

Knights, Knaves, and Logical Reasoning

Mechanising the Laws of Thought

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15 July 2015

¹Special thanks to Francis Southern

Introduction

Thinking



Formalising



Modelling



Computing

Thinking

A Puzzle

You are on a strange island where people are divided into

- Knights – always saying the truth
- Knaves – always saying lies

You meet two natives of the island Alice and Bob, and ask them

“Are you knights or knaves?”

Alice answers *“At least one of us is a knave”*

What are Alice and Bob?

Alice: "At least one of us is a knave"

Alice



Bob



Alice



Bob



Alice



Bob



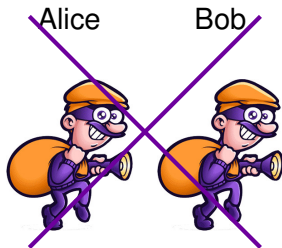
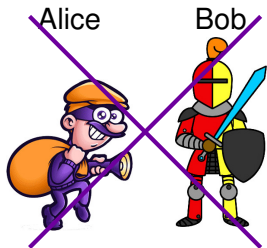
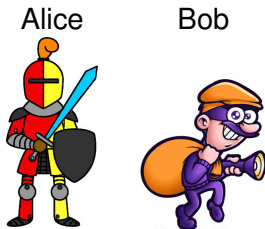
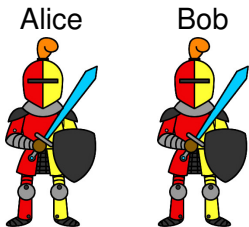
Alice



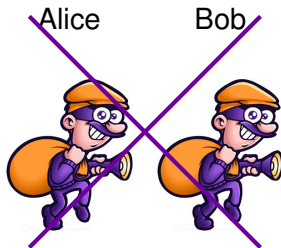
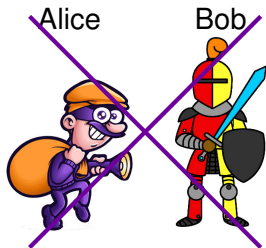
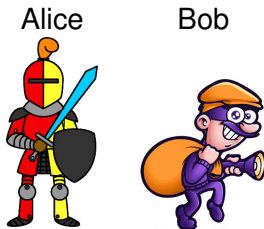
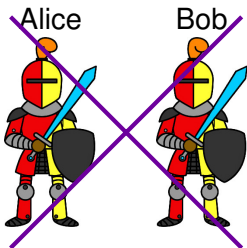
Bob



Alice: "At least one of us is a knave"



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Formalising

Formalising Correct Reasoning

A: Socrates is a man

B: All men are mortal

C: All men are Socrates

C: Socrates is mortal

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Woody Allen - Love and Death

Aristotle

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Aristotle

Linguistic, philosophical, or mathematical approaches to formalisation

Today: Propositional Logic

Propositions

An expression which is either true or false.

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Proposition test: Is it true that...?

- $2 + 2 = 5$
- Manchester
- Grass is green
- We're in Manchester
- What's your name?
- It's raining

Not – \neg , And – $\&$

Not

p	$\neg p$
F	T
T	F

It's *not* raining

Grass is *not* green.

Not – \neg , And – $\&$

Not

p	$\neg p$
F	T
T	F

It's *not* raining

Grass is *not* green.

And

p	q	$p \& q$
F	F	F
F	T	F
T	F	F
T	T	T

Grass is green *and* it's raining.

We're in Manchester *and* we're in France.

Or – \vee , Implication (If, then) – \rightarrow

Or

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
T	T	T

Take an aspirin *or* lie down.

You can have milk *or* sugar in your tea.

Or – \vee , Implication (If, then) – \rightarrow

Or

p	q	$p \vee q$
F	F	F
F	T	T
T	F	T
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Implication

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

If you get 90% on this assignment, *then* you'll pass the course.

If you're late, *then* you'll give me a fiver.

Biimplication (If and only if) – \leftrightarrow

Biimplication

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

I'll buy you a new wallet *if (and only if)* you need one.

He studies *if (and only if)* he can.

An Example: $(p \ \& \ q) \rightarrow r$

p	q	r	$(p \ \& \ q)$	$(p \ \& \ q) \rightarrow r$
F	F	F		
F	F	T		
F	T	F		
F	T	T		
T	F	F		
T	F	T		
T	T	F		
T	T	T		

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F	F	T	F	
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F	T	T	F	
T	F	F	F	
T	F	T	F	
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T	F	T	F	
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F	F	T	F	T
F	T	F	F	T
F	T	T	F	T
T	F	F	F	T
T	F	T	F	T
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Modelling

The Trick

k_A = Alice is a knight $\neg k_A$ = Alice is a knave

“Alice says X” is the same as $k_A \leftrightarrow X$

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Alice says

- “at least one of us is a knave”
- “I’m a knave or Bob is a knave”
- $\neg k_A \mid \neg k_B$

$$\Rightarrow k_A \leftrightarrow (\neg k_A \mid \neg k_B)$$

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F	F	T	T	T	
F	T	T	F	T	
T	F	F	T	T	
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From Solving to Modelling

Alice: "At least one of us is a knave"

Alice



Bob



Alice



Bob



~~Alice~~



~~Bob~~



~~Alice~~



~~Bob~~



From Solving to Modelling

Alice: “At least one of us is a knave”

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The trick: “Alice says X” is the same as $k_A \leftrightarrow X$

“At least one of us is a knave” = $\neg k_A \mid \neg k_B$

Alice says “At least one of us is a knave” = $k_A \leftrightarrow (\neg k_A \mid \neg k_B)$



From Solving to Modelling

Alice: “At least one of us is a knave”

k_A = Alice is a knight

The trick: “Alice says X” is the same as $k_A \leftrightarrow X$

“At least one of us is a knave” = $\neg k_A \mid \neg k_B$

Alice says “At least one of us is a knave” = $k_A \leftrightarrow (\neg k_A \mid \neg k_B)$

It can be (really) hard, but you only have to do it once!



Modelling a Sudoku

			7			4	1	
		3		2				6
1		7	4			5	2	3
4		1	6				8	
	2	9		7		6	3	
	7				4	2		1
7	5	2			6	3		9
3				4		1		
	1	4			3			

What propositions do we need?

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4		1	6				8	
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	7				4	2		1
7	5	2			6	3		9
3				4		1		
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What propositions do we need?

Number n is in row i and column j

- number 7 is in row 1 and column 4
- number 2 is in row 6 and column 7

$p_{1,1,1}, p_{1,1,2}, \dots, p_{9,9,8}, p_{9,9,9}$

729 propositions!

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What do we have to model?

- at least one number per cell ($p_{1,1,4} \mid \dots \mid p_{9,1,4}$)
- at most one number per cell ($p_{7,1,4} \rightarrow \neg p_{1,1,4}, p_{7,1,4} \rightarrow \neg p_{2,1,4}$)

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- no number can be repeated in a row

Modelling a Sudoku

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- no number can be repeated in a row/column

Modelling a Sudoku

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1		7	4			5	2	3
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- no number can be repeated in a row/column/region

Computing

Automating the Process

Truth table

- mechanical
- time consuming (2^n rows!)
- tedious

Automating the Process

Truth table

- mechanical
- time consuming (2^n rows!)
- tedious

Let a computer do it for you!

- ideal for mechanical tasks
- only needs an input formula
- more reliable than us
- much faster than us
- the output is easily customisable

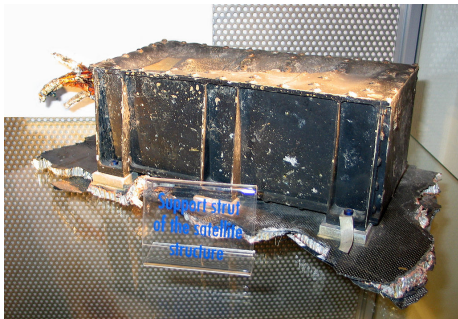
Automated Reasoning

Much more than solving puzzles!

- software and hardware verification
Intel and Microsoft
- information management
biomedical ontologies, Semantic Web, databases
- combinatorial reasoning
constraint satisfaction, planning, scheduling
- Internet security
- theorem proving in mathematics

Where Could Have Been Used

Ariane 5 rocket failure due to a software bug, cost \$370 million.



Where Has Been Used




To find and fix a bug in a widely used sorting algorithm!

 JDK / JDK-8072909



TimSort fails with `ArrayIndexOutOfBoundsException` on worst case long arrays

Agile Board More Actions ▾ Views ▾

▾ Details

Type:	 Bug	Status:	 Resolved
Priority:	 P3	Resolution:	Fixed
Affects Version/s:	7u76, 8, 9	Fix Version/s:	9
Component/s:	core-libs		
Labels:	Bu60 jdk-port regression webbug		
Subcomponent:	java.util		
Resolved In Build:	b51		
CPU:	x86		
OS:	linux		

▾ People

Assignee:	Lev Prima (Inactive)
Reporter:	Webbug Group
 Vote (0)	 Watch (4)

▾ Dates

Created:	2015-02-06 08:22
Updated:	2015-06-09 15:26
Resolved:	2015-02-12 07:44

▾ Agile

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Agile

Even Amazon and Facebook use automated reasoning techniques!

Automated Reasoning Competitions

- The CADE ATP System Competition (CASC)
- OWL Reasoning Competition (ORE)
- SAT-Race



Automated Reasoning Competitions

- The CADE ATP System Competition (CASC)
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You can bet on the winner!

Do You Want to Know More?

Look at the references on the handout!