula; i.e. a formula all of whose quantifiers have one of the forms \((\forall x \in t)\) or \((\exists x \in t)\) where \( t \) is a variable or parameter.
The Collection Principles

Definition: For classes $X, Y, R$

$R : X \rightarrow Y$ \text{ iff } (\forall x \in X)(\exists y \in Y) (x, y) \in R$

$R : X \rightarrow \leftarrow Y$ \text{ iff } [R : X \rightarrow Y] \& (\forall y \in Y)(\exists x \in X) (x, y) \in R$
The Collection Principles

**Definition:** For classes $X, Y, R$

\[ R : X \succ Y \quad \text{iff} \quad (\forall x \in X)(\exists y \in Y) \ (x, y) \in R \]

\[ R : X \succ\succ Y \quad \text{iff} \quad [R : X \succ Y] \& (\forall y \in Y)(\exists x \in X) \ (x, y) \in R \]

**Strong Collection Scheme:** For classes $Y, R$, if $a$ is a set such that $R : a \succ Y$ then there is a set $b \subseteq Y$ such that $R : a \succ\succ b$. 
The Collection Principles

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R : X \succ Y \quad \text{iff} \quad (\forall x \in X)(\exists y \in Y) \ (x, y) \in R
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R : X \succ\prec Y \quad \text{iff} \quad [R : X \succ Y] \& (\forall y \in Y)(\exists x \in X) \ (x, y) \in R
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**Subset Collection Axiom:** For all sets $a, b$ there is a set $c$ of subsets of $b$ such that for every set $r : a \succ b$ there is a set $b' \in c$ such that $r : a \succ\prec b'$.
The Collection Principles

Definition: For classes $X, Y, R$

\[ R : X \supseteq Y \iff (\forall x \in X)(\exists y \in Y) \ (x, y) \in R \]

\[ R : X \supseteq Y \iff [R : X \supseteq Y ] \& (\forall y \in Y)(\exists x \in X) \ (x, y) \in R \]

Strong Collection Scheme: For classes $Y, R$, if $a$ is a set such that $R : a \supseteq Y$ then there is a set $b \subseteq Y$ such that $R : a \supseteq b$.

Subset Collection Axiom: For all sets $a, b$ there is a set $c$ of subsets of $b$ such that for every set $r : a \supseteq b$ there is a set $b' \in c$ such that $r : a \supseteq b'$.

or: For all sets $a, b$ there is a set $c$ of subsets of $a \times b$ such that for every set $r : a \supseteq b$ there is a set $r' \in c$ such that $r \supseteq r' : a \supseteq b$. 
The Regular Extension Axiom

A set \( A \) is **transitive** if \( \forall x \in A \ x \subseteq A \).
The Regular Extension Axiom

- A set $A$ is **transitive** if $\forall x \in A \ x \subseteq A$.
- A transitive set $A$ is **regular** if it satisfies the second-order version of Strong Collection; i.e.
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  for all sets $Y, R$ such that $Y \subseteq A$ and $R \subseteq A \times A$, if $a \in A$ such that $R : a \succ Y$ then there is a set $b \in A$ such that $b \subseteq Y$ and $R : a \succ b$. 

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  for all sets $Y, R$ such that $Y \subseteq A$ and $R \subseteq A \times A$, if $a \in A$ such that $R : a \rightarrow Y$ then there is a set $b \in A$ such that $b \subseteq Y$ and $R : a \rightarrow<b$.

- **The Regular Extension Axiom (REA):** Every set is a subset of a regular set.
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- $uREA$: Every set is a subset of a union-closed regular set.
The Regular Extension Axiom

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- **The Regular Extension Axiom (REA):** Every set is a subset of a regular set.
- $A$ is **union-closed** if $\forall x \in A \cup x \in A$.
- **$uREA$:** Every set is a subset of a union-closed regular set.
- $CZF^{+} \equiv CZF^{\circ} + REA \ (\equiv CZF + REA)$. 
The Regular Extension Axiom

- A set $A$ is **transitive** if $\forall x \in A \ x \subseteq A$.
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- **The Regular Extension Axiom (REA):** Every set is a subset of a regular set.
- **$uREA$:** Every set is a subset of a union-closed regular set.
- $CZF^+ \equiv CZF^o + REA$ (≡ $CZF + REA$).
- $CZF^u \equiv CZF^o + uREA$ (≡ $CZF + uREA$).

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Some Choice Principles

Countable Choice ($CC$): For all sets $X$, $R$, if $R : \mathbb{N} \rightharpoonup X$ then there is a $f : \mathbb{N} \to X$ such that

$$(\forall n \in \mathbb{N}) \ (n, f(n)) \in R.$$
Some Choice Principles

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Dependent Choices (DC): For all sets $X, R$, if $R : X \rhd X$ and $a \in X$ then there is $f : \mathbb{N} \to X$ such that $f(0) = a$ and

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Some Choice Principles

Countable Choice ($CC$): For all sets $X, R$, if $R : \mathbb{N} \rightarrow X$ then there is a $f : \mathbb{N} \rightarrow X$ such that

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Relative Dependent Choices ($RDC$): For classes $X, R$, if $R : X \succ X$ and $a \in X$ then there is $f : \mathbb{N} \rightarrow X$ such that $f(0) = a$ and

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Relation Reflection

 Relation Reflection Scheme \((\text{RRS})\): For classes \(X, R\) such that \(R : X \succ X\), if \(a\) is a subset of \(X\) then there is a subset \(b\) of \(X\) such that \(a \subseteq b\) and \(R : b \succ b\).
Relation Reflection

- **Relation Reflection Scheme (RRS):** For classes $X, R$ such that $R : X \succ X$, if $a$ is a subset of $X$ then there is a subset $b$ of $X$ such that $a \subseteq b$ and $R : b \succ b$.

- A transitive set $A$ has the **Relation Reflection Property** if, for all sets $X, R$ if $a$ is a subset of $X$ such that $a \in A$ then there is a subset $b$ of $X$ such that $b \in A$, $a \subseteq b$ and $R : b \succ b$. 
Relation Reflection

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- A union-closed regular set is $\ast$-regular if it has the Relation Reflection Property.
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$*REA$: Every set is a subset of a $\ast$-regular set.
Relation Reflection

Relation Reflection Scheme ($\text{RRS}$): For classes $X, R$ such that $R : X \geq X$, if $a$ is a subset of $X$ then there is a subset $b$ of $X$ such that $a \subseteq b$ and $R : b \geq b$.

A transitive set $A$ has the Relation Reflection Property if, for all sets $X, R$ if $a$ is a subset of $X$ such that $a \in A$ then there is a subset $b$ of $X$ such that $b \in A$, $a \subseteq b$ and $R : b \geq b$.

A union-closed regular set is *-regular if it has the Relation Reflection Property.

*REA*: Every set is a subset of a *-regular set.

$CZF^* \equiv CZF^0 + *REA$. 

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Some Results

\[ CZF^o \ + \ RDC \vdash DC \]
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- $\text{CZF}^o + RDC \vdash DC$
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- $\text{CZF}^o + RDC \vdash DC$
- $\text{CZF}^o + DC \vdash CC$
- $\text{CZF}^o + RDC \vdash RRS$
- $\text{CZF}^o \vdash RDC \equiv [DC + RRS]$
- $ZF \vdash RRS$, but $ZF \not\vdash DC$
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- $\text{CZF}^o + \text{RDC} \vdash \text{DC}$
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- $\text{CZF}^o + \text{RRS} \vdash \"[V^\Omega \models \text{CZF}^o + \text{RRS}] \text{ for each set-presented } cH a \Omega\"$. 
Some Results

- $CZF^o + RDC \vdash DC$
- $CZF^o + DC \vdash CC$
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- $CZF^o + DC + RRS \vdash RDC$
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- $CZF^o + RRS \vdash \text{"[}V^\Omega \models CZF^o + RRS\text{]}$ for each set-presented $cHa\Omega$.
- $CZF^o + DC \vdash \text{"If } A \text{ is a union-closed regular set such that } \mathbb{N} \in A \text{ then } A \text{ has the relation reflection property}$. 
Some Results

- $\text{CZF}^o + \text{RDC} \vdash \text{DC}$
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- $\text{CZF}^o + \text{RRS} \vdash [V^\text{\it{H}} \models \text{CZF}^o + \text{RRS}]$ for each set-presented $cH a \text{\it{H}}$.
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- $\text{CZF}^o + \text{uREA} + \text{DC} \vdash \ast\text{REA}$. 
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Three results of $CZF + uREA + DC$

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For 1,2 $uREA + DC$ can be weakened to $*REA$.
For 3 something that seems a bit stronger than $*REA$, $*^2REA$, seems to be needed.
Inductive Definitions

Let $C$ be a covering system on a class $S$; i.e. an operation $C : S \rightarrow \text{Pow}(\text{Pow}(S))$.

$X/a$ is a $C$-step if $a \in S$ and $X \in C(a)$.

A class $U \subseteq S$ is $C$-closed if, for every $C$-step $X/a$,

$$X \subseteq U \Rightarrow a \in U.$$
Inductive Definitions

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A class $U \subseteq S$ is $C$-closed if, for every $C$-step $X/a$, $X \subseteq U \Rightarrow a \in U$.

Theorem ($CZF^o$): For each class $U \subseteq S$ there is a smallest $C$-closed class $AU$ that includes $U$. $AU$ is the class inductively defined by $C, U$. 
**Inductive Definitions**

Let $C$ be a covering system on a class $S$; i.e. an operation $C : S \rightarrow \text{Pow}(\text{Pow}(S))$.

- $X/a$ is a $C$-step if $a \in S$ and $X \in C(a)$.

- A class $U \subseteq S$ is $C$-closed if, for every $C$-step $X/a$,

  $$X \subseteq U \Rightarrow a \in U.$$

**Theorem ($\text{CZF}^o$):** For each class $U \subseteq S$ there is a smallest $C$-closed class $\mathcal{A}U$ that includes $U$. $\mathcal{A}U$ is the class inductively defined by $C, U$.

**Theorem ($\text{CZF}^o + \text{REA}$):** If $S$ is a set, for each set $U \subseteq S$ the class $\mathcal{A}U$ is a set. Moreover there is a covering system $D$ on $S$ such that such that, for all $U \in \text{Pow}(S)$, if $a \in S$ then

$$a \in \mathcal{A}U \iff (\exists V \in D(a))[V \subseteq U].$$
Coinductive Definitions

Let $C$ be a covering system on a class $S$. A class $U \subseteq S$ is $C$-progressive if, for all $C$-steps $X/a$,

$$a \in U \Rightarrow X \sac{U},$$

where $X \sac{U}$ if $X \cap U$ is inhabited.
Coinductive Definitions

Let $C$ be a covering system on a class $S$. A class $U \subseteq S$ is $C$-progressive if, for all $C$-steps $X/a$,

$$a \in U \Rightarrow X \setminus U,$$

where $X \setminus U$ if $X \cap U$ is inhabited.

**Theorem ($CZF^o + RDC$):** For each class $U \subseteq S$ there is a largest $C$-progressive class $JU$ included in $U$. $JU$ is the class coinductively defined by $C, U$.

- $RDC$ can be replaced by $RRS$. 

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Coinductive Definitions

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$$a \in U \Rightarrow X \backslash U,$$

where $X \backslash U$ if $X \cap U$ is inhabited.

Theorem ($CZF^o + RDC$): For each class $U \subseteq S$ there is a largest $C$-progressive class $J_U$ included in $U$. $J_U$ is the class coinductively defined by $C, U$.

- $RDC$ can be replaced by $RRS$.

Theorem ($CZF^o + uREA + DC$): If $S$ is a set then, for each set $U \subseteq S$, the class $J_U$ exists and is a set.

- $uREA + DC$ can be replaced by $\ast REA$. 

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An example avoiding Dependent Choices, 1

Theorem $[\text{CZF}^0+?]$: Let $C$ be a covering system on a class $S$ and let

$$J = \bigcup \{V \in Pow(S) \mid V \text{ is } C\text{-progressive}\}.$$ 

Then $J$ is the largest $C$-progressive class; i.e. (i) $J$ is $C$-progressive, (ii) If $B$ is a $C$-progressive class then $B \subseteq J$. 
An example avoiding Dependent Choices, 1

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Proof: (i) is easy. For (ii), for classes \( B_1, B_2 \) let

\[ B_1 \hookrightarrow B_2 \iff \forall a \in B_1 \forall X \in C(a) \ X \cap B_2. \]

Let \( B \) be \( C \)-progressive; i.e. \( B \hookrightarrow B \).
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(*) \quad \forall a \in B \ \forall Y \in \text{Pow}(B) \ \exists y \in Y \ y \in X.
\]

We must show that \(\forall a \in B \ \exists V \in \text{Pow}(S) [a \in V \hookrightarrow V]\).
Lemma: \( \forall U \in \text{Pow}(B) \ \exists V \in \text{Pow}(B) \ \ U \mapsto V. \)
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To prove: \( \forall a \in B \ \exists V \in \text{Pow}(S)[a \in V \mapsto V]. \)
Lemma: \( \forall U \in \text{Pow}(B) \ \exists V \in \text{Pow}(B) \ U \leftrightarrow V. \)

To prove: \( \forall a \in B \ \exists V \in \text{Pow}(S)[a \in V \leftrightarrow V]. \) Let \( a \in B. \)

Then \( \{a\} \in \text{Pow}(B) \) so that, by the lemma and RDC there is \( f : \mathbb{N} \to \text{Pow}(B) \) such that \( f(0) = \{a\} \) and, for \( n \in \mathbb{N}, \)

\[ f(n) \mapsto f(n + 1). \]
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f(n) \hookrightarrow f(n+1).
\]

Let \( V = \bigcup_{n \in \mathbb{N}} f(n) \in \text{Pow}(S). \) Then \( a \in V. \) Also \( V \hookrightarrow V \) as

\[
b \in V \quad \Rightarrow \quad b \in f(n) \text{ for some } n \in \mathbb{N}
\]

\[
\Rightarrow \forall X \in C(b) \quad X \downarrow f(n+1)
\]

\[
\Rightarrow \forall X \in C(b) \quad X \downarrow V,
\]
An example avoiding Dependent Choices, 2

Lemma: \( \forall U \in \text{Pow}(B) \ \exists V \in \text{Pow}(B) \ U \mapsto V. \)

To prove: \( \forall a \in B \ \exists V \in \text{Pow}(S)[a \in V \mapsto V] \). Let \( a \in B \).

Then \( \{a\} \in \text{Pow}(B) \) so that, by the lemma and RDC there is \( f : \mathbb{N} \to \text{Pow}(B) \) such that \( f(0) = \{a\} \) and, for \( n \in \mathbb{N}, \)

\[
f(n) \mapsto f(n + 1).
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Let \( V = \bigcup_{n \in \mathbb{N}} f(n) \in \text{Pow}(S) \). Then \( a \in V \). Also \( V \mapsto V \) as

\[
b \in V \Rightarrow b \in f(n) \text{ for some } n \in \mathbb{N} \Rightarrow \forall X \in C(b) \ X \cup f(n + 1) \Rightarrow \forall X \in C(b) \ X \cup V,
\]

Using \textit{RRS} instead of \textit{RDC}: there is a set \( \mathcal{X} \subseteq \text{Pow}(B) \) such that \( \{a\} \in \mathcal{X} \) and \( \forall U \in \mathcal{X} \ \exists V \in \mathcal{X} \ U \mapsto V. \) Let
\( V = \bigcup \mathcal{X} \in \text{Pow}(S) \). Then \( a \in V \mapsto V \).
An example avoiding Dependent Choices, 3

Proof of Lemma: \( \forall U \in \text{Pow}(B) \ \exists V \in \text{Pow}(B) \ U \leftrightarrow V. \)

Recall that

\[
(*) \quad \forall a \in B \ \exists Y \in \text{Pow}(B) \ \forall X \in C(a) \ Y(X).
\]
An example avoiding Dependent Choices, 3

Proof of Lemma: \( \forall U \in \text{Pow}(B) \ \exists V \in \text{Pow}(B) \ U \leftrightarrow V \).

Recall that

\[ (\ast) \quad \forall a \in B \ \exists Y \in \text{Pow}(B) \ \forall X \in C(a) \ Y \downarrow X. \]

Let \( U \in \text{Pow}(B) \). Then

\[ \forall a \in U \ \exists Y \in \text{Pow}(B) \ \forall X \in C(a) \ Y \downarrow X. \]
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Let \( U \in \text{Pow}(B) \). Then

\[ \forall a \in U \: \exists Y \in \text{Pow}(B) \quad \forall X \in C(a) \quad Y \downarrow X. \]

By Strong Collection there is a set \( Z \subseteq \text{Pow}(B) \) such that

\[ \forall a \in U \: \exists Y \in Z \quad \forall X \in C(a) \quad Y \downarrow X. \]

Let \( V = \cup Z \). Then \( U \leftrightarrow V. \)
Conclusion

Recall:

\[
\begin{align*}
CZF^+ & \equiv CZF^o + REA \\
CZF^u & \equiv CZF^o + uREA \\
CZF^* & \equiv CZF^o + \ast REA \\
CZF^{\ast 2} & \equiv CZF^o + \ast_2 REA
\end{align*}
\]

\(CZF^*\) and \(CZF^{\ast 2}\) can be used to prove certain results that had been previously proved in \(CZF^u + DC\).
Conclusion

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- \( CZF^* \) and \( CZF^{*2} \) can be used to prove certain results that had been previously proved in \( CZF^u + DC \).

**Conjecture:** \( CZF^* \) (and \( CZF^{*2} \)) are preserved in Heyting valued models over set-presented cHa’s.
Future Work:

- Construct a Heyting valued model over set-presented cHa’s and verify the conjecture.
- Explore alternative methods to avoid dependent choices in Heyting valued models.
- Investigate the implications of the conjecture on the applicability of $CZF^*$ in constructive topology.

**Conclusion:**

Recall:

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CZF^+ & \equiv CZF^o + REA \\
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- $CZF^*$ and $CZF^\ast^2$ can be used to prove certain results that had been previously proved in $CZF^u + DC$.
- **Conjecture:** $CZF^*$ (and $CZF^\ast^2$) are preserved in Heyting valued models over set-presented cHa’s.
- **But** DC does not generally hold in Heyting valued models.
Let $S = (S, ◁)$ be a fixed formal topology. I prefer to work with the positivity operators $J : \text{Pow}(S) \rightarrow \text{Pow}(S)$ where, for all $a, U$, $a \in JU \equiv a \bowtie U$.

Call $F \subseteq S$ completely prime if, whenever $a \bowtie U$, $a \in F \Rightarrow U \subseteq F$. 

A characterisation of the positivity relations $\bowtie$ on $(S, ◁)$
A characterisation of the positivity relations $\sqsubset$ on $(S, \sqsubset)$

Let $S = (S, \sqsubset)$ be a fixed formal topology. I prefer to work with the positivity operators $\mathcal{J} : \text{Pow}(S) \to \text{Pow}(S)$ where, for all $a, U, a \in \mathcal{J}U \equiv a \sqsubset U$.

Call $F \subseteq S$ completely prime if, whenever $a \sqsubset U$, $a \in F \Rightarrow U \backslash F$.

Let $I$ be a class of completely prime sets. For all $U$ let $\mathcal{J}_I U = \bigcup \{ F \in I \mid F \subseteq U \}$, and call $I$ standard if $\forall I' \in \text{Pow}(I) \cup I' \in I$ and $\forall U \in \text{Pow}(S)$ $\mathcal{J}_I U$ is a set.
A characterisation of the positivity relations \( \bowtie < \) on \((S, \bowtie)\)

Let \( S = (S, \bowtie) \) be a fixed formal topology. I prefer to work with the positivity operators \( J : \text{Pow}(S) \rightarrow \text{Pow}(S) \) where, for all \( a, U, a \in JU \equiv a \bowtie < U \).

- Call \( F \subseteq S \) completely prime if, whenever \( a \bowtie < U \),
  \[ a \in F \Rightarrow U \setminus F. \]

- Let \( I \) be a class of completely prime sets. For all \( U \) let
  \[ J_I U = \bigcup \{ F \in I \mid F \subseteq U \}, \]
  and call \( I \) standard if
  \[ (\forall I' \in \text{Pow}(I)) \cup I' \in I \text{ and } (\forall U \in \text{Pow}(S)) J_I U \text{ is a set.} \]

Proposition: The assignments \( J \mapsto \{ F \mid JF = F \} \) and \( I \mapsto J_I \) are inverse one-one correspondences between the standard classes of completely prime sets and the positivity operators.
A characterisation of the positivity relations $\prec$ on $(S, \prec)$

Let $S = (S, \prec)$ be a fixed formal topology. I prefer to work with the positivity operators $\mathcal{J} : \text{Pow}(S) \to \text{Pow}(S)$ where, for all $a, U$, $a \in \mathcal{J}U \equiv a \prec U$.

Call $F \subseteq S$ completely prime if, whenever $a \prec U$, $a \in F \Rightarrow U \subseteq F$.

Let $I$ be a class of completely prime sets. For all $U$ let

$$\mathcal{J}_I U = \bigcup \{ F \in I \mid F \subseteq U \},$$

and call $I$ standard if $(\forall I' \in \text{Pow}(I)) \cup I' \in I$ and $(\forall U \in \text{Pow}(S)) \mathcal{J}_I U$ is a set.

**Proposition:** The assignments $\mathcal{J} \mapsto \{ F \mid \mathcal{J}F = F \}$ and $I \mapsto \mathcal{J}_I$ are inverse one-one correspondences between the standard classes of completely prime sets and the positivity operators.

**Note:** $\mathcal{J}_I$ is a set-presented positivity operator for any set $I$ of completely prime sets.
On avoiding dependent choices in Formal Topology – p.20/20
x0

x1
  • x2
  • x3
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