Subset-Simulation as Minimality Criterion

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Abstract

We present a new kind of simulation with the aim of using it as minimality criterion for models of modal logics. We discuss such minimality criterion by comparing it with minimal modal Herbrand models. Even though our final goal is the automated generation of minimal models through a tableau-based method, this paper is on a theoretical level and does not aim to propose algorithms on how to generate such minimal models.

Keywords: modal logics, minimal models, model generation, automated reasoning.

1 Introduction

Model generation and minimal model generation have been studied for several logics in the field of automated reasoning. As Herbrand models are widely used in automated reasoning, it is not surprising that several minimality criteria for first-order logic are based on them, e.g. [1,2,4,5]. As many modal logics are translatable to fragments of first-order logic, minimal Herbrand models for modal logics have also been studied in a direct (without translating to first-order logic [6]) or indirect [7] way.

The calculus presented in [6] is able to generate minimal modal Herbrand models for the multi-modal logic $\mathbf{K}_{(m)}$ and its extension through reflexivity and symmetry. Such logics have the property that the Herbrand models are finite, which made the creation of the calculus easier. The approach in [6] shows its weaknesses when trying to introduce other well-known frame properties like transitivity, seriality and euclideanness.

In this short paper we discuss a new minimality criterion based on a slight variation of graph simulation, which we call *subset-simulation*. We think that this new criterion is a natural development of the minimality criterion used in [6], and it can be used for the extensions cited above.

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2 Preliminaries

We consider the basic propositional modal logic **K** possibly extended with well-known frame properties as reflexivity, seriality, symmetry, euclideanness and transitivity.

An *interpretation* $M = (W, R, V)$ for the basic propositional modal logic **K** is a triple composed of a non-empty set of worlds W, a binary relation R over W called the accessibility relation, and a labelling function that assigns a set of propositional variables to each world (i.e., $V : W \mapsto \Sigma^2$). Note that the labelling function takes as input a world and not a propositional variable. We define the labelling function in this way because it makes the presentation of subset-simulation easier and more compact.

Given an interpretation $M = (W, R, V)$ and a world $u \in W$, if a modal formula ϕ is such that $M, u \models \phi$, then M is a *model* for ϕ. We use the letters u and v to represent worlds.

In a previous paper [6], we presented a tableau calculus for the generation of *minimal modal Herbrand models*. The idea of (minimal) modal Herbrand model is our starting point to present the criterion based on subset-simulation.

Given a modal formula ϕ, a *modal Herbrand model* $H = (W, R, V)$ of ϕ is a model obtained by always creating new witnesses to satisfy diamond formulae.

Given a modal formula ϕ and a modal Herbrand model $H = (W, R, V)$, H is a *minimal modal Herbrand model* of ϕ iff for any other modal Herbrand model $H' = (W', R', V')$ and all $u \in W'$, if $V'(u) \subseteq V(u)$ and $R' \subseteq R$ then $H = H'$.

As minimality is based on a subset relationship between the labelling functions, it is important to create always the same witness for the same occurrence of a diamond formula.

The generation of minimal modal Herbrand models has been studied in [6] for the multi-modal logic $\mathbf{K}_{(m)}$ and its extension with reflexivity and symmetry, but it presents two main weaknesses. First, it is a minimisation based on syntax, this means that it considers as minimal also models that are not minimal from a semantic point of view. Second, creating always new witnesses for diamond formulae easily leads to infinite modal Herbrand model when the logic is equipped with properties like transitivity or seriality.

3 Subset-Simulation

We propose a new minimality criterion based on a slight modification to the notion of graph simulation.

Let $M = (W, R, V)$ and $M' = (W', R', V')$ be two models of a modal formula ϕ. A *simulation* is a total binary relation $S \subseteq W \times W'$, such that for any two worlds $u \in W$ and $u' \in W'$, uSu' iff

- $V(u) = V'(u')$ and
- if uRv then there exists a $v' \in W'$ such that vSv' and $u'R'v'$

It is important to note that we explicitly require the relations to be *total*, which means that every world in W is related to some other world via S.

As suggested by the name, the notion of simulation is as the notion of bisimulation except that one of the “zig-zag” conditions is omitted. This small difference is important, because it allows us to check if a model is embedded in another model, while bisimulation allows checking the equivalence of two models.

For our purpose the notion of simulation is still too strong. Minimal modal Herbrand models have the nice feature of comparing the propositional variables true in a specific world. This is not possible using simulation, because the labels are required to be equal (by the first condition of the definition). For this reason we propose the concept of subset-simulation.

Let $M = (W, R, V)$ and $M' = (W', R', V')$ be two models of a modal formula ϕ. A *subset-simulation* is a total binary relation $S_{\subseteq} \subseteq W \times W'$, such that for any two worlds $u \in W$ and $u' \in W'$, $uS_{\subseteq}u'$ iff

- $V(u) \subseteq V'(u')$
- if uRv then there exists a $v' \in W'$ such that $vS_{\subseteq}v'$ and $u'R'v'$

If such subset-simulation exists we say that M *subset-simulates* M'.

Similarly to minimal modal Herbrand models, we define minimality with respect to subset-simulation as follows.

Given a modal formula ϕ and a model M, M is a *minimal model modulo subset-simulation* of ϕ iff for any other model M' of ϕ, if M' subset-simulates M, then M subset-simulates M'.

It is important to note that minimal modal Herbrand models can be seen as a special case of minimal models modulo subset-simulation, where $S_{\subseteq} = \{(u, v) | u \in W, v \in W' \text{ and } u = v\}$. This implies that models that are not minimal modal Herbrand models are not even minimal modulo subset-simulation.

4 Subset-Simulation vs Modal Herbrand Models

The idea behind minimal models modulo subset-simulation is to overcome the two weaknesses of minimal modal Herbrand models. First, while comparing subset of propositional variables like minimal modal Herbrand models, minimality modulo subset-simulation is not restricted to a syntactic level. Specifically, while for minimal modal Herbrand models $V(u) \subseteq V'(v)$ is performed only when $u = v$, in the new minimality criterion u and v can be different as long as they are related by $S_{\subseteq}$. Second, as there are no restrictions on the generation of models, minimality modulo subset-simulation can be successfully applied on those modal logics enjoying the finite model property.

We present differences between minimality modulo subset-simulation and minimal modal Herbrand models by means of examples. In those examples minimal modal Herbrand models are obtained by following the tableau calculus in [6].

As first example, let us consider the modal formula $(\Diamond p \wedge \Diamond q) \vee \Diamond (p \wedge q)$. This formula has two minimal modal Herbrand models, but only one minimal model modulo subset-simulation, as shown in Figure 1. Specifically, it is possible to note that model (a) subset-simulates (b), but the other way around does not

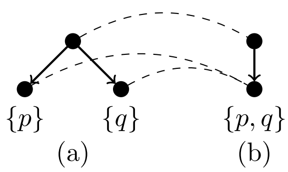

Fig. 1. Minimal modal Herbrand models of $(\Diamond p \wedge \Diamond q) \vee \Diamond(p \wedge q)$

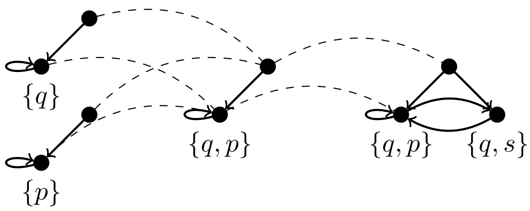

Fig. 2. Partial order of minimal models modulo subset-simulation of $(\Diamond(p \vee q) \wedge \Box\Diamond(p \vee q)) \vee (\Diamond(q \wedge (p \vee s)) \wedge \Box\Diamond(q \wedge (p \vee s)))$

hold because there is no world u in (a) such that $\{p, q\} \subseteq V(u)$.

From the minimal model modulo subset-simulation in Figure 1, it is interesting to note that our new minimality criterion tends to prefer models where labels of worlds are minimised to the detriment of minimising the accessibility relations. This highlight an important difference between minimal modal Herbrand models and the new minimality criterion, that is, the new criterion minimises only propositional variables and not the accessibility relation.

The next example shows how subset-simulation can be used for minimality in case that the modal Herbrand models are infinite, and it is also an example of the partial order resulting from subset-simulation. The formula under consideration is $(\Diamond(p \vee q) \wedge \Box\Diamond(p \vee q)) \vee (\Diamond(q \wedge (p \vee s)) \wedge \Box\Diamond(q \wedge (p \vee s)))$ in a transitive frame, and some possible model of it is shown in Figure 2.

The minimal models modulo subset-simulation in Figure 2 are the left most models. The figure is thought in such a way that there are subset-simulations from left models to right models, and not the other way around. This example is an instance of models that cannot be generated by using the approach in [6]. In fact, as no blocking is allowed in generating modal Herbrand models, transitivity easily results in infinite Herbrand models.

The second example also shows an instance of the partial ordering induced by subset-simulation. Specifically, subset-simulation induces a reflexive and transitive relation among models. Subset-simulation is clearly reflexive because any model is bisimilar to itself, implying that is subset-simulate itself. Subset-simulation is also transitive, because if a model M is embedded in a model M' and M' is embedded in a model M'', then M is also embedded in M''. It can be argued that subset-simulation does not induce an actual partial order, because subset-simulation is symmetric for bisimilar models. But two bisimilar models are equivalent. This means that one is semantically redundant, and we can omit bisimilar models and keep only one of them for minimisation purposes.

Regarding the relation between subset-simulation and bisimulation, we conjecture that if a model M subset-simulates a model M', and M' subset-simulates M, then M and M' are bisimilar. If such conjecture holds, than minimality modulo subset-simulation can be seen as some king of lifting from syntactic minimality to semantic minimality.

5 Conclusion

We have presented a possible new minimality criterion for models of modal logics. Even though the proposed criterion does not minimise the accessibility relation, we believe that it is a natural development of the syntactic minimality obtained by using Herbrand models towards a more semantic minimality.

We think that minimality based on subset-simulation is an interesting and promising kind of minimisation, and it deserves to be studied. Our final goal is to create a tableau-based method for the generation of minimal models modulo subset-simulation for multi-modal logics, but it is too early to present such a method, because there are still few open problems that need to be solved. First, a study of the complexity of performing subset-simulation. But it is worth noticing that is not difficult to think of an algorithm to perform it, meaning that the problem is at least decidable. Second, different blocking strategies results in different models, and they clearly have an impact on the resulting minimal models. Finally, it is not clear if there are techniques or tests to understanding if the model being generated is minimal (as in [1,2,3,5,6]). If such techniques do not exist, then it would be necessary to generate all the models and compare them. As subset-simulation results in a partial order among the models, the latter possibility is clearly valid and it is not difficult to think of an incremental computation of the minimal models without the need of comparing them all, but it may be practically unfeasible.

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