Separation Properties in Constructive Topology

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0.1 Introduction

Until the 1970s there was only a limited focus on the general notion of a topological space in constructive mathematics. An exception was the work of Anne Troelstra on intuitionistic general topology, starting with his thesis, [9]. Most attention was paid to metric space notions both in intuitionistic analysis and in Bishop style constructive analysis. But in later years, because of the development of topos theory, the study of sheaf models and work on point-free topology, including work on formal topology its more predicative version, the notions of general topology in constructive mathematics have received more attention. The papers $[5, 6]$ built on Troelstra's work to study general topological notions in sheaf models. In point-free topology the formal points of a locale form a (sober) topological space and it can be of interest to investigate how the properties of such a space of points relate to the properties of the locale it comes from. In formal topology the formal points of a formal topology still generally form a 'topological space' in some sense. But care is needed because in the more predicative context of formal topology the points do not always form a set. The limited aim of this paper is to present a fairly systematic survey of the main separation properties that topological spaces can have in constructive mathematics. These are perhaps the first kinds of properties to consider when moving from the study of metric spaces to the study of general topological spaces.

Because we want to relate our work to work in formal topology, but also be compatible with work in sheaf models we choose to present our definitions and results in the setting of constructive set theory. More specifically the work can be formalised in the formal system CZF of constructive ZF, see [1], but it should not be necessary for the reader to have a precise working knowledge of the axioms. It should suffice for the reader to be aware that CZF is a subsystem of ZF that uses intuitionistic rather than classical logic and does not have the powerset axiom and does not have the full separation scheme, but only the restricted separation scheme. This is the version of the scheme which only entails that $\{x \in a \mid \phi\}$ is a set when a is a set and ϕ is a restricted formula, i.e. a formula in which all quantifiers are restricted to sets. In CZF there is the axiom scheme of subset collection whose main consequence is Myhill's exponentiation axiom, which expresses that the class of all functions $f : a \to b$ from a set a to a set b form a set b^a.

We use standard set theoretic class terminology. Given a class X we write $Pow(X)$ for the *powerclass of* X; i.e. the class of all subsets of the class X. Even when X is a set we are not able to take $Pow(X)$ to be a set. In fact the Powerset axiom is a consequence in CZF of the assumption that just the powerclass of a singleton set is a set. By adding the Powerset axiom and the full separation scheme to CZF we obtain the much stronger theory IZF of Intuitionistic ZF. This theory has the same proof theoretic strength as ZF.

Some useful classically true statements that cannot be proved even in IZF are REM, $wREM$ and LPO. REM is the principle of restricted excluded middle; i.e. the scheme that $\phi \vee \neg\phi$ for all restricted formulae. $wREM$ is the principle of weak restricted excluded middle; i.e. the scheme that $\neg\neg\phi \vee \neg\phi$ for all restricted formulae. LPO is the Limited Principle of Omniscience. These principles may be expressed as the following axioms.

REM**:** $\forall p \in Pow(\{0\})[p = 1 \vee \neg(p = 1)]$.
$wREM$**:** $\forall p \in Pow(\{0\})[\neg\neg(p = 1) \vee \neg(p = 1)]$.
LPO**:** $(\forall f \in 2^{\mathbb{N}})[(\exists n \in \mathbb{N})(f(n) = 1) \vee (\forall n \in \mathbb{N})(f(n) = 0)]$.

In sections 2 and 3 we introduce the notions of constructive topological space (ct-space for short) and formal topology that we will use and give the construction of the ct-space of formal points of a formal topology. In section 4 we introduce three versions of the classical T_i separation properties and develop the relationships between them. In section 5 we relate these notions to some notions developed in [3]. In section 6 we consider a generalised notion of metric space. In section 7 we show that the ct-space of formal points of a regular formal topology is $T_3^\sharp$. We end in section 8 by considering the notion of a sober ct-space.

0.2 Constructive topological spaces (ct-spaces)

We formulate a notion of topological space, the notion of constructive topological space
(ct-space) below, that is general enough to include the spaces of formal points of a formal topology, but restricted enough so that in IZF we get essentially the usual notion. In order to be general enough we have to allow the points to form a class that might not be a set. In order to be restricted enough we assume certain smallness conditions. In particular we assume as part of the structure a set indexed base for the opens.

A *constructive topological space (*ct*-space)* consists of a class X of the points of the space together with a set S and a class $\Vdash\ \subseteq X \times S$ such that, letting $B_a = \{x \in X \mid x \Vdash a\}$ for each $a \in S$, $B_U = \bigcup_{a \in U} B_a$ for each $U \in Pow(S)$ and $\alpha_x = \{a \in S \mid x \Vdash a\}$ for each $x \in X$, the following conditions hold.

CS1 $X = B_S$,
CS2 If $a_1, a_2 \in S$ then $B_{a_1} \cap B_{a_2} \subseteq B_U$, where $U = \{a \in S \mid B_a \subseteq B_{a_1} \cap B_{a_2}\}$.
CS3 For each $x \in X$ the classes α_x and $\{y \in X \mid \alpha_y = \alpha_x\}$ are sets.

In a ct-space a class Y of points is *open* if $Y = B_U$ for some subclass U of S. So in a ct-space it is the open classes that form a 'topology' in something like the standard sense. But as part of the structure of a ct-space there is the base of open classes, B_a for $a \in S$, indexed by the set S.

A ct-space X is *small* if its class of points is a set. Then $\Vdash = \bigcup_{x \in X}\{x\} \times \alpha_x$ is a set so that each B_a is also a set. So a small ct-space is essentially just a topological space in the usual sense, except that the open sets form a class and, as part of the structure, there is a set-indexed base for the topology.

Proposition 0.1. (IZF) *Every* ct*-space is small.*

Proof. Given a ct-space , in IZF $Pow(S)$ is a set by the Powerset Axiom and so

$$Q = \{\ U \in Pow(S)\ \mid\ (\exists x \in X)(U = \alpha_x)\ \}$$

is a set by the full Separation Scheme. For each $U \in Q$ let $f(U) = \{y \in X \mid \alpha_y = U\}$. This is a set because $U = \alpha_x$ for some $x \in X$ and so $f(U) = \{y \in X \mid \alpha_y = \alpha_x\}$.

Now observe that $X = \bigcup_{U\in Q} f(U)$, which is a set by the Replacement scheme and the Union axiom. $\square$

We call a class F of points *closed* if $cl(F) \subseteq F$, where $cl(F)$ is the class of limit points of F. Here a *limit point* of F is a point x such that for every $a \in \alpha_x$ the class B_a has a point in common with F. Note that F is closed iff $F = \mathsf{rest}\, U$ for some subclass U of S, where $\mathsf{rest}\, U = \{y \in X \mid \alpha_y \subseteq U\}$. There is another notion that is also of interest. Call F *strongly closed* if it is the complement of an open class. Any strongly closed class is closed.

Proposition 0.2 *F is strongly closed iff $wcl(F) \subseteq F$, where $wcl(F)$ is the class of weak limit points of F. Here a point x is a weak limit point of F if for every $a \in \alpha_x$ the class B_a is not disjoint from F.*

Proof. Let $U_F = \bigcap_{y\in F}(S \backslash \alpha_y)$. Then $F \cap B_{U_F} = \emptyset$ and it is easy to check that x is a weak limit point of F iff $x \notin B_{U_F}$. It follows that every weak limit point of F is in F iff F is the complement of B_{U_F}. Thus the implication from right to left is proved. For the reverse implication assume that F is the complement of B_U. It follows that $U \subseteq U_F$ and so if x is a weak limit point of F then $x \notin B_U$ and so $x \in F$. Thus F is strongly closed. $\square$

0.3 The space of formal points of a formal topology

Local collections

We describe a simple general method for constructing ct-spaces that includes as a special case the construction of the ct-space of formal points of a formal topology. We first give a convenient representation of T_0 ct-spaces up to isomorphism. As defined in the next section, a ct-space is T_0 if, for all points x, y

$$[\ \alpha_x = \alpha_y \ \Rightarrow\ x = y\ ].$$

A class X of inhabited subsets of a set S is called a *local collection* of subsets of S if, for all $x \in X$ and all $a_1, a_2 \in x$ there is $a \in x$ such that

$$\forall y \in X [\ a \in y \ \Rightarrow\ a_1, a_2 \in y\ ].$$

Observe that if X is a local collection of subsets of a set S then it is the class of points of a T_0 ct-space where

$$\Vdash\ =\ \{\ (x, a) \in X \times S \ \mid\ a \in x\ \}.$$

Conversely every ct-space X determines a local collection $X' = \{\alpha_x \mid x \in X\}$. Moreover when X is T_0 then the map $x \mapsto \alpha_x$ is an isomorphism $X \cong X'$.

We now describe the general method to construct local collections that generalises the construction of a ct-space from a formal topology. Let R be any binary relation on the set S. Call an inhabited subset x of S an *R-point* if the following hold.

1. If $a_1, a_2 \in x$ then for some $a \in x$, aRa_1 and aRa_2.

2. If bRa then $[\ b \in x \ \Rightarrow \ a \in x\ ]$.

Any class of R-points forms a local collection. Conversely each local collection X is a class of R_X-points, where

$$R_X = \{\ (a,b) \in S \times S \ \mid \ \forall x \in X[\ a \in x \ \Rightarrow \ b \in x\ ]\ \}.$$

Note that R_X is always reflexive, transitive.

The notion of a formal topology

We will use the following version of the notion of a formal topology. A *formal topology* on a set S is an operator $\mathcal{A} : Pow(S) \to Pow(S)$ such that for all sets $U, V \subseteq S$,

1. $U \subseteq \mathcal{A}U$,
2. $U \subseteq \mathcal{A}V \Rightarrow \mathcal{A}U \subseteq \mathcal{A}V$,
3. $\mathcal{A}U \cap \mathcal{A}V \subseteq \mathcal{A}(U\downarrow \cap V\downarrow)$,

where $a \leq b \Leftrightarrow a \in \mathcal{A}\{b\}$ for $a, b \in S$ and $U\downarrow = \{a \in S \mid \exists b \in U\ a \leq b\}$ for sets $U \subseteq S$. Usually these axioms are given in terms of the relation $\lhd$ where

$$a \lhd U \Leftrightarrow a \in \mathcal{A}U$$

for $a \in S$ and $U \in Pow(S)$. Note that given a small ct-space we may obtain a formal topology on S by defining, for each set $U \in Pow(S)$.

$$\mathcal{A}U = \{a \in S \mid B_a \subseteq B_U\}.$$

We need the ct-space to be small in order to ensure that $\mathcal{A}U$ is a set.

The ct*-space of formal points*

Given a formal topology $\mathcal{A}$ on S a *formal point* of the formal topology is a $\leq$-point x such that for all subsets U of S

$$x \between \mathcal{A}U \Rightarrow x \between U,$$

where, for classes A, B we write $A \between B$ if A, B have a common element. By definition these form a class of $\leq$-points and hence a local collection and so form a T_0 ct-space.

0.4 Constructive separation properties

In this section we start by giving the constructive formulations of the T_i separation properties, for $i = 0, 1, 2, 3$ used in [5] and then go on to give the classically equivalent stronger versions $T_i^\sharp$, also considered there. At the same time we also introduce a new family of intermediate separation properties, T_i^+ that were inspired by a study of [3].

The separation properties for $i = 0, 1, 2$

The standard classical formulations of these separation properties is usually in terms of separating via open sets a pair of distinct points. It seems that a correct constructive formulation of these separation properties instead expresses that inseparable points are equal using the following notions of inseparability. For points x, y of a ct-space X we define $x \sim_i y$ as follows.

- $x \sim_0 y$ iff $\alpha_x = \alpha_y$.
- $x \sim_1 y$ iff $\alpha_x \subseteq \alpha_y$.
- $x \sim_2 y$ iff $(\forall a \in \alpha_x)(\forall b \in \alpha_y)(B_a \between B_b)$.

We now state our first family of separation properties

- For $i = 0, 1, 2$ the ct-space X is T_i if, for all points x, y

$$(x \sim_i y) \Rightarrow (x = y)$$

The original classical formulation of T_i for $i = 0, 1, 2$ has the form that for all points x, y

$$\neg(x = y) \Rightarrow (x \neq_i y),$$

where

- $x \neq_0 y$ iff $(\exists a \in \alpha_x)(a \notin \alpha_y) \vee (\exists b \in \alpha_y)(b \notin \alpha_x)$.
- $x \neq_1 y$ iff $(\exists a \in \alpha_x)(a \notin \alpha_y)$.
- $x \neq_2 y$ iff $(\exists a \in \alpha_x)(\exists b \in \alpha_y)[B_a \cap B_b = \emptyset]$

The classical T_i do not seem to be of constructive interest. Instead we consider the following classically equivalent notions.

For $i = 0, 1, 2$, a ct-space X is T_i^+ if, for all points x, y

$$\neg(x \neq_i y) \Rightarrow (x = y).$$

The following facts may be worth noting.

Proposition 0.3

1. *A* ct*-space is classical T_1 iff the complement of every point singleton is open.*
2. *A* ct*-space is T_1 iff every point singleton is closed.*
3. *A* ct*-space is T_1^+ iff every point singleton is strongly closed.*

We get a stronger family of separation notions as follows. For $i = 0, 1, 2$ a ct-space is a $T_i^\sharp$ ct-*space* if it is T_0 and, for all points x, y

$$(\forall a \in \alpha_x)[\ y \neq_i x \ \vee \ a \in \alpha_y\ ].$$

Notions of regular space

We formulate notions $R, R^+, R^\sharp$ corresponding to the classical notion of a regular space. A standard formulation of the classical notion states that a topological space is classical regular if whenever given a closed set and a point not in it there are two disjoint open sets the first including the closed set and the second containing the point. Let X be a ct-space. Our constructive versions replace the quantification over closed sets not containing the point by a quantification over (basic) open neighborhoods of the point.

- The space X is R if
$$\forall x \in X \forall a \in \alpha_x \exists b \in \alpha_x [cl(B_b) \subseteq B_a].$$
- The space X is R^+ if
$$\forall x \in X \forall a \in \alpha_x \exists b \in \alpha_x [wcl(B_b) \subseteq B_a].$$
- The space X is a $R^\sharp$ if
$$\forall x \in X \forall a \in \alpha_x \exists b \in \alpha_x [X \subseteq B_b^* \cup B_a],$$
where, for any class Z of points, Z^* is the largest open class disjoint from Z; i.e. it is B_U where $U = \{c \in S \mid B_c \cap Z = \emptyset\}$.

The space X is a T_3 *(T_3^+ or $T_3^\sharp$) ct-space* if it is T_0 and R (R^+ or $R^\sharp$ respectively).

Some implications

Clearly for all $x, y \in X$, $(x \sim_0 y) \Rightarrow (x \sim_1 y) \Rightarrow (x \sim_2 y)$, so $T_2 \Rightarrow T_1 \Rightarrow T_0$. Also $(x \neq_2 y) \Rightarrow (x \neq_1 y) \Rightarrow (x \neq_0 y)$, so $T_2^+ \Rightarrow T_1^+ \Rightarrow T_0^+$ and $T_2^\sharp \Rightarrow T_1^\sharp \Rightarrow T_0^\sharp$.

The implications $T_i^+ \Rightarrow T_i$ are easy to prove for $i = 0, 1, 2, 3$, as are the implications $T_i^\sharp \Rightarrow T_i^+$ for $i = 0, 2, 3$. $T_1^\sharp \Rightarrow T_1^+$ requires a little more work because the relation $\neq_1$ is not symmetric.

Proposition 0.4 *If X is $T_1^\sharp$ then X is T_1^+.*

Proof. Since X is $T_1^\sharp$, it must be $T_0^\sharp$ and hence T_0^+. We will show that $(\forall x, y \in X)[x \neq_0 y \rightarrow x \neq_1 y]$ so that X is T_1^+

Suppose that $x \neq_0 y$. If $a \in \alpha_x$ and $a \notin \alpha_y$ then $x \neq_1 y$, so assume instead that there is an $a \in \alpha_y$ such that $a \notin \alpha_x$. Since X is $T_1^\sharp$ either $x \neq_1 y$ or $a \in \alpha_y$, but the latter is false so $x \neq_1 y$. □

Proposition 0.5 *If X is T_3 (T_3^+ or $T_3^\sharp$) then X is T_2 (T_2^+ or $T_2^\sharp$ respectively).*

Proof. If X is T_0 then observe the following characterisations.

- X is T_2 iff $(\forall x \in X)(\forall a \in \alpha_x)[\{y \in X \mid x \sim_2 y\} \subseteq B_a]$.
- X is T_2^+ iff $(\forall x \in X)(\forall a \in \alpha_x)[\{y \in X \mid \neg(x \neq_2 y)\} \subseteq B_a]$.
- X is $T_2^\sharp$ iff $(\forall x \in X)(\forall a \in \alpha_x)[X \subseteq \{y \in X \mid x \neq_2 y\} \cup B_a]$.

In view of the definitions of T_3, T_3^+ and $T_3^\sharp$ it suffices to show that if $x \in X$ and $a, b \in \alpha_x$ then the following hold.

- $\{y \in X \mid x \sim_2 y\} \subseteq cl(B_b)$,
- $\{y \in X \mid \neg(x \neq_2 y)\} \subseteq wcl(B_b)$,
- $B_b^* \subseteq \{y \in X \mid x \neq_2 y\}$.

These are straightforward to check. □

Summary: We have the following implications between the separation axioms:

$$\begin{array}{ccccccc}
T_3^\sharp & \Longrightarrow & T_2^\sharp & \Longrightarrow & T_1^\sharp & \Longrightarrow & T_0^\sharp \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
T_3^+ & \Longrightarrow & T_2^+ & \Longrightarrow & T_1^+ & \Longrightarrow & T_0^+ \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
T_3 & \Longrightarrow & T_2 & \Longrightarrow & T_1 & \Longrightarrow & T_0
\end{array}$$

In the next subsection we show that none of the converses of these implications hold constructively.

Some counterexamples

We first show how some of the standard counterexamples from [8] can be constructed in CZF, and then consider the discrete topology on a set, which can provide weak counterexamples to the implications $T_i \Rightarrow T_i^+$ and $T_i^+ \Rightarrow T_i^\sharp$.

Example 0.6 Odd-even topology on the natural numbers ($R^\sharp$ but not T_0)

Let $X = \mathbb{N}$, $S = \mathbb{N}$ and $B_a = \{2a, 2a+1\}$ for each $a \in S$. The sets $(B_a)_{a\in S}$ form a partition of X, and it follows that X is a small ct-space.

X **is** $R^\sharp$**:** Let $x \in X$ and $a \in \alpha_x$. If we take $b = a$ then $B_a \cup B_b^* = B_a \cup (X \backslash B_a) = X$ since equality on $\mathbb{N}$ is decidable.

X is not T_0**:** $\alpha_0 = \alpha_1$ but $\neg(0 = 1)$, so X is not T_0. □

Example 0.7 Two-point[1] Sierpinski space ($T_0^\sharp$ but not T_1)

Let $X = \{0, 1\}$, $S = \{0, 1\}$, $B_0 = X$ and $B_1 = \{1\}$. Then X is a small ct-space.

X **is** T_0**:** Suppose that $\alpha_x = \alpha_y$. Then $x \in B_1$ iff $y \in B_1$, and hence $x = y$ by the decidability of equality on $\{0, 1\}$.

X **is** $T_0^\sharp$**:** Suppose that $a \in \alpha_x$ and $y \in X$. Either $a \in \alpha_y$ or $\neg(a \in \alpha_y)$. If $\neg(a \in \alpha_y)$ then $y \neq_0 x$.

X **is not** T_1**:** $\alpha_0 \subseteq \alpha_1$ but $\neg(0 = 1)$. □

Example 0.8 "Cofinite" topology on the natural numbers ($T_1^\sharp$ but not T_2)

Let $X = \mathbb{N}$, $S = \mathbb{N} \times \mathbb{N}$ and $B_{(a,b)} = \{x \in X \,|\, (x = a) \vee (b \leq x)\}$. Using the decidability of $=$ and $\leq$ on $\mathbb{N}$, it is straightforward to check that the intersection of two basic open sets is open, so X is a small ct-space. Note that although this space is classically the same as the finite complement topology, not every open class will be cofinite constructively.

X **is** T_0**:** Suppose that $x, y \in X$ and $\alpha_x = \alpha_y$. Let $a = (x, \max\{x, y\} + 1)$. Then $x \in B_a$, and hence $y \in B_a$, so $x = y$.

[1]This is not the same as the usual constructive version of the Sierpinski space, whose points are the elements of the class $Pow(0)$. The powerclass Sierpinski space is not T_0^+ constructively.

X **is** $T_1^\sharp$**:** Now suppose that $(a,b) \in \alpha_x$ and $y \in X$. Either $x = y$ or $\neg(x = y)$. If $x = y$ then $(a,b) \in \alpha_y$, and if $\neg(x = y)$ then $(y, \max\{x,y\}+1) \in \alpha_y \backslash \alpha_x$ so $x \neq_1 y$.

X **is not** T_2**:** Any two basic open sets have an inhabited intersection. □

Example 0.9 Irrational slope topology ($T_2^\sharp$ but not R)

In this example we will use the Dedekind reals, as defined in [1]. Given two real numbers x and y, write $x \neq y$ for $(x < y) \vee (y < x)$ (x is *apart* from y), and as in [2] we define an *irrational number* to be a real number which is apart from every rational number.

Choose an arbitrary irrational number θ (for example $\sqrt{2}$). Let $X = \mathbb{Q} \times \mathbb{Q}^{\geq 0}$, $S = \mathbb{Q} \times \mathbb{Q}^{\geq 0} \times \mathbb{Q}^{>0}$ and

$$B_{(a,b,r)} = \{(a,b)\} \cup (N_r(a - b\theta) \times \{0\}) \cup (N_r(a + b\theta) \times \{0\})$$

where $N_r(\zeta) = \{z \in \mathbb{Q} \mid |z - \zeta| < r\}$.

The intersection of two basic open sets is open:
Suppose that $(x,y) \in B_{(a,b,r)} \cap B_{(c,d,s)}$. If $y > 0$ then $(x,y) = (a,b) = (c,d)$, so $(x,y) \in B_{(x,y,\min\{r,s\})}$. If $y = 0$ then there is a $t \in \mathbb{Q}^{>0}$ such that $N_t(x) \times \{0\} \subseteq B_{(a,b,r)} \cap B_{(c,d,s)}$, and hence $(x,y) \in B_{(t,0)} \subseteq B_{(a,b,r)} \cap B_{(c,d,s)}$.

So X is a small ct-space.

X is T_0**:** Suppose that $\alpha_{(x,y)} = \alpha_{(x',y')}$. If $y > 0$ and $y' > 0$ then $(x,y,1) \in \alpha_{(x,y)} = \alpha_{(x',y')}$, so $(x',y') = (x,y)$. If $y = 0$, then $(x,y,r) \in \alpha_{(x',y')}$ for all $r \in \mathbb{Q}^{>0}$, so $y' = 0$ and $|x - x'| < r$ for all $r \in \mathbb{Q}^{>0}$, and so $x = x'$. Similarly if $y' = 0$ then $y = 0$ and $x = x'$.

X is $T_2^\sharp$**:** Suppose that $(a,b,r) \in \alpha_{(x,y)}$ and $(x',y') \in S$. Then $((x,y) = (x',y'))$ is decidable. If $(x,y) = (x',y')$, then $(x,y,r) \in \alpha_{(x',y')}$. Suppose that $\neg((x,y) = (x',y'))$. Since θ is irrational, we have $\theta \neq \frac{x-x'}{y'-y}$, so $(y'-y)\theta \neq x - x'$, and hence $x + y\theta \neq x' + y'\theta$. Similarly it can be shown that $x + y\theta \neq x' - y'\theta$, $x - y\theta \neq x' + y'\theta$ and $x - y\theta \neq x' - y'\theta$. So we can choose an $r \in \mathbb{Q}^{>0}$ such that $N_r(x - y\theta) \cup N_r(x + y\theta)$ is disjoint from $N_r(x' - y'\theta) \cup N_r(s' + y'\theta)$. $(x,y) \in B_{(x,y,r)}$, $(x',y') \in B_{(x',y',r)}$ and $B_{(x,y,r)} \cap B_{(x',y',r)} = \emptyset$, so $(x',y') \neq_2 (x,y)$.

X **is not** R**:** Suppose that X is R. Let $x = (0,1)$ and $a = (0,1,1) \in \alpha_x$. There must be a $b \in \alpha_x$ such that $cl(B_b) \subseteq B_a$. $b = (0,1,r)$ for some $r \in \mathbb{Q}^{>0}$.
Let $y = (r/2, 1)$. Then for all $\varepsilon \in \mathbb{Q}^{>0}$, $B_{(r/2,1,\varepsilon)} \cap B_b$ is inhabited, so $y \in cl(B_b) \subseteq B_a$. But $(r/2,1) \notin B_{(0,1,1)}$, so X cannot be R. □

The next example can be used to show that the converses of the implications $T_i \Rightarrow T_i^+ \Rightarrow T_i^\sharp$ cannot be proved in CZF.

Example 0.10 "Discrete" topology[2] on X

[2] Although this is defined in the same way as the classical discrete topology, the word "discrete" seems slightly inappropriate as the set X is not necessarily discrete.

Given a set X, let $S = X$ and $B_a = \{a\}$ for each $a \in S$.

Lemma 0.11 *Let X be a set with the discrete topology, as defined above. Then for $i = 0, 1, 2, 3$:*

(i) *X is T_i*

(ii) *(X is T_i^+) $\Leftrightarrow$ (X is R^+) $\Leftrightarrow$ $(\forall x, y \in X)[\neg\neg(x = y) \rightarrow (x = y)]$*

(iii) *(X is $T_i^\sharp$) $\Leftrightarrow$ (X is $R^\sharp$) $\Leftrightarrow$ $(\forall x, y \in X)[(x = y) \vee \neg(x = y)]$*

Proof. It suffices to prove the proposition for T_0, T_0^+, $T_0^\sharp$ and for R, R^+, $R^\sharp$. These cases follow immediately from the definitions, observing that

- $\alpha_x = \{x\}$
- $x \neq_0 y$ iff $\neg(x = y)$
- $cl(\{x\}) = \{x\}$ □

Corollary 0.12

(i) (Every T_i *space is* T_i^+) $\Rightarrow$ REM *for* $i = 0, 1, 2, 3$

(ii) (*Every* R *space is* R^+) $\Rightarrow$ REM

(iii) (*Every* T_i^+ *space is* $T_i^\sharp$) $\Rightarrow$ wREM *for* $i = 0, 1, 2, 3$

(iv) (*Every* R^+ *space is* $R^\sharp$) $\Rightarrow$ wREM

Proof.

(i) Given a restricted formula ϕ, let $p = \{0 \mid \phi \vee \neg\phi\}$, $1 = \{0\}$ and $X = \{1, p\}$ with the discrete topology. X is T_i. If X is T_i^+ then $\neg\neg(1 = p) \rightarrow 1 = p$, and so $\neg\neg(\phi \vee \neg\phi) \rightarrow (\phi \vee \neg\phi)$. But $\neg\neg(\phi \vee \neg\phi)$ is true in CZF, so $\phi \vee \neg\phi$.

(ii) Identical to (i), replacing T_i with R and T_i^+ with R^+.

(iii) Given a restricted formula ϕ, let $p = \{0 \mid \neg\phi\}$, $1 = \{0\}$ and $X = \{1, p\}$ with the discrete topology. $\neg\neg\neg\phi \rightarrow \neg\phi$, so for all $x, y \in X$ we have we have $\neg\neg(x = y) \rightarrow (x = y)$. So X is T_i^+. If X is $T_i^\sharp$ then $(1 = p) \vee \neg(1 = p)$, so $\neg\phi \vee \neg\neg\phi$.

(iv) Identical to (iii), replacing T_i^+ with R^+ and $T_i^\sharp$ with $R^\sharp$. □

0.5 Separation relative to an inequality

This section is based on ideas in [3], which assumes that a topological space comes with an inequality relation on it. We have reformulated the ideas to our context. We call a relation $\neq$ on a class X an *inequality* if, for all $x, y \in X$,

$$x = y \Rightarrow \neg(x \neq y)$$

and is *tight* if the reverse implication also holds.

Definition 0.13 *Let $\neq$ be an inequality on a* $\mathtt{ct}$*-space X. For each $a \in S$ let*

$$B_a^{\neq} = \{y \in X \mid \forall z \in B_a\ z \neq y\}.$$

- *X is $T_1^{B\&V}$ wrt $\neq$ if, for all $x, y \in X$,*

$$x \neq y \Rightarrow \exists a \in \alpha_x\ y \in B_a^{\neq}.$$

- *X is $T_2^{B\&V}$ wrt $\neq$ if, for all $x, y \in X$,*

$$x \neq y \Rightarrow \exists a \in \alpha_x \exists b \in \alpha_y B_b \subseteq B_a^{\neq}.$$

- *X is topologically cotransitive wrt $\neq$ if, for all $x, y \in X$,*

$$\forall a \in \alpha_x [x \neq y \vee a \in \alpha_y].$$

- *X is locally decomposable wrt $\neq$ if, for all $x \in X$,*

$$\forall a \in \alpha_x \exists b \in \alpha_x [X \subseteq B_b^{\neq} \cup B_a].$$

Note that, for $i = 0, 1, 2$, the definition of a $T_i^\sharp$ ct-space can be formulated as follows. A ct-space is $T_i^\sharp$ if it is T_0 and topologically cotransitive wrt $\neq_i$.

Proposition 0.14 *Let X be a* ct*-space .*

1. *For $i = 1, 2$ the* ct*-space X is T_i^+ iff X is $T_i^{B\&V}$ wrt some tight inequality.*
2. *If X is $R^\sharp$ then X is locally decomposable wrt $\neq_2$.*
3. *If X is T_1 and X is topologically cotransitive wrt an inequality $\neq$ then $\neq$ is tight.*

Proof. Let X be a ct-space.

1. For $i = 1, 2$ let X be $T_i^{B\&V}$ wrt a tight inequality $\neq$.

   Claim: For $x, y \in X$, $x \neq y \Rightarrow x \neq_i y$.

   Proof. Note that $B_a^{\neq} \subseteq X \backslash B_a$.

$$\begin{aligned} i = 1 : x \neq y &\Rightarrow \exists a \in \alpha_x \; y \in B_a^{\neq} \\ &\Rightarrow \exists a \in \alpha_x \backslash \alpha_y \\ &\Rightarrow \; x \neq_1 y \\ i = 2 : x \neq y &\Rightarrow \exists a \in \alpha_x \exists b \in \alpha_y B_b \subseteq B_a^{\neq} \\ &\Rightarrow \exists a \in \alpha_x \exists b \in \alpha_y B_a \cap B_b = \emptyset \\ &\Rightarrow \; x \neq_2 y \end{aligned}$$

   By the claim

$$\begin{aligned} \neg(x \neq_i y) &\Rightarrow \neg(x \neq y) \\ &\Rightarrow x = y \end{aligned}$$

   as $\neq$ is tight. So X is T_i^+.

   For the converse let X be T_i^+. Then $\neq_i$ is tight and it suffices to show that X is $T_i^{B\&V}$ wrt $\neq_i$.

   $i = 1$**:** For $x, y \in X$

$$\begin{aligned} (x \neq_1 y) &\Rightarrow \exists a \in \alpha_x \backslash \alpha_y \\ &\Rightarrow \exists a \in \alpha_x \forall z \in B_a a \in \alpha_z \backslash \alpha_y \\ &\Rightarrow \exists a \in \alpha_x \forall z \in B_a z \neq_1 y \\ &\Rightarrow \exists a \in \alpha_x \; y \in B_a^{\neq_1} \end{aligned}$$

   So X is $T_1^{B\&V}$ wrt $\neq_1$.

$i = 2$: First observe that if $B_a \cap B_b = \emptyset$ then $x \in B_a$ & $y \in B_b \Rightarrow x \neq_2 y$ so that $B_b \subseteq B_a^{\neq}$. So

$$\begin{aligned} x \neq_2 y &\Rightarrow \exists a \in \alpha_x \exists b \in \alpha_y B_a \cap B_b = \emptyset \\ &\Rightarrow \exists a \in \alpha_x \exists b \in \alpha_y B_b \subseteq B_a^{\neq}; \end{aligned}$$

i.e. X is $T_2^{B\&V}$ wrt $\neq_2$.

2. It is enough to observe that $B_b^* \subseteq B_b^{\neq_2}$.
3. Let X be T_1 and topologically cotransitive wrt $\neq$. Then

$$\begin{aligned} \neg(x \neq y) &\Rightarrow \forall a \in \alpha_x\ a \in \alpha_y \\ &\Rightarrow \alpha_x \subseteq \alpha_y \\ &\Rightarrow x = y \end{aligned}$$

as X is T_1. Thus X is tight. □

0.6 $\mathbb{R}^u$-metric spaces

The usual definition of a metric space is a set X equipped with a function from $X \times X$ to the real numbers which satisfies certain axioms. For our purposes, we will allow the metric to take values in the class of *upper reals*, $\mathbb{R}^u$, defined below, to allow a wider class of spaces to be metrized. This version of the real numbers is used by Vickers in [11] in his generalized metric spaces, and it is also used in the constructive theory of metric formal topologies. Some other axioms that can be added to this definition are given in [10], and in particular we will consider the strongly monotone and Dedekind reals.

There are, however, some drawbacks in using $\mathbb{R}^u$ instead of $\mathbb{R}$. Since $\mathbb{R}^u$ is a proper class, it cannot itself be an $\mathbb{R}^u$-metric space by our definition, because we shall require the points to be a set. In fact when $\mathbb{R}^u$ is given the Euclidean "metric", it does not even have a dense subset.

Definition 0.15 *An upper real is a subset R of $\mathbb{Q}^{>0}$ such that*

$$(\forall r \in \mathbb{Q}^{>0})(r \in R \leftrightarrow (\exists r' \in R)(r' < r))$$

$\mathbb{R}^u$ denotes the class of upper reals. We say that $R \in \mathbb{R}^u$ is:

- *finite if $(\exists r)(r \in R)$*
- *strongly monotone if $(r < s \;\&\; \neg\neg(r \in R)) \rightarrow s \in R$ for all $r, s \in \mathbb{Q}^{>0}$*
- *located or Dedekind if $r < s \rightarrow (r \notin R \vee s \in R)$ for all $r, s \in \mathbb{Q}^{>0}$*

Note that every located upper real is strongly monotone.

Given $R, S \in \mathbb{R}^u$, define $R + S = \{r + s \mid r \in R \;\&\; s \in S\}$. If $r \in \mathbb{Q}^{\geq 0}$, then let $r^* = \{s \in \mathbb{Q}^{>0} \mid r < s\}$. Write $R \leq S$ iff $S \subseteq R$ and $R < S$ iff $(\exists r \in \mathbb{Q}^{>0})(R + r^* \leq S)$. It is easy to show that $R < r^*$ iff $r \in R$. From now on we will write r instead of r^* provided the meaning is clear.

Definition 0.16 *An $\mathbb{R}^u$-metric space is a pair (X, d), consisting of a set X together with a function $d : X \times X \rightarrow \mathbb{R}^u$ such that for all $x, y, z \in X$:*

- $d(x, y) = 0 \Leftrightarrow x = y$
- $d(x, y) = d(y, x)$
- $d(x, y) + d(y, z) \leq d(x, z)$

We say that the space is finitary if $d(x, y)$ is finite for all $x, y \in X$, strongly monotone if $d(x, y)$ is strongly monotone and Dedekind if $d(x, y)$ is Dedekind. A metric space is a finitary Dedekind $\mathbb{R}^u$-metric space.

If X is an $\mathbb{R}^u$-metric space, we can define a topology on X by taking $S = X \times \mathbb{Q}^{>0}$ and $B_{(x,\varepsilon)} = \{y \in X \mid d(x, y) < \varepsilon\}$.

Proposition 0.17 *Let X be an $\mathbb{R}^u$-metric space.*
(i) *X is T_3.*
(ii) *If X is strongly monotone then X is T_3^+.*
(iii) *If X is Dedekind then X is $T_3^\sharp$.*

Proof.

(i) First we have to show that X is T_0. Let $x, y \in X$ and suppose that $\alpha_x = \alpha_y$. It follows that $y \in B_{(x,\varepsilon)}$ for all $\varepsilon \in \mathbb{Q}^{>0}$. So $d(x, y) = 0$, and hence $x = y$.
Next we show that X is R. Suppose that $x \in X$ and $(y, \varepsilon) \in \alpha_x$. Then $(y, \varepsilon') \in \alpha_x$ for some $\varepsilon' < \varepsilon$. If $z \in cl(B_{(y,\varepsilon')})$ then $B_{(z,(\varepsilon-\varepsilon'))} \between B_{(y,\varepsilon')}$, and so by the triangle inequality $z \in B_{(y,\varepsilon)}$. So $cl(B_{(y,\varepsilon')}) \subseteq B_{(y,\varepsilon)}$.

(ii) X is T_0 by part (i), so it remains to prove that X is R^+. Given $x \in X$ and $(y, \varepsilon) \in \alpha_x$, there exist positive rationals $\varepsilon'' < \varepsilon' < \varepsilon$ such that $(y, \varepsilon'') \in \alpha_x$. Suppose that $z \in wcl(B_{(y,\varepsilon'')})$. Then $B_{(z,(\varepsilon'-\varepsilon''))} \cap B_{(y,\varepsilon'')} \neq \emptyset$, so by the triangle inequality $\neg\neg(d(y, z) < \varepsilon')$. Since X is strongly monotone $d(y, z) < \varepsilon$, so $z \in B_{(x,\varepsilon)}$.

(iii) It remains to prove that X is $R^\sharp$. Given $x \in X$ and $(y, \varepsilon) \in \alpha_x$, choose an $\varepsilon' < \varepsilon$ such that $(y, \varepsilon') \in \alpha_x$. For all $z \in X$, either $d(y, z) < \varepsilon$ or $d(y, z) > \varepsilon'$. In the first case $z \in B_{(y,\varepsilon)}$, and in the second $z \in B^*_{(y,\varepsilon')}$. □

Example 0.18 Discrete $\mathbb{R}^u$-metric on X

Given a set X, we can define the discrete $\mathbb{R}^u$-metric on X by setting

$$d(x, y) = \{r \in \mathbb{Q}^{>0} \mid (x = y) \vee (r > 1)\}$$

It is easily verified that $d(x, y)$ is a finite upper real, and that d is an $\mathbb{R}^u$-metric on X.

If $\varepsilon \leq 1$ then $B_{(x,\varepsilon)} = \{x\}$, and if $\varepsilon > 1$ then $B_{(x,\varepsilon)} = X$. So the topology induced by this $\mathbb{R}^u$-metric has the same open sets as the discrete topology on X. □

Proposition 0.19 *Let X be a set with the discrete $\mathbb{R}^u$-metric.*
(i) *X is strongly monotone iff $(\forall x, y \in X)(\neg\neg(x = y) \to x = y)$.*
(ii) *X is Dedekind iff $(\forall x, y \in X)(x = y \vee \neg(x = y))$.*

Proof.

(i) The implication from left to right follows from Lemma 0.11 and Proposition 0.17, so assume that $\neg\neg(x = y) \to x = y$ holds for all $x, y \in X$. Let $r, s \in \mathbb{Q}^{>0}$ with $r < s$, and suppose that $\neg\neg(r \in d(x, y))$. If $s > 1$ then $s \in d(x, y)$, so suppose that $s \leq 1$. Then $r < 1$, so $\neg\neg(x = y)$, hence $x = y$, so $s \in d(s, y)$. So X is strongly monotone.

(ii) Again the implication from left to right follows from Lemma 0.11 and Proposition 0.17, so assume that $=$ is decidable on X. Let $r, s \in \mathbb{Q}^{>0}$ with $r < s$, and let $x, y \in X$. If $s > 1$ then $s \in d(x, y)$, so suppose that $r < s \leq 1$. If $x = y$ then $s \in d(x, y)$, and if $\neg(x = y)$ then $r \notin d(x, y)$. So X is Dedekind. □

0.7 Regular formal topologies

Definition 0.20 *A regular formal topology is a formal topology such that $a \lhd W_a$ for all $a \in S$, where*

$$W_a = \{b \in S \mid (\forall c \in S)\ c \lhd \{a\} \cup b^*\}.$$

Here $b^* = \{c \in S \mid \forall d \in S\ [d \leq b, c \Rightarrow d \lhd \emptyset]\}$.

Theorem 0.21 *The* ct*-space of formal points of a regular formal topology is $T_3^\sharp$.*

Proof. Let X be the space of formal points of a formal topology on S. Then X is T_0. To show that X is $R^\sharp$, given $\alpha \in X$ and $a \in \alpha$ we want $b \in \alpha$ such that $X \subseteq B_b^* \cup B_a$. As $a \in \alpha$ and $a \lhd W_a$ there is $b \in \alpha$ such that $b \in W_a$. We show that $X \subseteq B_b^* \cup B_a$.

Let $\beta \in X$. As β is inhabited we may choose some $d \in \beta$. It follows that $d \lhd \{a\} \cup b^*$ so that, as $d \in \beta$, there is some $c \in \beta$ such that $c \in \{a\} \cup b^*$. So either $a = c \in \beta$ so that $\beta \in B_a$ or $c \in b^*$.

It only remains to show that, assuming $c \in b^*$, we have $\beta \in B_b^*$. If $\gamma \in B_c \cap B_b$ then $b, c \in \gamma$ so that there is $d \in \gamma$ such that $d \leq b, c$, and hence, as $c \in b^*$, $d \lhd \emptyset$ which contradicts that γ is a formal point. So $B_c \cap B_b = \emptyset$. As $\beta \in B_c$, $\beta \in B_b^*$. □

0.8 Sober spaces

The notion of a sober topological space comes from point-free topology. Classically a sober space is one that is isomorphic to the space of formal points of a locale. The notion is classically a separation property lying somewhere between classical T_0 and classical T_2. The following seems an appropriate constructive formulation.

Definition 0.22 *A* ct*-space is sober if every ideal point of X is equal to α_x for a unique point $x \in X$, where an ideal point of X is a subset α of S such that:*

1. $\exists a \in \alpha$
2. $(\forall a, b \in \alpha)(\exists c \in \alpha)[B_c \subseteq B_a \cap B_b]$
3. $(\forall a \in \alpha)(\forall U \in Pow(S))[B_a \subseteq B_U \to \alpha \between U]$

Note that for every point x of a ct-space X the set α_x is an ideal point of X and every sober space is T_0.

Proposition 0.23 *The* ct*-space of formal points of a formal topology is sober.*

Proof. Let X be the ct-space of formal points of a formal topology on S. Observe that for any formal point β of $\mathcal{A}$,

$$\alpha_\beta = \{a \in S \mid \beta \in B_a\} = \{a \in S \mid a \in \beta\} = \beta$$

Thus if α is an ideal point of X, to show that there is a unique $\beta \in X$ such that $\alpha = \alpha_\beta$ we have to show that α is itself a formal point of $\mathcal{A}$. To do this, we must prove the following.

1. $a, b \in \alpha \;\Rightarrow\; \exists c \in \alpha\; c \leq a, b$
2. $a \in \alpha \;\&\; a \lhd U \;\Rightarrow\; \alpha \between U$.

1. Given $a, b \in \alpha$, let $U = \{d \in S \mid d \leq a, b\}$. Since α is an ideal point there is a $c \in \alpha$ such that $B_c \subseteq B_a \cap B_b$. Given any $\beta \in B_c$, $a, b \in \beta$ and since β is a $\leq$-point there is a $d \in \beta$ such that $d \in U$. Thus $B_c \subseteq B_U$, so $\alpha \between U$.
2. Suppose that $a \in \alpha$ and $a \lhd U$. If $\beta \in B_a$ then $a \in \beta$ and so $\beta \between U$ since β is a formal point. So $B_a \subseteq B_U$ and hence $a \between U$. $\square$

Classically it is true that every Hausdorff (T_2) space is sober. Constructively we can show that every complete metric space is sober, and more generally that every complete uniform space is sober. Constructively we cannot expect to do much better than this, as the space of rational numbers with the Euclidean topology cannot be shown to be sober. This was proved in [4] using sheaf models, and we give an alternative proof below.

Example 0.24 The rational numbers

The usual topology on the rational numbers can be given by taking $X = \mathbb{Q}$, $S = \mathbb{Q} \times \mathbb{Q}^{>0}$ and $B_{(x,\varepsilon)} = \{y \in \mathbb{Q} \mid |x - y| < \varepsilon\}$. $\square$

Proposition 0.25 *If $\mathbb{Q}$ is sober then LPO holds.*

Proof. Given a function $f : \mathbb{N} \to \{0, 1\}$, we may define a sequence (x_n) of rationals such that

$$x_n = \begin{cases} n^{-1} \text{ if } (\forall k \leq n)(f(k) = 0) \\ k^{-1} \text{ if } (k \leq n) \;\&\; (f(k) = 1) \;\&\; (\forall j < k)(f(j) = 0) \end{cases}$$

Let

$$\alpha = \{\; (x, \varepsilon) \in S \;\mid\; (\exists \varepsilon' < \varepsilon)(\exists N \in \mathbb{N})(\forall n \geq N)[x_n \in B_{(x,\varepsilon')}] \;\}$$

We claim that α is an ideal point.

1. For all n, $0 < x_n \leq 1$, so $(1, 2) \in \alpha$.
2. If $(x, \varepsilon), (y, \delta) \in \alpha$, then there exists an $\eta \in \mathbb{Q}^{>0}$ and an $N \in \mathbb{N}$ such that $(\forall n \geq N)(x_n \in B_{(x,\varepsilon-\eta)} \cap B_{(y,\delta-\eta)})$. In particular $B_{(x,\varepsilon-\eta)} \cap B_{(y,\delta-\eta)}$ is inhabited, so by the standard properties of the rational numbers it is an open interval, and so there exists $(z, \zeta) \in S$ with $B_{(z,\zeta)} = B_{(x,\varepsilon-\eta)} \cap B_{(y,\delta-\eta)}$. Now if we take $a = (z, \zeta + \eta)$, then $B_a \in \alpha$ and $B_a = B_{(x,\varepsilon)} \cap B_{(y,\delta)}$.

3. Suppose that $(x, \varepsilon) \in \alpha$, $U \subseteq S$ and $B_{(x,\varepsilon)} \subseteq B_U$. Then there exists an $\varepsilon' < \varepsilon$ and an $N \in \mathbb{N}$ such that $(\forall n \geq N)(x_n \in B_{(x,\varepsilon')})$.
Either (i) $(x - \varepsilon' > 0)$ or (ii) $(x - \varepsilon' \leq 0)$, so we can treat the two cases separately.
   (i) If $x - \varepsilon' > 0$, then there is a positive integer $n \geq N$ with $n^{-1} < x - \varepsilon'$. $x_n \in B_{(x,\varepsilon')}$, so $\neg(x_n = n^{-1})$, so we must have $x_n = k^{-1}$ for some $k \leq n$ such that $f(k) = 1$ and $f(j) = 0$ for all $j < k$. $k^{-1} \in B_{(x,\varepsilon)}$, so $k^{-1} \in B_{(y,\delta)}$ for some $(y, \delta) \in U$, and hence $k^{-1} \in B_{(y,\delta')}$ for some $\delta' < \delta$. So $x_m \in B_{(y,\delta')}$ for all $m \geq k$, hence $(y, \delta) \in \alpha$.
   (ii) If $x - \varepsilon' \leq 0$, then $x + \varepsilon' > 0$, because otherwise no x_n would lie in $B_{(x,\varepsilon')}$. So $0 \in B_{(x,\varepsilon)}$, and so there is a neighbourhood $(y, \delta) \in U$ with $0 \in B_{(y,\delta)}$. Now choose $n \in \mathbb{N}$ with $n^{-1} < y + \delta$. Either $x_n \in B_{(y,\delta)}$ or $x_n \notin B_{(y,\delta)}$. In the first case we can find a $\delta' < \delta$ with $0, x_n \in B_{(y,\delta')}$, so $x_m \in B_{(y,\delta')}$ for all $m \geq n$, and so $(y, \delta) \in \alpha$. In the second case $x_n \geq y + \delta > n^{-1}$, so $f(k) = 1$ for some $k \leq n$ and we can proceed as in case (i).

This completes the proof that α is an ideal point. So, by the assumption that X is sober, $\alpha = \alpha_x$ for some $x \in \mathbb{Q}$. Note that $(x = 0) \vee (x < 0) \vee (x > 0)$.

If $x = 0$**,** then for all $n \in \mathbb{N}$ $B_{(0,n^{-1})} \in \alpha$ so there exists an $m \in \mathbb{N}$ such that $x_m < n^{-1}$, hence $(\forall n \in \mathbb{N})(f(n) = 0)$.

If $x < 0$**,** we get a contradiction since no x_n lies in $B_{(x,x/2)}$.

If $x > 0$**,** then $(x, x/2) \in \alpha$. Choose $n \in \mathbb{N}$ with $n^{-1} < \frac{1}{2}x$. Then $n^{-1} \notin (x, x/2)$, so $x_n = k^{-1}$ for some $k \in \mathbb{N}$ with $f(k) = 1$.

So in each case $(\forall n \in \mathbb{N})(f(n) = 0) \vee (\exists n \in \mathbb{N})(f(n) = 1)$. □

Strong ideal points and weak sobriety The usual definition of an ideal point arises from the functors Pt and Ω between the categories of topological spaces and formal topologies. In [7], Sambin defines a *balanced formal topology* to be a formal topology with a *binary positivity predicate* $\ltimes$ added, which must satisfy certain axioms. The intuitive meaning of $a \ltimes U$ for $a \in S$ and $U \in Pow(S)$ is that the formal neighbourhood a touches some "formal closed set" determined by U.

Given a small ct-space X, we obtain a balanced formal topology on S by taking $a \lhd U$ iff $B_a \subseteq B_U$, and $a \ltimes U$ iff $B_a \between \mathsf{rest}\, U$. Sambin's definition of the points of a balanced formal topology leads us to a new definition of ideal points:

Definition 0.26 *A strong ideal point of a* ct*-space* X *is a subset* α *of* S *such that:*

1. $\exists a \in \alpha$
2. $(\forall a, b \in \alpha)(\exists c \in \alpha)[B_c \subseteq B_a \cap B_b]$
3. $(\forall a \in \alpha)(\exists x \in B_a)[\alpha_x \subseteq \alpha]$

Observe that every strong ideal point is an ideal point and α_x is a strong ideal point for each $x \in X$. We say that X is *weakly sober* if for every strong ideal point α, there is a unique $x \in X$ such that $\alpha = \alpha_x$.

Proposition 0.27 *Every* T_2 *space is weakly sober.*

Proof. Let X be T_2, and let α be a strong ideal point of X. By (1) and (3) there is an $x \in X$ such that $\alpha_x \subseteq \alpha$.

Given $y \in X$ such that $\alpha_y \subseteq \alpha$, we will show that $x = y$. Given $a \in \alpha_x$ and $b \in \alpha_y$ we have $a, b \in \alpha$, so by (2) and (3) there is a $z \in B_a \cap B_b$. Hence $x \sim_2 y$, so $x = y$.

It remains to prove that $\alpha \subseteq \alpha_x$. If $a \in \alpha$, then by (3) there is a $y \in B_a$ such that $\alpha_y \subseteq \alpha$, and by the previous paragraph $y = x$, so $a \in \alpha_x$. $\square$

REFERENCES

[1] P. Aczel and M. Rathjen. Notes on constructive set theory. Technical Report 40, 2000/2001, Institut Mittag-Leffler, The Royal Swedish Academy of Sciences, 2001.

[2] E. Bishop and D. Bridges. *Constructive Analysis*. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.

[3] D. Bridges and L. Vita. Separatedness in constructive topology. Draft preprint, November 2002.

[4] M. Fourman and D. Scott. Sheaves and logic. *Applications of Sheaves, LNM*, 753:302–401, 1979.

[5] R. J. Grayson. Concepts of general topology in constructive mathematics and in sheaves. *Annals of Math. Logic*, 20:1–41, 1981.

[6] R. J. Grayson. Concepts of general topology in constructive mathematics and in sheaves, II. *Annals of Math. Logic*, 23:55–98, 1982.

[7] G. Sambin. Some points in formal topology. To appear in Theoretical Computer Science.

[8] L. A. Steen and J. A. Seebach, Jr. *Counterexamples in Topology*. Holt, Rinehart and Winston, Inc., 1970.

[9] A. S. Troelstra. Intuitionistic general topology. Thesis, Amsterdam, 1966.

[10] A. S. Troelstra and D. van Dalen. *Constructivism in Mathematics*. North-Holland publ. co., 1998.

[11] S. J. Vickers. Localic completion of generalized metric spaces I. Draft available at
`http://www.cs.bham.ac.uk/~sjv`.