Models Minimal Modulo Subset-Simulation for Expressive Propositional Modal Logics

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(Minimal) Model Generation

Useful for several tasks:

- hardware and software verification
- fault analysis
- commonsense reasoning
- query answering
- . . .

Minimality criteria:

- domain minimality
- minimisation of a certain set of predicates
- minimal Herbrand models
- In this talk: models minimal modulo subset-simulation

Done and to Do (Aims)

IJCAR 2014

Minimal model procedures for all the sublogics of $\mathbf{S5}$

- sound
- refutationally complete
- minimal model sound
- minimal model complete
- terminating

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Aims

Discuss how to generalise to more expressive modal logics

- multi-modal logics
- inclusion axioms
- universal modalities

Propositional Modal Logic

Syntax: $\phi ::= \bot \mid \top \mid p_i \mid \neg\phi \mid \phi_1 \vee \phi_2 \mid \phi_1 \wedge \phi_2 \mid \Box\phi \mid \Diamond\phi$

Kripke Semantics: An interpretation $\mathcal{I}$ is a tuple (W, R, V).

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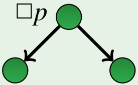
Box semantics



V assigns a set of propositional symbols to each element of W.

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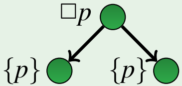

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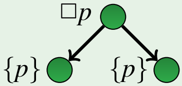

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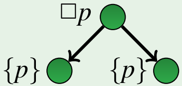

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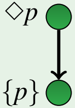

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Frame Properties

$\Box$	Axiom	Frame condition	First-order representation
$\mathbf{K}$			
$\mathbf{T}$	$\Box p \to p$	reflexivity	$\forall x R(x, x)$
$\mathbf{B}$	$p \to \Box\Diamond p$	symmetry	$\forall x \forall y (R(x, y) \to R(y, x))$
$\mathbf{D}$	$\Box p \to \Diamond p$	seriality	$\forall x \exists y R(x, y)$
$\mathbf{4}$	$\Box p \to \Box\Box p$	transitivity	$\forall x \forall y \forall z (R(x, y) \wedge R(y, z) \to R(x, z))$
$\mathbf{5}$	$\Diamond p \to \Box\Diamond p$	Euclideanness	$\forall x \forall y \forall z (R(x, y) \wedge R(x, z) \to R(y, z))$

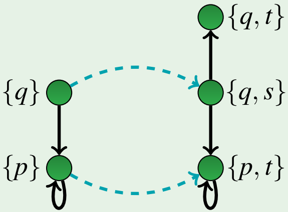
Fifteen possible logics

$\mathbf{K}$, $\mathbf{KT}$, $\mathbf{KB}$, $\ldots$, $\mathbf{K45}$, $\mathbf{KD45}$, $\mathbf{KB4}$ and $\mathbf{KT5}(= \mathbf{S5})$

Subset-Simulation $S_{\subseteq}$

Relation between nodes of two models $\mathcal{I} = (W, R, V)$ and $\mathcal{I}' = (W', R', V')$ s.t. for any two worlds $u \in W$ and $u' \in W'$, if uSu' then the following hold.

- $V(u) \subseteq V'(u')$, and
- if uRv, then there exists a $v' \in W'$ such that $u'R'v'$ and vSv'.



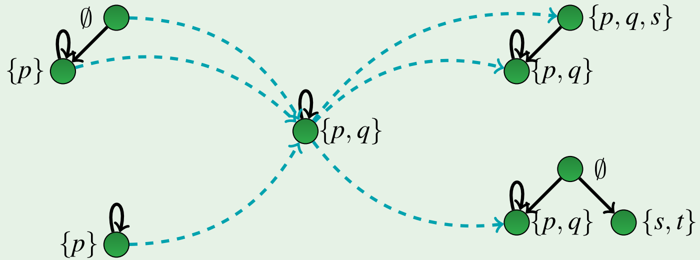
If for all $u \in W$ there is at least one $u' \in W'$ such that uSu', then we call $S_{\subseteq}$ a full subset-simulation from $\mathcal{I}$ to $\mathcal{I}'$ ($\mathcal{I} \leq_{\subseteq} \mathcal{I}'$).

Models Minimal Modulo Subset-Simulation

Subset-simulation is a preorder on models.

Definition

A model $\mathcal{I}$ of a modal formula ϕ is minimal modulo subset-simulation iff for any model $\mathcal{I}'$ of ϕ, if $\mathcal{I}' \leq_{\subseteq} \mathcal{I}$, then $\mathcal{I} \leq_{\subseteq} \mathcal{I}'$.



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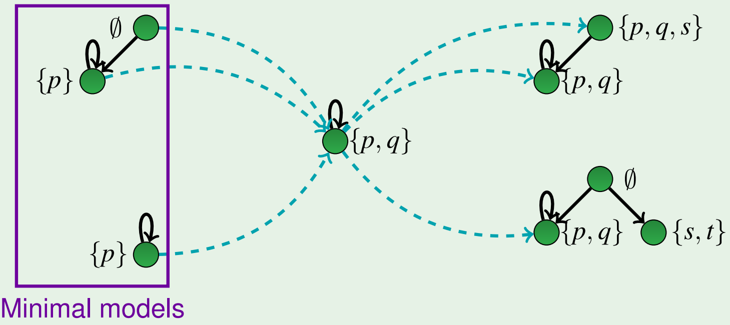

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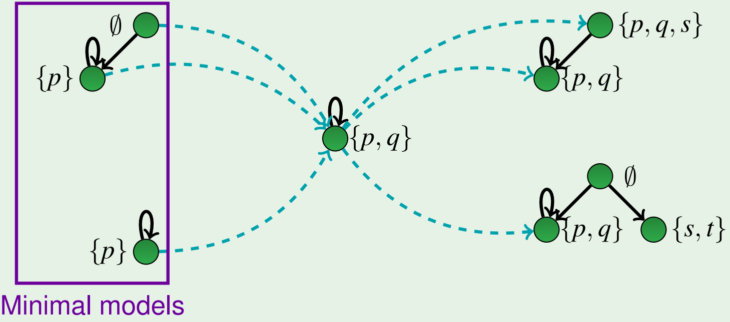


Infinitely many minimal models can belong to a symmetry class.

Minimal Model Soundness and Completeness

Minimal Model Soundness

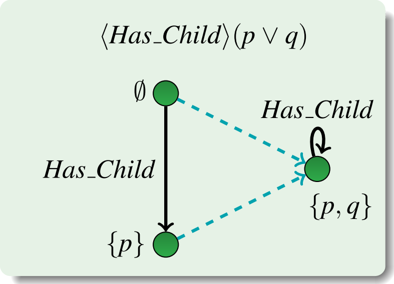
A procedure is minimal model sound if it generates only models minimal modulo subset-simulation.

Minimal Model Completeness

A procedure is minimal model complete if it generates at least one model minimal modulo subset-simulation per symmetry class.

Properties of the Minimality Criterion

- loop free models are preferred
- syntax independent
- minimisation of the valuation function
- suitable for many non-classical logics



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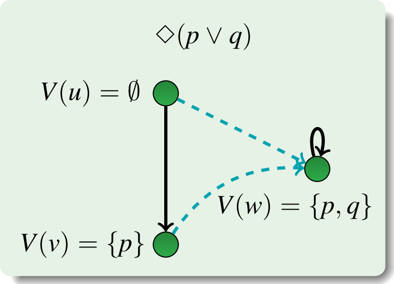

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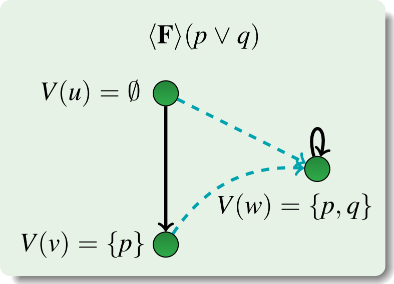

Procedures for Computing Minimal Models

Combination of tableaux calculi and a minimality test.

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- goal-oriented rules
- modularity
- termination
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Subset-simulation test

- closes unwanted branches of a tableau
- logic independent
- ensures minimal model soundness

Generalisations

Multi-modal logics

$$(W, R, V) \qquad \Rightarrow \qquad (W, R_1, \ldots, R_n, V)$$

$$[R_1], \langle R_1 \rangle, \ldots, [R_n], \langle R_n \rangle$$

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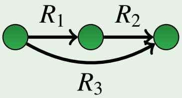
Inclusion axioms

Syntax: $[R_i]\phi \to [R_1] \ldots [R_n]\phi$

Semantics: $R_1 \circ \ldots \circ R_n \subseteq R_i$

$$[R_3]\phi \to [R_1][R_2]\phi$$



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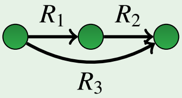

Universal modalities

$[\mathcal{U}]\phi$
ϕ holds in every world

$\langle \mathcal{U} \rangle \phi$
ϕ holds in some world

Challenges

- incorporate the semantics into the procedures
  - new rules
- preserve properties of the procedures
  - minimal model completeness
  - minimal model soundness
  - termination

New Rules

Multi-modal logics

Inclusion axioms

Universal modalities

New Rules

Multi-modal logics

- modification of existing rules

$$(\Box)\ \frac{(u,v) : R \qquad u : \Box\phi}{v : \phi} \quad \Rightarrow \quad (\Box)^i\ \frac{(u,v) : R_i \qquad u : [R_i]\phi}{v : \phi}$$

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$$[R_i]\phi \to [R_1]\dots[R_n]\phi \qquad \frac{(u_1,u_2) : R_1,\ \dots\ ,\ (u_n,u_{n+1}) : R_n}{(u_1,u_{n+1}) : R_i}$$

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Universal modalities

$\langle \mathcal{U} \rangle \phi$ $\quad \dfrac{u : \langle \mathcal{U} \rangle \phi}{v_{\langle \mathcal{U} \rangle \phi} : \phi}$ where $v_{\langle \mathcal{U} \rangle \phi}$ is uniquely assigned to $\langle \mathcal{U} \rangle \phi$

$[\mathcal{U}]\phi$ $\quad \dfrac{u : [\mathcal{U}]\phi}{v : \phi}$ for any v appearing on the branch

Minimal Model Soundness and Completeness

Minimal model completeness

Adaptation of our previous proof

- take any minimal model M
- the tableau generates at least a model M' s.t. $M' \leq_{\subseteq} M$
- minimality of $M \Rightarrow M$ and M' same symmetry class

$\Rightarrow$ minimal model completeness

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Minimal model soundness

Obtained by the application of the subset-simulation test.

Termination

Decision procedures exist $\Rightarrow$ blocking techniques exist

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Main Challenge

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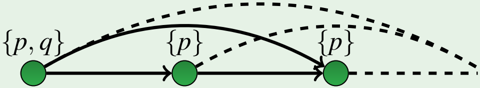

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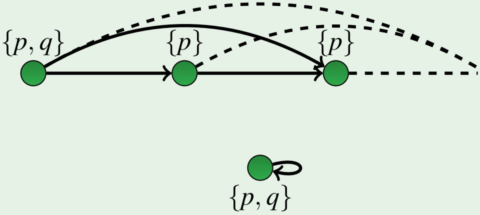

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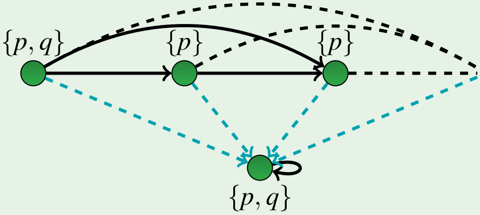

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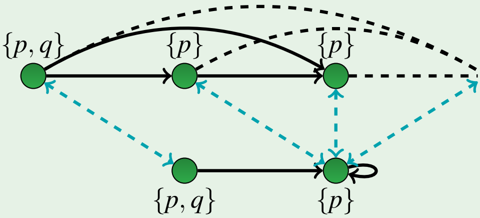

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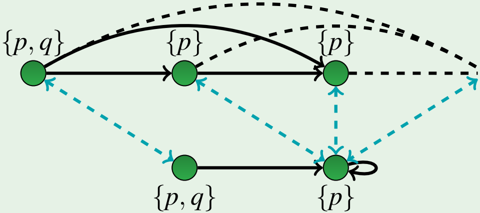

Most probable solution: variations of equality blocking

Benefits of Termination

- new decision procedures
- theoretical implications (e.g., finitely many symmetry classes)
- effective implementations

Where Are We Now?

- new rules
- minimal model soundness and completeness
- termination
  - multi-modal logics
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  - inclusion axioms – more investigation needed
  - universal modalities – working on the proof

Conclusion and Further Work

- generalisations of the procedures are possible
- termination is the hardest challenge
  - almost solved for multi-modal logics and universal modalities
  - unsolved for inclusion axioms

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- generalisations of the procedures are possible
- termination is the hardest challenge
  - almost solved for multi-modal logics and universal modalities
  - unsolved for inclusion axioms
- there is no limit to generalisations!
  - converse relations
  - dynamic modal logics
  - other non-classical logics
- fragments of first-order logic
- implementation