Minimal Models Modulo Subset-Simulation for Modal Logics

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Abstract: In this abstract we present a novel minimality criterion for models of propositional modal logics, and we sketch a tableau calculus for the generation of minimal models based on this new minimality criterion.

1 Introduction

In a previous work [5], we presented a tableau calculus for the generation of minimal Herbrand model for the multi-modal logic $\mathbf{K}_{(m)}$ extended with reflexivity and symmetry. Minimal Herbrand models have the characteristic of being syntactic models. Hence, the models generated in [5] could be semantically redundant, or semantically not minimal. For this reason, in this abstract we present a novel semantic minimality criterion, and we sketch a tableau calculus for the basic modal logic $\mathbf{K}$ for the generation of minimal models satisfying this criterion.

2 Subset-Simulation

An *interpretation* M is a triple (W, R, V) where W is a non-empty set of worlds, R is a binary relation over W, and V is an interpretation function that assigns to each $u \in W$ a subset of the propositional variables in the signature. It is important to note that our definition of the interpretation function differs from the more common one. This allows us to simplify the presentation of subset-simulation.

Let ϕ be a modal formula, $M = (W, R, V)$ an interpretation and $u \in W$ a world. If $M, u \models \phi$ then M is a *model* of ϕ.

Our novel minimality criterion is based on subset-simulation.

Let $M = (W, R, V)$ and $M' = (W', R', V')$ be two models of a modal formula ϕ. A *subset-simulation* is a total binary relation $S_\subseteq \subseteq W \times W'$ such that for any two worlds $u \in W$ and $u' \in W'$, $uS_\subseteq u'$ if the following hold.

- $V(u) \subseteq V'(u')$ and
- if uRv then there exists a $v' \in W'$ such that $vS_\subseteq v'$ and $u'R'v'$.

If such a subset-simulation exists we say that M *subset-simulates* M'.

Subset-simulation is a slight variation of the notion of simulation in model checking [2]. It is also possible to find in literature [1, 4] a definition of simulation for the description logic $\mathcal{EL}$, where the notion of simulation corresponds to our definition of subset-simulation.

As it is defined, subset-simulation is not a minimality criterion, but it has properties allowing us to use subset-simulation for generating minimal models. Subset-simulation is a reflexive and transitive relation over models, hence it is a preorder. As subset-simulation is a preorder, we can consider as minimal models all the minimal elements of the preorder. Such a minimality criterion would result in a huge number of minimal models due to models that are symmetric with respect to subset-simulation. To avoid this situation, our minimality criterion is based on the concepts of maximal subset-simulation and uncovering. A subset-simulation $S_\subseteq$ between two models M and M' is a *maximal subset-simulation* if for any other subset-simulation $S'_\subseteq$ between M and M' we have that $S'_\subseteq \subseteq S_\subseteq$. Given a maximal subset-simulation $S_\subseteq$ between two models $M = (W, R, V)$ and $M' = (W', R', V')$, we refer to the set $U = W' \setminus range(S_\subseteq)$ as the *uncovered worlds of* $S_\subseteq$.

All the above notions allow us to formulate a minimality criterion based on subset-simulation as follows. Let M be a model of a modal formula ϕ, M is a *minimal model modulo subset-simulation* if for any other model M' of ϕ, if M' subset-simulates M, then M subset-simulates M' and $|U| \geq |U'|$. In other words, non-bisimilar models with a smaller domain are preferred over symmetric models with a bigger domain.

Figure 1 shows an example of the defined minimality criterion. Both models subset-simulate each other, but the set of uncovered worlds of the simulation from left to right (red arrow) has a bigger cardinality than the uncovered worlds of the other subset-simulation (from right to left, blue arrows). Hence, the model on the left is considered minimal with respect the model on the right.

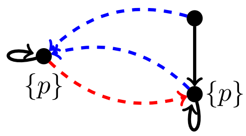

Figure 1: Example of the minimality criterion based on subset-simulation

The minimality criterion we propose has several important properties. First, it is a criterion based on the semantics of models. This means that the comparison does not suffer of strong syntactic restrictions such as worlds names, and the resulting frame does not need to be as rigid as for minimal modal Herbrand models in [5]. Second, finite models are preferred over infinite models, where we consider as infinite models even finite models resulting in infinite ones once unravelled. As the criterion is based on a subset-simulation relation over graphs, it is scalable and applicable to many modal logics (e.g., to generalisations such as universal modalities or other common frame properties). Finally, an algorithm for computing maximal subset-situation is obtainable via a modification of the algorithm for computing maximal auto-simulation in [3].

3 Tableau Calculus

The main idea of our approach is to close branches selectively when the extracted model is not minimal from an already sound and complete tableau calculus.

As many modal logics can lead to the generation of infinite models, we have studied several blocking techniques. Blocking techniques are usually used to achieve termination of sound and complete tableaux calculi, but may not be suited for particular kind of model generation. This happens to be the case for minimal model generation with respect to our criterion. For this reason, we decided to avoid any blocking technique and to require the calculus to have the following rule for the expansion of diamond formulae.

$$(\Diamond)\ \frac{u : \Diamond\phi}{\begin{array}{l|c|l|l} (u, v_1) : R & \ldots & (u, v_n) : R & (u, w) : R \\ v_1 : \phi & & v_n : \phi & w : \phi \end{array}}$$

where all v_i are worlds appearing on the current branch, and w is a fresh world.

Such a $(\Diamond)$ rule is clearly expensive, but it is necessary to achieve minimal model completeness.

A general principle to reduce the search space is to avoid the expansion of disjunctions that have among the disjuncts negated propositional variable if the complement of those does not appear on the branch. This is because the disjunctions would be trivially satisfied even without contributing to the model, and branching on them would only create non-minimal models.

Another observation to reduce the search space and to increase the possibility to have a minimal model on the left-most branch is the use of the following complement splitting rule.

$$(CS)\ \frac{u : \mathcal{A} \vee \phi^+}{\begin{array}{l|l} u : \mathcal{A} & u : \phi^+ \\ neg(\phi^+) & \end{array}}$$

where $\mathcal{A}$ is of the form p_i, $\Diamond\phi$, or $\Box\phi$; ϕ^+ is a disjunction where no disjunct is a negated propositional variable; and $neg(\phi^+) = \neg p_1 \wedge \ldots, \wedge\neg p_n$, where $p_1, \ldots, p_n$ are all the propositional variables appearing as disjuncts of ϕ^+.

To avoid the application of unnecessary inference steps, we want to be able to close a branch as soon as the model extracted from it is not minimal. As subset-simulation can be computed only if at least one of the models under consideration is a complete model and not a partial one, the branch selection strategy can neither be a depth-first strategy nor a breadth-first strategy. The branch selection strategy we use is a left-to-right strategy that selects the branch with the least number of worlds. In this way, the first generated models are models with minimal domains, and they can be used to close not fully expanded branches where the partial model is not minimal with respect to the already extracted ones.

All the other rules of the calculus are standard rules for the expansion of semantic ground tableaux for modal logics and, for space reasons, are omitted.

As the proposed tableau calculus is currently under study, we have not formally proved minimal model soundness and completeness yet. We are, however, quite confident that such properties hold. A desirable property that does not hold for the proposed calculus is termination. This is because minimal models modulo subset-simulation can be infinite, even in case of modal logics with the finite model property. For this reason, an implementation of such a calculus requires some heuristics to improve the generation of minimal models and to guarantee termination.

4 Conclusion

We presented a novel minimality criterion for propositional modal logics. Thanks to the properties of such a minimality criterion, it might be used for a wide variety of propositional modal logics.

We also sketched a tableau calculus for the generation of minimal model modulo subset-simulation. The calculus was presented as a calculus for the basic modal logic $\mathbf{K}$, but its generalisation to multi-modal logic is immediate. Furthermore, the presented rules are just the core rules of the calculus, and they can be enriched with other rules to cover all the well-known frame properties while preserving minimal model soundness and completeness.

References

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