$\mathcal{E}$-Connections and Logics of Distance

$\mathcal{E}$-CONNECTIONS AND LOGICS OF DISTANCE

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Contents

Overview

This thesis intends to contribute to two different strands of logic-based Artificial Intelligence (AI) research.

In Part 1 of the thesis, we investigate logics of distance spaces: a family of knowledge representation formalisms aimed to bring a numerical, quantitative concept of distance into the conventional qualitative representation and reasoning.

The second part is complementary to the first: since knowledge representation requires to capture different aspects of the world, like temporal, spatial, or epistemic aspects, there is a strong need, having specialised formalisms for each of these aspects at hand, to reintegrate them in a way that allows those different aspects to interact, and, ideally, such that reasoning mechanisms for the component formalisms can be used to support reasoning within the combined formalism. Combining knowledge representation formalisms, however, is difficult, since the computational behaviour of the resulting hybrids is often much worse than the behaviour of the combined components.

Thus, in Part 2 of the thesis, we are concerned with the study of $\mathcal{E}$-connections, a combination methodology for logical formalism that is widely applicable, and which is very well-behaved computationally.

To conclude the thesis, we will summarise our main achievements, mention a number of open problems, and discuss directions for future research.

The two sections that follow contain brief descriptions of the contents of (the two parts of) this thesis. More detailed introductions can be found, respectively, in Sections 1.1 and 4.1.

Logics of Distance

In Part 1 of the thesis, we systematically investigate first-order, modal, and Boolean modal languages intended for reasoning about distances, where the concept of 'distance' is understood in a wide, not necessarily spatial sense. The structures in which these languages are interpreted are metric spaces, or more general classes of distance spaces satisfying only a subset of the conditions of metric spaces.

Chapter 1 introduces the first-order languages $\mathcal{LF}[M]$, their two-variable fragments $\mathcal{LF}_2[M]$, as well as a family $\mathcal{LO}_O[M]$ of modal languages parametrised by sets

O of distance operators being primitive in the language, and by parameter sets M of subsets of the reals, i.e., the distances that formulae can explicitly refer to. Over different classes of distance spaces, we compare the expressive power of the first-order languages with the modal distance languages, as well as with a variant $\mathcal{LOB}[M]$ of Boolean modal languages. It is shown that the modal language $\mathcal{LO}[M]$ is expressively complete over metric spaces for the two-variable fragment $\mathcal{LF}_2[M]$.

Chapter 2 investigates the computational behaviour of these languages. We show the two-variable fragment of first-order logic to be undecidable when interpreted in metric spaces, and single out an expressive and decidable fragment $\mathcal{LO}_D[M]$ of $\mathcal{LO}[M]$ that has the finite model property.

In Chapter 3, we study logical properties of the modal distance logics introduced. We give complete axiomatisations of modal distance logics, amongst them the counterpart of two-variable first-order distance logic interpreted in metric spaces, and discuss compactness, and the (mostly failing) interpolation property.

$\mathcal{E}$-Connections

Part 2 of the thesis is concerned with a new combination technique for logics, called $\mathcal{E}$-connections.

In Chapter 4, we introduce abstract description systems as a framework for studying combinations of logics, introduce the methodology of $\mathcal{E}$-connections, and provide a number of examples.

Chapter 5 studies the computational behaviour of $\mathcal{E}$-connections. It is shown that, unlike for instance products of logics, $\mathcal{E}$-connections exhibit an extremely stable computational behaviour: the basic $\mathcal{E}$-connection of any number of decidable logical systems is again decidable. This result can be refined to show that $\mathcal{E}$-connections of certain subclasses of decidable logics remain decidable, even if the interaction between the component logics is enriched in various ways.

In Chapter 6, we begin by comparing $\mathcal{E}$-connections with the related combination methodology of distributed description logics (DDLs). We show that DDLs can be understood as a special case of $\mathcal{E}$-connections. We then briefly start an investigation into the expressiveness of $\mathcal{E}$-connections by lifting the concept of bisimulations to $\mathcal{E}$-connections, and apply this theory to show that certain properties are not definable by basic $\mathcal{E}$-connections. It is shown, however, that such undefinable properties can be added as 'first-order constraints' to $\mathcal{E}$-connections in a way that, again, preserves decidability. Finally, we discuss the relationship between $\mathcal{E}$-connections and other combination methodologies such as multi-dimensional products of logics, independent fusions, fibrings, and description logics with concrete domains.

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Part 1

Logics of Distance

CHAPTER 1

Languages, Logics, and Expressivity

1.1. Introducing Logics of Distance

Logics of distance spaces were conceived as knowledge representation formalisms aimed to bring a numerical, quantitative concept of distance into the conventional qualitative representation and reasoning [Sturm et al., 2000, Kutz et al., 2003b]. The main application area of these formalisms envisaged was spatial reasoning. However, the notion of 'distance' allows a wide variety of interpretations.

Distances can be induced by different measures. We may be interested in the physical distance between two cities a and b, i.e., in the length of the straight (or geodesic) line between a and b. More pragmatic would be to be concerned about the length of the railroad connecting a and b, or even better, the time it takes to go from a to b by train (plane, ship, etc.). But we can also define the distance as the number of cities (stations, friends to visit, etc.) on the way from a to b, as the difference in altitude between a and b, and so forth. A more abstract notion of distance is obtained by assuming the distance between two points to be induced by a **similarity measure**: we may say that two points have distance 1 if they share a certain number of properties, distance 2 if they share a certain smaller number of properties, etc.

The standard mathematical models, capturing common features of various notions of distance, are known as metric spaces. A **metric space** is a pair $\langle W, d\rangle$, where W is a set (of points) and d a function from $W \times W$ into the set $\mathbb{R}^+$ (of non-negative real numbers) satisfying, for all $x, y, z \in W$, the following axioms:

$$d(x,y) = 0 \iff x = y; \tag{1}$$

$$d(x,y) = d(y,x); \tag{2}$$

$$d(x,z) \leq d(x,y) + d(y,z). \tag{3}$$

We refer to (2) as **symmetry** of the metric, to (3) as **triangularity**, and call the value $d(x,y)$ the **distance** from the point x to the point y. Axiom (1) is related to the Leibnizian principle of the *indiscernibility of identicals* and is assumed throughout. Clearly, the distance from a point to itself should be zero in any sensible interpretation of 'distance'.

Note, however, that the axiom also implies the converse, namely the *identity of indiscernibles*: if we assume the distance function to measure similarity, perfect similarity, i.e. distance zero, implies identity.[1]

Perhaps the most well-known metric spaces are the n-dimensional Euclidean spaces $\langle \mathbb{R}^n, d_n \rangle$ with the metric

$$d_n(\vec{x}, \vec{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}.$$

Although acceptable in many cases, the concept of metric space is not universally applicable to all interesting measures of distance between points, especially those used in everyday life. Consider, for instance, the following three examples:

(i) $d(x, y)$ measures the flight-time from location x to location y;

(ii) $d(x, y)$ measures the 'fuzzy distance' between locations x and y, i.e., is one of '*short*', '*medium*', and '*long*';

(iii) $d(x, y)$ measures the similarity of scientific topics x and y identified with some subset of keywords from a list K, i.e., computes the ratio of non-shared to shared keywords.

In (i), d is clearly not necessarily symmetric, just book an arbitrary flight from x to y and back, and stop the time. In (ii), note that to represent these measures we can, of course, take functions d from $W \times W$ into the subset $\{1, 2, 3\}$ of $\mathbb{R}^+$ and define *short* $:= 1$, *medium* $:= 2$, and *long* $:= 3$. So we can still regard these distances as real numbers. However, for measures of this type the triangular inequality (3) usually does not hold: short plus short can still be short, but it can also be medium or long. As concerns (iii), assume K is some list of keywords, and topics x, y are identified with some non-empty subset of K. If $x \cap y = \varnothing$, we may want to define $d(x, y) := |K|$. Otherwise, we may set

$$d(x, y) := \frac{|(x \cup y) \setminus (x \cap y)|}{|x \cap y|},$$

using the set-theoretic symmetric difference to count the number of keywords on which x and y disagree, and intersection to count the number of keywords on which they agree. This measure clearly satisfies (1) and (2), but it does not satisfy the triangular inequality (3). For instance, take K to be the index of terms of this thesis and

x = $\{$*Euclidean spaces*, *metric spaces*$\}$;
y = $\{$*metric spaces*, *frame-companions*$\}$;
z = $\{$*frame-companions*, *compactness*$\}$.

[1]On the other hand, it does make sense to allow for the situation where the distance between points x and y is zero (because, e.g., they share all properties in question) and where x and y still denote distinct points. The investigation of such kinds of models is left for future work.

Then $d(x,y) = \frac{2}{1}$, $d(y,z) = \frac{2}{1}$, but $d(x,z) = 182$, since x and z share no keywords at all and $|K| = 182$.

Metric spaces as well as more general **distance spaces** $\langle W, d \rangle$ satisfying only axiom (1) are the intended models of the languages we will construct and investigate. We will denote the class of all metric spaces by $\mathcal{MS}$ or $\mathcal{D}^m$; the class of **symmetric distance spaces** satisfying (1) and (2) by $\mathcal{D}^s$; the class of all **triangular distance spaces** satisfying (1) and (3) by $\mathcal{D}^t$; and, the class of all **distance spaces**, still satisfying (1), by $\mathcal{D}^d$, or simply by $\mathcal{D}$.

This chapter is mostly concerned with the introduction of the syntax and semantics of first-order, modal, and Boolean modal languages intended for reasoning about distances, and with studying and comparing their respective expressive power. We begin, in Section 1.2, by formally introducing the **first-order distance languages** $\mathcal{LF}[M]$, where $M \subseteq \mathbb{R}^+$,[2] containing monadic predicates (for subsets of W), individual constants (for points in W), and the binary predicates $\delta(x,y) < a$ and $\delta(x,y) = a$, $a \in M$, saying that the distance between x and y is smaller than a or equal to a, respectively. Typical sets M of possible distances appearing in formulae—called **parameter sets**—will be $\mathbb{Q}^+$ (the non-negative rational numbers) and $\mathbb{N}$ (the natural numbers including 0). The following example will be used to illustrate the expressive power of our languages.

EXAMPLE 1.1. Imagine that you are going to buy a house in London. You then inform your estate agent about your intention and provide her with a number of constraints:

(A) The house should not be too far from your college, say, not more than 10 miles.

(B) The house should be close to shops and restaurants; they should be reachable, say, within 1 mile.

(C) There should be a 'green zone' around the house, at least within 2 miles in each direction.

(D) Factories and motorways must be far from the house, not closer than 5 miles.

(E) There must be a sports centre around, and moreover, all sports centres of the district should be reachable on foot, i.e., they should be within, say, 3 miles.

(F) Public transport should be easily accessible: whenever you are within 8 miles from your home, there should be a bus stop or a tube station within a distance of 2 miles.

[2]Where M satisfies certain closure conditions, compare Definition 1.2 on Page 15.

(G) And, of course, there must be a tube station around, not too close, but not too far either—somewhere between 0.5 and 1 mile.

The constraints in Example 1.1 can be formalised in $\mathcal{LF}[\mathbb{Q}^+]$ by the (conjunction of the) formulae (A′)–(G′) listed below. We use first-order constants *college* and *house*, as well as unary predicates shop, restaurant, green_zone, factory, motorway, district_sports_centre, public_transport, and tube_station.

(A′) $\delta(college, house) \leq 10$;
(B′) $\exists x\big(\delta(house, x) \leq 1 \wedge \mathsf{shop}(x)\big)$, and
$\exists x\big(\delta(house, x) \leq 1 \wedge \mathsf{restaurant}(x)\big)$;
(C′) $\forall x\big(\delta(house, x) \leq 2 \to \mathsf{green\_zone}(x)\big)$;
(D′) $\forall x\big(\mathsf{factory}(x) \vee \mathsf{motorway}(x) \to \delta(house, x) > 5\big)$;
(E′) $\exists x\big(\delta(house, x) \leq 3 \wedge \mathsf{district\_sports\_center}(x)\big)$, and
$\forall x\big(\delta(house, x) > 3 \to \neg\mathsf{district\_sports\_center}(x)\big)$;
(F′) $\forall x\big(\delta(house, x) \leq 8 \to \exists y\big(\delta(x, y) \leq 2 \wedge \mathsf{public\_transport}(y)\big)\big)$;
(G′) $\exists x\big(\delta(house, x) > 0.5 \wedge \delta(house, x) \leq 1 \wedge \mathsf{tube\_station}(x)\big)$.

Similar to the fields of temporal, modal, and description logics, which avoid the use of first-order quantifiers by replacing them with various kinds of modal operators like ‘sometime in the future’, ‘it is possible’, and so on, the **modal distance languages** we introduce are variable-free languages $\mathcal{LO}_O[M]$, $M \subseteq \mathbb{R}^+$, which, instead of first-order quantifiers, use some set O of **distance operators** from the list

$$\{\mathsf{A}^{<a}, \mathsf{A}^{\leq a}, \mathsf{A}^{>a}, \mathsf{A}^{\geq a}, \mathsf{A}^{=a}, \mathsf{A}^{>a}_{<b}, \mathsf{A}^{\geq a}_{<b}, \mathsf{A}^{>a}_{\leq b}, \mathsf{A}^{\geq a}_{\leq b} : a, b \in M\},$$

where, e.g., $\mathsf{A}^{\leq a}$ is understood as ‘everywhere in the circle of radius a’ and $\mathsf{A}^{>a}$ as ‘everywhere outside the circle of radius a’, and which, instead of first-order constants, use **nominals**, i.e., atomic formulae that are interpreted by singleton subsets. The language containing all these operators will be called $\mathcal{LO}[M]$, and, if nominals are not present in the language, we denote the respective languages by $\mathcal{L}_O[M]$.

The constraints in Example 1.1 can be formulated in the language $\mathcal{LO}[\mathbb{N}]$ as follows. As before, we treat ‘*house*’ and ‘*college*’ as constants representing certain points in the space. In the modal context, this can be done with the help of nominals. Further, ‘shop,’ ‘restaurant’ and other unary predicates are now understood as **propositional variables** interpreted as subsets of the domain of the distance space, and the operators $\mathsf{E}^{\leq a}$ are, as in modal logic, the **duals** of the operators $\mathsf{A}^{\leq a}$, i.e., $\mathsf{E}^{\leq a}\,\_\_ = \neg\mathsf{A}^{\leq a}\neg\,\_\_$.

(A″) *house* $\rightarrow \mathsf{E}^{\leq 10}$*college*;
(B″) *house* $\rightarrow (\mathsf{E}^{\leq 1}\mathsf{shop} \wedge \mathsf{E}^{\leq 1}\mathsf{restaurant})$;
(C″) *house* $\rightarrow \mathsf{A}^{\leq 2}\mathsf{green\_zone}$;
(D″) *house* $\rightarrow \neg\mathsf{E}^{\leq 5}(\mathsf{factories} \vee \mathsf{motorways})$;
(E″) *house* $\rightarrow (\mathsf{E}^{\leq 3}\mathsf{district\_sports\_center} \wedge \mathsf{A}^{>3}\neg\mathsf{district\_sports\_center})$;
(F″) *house* $\rightarrow \mathsf{A}^{\leq 8}\mathsf{E}^{\leq 2}\mathsf{public\_transport}$;
(G″) *house* $\rightarrow \mathsf{E}^{>0.5}_{\leq 1}\mathsf{tube\_station}$.

The intended meaning of the distance operators used in the example above is as follows. The formula $\mathsf{E}^{\leq 1}\mathsf{shop}$ is true at exactly those points in the domain from which at least one shop is reachable within 1 mile. Likewise, $\mathsf{A}^{\leq 2}\mathsf{green\_zone}$ is true at point x, if the whole 'neighbourhood' of radius 2 around x belongs to the green zone, whereas $\mathsf{E}^{>0.5}_{\leq 1}\mathsf{tube\_station}$ is true at those points that are located in a distance between 0.5 and 1 mile (excluding 0.5) from at least one tube station.[3]

By replacing quantifiers with distance operators we do not loose expressive power as compared with the **two-variable distance language** $\mathcal{LF}_2[M]$, i.e., the two-variable fragment of $\mathcal{LF}[M]$ consisting of all $\mathcal{LF}[M]$-formulae with the variables x and y only. In fact, after introducing and discussing the necessary terminology in Section 1.3, we show in Section 1.4 that the language $\mathcal{LO}_F[M]$—containing distance operators $\mathsf{A}^{=a}, \mathsf{A}^{<a}, \mathsf{A}^{>a}, \mathsf{A}^{>a}_{<b}$ and nominals—is **expressively complete** for $\mathcal{LF}_2[M]$ in the class $\mathcal{MS}$ of all metric spaces, for any $M \subseteq \mathbb{R}^+$ (Theorem 1.17). The proof of this theorem, which is similar to proofs in Etessami et al. [1997] and Lutz et al. [2001b], and the expressive completeness of Boolean modal languages enriched with converse modal operators and the difference operator for (standard) two-variable first-order logic, as shown in Lutz et al. [2001b], suggest that there is a close relationship between modal distance logic and Boolean modal logic. Indeed, in Section 1.5 we introduce a natural **Boolean modal distance language** and show it to be expressively complete for the language $\mathcal{LO}_F[M]$ and thus for the two-variable first-order distance language interpreted in metric spaces. At the end of Section 1.5, the reader can find a table summarising the relative expressiveness results obtained in this chapter.

In Chapter 2, we turn to an analysis of the **computational behaviour** of the formalisms constructed. As one might expect, the **satisfiability problem** for $\mathcal{LF}[\mathbb{Q}^+]$ and $\mathcal{LF}[\mathbb{N}]$-formulae in any class of distance spaces containing the class $\mathcal{MS}$ of all metric

[3] By the way, the end of the imaginary story about buying a house in London was not satisfactory. Having checked her knowledge base, the estate agent said: 'Unfortunately, your constraints (A)–(G) are not satisfiable in London, where we have

$$\mathsf{tube\_station} \rightarrow \mathsf{E}^{\leq 3.5}(\mathsf{factory} \vee \mathsf{motorway}).$$

In view of the triangular inequality, this contradicts constraints (D″) and (G″).'

spaces is undecidable. This is shown by a reduction to the hereditarily undecidable theory of graphs in Theorem 2.1 (i). In order to find decidable but still reasonably expressive sublanguages of $\mathcal{LF}[\mathbb{Q}^+]$, we turn our attention to its two-variable fragment $\mathcal{LF}_2[\mathbb{Q}^+]$. Note that all formulae in the example above belong to this fragment. The two-variable fragment of classical first-order logic is known to be decidable[4] and NEXPTIME-complete[5]. We use this result to show, in Theorem 2.15, that the satisfiability problem for $\mathcal{LF}_2[\mathbb{Q}^+]$-formulae is decidable

- in the class $\mathcal{D}$ of arbitrary distance spaces, and
- in the class $\mathcal{D}^s$ of all symmetric distance spaces.

Unfortunately, this does not hold any more as soon as we add the triangular inequality (3): we show in Theorem 2.1 (ii) that the satisfiability problem for $\mathcal{LF}_2[\mathbb{N}]$-formulae is undecidable both in

- the class $\mathcal{MS}$ of all metric spaces and in
- the class $\mathcal{D}^t$ of triangular distance spaces.

The proof shows that the undecidable $\mathbb{N} \times \mathbb{N}$-tiling problem can be reduced to a satisfiability problem in the language $\mathcal{L}_F[\mathbb{N}]$ (not using nominals), and uses the expressive completeness theorem (effectively) relating first-order and modal distance languages. In fact, the proof makes it clear that seemingly weak fragments of $\mathcal{L}_F[\mathbb{N}]$ are already undecidable. Roughly speaking, we loose decidability as soon as we are able to speak about 'rings', as in constraint (G).

Note that the expressive completeness results also have interesting consequences as concerns the computational behaviour of various sublanguages $\mathcal{LO}_O[M]$ of $\mathcal{LO}[M]$, i.e., languages containing different sets of distance operators. First, any (decidable) fragment of $\mathcal{LF}_2[M]$ can be obtained as a (decidable) fragment of $\mathcal{LO}_F[M]$. And second, since the translation from $\mathcal{LF}_2[M]$ into $\mathcal{LO}_F[M]$ is effective, decidable fragments of $\mathcal{LO}_F[M]$ have to be proper. In particular, $\mathcal{LO}_F[\mathbb{N}]$ itself is undecidable when interpreted in distance spaces satisfying the triangular inequality.

The main result concerning fragments of $\mathcal{LO}[M]$ is in Section 2.3. We show that the rather expressive and natural fragment $\mathcal{LO}_D[M]$—containing distance operators $\mathsf{A}^{\leq a}$, $\mathsf{A}^{>a}$, and nominals—has the finite model property (even for parameters from $\mathbb{R}^+$, Theorem 2.8) and is decidable (if parameters are taken from $\mathbb{Q}^+$, Theorem 2.14). Note that this language can still express all the constraint from Example 1.1, except constraint (G).

[4]The decidability of the two-variable fragment without equality was proved by Scott [1962], and for the language with equality in Mortimer [1975].

[5]Consult Fürer [1984], Grädel et al. [1997], and, for more information, Grädel and Otto [1999], Börger et al. [1997].

To prove these results, we first give a relational representation of metric spaces with respect to the language $\mathcal{LO}_D[M]$ in Theorem 2.7, which shows that a formula from the language $\mathcal{LO}_D[M]$ is satisfiable in some metric space if and only if it is satisfiable in a certain kind of Kripke frame, called D-metric frames. By performing a complex filtration and 'frame-repair' procedure, we then show that this class of Kripke frames has the finite frame property, and that we can construct an adequate metric space from a frame. Finally, analogous results are shown for weaker classes of distance spaces not necessarily satisfying the symmetry condition in Theorem 2.17.

Table 1.1 summarises the main decidability results we obtain in Chapter 2: $+$ $(-)$ means that the satisfiability problem for the corresponding language in the corresponding class of structures is decidable (undecidable). The results do not depend on whether the parameters are from $\mathbb{N}$ or $\mathbb{Q}^+$.

	$\mathcal{D}$	$\mathcal{D}^s$	$\mathcal{D}^t$	$\mathcal{MS}$
$\mathcal{LF}[\mathbb{Q}^+/\mathbb{N}]$	$-$	$-$	$-$	$-$
$\mathcal{LF}_2[\mathbb{Q}^+/\mathbb{N}]$	$+$	$+$	$-$	$-$
$\mathcal{LO}_F[\mathbb{Q}^+/\mathbb{N}]$	$+$	$+$	$-$	$-$
$\mathcal{LO}_D[\mathbb{Q}^+/\mathbb{N}]$	$+$	$+$	$+$	$+$

Table 1.1: The satisfiability problem for distance logics.

In Chapter 3, we investigate **logical properties** of distance logics like axiomatisability, compactness, and interpolation. **Logics of distance spaces** are understood semantically, that is, we identify the logic $\mathcal{MSO}_O^i[M]$ ($\mathcal{MS}_O^i[M]$), $i \in \{d, s, t, m\}$, with the set of **validities** in the language $\mathcal{LO}_O[M]$ ($\mathcal{L}_O[M]$) interpreted in the class $\mathcal{D}^i$ of distance spaces. In Section 3.1, we present Hilbert-style axiomatisations of the logics $\mathcal{MS}_D^i[M]$, $i \in \{d, s, t, m\}$, in the languages $\mathcal{L}_D[M]$ not containing nominals. To prove completeness in Theorem 3.3, we employ the relational representation for these languages given in Section 2.3 and use Sahlqvist completeness theory as well as variants of the filtration and 'frame-repair' techniques used to prove the decidability of the satisfiability problem for these languages in Chapter 2.

In Section 3.2, we draw our attention to the modal distance logic $\mathcal{MSO}_F[M]$, whose language we show in Section 1.4 to be expressively complete for the two-variable fragment $\mathcal{LF}_2[M]$ interpreted in metric spaces. We show that even for this (undecidable) language, an elementary relational representation of metric spaces can be given that captures theoremhood (Theorem 3.12). To axiomatise the corresponding class of frames—called F-metric frames—in the language $\mathcal{LO}_F[M]$, we make use of some rather general completeness results from hybrid completeness theory (Theorem 3.24).

Finally, Sections 3.3 and 3.4 discuss several themes related to the (frame) representation theorems for languages $\mathcal{LO}_D[M]$ and $\mathcal{LO}_F[M]$. We start by showing that while F-metric frames can capture theoremhood in $\mathcal{LO}_F[M]$, the local consequence relations with respect to metric spaces and F-metric frames differ in that the latter is compact, while the former is not. We proceed by showing that the frame representations can be used to derive corresponding representation theorems for a variety of sublanguages of the full modal distance language $\mathcal{LO}[M]$, and provide sound and complete axiomatic systems for the respective logics. We close our investigations on distance logics by deriving a few results on Craig interpolation: we show that while languages containing some subset of $\{\mathsf{A}^{\leq a}, \mathsf{A}^{<a}\}$ as distance operators have Craig interpolation (Theorem 3.29), failure of interpolation is the norm in distance logics. When operators of the form $\mathsf{A}^{>0}$ are present in the language, failure of interpolation can be traced back to the failure of interpolation in languages containing the **difference operator** D, where a formula $D\varphi$ is true at a point w of a Kripke model if φ is true at some point $v \neq w$ [de Rijke, 1992]. But even if we leave out this operator, languages like $\mathcal{L}_D[M]$ and $\mathcal{L}_F[M]$ fail to have interpolation, which can be shown by an argument similar to the proof of failure of interpolation in Humberstone's inaccessibility logic [Humberstone, 1983, Areces and Marx, 1998] (Theorem 3.34).

The idea of constructing logical formalisms capable of speaking about distances is not new. For example, somewhat weaker spatial 'modal logics of distance' were introduced in Rescher and Garson [1968], von Wright [1979], Segerberg [1980], Jansana [1994], and Lemon and Pratt [1998]. However, their computational behaviour has remained unexplored. More attention has recently been devoted to metric (or quantitative) temporal logics,[6] which clearly reflects the fact that temporal logic in general is more developed than spatial logic. For example, starting with Kamp's classical result on the expressive completeness of temporal logic with respect to monadic first-order logic [Kamp, 1968], an elegant theory comparing the expressive power of first-order, second-order and temporal languages for trees and linear orderings has been developed [Gabbay et al., 1994, Rabinovich, 2000]. To our knowledge, nothing like this has been done for spatial logics.

We hope our investigations—grown up from earlier work in Suzuki [1997], Sturm et al. [2000], Kutz et al. [2002a] and Kutz et al. [2003b]—will help to fill the gap.

[6]See, e.g., Alur and Henzinger [1992], Montanari [1996], Henzinger [1998], and Hirshfeld and Rabinovich [1999].

1.2. First-Order and Modal Languages

All the languages we introduce will be defined with respect to some set of parameters from $\mathbb{R}^+$ that may appear in formulae. These sets are called *parameter sets* and are defined thus:

DEFINITION 1.2 (PARAMETER SETS). *Let $M \subseteq \mathbb{R}^+$ be a set of non-negative real numbers. M is called a* ***parameter set****, if the following two conditions are satisfied:*

(PS1) $0 \in M$;

(PS2) *If $a, b \in M$ and there exists $c \in M$ such that $a + b < c$, then $a + b \in M$;*

The sets $\mathbb{R}^+$, $\mathbb{Q}^+$ and $\mathbb{N}$ are natural candidates for parameter sets, but they are all infinite. It should be obvious, however, that finite parameter sets do exist, for instance, $M = \{0, 1, 2, \ldots, n\}$, for $n \in \mathbb{N}$.

As for the semantic structures, the languages will be interpreted in different classes of distance spaces as introduced in the last section. We are going to use the following abbreviations:

- $\mathcal{D}^d$ or simply $\mathcal{D}$ denote the class of all distance spaces, satisfying (1);
- $\mathcal{D}^s$ denotes the class of all symmetric distance spaces, satisfying (1) and (2);
- $\mathcal{D}^t$ denotes the class of all triangular distance spaces, satisfying (1) and (3);
- $\mathcal{D}^m$ or simply $\mathcal{MS}$ stand for the class of all metric spaces, satisfying (1)–(3).

1.2.1. First-Order Distance Logics. The most obvious choice of a language to talk about metric or weaker distance spaces is probably a standard first-order language. Let M be a parameter set. The language $\mathcal{LF}[M]$ of **first-order distance logic** contains a countably infinite set $c_1, c_2, \ldots$ of **constant symbols**, a countably infinite set $x_1, x_2, \ldots$ of **individual variables**, a countably infinite set $P_1, P_2, \ldots$ of **unary predicate symbols**, the **equality symbol** $\doteq$, two (possibly infinite) sets of **binary predicates**

$$\delta(\_\_, \_\_) < a \quad \text{and} \quad \delta(\_\_, \_\_) = a \quad (a \in M),$$

the **Booleans** (including the **propositional constants** $\top$ for **verum** and $\bot$ for **falsum**), and the quantifier $\exists x_i$ for every variable x_i. Thus, the **atomic formulae** of $\mathcal{LF}[M]$ are of the form

$$\top, \quad \bot, \quad \delta(t, t') < a, \quad \delta(t, t') = a, \quad t \doteq t', \quad \text{and} \quad P_i(t),$$

where t and t' are **terms**, i.e., variables or constants, and $a \in M$. The **compound $\mathcal{LF}[M]$-formulae** are obtained from the atomic ones by applying the Booleans and quantifiers in the usual way:

$$\varphi ::= \text{atom} \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid \exists x_i \varphi.$$

We use $\delta(t_1, t_2) > a$ as an abbreviation for $\neg\,\delta(t_1, t_2) < a \wedge \neg\,\delta(t_1, t_2) = a$. $\mathcal{LF}_2[M]$ denotes the **two-variable fragment** of $\mathcal{LF}[M]$, that is, the set of all $\mathcal{LF}[M]$-formulae containing occurrences of at most two variables, say, x and y. Sometimes we are interested in only those formulae in $\mathcal{LF}_2[M]$ that contain precisely one free variable, denoted by $\mathcal{LF}_2^1[M]$, or those that contain no free variables, denoted by $\mathcal{LF}_2^0[M]$. The formulae in $\mathcal{LF}_2^0[M]$ are also called **sentences**.

$\mathcal{LF}[M]$-formulae are interpreted in structures of the form

$$\mathfrak{A} = \left\langle W, d, P_1^{\mathfrak{A}}, \ldots, c_1^{\mathfrak{A}}, \ldots \right\rangle,$$

where $\langle W, d\rangle$ is a distance space, the $P_i^{\mathfrak{A}}$ are subsets of W interpreting the unary predicates P_i, and the $c_i^{\mathfrak{A}}$ are elements of W interpreting the constants c_i. An **assignment** $\mathfrak{a}$ in $\mathfrak{A}$ is a function assigning elements of W to variables. The pair $\mathfrak{M} = \langle \mathfrak{A}, \mathfrak{a}\rangle$ will be called an $\mathcal{LF}[M]$**-model**, or simply a **first-order model**. For a term t, let $t^{\mathfrak{M}}$ denote $c_i^{\mathfrak{A}}$ if t is the constant c_i, and $\mathfrak{a}(x)$ if t is the variable x. Now, the **truth-relation** $\mathfrak{M} \models \varphi$, for an $\mathcal{LF}[M]$-formula φ, is defined inductively as follows:

- $\mathfrak{M} \models \top$ and $\mathfrak{M} \not\models \bot$;
- $\mathfrak{M} \models \delta(t_1, t_2) < a \iff d(t_1^{\mathfrak{M}}, t_2^{\mathfrak{M}}) < a$;
- $\mathfrak{M} \models \delta(t_1, t_2) = a \iff d(t_1^{\mathfrak{M}}, t_2^{\mathfrak{M}}) = a$;
- $\mathfrak{M} \models t_1 \doteq t_2 \iff t_1^{\mathfrak{M}} = t_2^{\mathfrak{M}}$;
- $\mathfrak{M} \models P_i(t) \iff t^{\mathfrak{M}} \in P_i^{\mathfrak{A}}$;
- $\mathfrak{M} \models \exists x_i\, \varphi \iff \langle \mathfrak{A}, \mathfrak{b}\rangle \models \varphi$ for some assignment $\mathfrak{b}$ in $\mathfrak{A}$ that may differ from $\mathfrak{a}$ only on x_i;
- $\mathfrak{M} \models \neg\varphi \iff \mathfrak{M} \not\models \varphi$;
- $\mathfrak{M} \models \varphi \wedge \psi \iff \mathfrak{M} \models \varphi$ and $\mathfrak{M} \models \psi$.

The notions of *validity* and *satisfiability* are defined as in standard first-order logic, but respect the class of intended models.

DEFINITION 1.3 (VALIDITY AND SATISFIABILITY). *Let $\mathcal{D}^i$, $i \in \{d, s, t, m\}$, be a class of distance spaces, M a parameter set, and $\varphi \in \mathcal{LF}[M]$. The formula φ is called **satisfiable in** $\mathcal{D}^i$, if there is a model $\mathfrak{M}$ based on a space $\langle W, d\rangle \in \mathcal{D}^i$ such that $\mathfrak{M} \models \varphi$. φ is **valid on a distance space** $\langle W, d\rangle \in \mathcal{D}^i$, if for all models $\mathfrak{M}$ based on $\langle W, d\rangle$ we have $\mathfrak{M} \models \varphi$. Finally, φ is **valid in the class** $\mathcal{D}^i$, if φ is valid on every $\langle W, d\rangle$ in $\mathcal{D}^i$.*

We can now introduce first-order logics of classes of distance spaces:

DEFINITION 1.4 (FIRST-ORDER DISTANCE LOGICS). *Let $\mathcal{D}^i$, $i \in \{d, s, t, m\}$, be a class of distance spaces and M a parameter set. Then $\mathcal{FM}^i[M]$ denotes the **first-order distance logic of the class** $\mathcal{D}^i$, i.e., the set of formulae of $\mathcal{LF}[M]$ that are valid in the class D^i. Likewise, the logics $\mathcal{FM}_2^i[M]$ are defined as the sets of formulae of the two-variable fragment $\mathcal{LF}_2[M]$ of $\mathcal{LF}[M]$ that are valid in the class $\mathcal{D}^i$.*

Unfortunately, from the computational point of view, the constructed logics turn out to be too expressive. In the next chapter, more precisely in Theorem 2.1, we will prove that the satisfiability problem for $\mathcal{LF}[\mathbb{N}]$-formulae is undecidable in any class of distance spaces containing the class $\mathcal{MS}$, and, moreover, that even the satisfiability problem for the two-variable fragment $\mathcal{LF}_2[\mathbb{N}]$ is undecidable in the class of distance spaces satisfying the triangular inequality.

For this reason, we need more fine-tuned languages, languages that are still reasonably expressive, but less expressive than the full first-order language. We follow this idea in the next section by introducing 'modal' languages that allow only for some kind of restricted quantification.

1.2.2. Modal Distance Logics. We begin by introducing a number of propositional modal and hybrid languages. Let M be a parameter set. The languages we define, then, will depend on such a set M, on the collection of distance operators being considered as primitive, as well as on the presence of nominals and the universal modality. Following similar conventions from the field of description logics, we use '$\mathcal{O}$' to indicate that nominals are present in the language. To avoid confusion we should note, however, that in the languages we define nominals always come with the universal modality as a 'binding device'. Here are the details:

DEFINITION 1.5 (SYNTAX). *Suppose $M \subseteq \mathbb{R}^+$ is a parameter set. The alphabet of a language $\mathcal{LO}_O[M]$ consists of a denumerably infinite list $\mathsf{Var} = \{p_l : l < \omega\}$ of **propositional variables**, a denumerably infinite list $\mathsf{Nom} = \{i_l : l < \omega\}$ of **nominals**, the **Boolean connectives** $\wedge$ and $\neg$, the **propositional constants** $\top$ and $\bot$, the **universal modality** $\blacksquare$, as well as a subset $O \subseteq \mathfrak{D}[M]$, called **operator set**, of the following set $\mathfrak{D}[M]$ of **distance operators**, depending on M:*

$$\{\mathsf{A}^{<a}, \mathsf{A}^{\leq a}, \mathsf{A}^{>a}, \mathsf{A}^{\geq a}, \mathsf{A}^{=a}, \mathsf{A}^{>a}_{<b}, \mathsf{A}^{\geq a}_{<b}, \mathsf{A}^{>a}_{\leq b}, \mathsf{A}^{\geq a}_{\leq b} \mid a, b \in M\}.$$

The set of well-formed formulae of such a language is constructed in the standard way; it will be identified with $\mathcal{LO}_O[M]$. Furthermore, by $\mathcal{L}_O[M]$ we shall denote the language that is constructed just like $\mathcal{LO}_O[M]$, with the exception that there are neither nominals nor a universal modality in the language. The full language $\mathcal{LO}_{\mathfrak{D}[M]}[M]$ will simply be called $\mathcal{LO}[M]$.

Usually, when a parameter set M has been fixed, we will omit it from the notation since most of the questions we are interested in do not depend on the particular choice of M. Some logical properties, like axiomatisability, do not depend on the choice of M.[7] However, compactness does depend on whether or not M is infinite, and to prove decidability, we obviously require that M is a recursive set.

[7]That is, when parameter sets satisfy minimal closure conditions as specified in Definition 1.2.

Other Booleans as well as the dual distance operators $\mathsf{E}^{\leq a}$, $\mathsf{E}^{<a}$ etc. and the universal diamond $\blacklozenge$ are defined as abbreviations, e.g., $\mathsf{E}^{\leq a} = \neg\mathsf{A}^{\leq a}\neg$, etc. We use lower case Latin letters $p, q, r, \ldots$ to denote propositional variables, i, j, k to denote nominals, lower case Greek letters $\chi, \varphi, \psi, \ldots$ to denote formulae, and upper case Greek letters $\Delta, \Sigma, \Theta, \ldots$ to denote sets of formulae. Next, we will give the semantics for these languages:

DEFINITION 1.6 (SEMANTICS). *We distinguish two kinds of models: Firstly, a* ***full model****, or simply a* ***model****, is a structure of the form:*

$$\mathfrak{B} = \left\langle W, d, p_0^{\mathfrak{B}}, p_1^{\mathfrak{B}}, \ldots, i_0^{\mathfrak{B}}, i_1^{\mathfrak{B}}, \ldots \right\rangle,$$

where $\langle W, d \rangle$ is a distance space, the $p_l^{\mathfrak{B}}$ are subsets of W and nominals i_l are interpreted by singleton subsets $i_l^{\mathfrak{B}}$. Secondly, a ***nominal-free model*** *is a structure*

$$\mathfrak{B} = \left\langle W, d, p_0^{\mathfrak{B}}, p_1^{\mathfrak{B}}, \ldots \right\rangle,$$

as above, but without providing interpretations for nominals. Obviously, full models will be used to interpret languages of type $\mathcal{LO}_O[M]$ for some O, while nominal-free structures suffice for languages $\mathcal{L}_O[M]$.

The ***truth-relation*** *$\langle \mathfrak{B}, w \rangle \models \varphi$, for $\mathfrak{B}$ a full model, $\mathcal{LO}[M]$ formulae φ and a point $w \in W$, is defined inductively as follows:*

- $\langle \mathfrak{B}, w \rangle \models p \iff w \in p^{\mathfrak{B}}$;
- $\langle \mathfrak{B}, w \rangle \models i \iff \{w\} = i^{\mathfrak{B}}$;
- $\langle \mathfrak{B}, w \rangle \models \varphi \wedge \psi \iff \langle \mathfrak{B}, w \rangle \models \varphi$ *and* $\langle \mathfrak{B}, w \rangle \models \psi$;
- $\langle \mathfrak{B}, w \rangle \models \neg\varphi \iff \langle \mathfrak{B}, w \rangle \not\models \varphi$;
- $\langle \mathfrak{B}, w \rangle \models \blacksquare\varphi \iff \langle \mathfrak{B}, u \rangle \models \varphi$ *for all* $u \in W$.
- $\langle \mathfrak{B}, w \rangle \models \mathsf{A}^{<a}\varphi \iff \langle \mathfrak{B}, u \rangle \models \varphi$ *for all* $u \in W$ *with* $d(w, u) < a$;
- $\langle \mathfrak{B}, w \rangle \models \mathsf{A}^{\leq a}\varphi \iff \langle \mathfrak{B}, u \rangle \models \varphi$ *for all* $u \in W$ *with* $d(w, u) \leq a$;
- $\langle \mathfrak{B}, w \rangle \models \mathsf{A}^{>a}\varphi \iff \langle \mathfrak{B}, u \rangle \models \varphi$ *for all* $u \in W$ *with* $d(w, u) > a$;
- $\langle \mathfrak{B}, w \rangle \models \mathsf{A}^{\geq a}\varphi \iff \langle \mathfrak{B}, u \rangle \models \varphi$ *for all* $u \in W$ *with* $d(w, u) \geq a$;
- $\langle \mathfrak{B}, w \rangle \models \mathsf{A}^{=a}\varphi \iff \langle \mathfrak{B}, u \rangle \models \varphi$ *for all* $u \in W$ *with* $d(w, u) = a$;
- $\langle \mathfrak{B}, w \rangle \models \mathsf{A}^{>a}_{<b}\varphi \iff \langle \mathfrak{B}, u \rangle \models \varphi$ *for all* $u \in W$ *with* $a < d(w, u) < b$;
- $\langle \mathfrak{B}, w \rangle \models \mathsf{A}^{\geq a}_{<b}\varphi \iff \langle \mathfrak{B}, u \rangle \models \varphi$ *for all* $u \in W$ *with* $a \leq d(w, u) < b$;
- $\langle \mathfrak{B}, w \rangle \models \mathsf{A}^{>a}_{\leq b}\varphi \iff \langle \mathfrak{B}, u \rangle \models \varphi$ *for all* $u \in W$ *with* $a < d(w, u) \leq b$;
- $\langle \mathfrak{B}, w \rangle \models \mathsf{A}^{\geq a}_{\leq b}\varphi \iff \langle \mathfrak{B}, u \rangle \models \varphi$ *for all* $u \in W$ *with* $a \leq d(w, u) \leq b$;

$\langle \mathfrak{B}, w \rangle$ is called a ***pointed model****.*

As usual, a formula φ is said to be ***valid in a model****, $\mathfrak{B} \models \varphi$, if it is true at every point of the model; φ is* ***valid in a distance space*** *$\langle W, d \rangle$, if it is valid in every model based on $\langle W, d \rangle$. We sometimes use $\varphi^{\mathfrak{B}}$ to denote the* ***truth-set*** *or the* ***extension*** *of φ, that is, the set* *of all $w \in W$ such that $\langle \mathfrak{B}, w\rangle \models \varphi$. Finally, φ is **valid in a class $\mathbf{K}$** of models (or distance spaces), if it is valid in every model (respectively, distance space) of $\mathbf{K}$.*

Since the notion of truth of a formula is defined pointwise, there are two different versions of consequence, namely **local** and **global consequence**. Both will play a role, but mostly we will be interested in global consequence since this is more natural in the context of spatial reasoning. The definition is completely analogous to the corresponding definition in modal logic.

DEFINITION 1.7 (GLOBAL AND LOCAL). *Let $\mathcal{D}^i$, $i \in \{d,s,t,m\}$, be a class of distance spaces, M a parameter set, and let Γ be a set of formulae and φ a formula of $\mathcal{LO}[M]$. We say that φ **follows locally** from Γ with respect to $\mathcal{D}^i$, in symbols $\Gamma \models^i_l \varphi$, if for every model $\mathfrak{B}$ based on some $\langle W, d\rangle \in \mathcal{D}^i$ and every point w in $\mathfrak{B}$ it holds that:*

$$\langle \mathfrak{B}, w\rangle \models \Gamma \implies \langle \mathfrak{B}, w\rangle \models \varphi.$$

*Further, φ is said to **follow globally** from Γ with respect to $\mathcal{D}^i$, $\Gamma \models^i_g \varphi$, if for every model $\mathfrak{B}$ based on some $\langle W, d\rangle \in \mathcal{D}^i$ it holds that:*

$$\mathfrak{B} \models \Gamma \implies \mathfrak{B} \models \varphi.$$

*$\models^i_l$ is called the $\mathcal{D}^i$-**local consequence relation** and $\models^i_g$ the $\mathcal{D}^i$-**global consequence relation**. If we write just $\Gamma \models^i \varphi$, we always mean global consequence. If the class of distance spaces is clear from the context, we leave out the qualifying superscript i.*

Clearly, if φ follows locally from Γ, it also follows globally, but not conversely. Note that the full language $\mathcal{LO}[M]$ is clearly redundant in the sense that some distance operators are definable[8] from others. For instance, the operator $\mathsf{A}^{\geq a}_{\leq b}$ can be defined by

$$\mathsf{A}^{\geq a}_{\leq b} \_ := \mathsf{A}^{> a}_{< b} \_ \wedge \mathsf{A}^{=a} \_ \wedge \mathsf{A}^{=b} \_ .$$

However, we have given the semantics for the full set $\mathfrak{O}[M]$ of distance operators since we are primarily investigating languages where different subsets of $\mathfrak{O}[M]$ are taken as primitive, and where nominals and the universal modality may or may not be present. Let us define some of those fragments. According to Definition 1.5, all languages we define agree on the presence of the propositional variables and all Boolean connectives. To determine a language, we thus just need to specify interesting operator sets, i.e., subsets of $\mathfrak{O}[M]$.

DEFINITION 1.8 (THE LANGUAGES $\mathcal{LO}_F$, $\mathcal{L}_F$, $\mathcal{LO}_D$, AND $\mathcal{L}_D$). *Fix a parameter set M and let:*

(1) *$F[M] := \{\mathsf{A}^{<a}, \mathsf{A}^{>a}, \mathsf{A}^{=a}, \mathsf{A}^{>a}_{<b} \mid a, b \in M\}$;*

[8] We will discuss **definability of operators** in a precise way in Section 1.3.

(2) $D[M] := \{\mathsf{A}^{\leq a}, \mathsf{A}^{>a} \mid a \in M\}$;

This defines the languages $\mathcal{LO}_{F[M]}$, $\mathcal{L}_{F[M]}$, $\mathcal{LO}_{D[M]}$, *and* $\mathcal{L}_{D[M]}$. *Again, we will usually suppress the dependence on the parameter set* M *and will talk about operator sets* F, D, *etc.*

Note that in $\mathcal{LO}_D[\mathbb{Q}^+]$ we can express all constraints from Example 1.1, save (G). The formula

(G)′ $house \rightarrow \mathsf{E}^{>0.5}\mathsf{tube\_station} \wedge \mathsf{E}^{\leq 1}\mathsf{tube\_station}$

is clearly not equivalent to

(G) $house \rightarrow \mathsf{E}^{>0.5}_{\leq 1}\mathsf{tube\_station}$,

for (G)′ is already satisfied if there are *two* tube stations located, say, at distances 1.2 and 0.2, while (G) requires a single tube station being located at a distance *between* 0.5 and 1.0.

As with first-order distance logics, **modal distance logics** are defined semantically, that is, as sets of formulae of some language that are valid in a specified class of spaces:

DEFINITION 1.9 (MODAL DISTANCE LOGICS). *Given a parameter set* $M \subseteq \mathbb{R}^+$ *and an operator set* O, *we define the logics* $\mathcal{MSO}^i_O[M]$ ($\mathcal{MS}^i_O[M]$) *as the sets of all* $\mathcal{LO}_O[M]$*-formulae* ($\mathcal{L}_O[M]$*-formulae*) *that are valid in the class* $\mathcal{D}^i$, *for* $i \in \{m, t, s, d\}$.

For the most important case where $D^i = \mathcal{MS}$, *we denote the logics simply by* $\mathcal{MSO}_O[M]$, *respectively,* $\mathcal{MS}_O[M]$.

Obviously, we can also use standard Kripkean possible worlds semantics to interpret the languages at hand. Let M be a parameter set. A polymodal M-frame for the languages $\mathcal{L}_O$ or $\mathcal{LO}_O$ is a structure of the form

$$\mathfrak{f} = \left\langle W, \{(R^o_{a|b})_{a,b \in M} \mid o \in O\} \right\rangle$$

which consists of a set W (whose members are called 'points') and families $(R^o_{a|b})_{a,b\in M}$ of binary relations on $W \times W$ for each operator symbol $o \in O$. The notation $R^o_{a|b}$ is shorthand for the fact that some operators, e.g. $\mathsf{A}^{>a}$, are indexed by one parameter $a \in M$, while others, e.g. $\mathsf{A}^{>a}_{<b}$, are indexed by two parameters, a and b. A model based on a frame is of the form

$$\mathfrak{M} = \left\langle \mathfrak{f}, p^{\mathfrak{M}}_0, p^{\mathfrak{M}}_1, \ldots i^{\mathfrak{M}}_0, i^{\mathfrak{M}}_1, \ldots \right\rangle,$$

where the $p^{\mathfrak{M}}_n$ are subsets of W and the $i^{\mathfrak{M}}_m$ singleton subsets. If we work in languages without nominals, the interpretations for nominals are omitted. The notions of truth (in a pointed model) and validity in M-models and M-frames are the usual Kripkean ones, with the addition that nominals are interpreted as singleton sets of worlds. For instance,

$$\langle \mathfrak{M}, w\rangle \models \mathsf{A}^{>a}_{<b}\varphi \iff \langle \mathfrak{M}, u\rangle \models \varphi \text{ for all } u \in W \text{ such that } wR^{>a}_{<b}u;$$
$$\langle \mathfrak{M}, w\rangle \models \blacksquare\varphi \iff \langle \mathfrak{M}, u\rangle \models \varphi \text{ for all } u \in W;$$
$$\langle \mathfrak{M}, w\rangle \models i \iff i^{\mathfrak{M}} = \{w\}.$$

Similarly for the other operators. It should be clear that the truth or falsity of a formula at a point depends only on the propositional variables, nominals, and operators appearing in it. Thus, given a sublanguage $\mathcal{LO}_{O'}[M'] \subset \mathcal{LO}_O[M]$ with $O' \subset O$ and $M' \subset M$ and a frame

$$\mathfrak{f} = \left\langle W, \{({}^{\mathfrak{f}}R^{o}_{a|b})_{a,b\in M} \mid o \in O\}\right\rangle$$

for $\mathcal{LO}_O[M]$, we may define the **frame-reduct** $\mathfrak{f}\restriction_{(O',M')}$ of $\mathfrak{f}$ as

$$\mathfrak{f}\restriction_{(O',M')} := \left\langle W, \{({}^{\mathfrak{f}}R^{o'}_{a'|b'})_{a',b'\in M'} \mid o' \in O'\}\right\rangle.$$

We then have for every formula φ of $\mathcal{LO}_{O'}[M']$ and every model $\mathfrak{M}$ based on $\mathfrak{f}$:

$$\left\langle \mathfrak{f}, p_0^{\mathfrak{M}}, \dots, i_0^{\mathfrak{M}}, w\right\rangle \models \varphi \iff \left\langle \mathfrak{f}\restriction_{(O',M')}, p_0^{\mathfrak{M}}, \dots, i_0^{\mathfrak{M}}, w\right\rangle \models \varphi.$$

This observation does not depend on the presence of nominals.

Surely, frames in general do not reflect any properties of distance or metric spaces. However, as the following proposition shows, all the logics defined above can also be characterised by classes of frames. But let us first introduce some helpful concepts. For the following definition, recall that a normal modal hybrid logic, whose language contains nominals, is defined just like a normal modal logic, except that it is closed not under the usual **uniform substitution** of formulae for propositional variables, but under **sorted substitution**, that is, arbitrary formulae may be substituted for propositional variables, and nominals for nominals.

DEFINITION 1.10 (FRAMES AND THEORY). *Let L be a normal modal (hybrid) logic in language $\mathcal{L}$ and $\boldsymbol{F}$ a class of frames for $\mathcal{L}$. We define the expression* $\mathsf{Fr}(L)$*, the* ***frames of*** *L, as*

$$\mathsf{Fr}(L) := \{\mathfrak{f} \mid \mathfrak{f} \models L\}.$$

Similarly, we define $\mathsf{Th}(\boldsymbol{F})$*, the* ***theory of $\boldsymbol{F}$****, as*

$$\mathsf{Th}(\boldsymbol{F}) := \{\varphi \in \mathcal{L} \mid \boldsymbol{F} \models \varphi\}.$$

It is well known from standard modal logic that the theory $\mathsf{Th}(\boldsymbol{F})$ of a class $\boldsymbol{F}$ of frames determines a normal modal logic. Also, if the logics L and L' coincide, where $L' = \mathsf{Th}(\mathsf{Fr}(L))$, then L is frame-complete with respect to standard Kripke semantics. In a similar fashion we can show that all distance logics are complete with respect to frame semantics.

DEFINITION 1.11 (FRAME-COMPANION). *Let $S = \langle W, d\rangle$ be a distance space and M a parameter set. We define the M-**frame-companion** of S for language $\mathcal{LO}_O$ or $\mathcal{L}_O$ as*

$$\mathfrak{f}_{O,M}(S) = \Big\langle W', \{(R^o_{a|b})_{a,b\in M} \mid o \in O\}\Big\rangle,$$

by setting $W' := W$ and, for all $u, v \in W$:

$$uR_{<a}v :\iff d(u,v) < a, \qquad uR_{>a}v :\iff d(u,v) > a,$$
$$uR_{=a}v :\iff d(u,v) = a, \qquad uR^{>a}_{<b}v :\iff a < d(u,v) < b,$$

etc., for those operators appearing in O.

Further, if $\mathfrak{B} = \langle W, d, p^{\mathfrak{B}}_0, p^{\mathfrak{B}}_1, \ldots, i^{\mathfrak{B}}_0, i^{\mathfrak{B}}_1, \ldots\rangle$ is a full model based on the distance space $S = \langle W, d\rangle$, then the Kripke model $\mathfrak{M}_{O,M}(\mathfrak{B})$ based on the M-frame $\mathfrak{f}_{O,M}(S)$ is the structure

$$\mathfrak{M}_{O,M}(\mathfrak{B}) = \Big\langle \mathfrak{f}_{O,M}(S), p_0^{\mathfrak{M}_{O,M}(\mathfrak{B})}, \ldots, i_0^{\mathfrak{M}_{O,M}(\mathfrak{B})}, \ldots \Big\rangle,$$

*with $p_n^{\mathfrak{M}_{O,M}(\mathfrak{B})} := p^{\mathfrak{B}}_n$ and $i_m^{\mathfrak{M}_{O,M}(\mathfrak{B})} := i^{\mathfrak{B}}_m$, for all $n, m < \omega$. $\mathfrak{M}_{O,M}(\mathfrak{B})$ is called the M-**frame-companion model** of $\mathfrak{B}$ for language $\mathcal{LO}_O[M]$. The same definition applies to languages $\mathcal{L}_O$, but nominals are left out. As usual, if the parameter set M is fixed, we leave it out in the notation.*

PROPOSITION 1.12 (FRAME CHARACTERISATION). *All the logics $\mathcal{MS}^i_O[M]$ and $\mathcal{MSO}^i_O[M]$ for $O \subseteq \mathfrak{O}[M]$, $i \in \{m, t, s, d\}$, are characterised by classes of frames, i.e., for some fixed parameter set M:*

$$\mathcal{MS}^i_O = \mathsf{Th}(\mathsf{Fr}(\mathcal{MS}^i_O)) \quad \textit{and} \quad \mathcal{MSO}^i_O = \mathsf{Th}(\mathsf{Fr}(\mathcal{MSO}^i_O)).$$

PROOF. That $\mathcal{MS}^i_O \subseteq \mathsf{Th}(\mathsf{Fr}(\mathcal{MS}^i_O))$ and $\mathcal{MSO}^i_O \subseteq \mathsf{Th}(\mathsf{Fr}(\mathcal{MSO}^i_O))$ hold should be clear. So it suffices to show that if $\varphi \notin \mathcal{MS}^i_O$ ($\notin \mathcal{MSO}^i_O$) then φ can be refuted in a frame for $\mathcal{MS}^i_O$ ($\mathcal{MSO}^i_O$).

First, by a straightforward structural induction we can show that for all (full) models $\mathfrak{B}$ based on some distance space $S = \langle W, d\rangle$, their frame-companion models $\mathfrak{M}_{O,M}(\mathfrak{B})$, all formulae $\psi \in \mathcal{LO}_O[M]$ and points $w \in W$:

$$\langle \mathfrak{B}, w\rangle \models \psi \iff \langle \mathfrak{M}_{O,M}(\mathfrak{B}), w\rangle \models \psi.$$

Hence, for every distance space $S \in \mathcal{D}^i$ we have $\mathfrak{f}_{O,M}(S) \in \mathsf{Fr}(\mathcal{MS}^i_O[M])$ and if $\mathfrak{B}$ is a (full) model such that $\langle \mathfrak{B}, w\rangle \not\models \varphi$ then $\langle \mathfrak{M}_{O,M}(\mathfrak{B}), w\rangle \not\models \varphi$. □

Thus, every class of distance spaces induces in a canonical way a corresponding class of frames which generates the same set of tautologies.

We will use Kripke semantics in some detail when proving completeness results in Chapter 3. In particular, it will turn out that the logics $\mathcal{MS}_D$ and $\mathcal{MSO}_F$ are complete with respect to first-order definable frame classes.

1.3. Comparing Languages

To compare the different languages introduced in some more detail and to transfer results about decidability or the finite model property from one language to another, we require precise definitions of notions like 'definability', 'simulation of nominals', 'expressiveness', and 'satisfiability equivalence'; compare also Goranko [1990b], Borgida [1996] and de de Rijke [1992]. We clarify all these notions and fix our terminology in a series of definitions. Mainly, we will need to distinguish between two different notions of 'expressiveness', one that is defined with respect to classes of models, and one that is defined with respect to classes of distance spaces—all the other notions mentioned above are then reducible to one of those two. In particular, note the analogy to the model/frame distinction frequently employed in standard modal logic to analyse expressivity etc.[9]

We start with the notion of expressiveness on the level of models based on distance spaces:

DEFINITION 1.13 (EXPRESSIVE COMPLETENESS). *Let $\boldsymbol{M}$ be some class of models based on distance spaces. We say that a language $\mathcal{L}_1$ is* ***as expressive as*** *a language $\mathcal{L}_2$ with respect to a class $\boldsymbol{M}$ of models for $\mathcal{L}_1$ and $\mathcal{L}_2$ if there is a translation $.^\sharp : \mathcal{L}_2 \longrightarrow \mathcal{L}_1$ such that, for every $\mathfrak{B} \in \boldsymbol{M}$, point w in $\mathfrak{B}$, and formula $\varphi \in \mathcal{L}_2$:*

$$\langle \mathfrak{B}, w \rangle \models \varphi \iff \langle \mathfrak{B}, w \rangle \models \varphi^\sharp.$$

Further, we say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are ***equally expressive*** *or that $\mathcal{L}_1$ is* ***expressively complete*** *for $\mathcal{L}_2$ with respect to $\boldsymbol{M}$, if $\mathcal{L}_1$ is as expressive as $\mathcal{L}_2$ and $\mathcal{L}_2$ is as expressive as $\mathcal{L}_1$.*

Sometimes, we say that a language $\mathcal{L}_1$ can **define an operator** $\bigcirc$ in a class of models $\boldsymbol{M}$. This is understood to be synonymous to the assertion that the language $\mathcal{L}_1$ is as expressive as the language $\mathcal{L}_2$ that results from $\mathcal{L}_1$ by adding the operator $\bigcirc$.

If two languages $\mathcal{L}_1$ and $\mathcal{L}_2$ are equally expressive over a class $\boldsymbol{M}$ and the translations are computable, we have established a very strong link between the two languages, making them basically interchangeable when interpreted in $\boldsymbol{M}$ (apart from questions concerning the complexity of the translation and the related complexity of reasoning problems for those languages). For instance, the satisfiability problem for $\mathcal{L}_1$ in $\boldsymbol{M}$ is decidable if and only if the satisfiability problem in $\boldsymbol{M}$ is decidable for $\mathcal{L}_2$, and $\mathcal{L}_1$ has the finite model property (FMP) with respect to $\boldsymbol{M}$ if and only if $\mathcal{L}_2$ has the FMP with respect to $\boldsymbol{M}$. One important example for such a strong link are the languages $\mathcal{LO}[M]$ and $\mathcal{LO}_F[M]$:

[9] For an extensive discussion of this distinction in modal logic, compare Blackburn et al. [2001].

PROPOSITION 1.14 (EXPRESSIVE COMPLETENESS OF $\mathcal{LO}_F[M]$ AND $\mathcal{LO}[M]$). *The languages $\mathcal{LO}_F[M]$ and $\mathcal{LO}[M]$ are equally expressive over the class of all models based on distance spaces.*

PROOF. It suffices to notice that, clearly, the additional distance operators of $\mathcal{LO}[M]$ are definable by the equivalences

$$\mathsf{A}^{\leq a}\varphi \leftrightarrow \mathsf{A}^{<a}\varphi \wedge \mathsf{A}^{=a}\varphi,$$

$$\mathsf{A}^{\geq a}_{\leq b}\varphi \leftrightarrow \mathsf{A}^{>a}_{<b}\varphi \wedge \mathsf{A}^{=a}\varphi \wedge \mathsf{A}^{=b}\varphi,$$

etc., which are valid in all distance spaces. □

For many languages that are of interest, however, the notion of expressive completeness over models is too strong. In particular, note that the notion of equal expressiveness over models actually makes no sense when one language contains nominals and the other does not. Since nominals can be understood as 'proper names', there is no means for the language $\mathcal{L}$ without nominals 'to know' to which point a nominal i refers to in a given model $\mathfrak{B}$. In fact, we cannot expect any expressive completeness result even for languages $\mathcal{L}_1$ and $\mathcal{L}_2$ that both have nominals, but which use syntactically different sets of names. We thus require a weaker notion of expressive completeness, namely, one that abstracts from interpretations and is defined over structures:

DEFINITION 1.15 (STRUCTURAL EXPRESSIVE COMPLETENESS). *Let $\boldsymbol{K}$ denote some class of distance spaces. $\mathcal{L}_1$ is **structurally as expressive as** a language $\mathcal{L}_2$ with respect to $\boldsymbol{K}$ if there is a translation $.^\sharp : \mathcal{L}_2 \longrightarrow \mathcal{L}_1$ such that, for every $S \in \boldsymbol{K}$, formula $\varphi \in \mathcal{L}_2$, and point $w \in S$ we have:*

$$\langle S, w\rangle \models \varphi \iff \langle S, w\rangle \models \varphi^\sharp.$$

*We then say that $\mathcal{L}_1$ and $\mathcal{L}_2$ are **structurally equally expressive** or that $\mathcal{L}_1$ is **structurally expressively complete** for $\mathcal{L}_2$ with respect to $\boldsymbol{K}$ if $\mathcal{L}_1$ is structurally as expressive as $\mathcal{L}_2$ and $\mathcal{L}_2$ is structurally as expressive as $\mathcal{L}_1$.*

Clearly, expressive completeness over a class $\boldsymbol{M}$ of models based on structures from a class $\boldsymbol{K}$ implies structural expressive completeness over $\boldsymbol{K}$. Also, we clearly have that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are (structurally) equally expressive over some class $\boldsymbol{M}$ ($\boldsymbol{K}$), then they are (structurally) equally expressive over any class $\widetilde{\boldsymbol{M}} \subseteq \boldsymbol{M}$ ($\widetilde{\boldsymbol{K}} \subseteq \boldsymbol{K}$).

Yet, it is important to note that structural expressive completeness is a strictly weaker notion than expressive completeness over models. For instance, we show below that the languages $\mathcal{LO}_F$ and $\mathcal{L}_F$ are structurally equally expressive over the class $\mathcal{D}$ of all distance spaces, but they are not equally expressive simpliciter, that is, over the class of pointed models based on distance spaces. This fact is closely related to the presence of nominals in the language $\mathcal{LO}_F$.

On the other hand, structural expressive completeness of two languages $\mathcal{L}_1$ and $\mathcal{L}_2$ with respect to computable translations still allows us to transfer, e.g., decidability and FMP with respect to a class of structures from one language to the other. More specifically, we can transfer results about properties of satisfying (refuting) models, i.e., if we can refute a formula φ in a specific point w of a structure S, then we can refute its translation $\varphi^\sharp$ in the same point of the same structure.

If a language $\mathcal{L}_1$ is structurally as expressive over $\boldsymbol{K}$ as the language $\mathcal{L}_1^+$ that results from $\mathcal{L}_1$ by adding nominals, we sometimes say that $\mathcal{L}_1$ **can simulate nominals** over $\boldsymbol{K}$. The following proposition gives two important examples of structural expressive completeness, namely, for distance logics containing the F and D operator sets.

PROPOSITION 1.16 (SIMULATION OF NOMINALS).
(i) *The languages $\mathcal{LO}_D$ and $\mathcal{L}_D$ are structurally equally expressive over the class $\mathcal{D}$ of all distance spaces. In particular, $\mathcal{L}_D$ can simulate nominals and define the universal modality and the difference operator.*
(ii) *The languages $\mathcal{LO}_F$ and $\mathcal{L}_F$ are structurally equally expressive over the class $\mathcal{D}$ of all distance spaces. In particular, $\mathcal{L}_F$ can simulate nominals and define the universal modality and the difference operator.*

PROOF. We prove (i) and (ii) simultaneously.

Let us first show that the universal modality can be defined in the languages $\mathcal{L}_F$ and $\mathcal{L}_D$. Fix an $a \in M$ and define for each formula $\varphi \in \mathcal{L}_F(\mathcal{L}_D)$:

$$\varphi_{\blacksquare} := \mathsf{A}^{\leq a}\varphi \wedge \mathsf{A}^{>a}\varphi,$$

where the operator $\mathsf{A}^{\leq a}$ is primitive in $\mathcal{L}_D$ and defined as $\mathsf{A}^{<a}\_ \wedge \mathsf{A}^{=a}\_$ in $\mathcal{L}_F$. We then have for any nominal-free model $\mathfrak{B}$ and point w:

$$\langle \mathfrak{B}, w\rangle \models \blacksquare\varphi \iff \langle \mathfrak{B}, w\rangle \models \varphi_{\blacksquare}.$$

Similarly, the difference operator can be defined by setting $\varphi_D := \mathsf{E}^{>0}\varphi$. Then

$$\langle \mathfrak{B}, w\rangle \models D\varphi \iff \langle \mathfrak{B}, w\rangle \models \varphi_D.$$

Next, since $\mathcal{L}_F \subset \mathcal{LO}_F$ and $\mathcal{L}_D \subset \mathcal{LO}_D$, to prove that $\mathcal{LO}_F$ $(\mathcal{LO}_D)$ is structurally as expressive as $\mathcal{L}_F$ $(\mathcal{L}_D)$, we can pick the constant maps as translations from $\mathcal{L}_F(\mathcal{L}_D)$ to $\mathcal{LO}_F(\mathcal{LO}_D)$. To prove that $\mathcal{L}_F$ $(\mathcal{L}_D)$ is structurally as expressive as $\mathcal{LO}_F$ $(\mathcal{LO}_D)$, we have to give translations:

$$.^{\dagger_F} : \mathcal{LO}_F \longrightarrow \mathcal{L}_F \text{ and } .^{\dagger_D} : \mathcal{LO}_D \longrightarrow \mathcal{L}_D$$

such that for all $\varphi_k \in \mathcal{LO}_k(k = F, D)$, distance spaces S and points $w \in S$:

$$(\clubsuit) \quad \langle S, w\rangle \models \varphi_k \iff \langle S, w\rangle \models \varphi^{\dagger_k}.$$

We show ($\clubsuit$) just for $k = F$; the argument for $k = D$ is the same. Let us introduce the abbreviations $i_{p_j} := p_j \wedge \mathsf{A}^{>0}\neg p_j$, where p_j is some propositional variable. Clearly, we can single out an infinite set $\{i_{p_j} \mid p_j \in \mathsf{Var}^N\}$ such that $\mathsf{Var} \setminus \mathsf{Var}^N$ is still infinite. E.g., set $\mathsf{Var}^N := \{p_i \mid i = 2n \text{ for some } n \in \omega\}$. Now, by definition of the operator $\mathsf{A}^{>0}$, we clearly have, for any nominal-free model $\mathfrak{B}^*$ and point w:

$$\langle \mathfrak{B}^*, w\rangle \models i_p \iff (i_p)^{\mathfrak{B}^*} = \{w\}.$$

Hence, the formulae i_p , $p \in \mathsf{Var}^N$, can be used, in a sense, as nominals in $\mathcal{L}_F$ and $\mathcal{L}_D$.

But note that this definition still allows for 'non-denoting' nominals. That is, we may have that for some model $\mathfrak{B}^*$, $(i_p)^{\mathfrak{B}^*} = \varnothing$, in which case $\langle \mathfrak{B}^*, w\rangle \not\models i_p$ for any w. So we have to be careful when treating the i_p as proper logical nominals. For instance, on the level of validity on distance spaces, we have that for any distance space S, point w and nominal i:

$$S \models \blacklozenge i, \text{ but } S \not\models \blacklozenge i_p.$$

Given a formula $\varphi \in \mathcal{LO}_F$, we write $\varphi(\overline{i_n})$, with $\overline{i_n} = \langle i_0, \dots, i_{n-1}\rangle$, to indicate that the nominals appearing in φ are among the n nominals in $\overline{i_n}$. Denote by $\varphi(i_{p(i_k)}/i_k)$ the result of substituting the formulae $i_{p(i_k)}$ for the nominals i_k appearing in φ, where we assume without loss of generality that the $i_{p(i_k)}$ contain no propositional variables appearing in φ. We can now define the translation, for $\varphi = \varphi(\overline{i_n}) \in \mathcal{LO}_F$, as

$$\varphi^{\dagger_F} = \big(\bigwedge_{i \in \overline{i_n}} \blacklozenge i_{p(i)}\big) \to \varphi(i_{p(i_k)}/i_k).$$

Clearly, we have that for any full model $\mathfrak{B}$ and formula $\varphi \in \mathcal{LO}_F[M]$ there is a nominal-free model $\mathfrak{B}^*$ based on the same metric space such that for all w:

$$\langle \mathfrak{B}, w\rangle \models \varphi \iff \langle \mathfrak{B}^*, w\rangle \models \varphi^{\dagger_F},$$

by setting, for every nominal i appearing in φ, $(p(i))^{\mathfrak{B}^*} := i^{\mathfrak{B}}$ and $p^{\mathfrak{B}^*} := p^{\mathfrak{B}}$ for those propositional variables appearing in φ. This shows the implication from right to left in ($\clubsuit$). Conversely, if $\mathfrak{B}^*$ is a nominal-free model based on S such that, for some w, we have $\langle \mathfrak{B}^*, w\rangle \not\models \varphi^{\dagger_F}$, then

$$\langle \mathfrak{B}^*, w\rangle \models \big(\bigwedge_{i \in \overline{i_n}} \blacklozenge i_{p(i)}\big) \wedge \neg\varphi(i_{p(i_k)}/i_k),$$

and so the $i_{p(i)}$ 'denote', i.e., $i_{p(i)}^{\mathfrak{B}^*} = \{w_i\}$, and we can again set $i^{\mathfrak{B}} := (p(i))^{\mathfrak{B}^*}$ and $p^{\mathfrak{B}} := p^{\mathfrak{B}^*}$, thus obtaining a full model $\mathfrak{B}$ such that $\langle \mathfrak{B}, w\rangle \not\models \varphi$, which proves the implication from left to right in ($\clubsuit$).

The same construction can be carried out for the languages $\mathcal{L}_D$ and $\mathcal{LO}_D$. In fact, by Proposition 1.14, all the languages $\mathcal{LO}$, $\mathcal{LO}_F$ and $\mathcal{L}_F$ are structurally expressively equivalent over $\mathcal{D}$, by appropriately replacing any of the distance operators of $\mathcal{LO}$ not appearing in $\mathcal{L}_F$. □

As a consequence of Propositions 1.14 and 1.16, every formula φ of $\mathcal{LO}$ is logically equivalent to a formula $\varphi^\dagger$ of $\mathcal{LO}_F$ and to a formula $\varphi^\ddagger$ of $\mathcal{L}_F \subset \mathcal{LO}_F$ not containing nominals or the universal modality. So the corresponding logics are equivalent in the sense that, for every formula $\varphi \in \mathcal{LO}$, we have

$$\varphi \in \mathcal{MSO}^i \iff \varphi^\dagger \in \mathcal{MSO}^i_F \iff \varphi^\ddagger \in \mathcal{MS}^i_F, \text{ for } i \in \{d,s,t,m\}.$$

Again, a similar equivalence holds true for the languages $\mathcal{LO}_D$ and $\mathcal{L}_D$, i.e.

$$\varphi \in \mathcal{MSO}^i_D \iff \varphi^\sharp \in \mathcal{MS}^i_D, \text{ for } i \in \{d,s,t,m\}.$$

Hence, since the respective translations $.^\dagger$, $.^\ddagger$ and $.^\sharp$ are computable, we can, for instance, give an axiomatisation of $\mathcal{MSO}_F$ and obtain 'axiomatisations' for $\mathcal{MSO}$ and $\mathcal{MS}_F$ automatically in the sense that we can enumerate the tautologies. We will axiomatise the logic $\mathcal{MSO}_F$ using hybrid completeness theory in Section 3.2 and the logic $\mathcal{MS}_D$ using more traditional techniques from standard modal logic in Section 3.1.

A similar equivalence holds true for definability of classes of distance spaces, where we call a class $\boldsymbol{K}_1$ of distance spaces **definable relative to a class $\boldsymbol{K}_2 \supset \boldsymbol{K}_1$** in language $\mathcal{L}$, if there is a set Γ of formulae of $\mathcal{L}$ such that

$$\boldsymbol{K}_1 = \{S \mid S \models \Gamma\} \cap \boldsymbol{K}_2;$$

It should be clear from the definitions that if two languages $\mathcal{L}_1$ and $\mathcal{L}_2$ are (structurally) equally expressive, then they can define precisely the same classes $\boldsymbol{K}_i$ of structures relative to a class $\boldsymbol{K}$. E.g., if the set Γ of formulae of $\mathcal{L}_2$ defines $\boldsymbol{K}_1$, then the set $\Gamma^\sharp$ of formulae of $\mathcal{L}_1$ defines $\boldsymbol{K}_1$ as well, and conversely. We will make some observations about definability of distance spaces in connection with definability of frames in Section 2.3.1.

1.4. Expressive Completeness of Modal Distance Logic

Let us now turn to a comparison of the expressive power of the first-order language $\mathcal{LF}$, more precisely, its two-variable fragment $\mathcal{LF}_2$, with the modal language $\mathcal{LO}_F$ (which we have just shown in Proposition 1.14 to be expressively equivalent to $\mathcal{LO}$ over the class of models based on distance spaces). Every $\mathcal{LF}[M]$-structure

$$\mathfrak{A} = \left\langle W, d, P_1^{\mathfrak{A}}, \ldots, c_1^{\mathfrak{A}}, \ldots \right\rangle$$

gives rise to its $\mathcal{LO}[M]$ counterpart

$$\mathfrak{A}_* = \left\langle W, d, p_1^{\mathfrak{A}_*}, \dots, i_{c_1}^{\mathfrak{A}_*}, \dots \right\rangle,$$

where $p_k^{\mathfrak{A}_*} := P_k^{\mathfrak{A}}$ and $i_{c_k}^{\mathfrak{A}_*} := \{c_k^{\mathfrak{A}}\}$ for all k. This correspondence is clearly bijective, whence we can safely assume that these languages are interpreted in the same classes of structures, thus, circumventing the problem of syntactically different sets of nominals by identifying the first-order constant c_k with the nominal i_{c_k}. If an $\mathcal{LF}[M]$-structure (or an $\mathcal{LF}[M]$-model) is based on a metric (symmetric) space, we call it a **metric** (**symmetric**) $\mathcal{LF}[M]$-**structure** ($\mathcal{LF}[M]$-**model**).

The theorem we are about to prove shows that, when speaking about metric or symmetric spaces, $\mathcal{LO}_F[M]$ is expressively complete for the (sentences of the) two-variable fragment $\mathcal{LF}_2$ with respect to metric or symmetric $\mathcal{LF}[M]$-*structures*. But note that we can translate any $\mathcal{LO}_F[M]$ formula into an equivalent $\mathcal{LF}_2$ sentence with respect to arbitrary $\mathcal{LF}[M]$-*structures* $\mathfrak{A}$, whereas we need the additional condition that $\mathfrak{A}$ is a symmetric structure for the converse translation of an $\mathcal{LF}_2$ sentence into an $\mathcal{LF}[M]$-formula.

By w/x we denote any assignment $\mathfrak{a}$ with $\mathfrak{a}(x) = w$.

THEOREM 1.17 (EXPRESSIVE COMPLETENESS OF $\mathcal{LO}_F$ FOR $\mathcal{LF}_2$).
(i) *For every $\mathcal{LO}_F[M]$-formula φ there exists a $\mathcal{LF}_2^1[M]$-formula $\varphi^\dagger$ with at most one free variable z such that its length is linear in the length of φ and, for any $\mathcal{LF}[M]$-structure $\mathfrak{A}$ and point w in $\mathfrak{A}$, we have*

$$\langle \mathfrak{A}, w/z \rangle \models \varphi^\dagger \quad \iff \quad \langle \mathfrak{A}_*, w \rangle \models \varphi.$$

(ii) *For every $\mathcal{LF}_2^1[M]$-formula φ with at most one free variable z there is an $\mathcal{LO}_F[M]$-formula $\varphi^\ddagger$ such that its length is exponential in the length of φ and, for any symmetric $\mathcal{LF}$-structure $\mathfrak{A}$ and point w in $\mathfrak{A}$, we have*

$$\langle \mathfrak{A}, w/z \rangle \models \varphi \quad \iff \quad \langle \mathfrak{A}_*, w \rangle \models \varphi^\ddagger.$$

(iii) *The languages $\mathcal{LO}_F[M]$ and $\mathcal{LF}_2^1$ are equally expressive over the class of symmetric models, and, a fortiori, $\mathcal{LO}_F[M]$ and $\mathcal{LF}_2^0$ are structurally equally expressive over the classes of symmetric and metric spaces, respectively.*

PROOF. We fix some parameter set M.

(i) The proof of the first claim is pretty standard; cf. Gabbay [1981]. We inductively translate all formulae of $\mathcal{LO}_F$ into $\mathcal{LF}_2$-formulae with at most one free variable and which use only two variables, say, x and y.

Let z and z' be metavariables ranging over $\{x, y\}$. The notation $[z'/z]$ denotes the operation of substituting the variable z' for the variable z, if z is free, and doing nothing otherwise. The translation $\cdot^{\dagger}$ is defined inductively as follows:

$$
\begin{aligned}
(p_k)^{\dagger} &= P_k(x);\\
(i_k)^{\dagger} &= (c_k \doteq x);\\
(\psi_1 \wedge \psi_2)^{\dagger} &= \psi_1^{\dagger}[x/z] \wedge \psi_2^{\dagger}[x/z']);\\
(\neg\psi)^{\dagger} &= \neg\psi^{\dagger};\\
(\mathsf{E}^{<a}\psi)^{\dagger} &= \exists z\,(\delta(z',z) < a \wedge \psi^{\dagger}(z));\\
(\mathsf{E}^{>a}\psi)^{\dagger} &= \exists z\,(\delta(z',z) > a \wedge \psi^{\dagger}(z));\\
(\mathsf{E}^{=a}\psi)^{\dagger} &= \exists z\,(\delta(z',z) = a \wedge \psi^{\dagger}(z));\\
(\mathsf{E}^{>a}_{<b}\psi)^{\dagger} &= \exists z\,(a < \delta(z',z) < b \wedge \psi^{\dagger}(z));\\
(\blacksquare\psi)^{\dagger} &= \forall z\,\psi^{\dagger}(z).
\end{aligned}
$$

To be precise about variable usage, if z is free in a formula $\psi^{\dagger}(z)$, we choose $z \neq z'$, and if $\psi^{\dagger}(z)$ contains no free variable, we choose $z = y$ and $z' = x$.

We can now check inductively that for all points w:

$$\langle \mathfrak{A}, w/z\rangle \models \varphi^{\dagger}(z) \quad\iff\quad \langle \mathfrak{A}_*, w\rangle \models \varphi.$$

First, the atomic and Boolean cases are trivial. If $\varphi = \mathsf{E}^{<a}\psi$, then $\langle \mathfrak{A}_*, w\rangle \models \varphi$ if and only if there is a u such that $d(w,u) < a$ and $\langle \mathfrak{A}_*, u\rangle \models \psi$ if and only if $\langle \mathfrak{A}, w/z'\rangle \models \exists z(\delta(z',z) < a \wedge \psi^{\dagger}(z))$, where the variable z' is the (possibly) free variable of $\psi^{\dagger}$. The other distance operators are treated similarly. So suppose $\langle \mathfrak{A}_*, w\rangle \models \blacksquare\psi$. This is the case if and only if ψ holds globally in $\mathfrak{A}_*$, i.e. we have $\langle \mathfrak{A}_*, u\rangle \models \psi$ for all u, which, by induction hypotheses, is equivalent to $\langle \mathfrak{A}, u/z\rangle \models \psi^{\dagger}(z)$ for all u which in turn is equivalent to $\langle \mathfrak{A}, w/z\rangle \models \forall z(\psi^{\dagger}(z))$.

Finally, it is obvious that the size of $\varphi^{\dagger}$ is linear in the size of φ.

(ii) To define the converse translation, we first observe that the following transformations of an $\mathcal{LF}_2$-formula $\varphi(x,y)$ result in an equivalent formula with respect to symmetric spaces.

Every occurrence of equality $t_1 \doteq t_2$ can be replaced by $\delta(t_1,t_2) = 0$. Further:

$\delta(t,t) = 0$ by $\top$;	$\delta(t,t) < 0$ by $\bot$;
$\delta(t,t) = a$ by $\bot$ if $a > 0$;	$\delta(t,t) < a$ by $\top$ if $a > 0$;
$\delta(y,t) = a$ by $\delta(t,y) = a$;	$\delta(y,t) < a$ by $\delta(t,y) < a$;
$\delta(t,x) = a$ by $\delta(x,t) = a$;	$\delta(t,x) < a$ by $\delta(x,t) < a$.

In what follows, we assume that these transformations have been applied to all our formulae, in particular, to φ. As a result, we can assume that φ does not contain equality and that in occurrences of $\delta(t_1,t_2)$ we always have $t_1 \neq t_2$, and the variable x

always appears first, y second. Note that we needed only symmetry and Axiom (1) of metric spaces, but not the triangular inequality, to perform these transformations.

We distinguish between three types of atomic formulae in $\mathcal{LF}_2$: **binary atoms** are of the form $\delta(x,y) < a$ or $\delta(x,y) = a$ (they have two free variables); **unary atoms** are of the form $\delta(x,c_k) < a$, $\delta(x,c_k) = a$, $P_i(x)$, $P_i(y)$, $\delta(c_k,y) < a$ or $\delta(c_k,y) = a$ (having only one free variable); atoms without free variables, $P_i(c_k)$, $\delta(c_k,c_l) < a$ or $\delta(c_k,c_l) = a$, can be called **nullary**.

We inductively translate all $\mathcal{LF}_2$-formulae ψ with at most one free variable to $\mathcal{LO}_F$-formulae $\psi^\ddagger$:

(1) If $\psi = P_i(c_k)$ then $\psi^\ddagger = \blacksquare(i_k \to p_i)$;
(2) If ψ is $\delta(c_k,c_l) = a$ $(k,l < \omega)$ then $\psi^\ddagger = \blacksquare(i_k \to \mathsf{E}^{=a} i_l)$.
(3) If ψ is $\delta(c_k,c_l) < a$ $(k,l < \omega)$ then $\psi^\ddagger = \blacksquare(i_k \to \mathsf{E}^{<a} i_l)$.
(4) If $\psi \in \{P_i(x), P_i(y)\}$ then $\psi^\ddagger = p_i$;
(5) If $\psi \in \{\delta(x,c_k) = a,\ \delta(c_k,y) = a \mid k < \omega\}$ then $\psi^\ddagger = \mathsf{E}^{=a} i_k$;
(6) If $\psi \in \{\delta(x,c_k) < a,\ \delta(c_k,y) < a \mid k < \omega\}$ then $\psi^\ddagger = \mathsf{E}^{<a} i_k$.
(7) If $\psi = \chi_1 \wedge \chi_2$ then $\psi^\ddagger = \chi_1^\ddagger \wedge \chi_2^\ddagger$.
(8) If $\psi = \neg\chi$ then $\psi^\ddagger = \neg(\chi^\ddagger)$.

The remaining cases of $\psi = \exists y \chi(x,y)$ and $\psi = \exists x \chi(x,y)$ are more sophisticated. We consider only the former. The formula $\chi(x,y)$ can be regarded as a Boolean combination of binary atoms β_i and formulae $\upsilon_i(x)$ (including the nullary atoms) and $\xi_i(y)$ with at most one free variable. Denote this Boolean combination by κ, i.e.,

$$\chi(x,y) = \kappa(\beta_1,\dots,\beta_r,\upsilon_1(x),\dots,\upsilon_l(x),\xi_1(y),\dots,\xi_s(y)).$$

Let us first move all components in κ without free y out of the scope of the outmost $\exists y$ in ψ. Then ψ can be equivalently rewritten as

$$\bigvee_{\langle \nu_1,\dots,\nu_l\rangle \in \{\top,\bot\}^l} \Big(\exists y\, \kappa(\beta_1,\dots,\beta_r,\nu_1,\dots,\nu_l,\xi_1(y),\dots,\xi_s(y)) \wedge \bigwedge_{1\le i\le l} (\upsilon_i(x) \leftrightarrow \nu_i)\Big).$$

Now let $0 = a_0 < a_1 < \cdots < a_n$ be the list of all numbers occurring in ψ together with 0. So this list is non-empty. Consider the set $\mathcal{R}_\psi$ containing the following formulae:

- $\delta(x,y) = a_i$, for $i \le n$;
- $a_i < \delta(x,y) < a_{i+1}$, for $i < n$;
- $\delta(x,y) > a_n$.

For every $\beta \in \mathcal{R}_\psi$ and every binary atom β_i in ψ, we have either $\beta \models \beta_i$ or $\beta \models \neg\beta_i$. In other words, by assigning a truth-value to some β in $\mathcal{R}_\psi$, we fix the truth values of all binary atoms in ψ. Let $\beta_i^\beta = \top$ if $\beta \models \beta_i$, and $\beta_i^\beta = \bot$ otherwise. Then ψ is equivalent to the formula

$$\bigvee_{\langle \nu_1,\dots,\nu_l\rangle \in \{\top,\bot\}^l} \Big(\bigvee_{\beta\in\mathcal{R}_\psi} \exists y\,(\beta \wedge \kappa(\beta_1^\beta,\dots,\beta_r^\beta,\nu_1,\dots,\nu_l,\xi_1,\dots,\xi_s)) \wedge \bigwedge_{1\le i\le l} (\upsilon_i(x) \leftrightarrow \nu_i)\Big).$$

Next, we replace each $\beta \in \mathcal{R}_\psi$ with the distance operator $\beta^\star$ defined by taking

- $(\delta(x,y) = a_i)^\star = \mathsf{E}^{=a_i}$, for $i \leq n$;
- $(a_i < \delta(x,y) < a_{i+1})^\star = \mathsf{E}^{>a_i}_{<a_{i+1}}$, for $i < n$;
- $(\delta(x,y) > a_n)^\star = \mathsf{E}^{>a_n}$,

delete the quantifier $\exists y$ and recursively compute the values of $v_i^\ddagger$ and $\xi_i^\ddagger$. This yields the formula $\psi^\ddagger$ which is

$$\bigvee_{\langle \nu_1,\dots,\nu_l\rangle \in \{\top,\bot\}^l} \Big(\bigvee_{\beta\in\mathcal{R}_\psi} \beta^*\big(\kappa(\beta_1^\beta,\dots,\beta_r^\beta,\nu_1,\dots,\nu_l,\xi_1^\ddagger,\dots,\xi_s^\ddagger)\big) \wedge \bigwedge_{1\leq i\leq l} (v_i^\ddagger \leftrightarrow \nu_i)\Big).$$

It should be clear from the construction that, in the worst case, the size of the formula $\varphi^\ddagger$ is exponential in the size of φ and that, for every metric structure $\mathfrak{A}$ and point w in $\mathfrak{A}$ we have

$$\langle \mathfrak{A}, w/z\rangle \models \varphi \quad\iff\quad \langle \mathfrak{A}_*, w\rangle \models \varphi^\ddagger,$$

which proves (ii). Item (iii) is a consequence of (i) and (ii). □

The following examples illustrate the translation given in the proof of (ii).

EXAMPLE 1.18. Consider the $\mathcal{LF}_2$-sentence

$$\varphi = \exists y\Big(\exists x\big(\delta(x,y) > 0 \wedge P_i(x)\big) \wedge \neg P_i(y)\Big).$$

Let $\xi_1(y) = \exists x\big(\delta(x,y) > 0 \wedge P_i(x)\big)$ and $\xi_2(y) = \neg P_i(y)$. Then we represent φ as

$$\exists y\big(\xi_1(y) \wedge \xi_2(y)\big)$$

which is equivalent to

$$\exists y\big(\delta(x,y) = 0 \wedge \xi_1(y) \wedge \xi_2(y)\big) \vee \exists y\big(\delta(x,y) > 0 \wedge \xi_1(y) \wedge \xi_2(y)\big),$$

since $\mathcal{R}_\psi = \{\delta(x,y) = 0, \delta(x,y) < 0, \delta(x,y) > 0\}$ and the missing formula $\exists y\big(\delta(x,y) < 0 \wedge \xi_1(y) \wedge \xi_2(y)\big)$ is inconsistent. Thus, we obtain the $\mathcal{LO}_F$-formula

$$\varphi^\ddagger = \mathsf{E}^{=0}(\xi_1^\ddagger \wedge \xi_2^\ddagger) \vee \mathsf{E}^{>0}(\xi_1^\ddagger \wedge \xi_2^\ddagger),$$

where $\xi_1^\ddagger = \mathsf{E}^{=0}(\bot \wedge p_i) \vee \mathsf{E}^{>0}(\top \wedge p_i)$ or, equivalently, $\xi_1^\ddagger = \mathsf{E}^{>0}p_i$, and $\xi_2^\ddagger = \neg p_i$. So the resulting translation (with rearranged conjunctions for better readability) is:

$$\varphi^\ddagger = \mathsf{E}^{=0}(\neg p_i \wedge \mathsf{E}^{>0}p_i) \vee \mathsf{E}^{>0}(\neg p_i \wedge \mathsf{E}^{>0}p_i).$$

Using the universal $\blacklozenge$, we finally obtain

$$\varphi^\ddagger = \blacklozenge(\neg p_i \wedge \mathsf{E}^{>0}p_i).$$

The reader can easily check that φ and $\varphi^\ddagger$ indeed say the same when the distance function is assumed to satisfy symmetry. Similarly, the formula

$$\varphi(y) = \exists x\big(\delta(x,y) > 0 \wedge P_i(x)\big) \wedge \neg P_i(y)$$

with one free variable y corresponds to $\neg p_i \wedge \mathsf{E}^{>0} p_i$, for we have, for all models $\mathfrak{A}$ and points w

$$\langle \mathfrak{A}, w/y \rangle \models \varphi(y) \iff \langle \mathfrak{A}_*, w \rangle \models \neg p_i \wedge \mathsf{E}^{>0} p_i.$$

However, note that in non-symmetric spaces $\varphi^\ddagger$ does not correspond to φ but to

$$\varphi^- = \exists y \Big(\exists x \big(\delta(y,x) > 0 \wedge P_i(x) \big) \wedge \neg P_i(y) \Big),$$

where we have swapped the variables in $\delta(x,y) > 0$. We indicate in the next section how this deficit in expressivity of the language $\mathcal{LO}_F$ can be overcome.

1.5. Boolean Distance Logics

An analysis of the expressive completeness result from the last section and in particular the expressiveness of the 'ring operators' of the form $\mathsf{A}^{>a}_{<b}$ suggests that the language $\mathcal{LO}_F$ is closely related to Boolean modal logic. For instance, while the formula $\mathsf{A}^{>a}_{<b}\varphi$ is clearly not equivalent to a conjunction of the form $\mathsf{A}^{>a}\varphi \wedge \mathsf{A}^{<b}\varphi$, the operator $\mathsf{A}^{>a}_{<b}$ does coincide with a Boolean modal operator $[\succ_a \wedge \prec_b]$ build from a conjunction of the symbols $\succ_a$ and $\prec_b$ being interpreting by the distance function d in the obvious way, i.e., given a model $\mathfrak{B}$ based on a distance space $\langle W, d \rangle$, we have, for all points $u \in W$:

$$\langle \mathfrak{B}, u \rangle \models [\succ_a \wedge \prec_b]\varphi \iff \langle \mathfrak{B}, v \rangle \models \varphi \text{ for all } v \in W \text{ with } d(u,v) > a \text{ and } d(u,v) < b.$$

Yet, unlike in the case of 'standard' Boolean modal logic, the natural ordering of parameters from $\mathbb{R}^+$ imposes additional structure on Boolean distance operators defined by allowing arbitrary Boolean combinations of symbols from the set

$$\{\approx_a, \prec_a, \succ_a \mid a \in M\},$$

and hence restricts the number of 'new' operators obtained in this way.

Furthermore, Boolean modal languages with similar expressive capabilities as the language $\mathcal{LO}_F$, namely Boolean modal logic enriched with converse modal operators and the difference operator, have been shown to be expressively equivalent to the two-variable fragment of first-order logic, compare Lutz et al. [2001a] and Lutz et al. [2001b].

In this section we show that, indeed, the language $\mathcal{LO}_F$ is expressively equivalent to a natural variant of Boolean modal logic (over the class of all distance spaces) and thus expressively equivalent to two-variable first-order logic interpreted on symmetric distance spaces.

Let us start by defining the language of Boolean distance logic:

DEFINITION 1.19 (BOOLEAN DISTANCE LOGIC). *Let M be a parameter set and define a set of **modal parameters** as follows:*

$$\mathbb{M} := \{\approx_a, \prec_a \mid a \in M\}.$$

*Then, let $\mathcal{B}(\mathbb{M})$ be the set of all Boolean combinations of symbols from $\mathbb{M}$. Now, the language $\mathcal{LOB}[M]$ consists of a denumerably infinite list $\{p_l : l < \omega\}$ of **propositional variables**, a denumerably infinite list $\{i_l : l < \omega\}$ of **nominals**, the **Boolean connectives** $\wedge$ and $\neg$, the **propositional constants** $\top$ and $\bot$, the **universal modality** $\blacksquare$, as well as the following set of **Boolean distance operators** depending on M:*

$$\{[\delta] : \delta \in \mathcal{B}(\mathbb{M})\}.$$

The set of well-formed formulae of this language is constructed in the standard way; it will be identified with $\mathcal{LOB}[M]$. As usual, we often omit the parameter set M in the notation.

Again, other Booleans as well as the dual Boolean distance operators $\langle\delta\rangle$ and the universal diamond $\blacklozenge$ are defined as abbreviations.

DEFINITION 1.20 (SEMANTICS FOR BOOLEAN DISTANCE LOGIC). *As before, the models for the language $\mathcal{LOB}$ are **full models** of the form:*

$$\mathfrak{B} = \left\langle W, d, p_0^{\mathfrak{B}}, p_1^{\mathfrak{B}}, \dots, i_0^{\mathfrak{B}}, i_1^{\mathfrak{B}}, \dots \right\rangle,$$

where $\langle W, d\rangle$ is a metric space, the $p_i^{\mathfrak{B}}$ are subsets of W and nominals i_l are interpreted by singleton subsets $i_l^{\mathfrak{B}}$.

*We just have to define the truth-relation for the new distance operators: let $\mathfrak{B}$ be a full model, w a point in W and $[\delta]\varphi$ a $\mathcal{LOB}$ formula with $\delta \in \mathcal{B}(\mathbb{M})$. First, we define the **extension** of δ with respect to a point $w \in W$, abbreviated as $\|\delta\|^w$, as follows:*

$$\|\delta\|^w := \{v \in W \mid \langle w, v\rangle \models \delta\},$$

where $\langle w, v\rangle \models \delta$ is inductively defined as follows:

- $\langle w, v\rangle \models \approx_a \iff w = v$;
- $\langle w, v\rangle \models \prec_a \iff d(w, v) < a$;
- $\langle w, v\rangle \models \delta \wedge \gamma \iff \langle w, v\rangle \models \delta$ *and* $\langle w, v\rangle \models \gamma$;
- $\langle w, v\rangle \models \neg\gamma \iff \langle w, v\rangle \not\models \gamma$.

Now set:

- $\langle \mathfrak{B}, w\rangle \models [\delta]\varphi \iff$ *for all* $v \in \|\delta\|^w$ *we have* $\langle \mathfrak{B}, v\rangle \models \varphi$.

Let us introduce the notion of **satisfiability** for the Boolean distance operators. A combination δ can be called **satisfiable**, if there is a model $\mathfrak{B}$ and a point w such that $\|\delta\|^w \neq \varnothing$. Deciding satisfiability of a given δ is a very simple problem, we just have to check whether or not a system of equalities and inequalities in one variable, i.e. of the form $x = a_i, x \neq a_j, x < a_k, x \geq a_l$, has a solution.

Clearly, all the distance operators of $\mathcal{LO}_F[M]$ and the universal modality are definable in $\mathcal{LOB}[M]$ with respect to all distance spaces, namely we can translate them as follows:

- $(\mathsf{A}^{=a}\varphi)^\star = [\approx_a]\varphi^\star$;
- $(\mathsf{A}^{<a}\varphi)^\star = [\prec_a]\varphi^\star$;
- $(\mathsf{A}^{>a}\varphi)^\star = [\neg(\prec_a \vee \approx_a)]\varphi^\star = [\neg \prec_a \wedge \neg \approx_a]\varphi^\star$;
- $(\mathsf{A}^{>a}_{<b}\varphi)^\star = [\neg(\prec_a \vee \approx_a) \wedge \prec_b]\varphi^\star = [\neg \prec_a \wedge \neg \approx_a \wedge \prec_b]\varphi^\star$;
- $(\blacksquare\varphi)^\star = (\mathsf{A}^{=a}\varphi)^\star \wedge (\mathsf{A}^{<a}\varphi)^\star \wedge (\mathsf{A}^{>a}\varphi)^\star$.

Note that, in the translation of the $\mathcal{LO}_F[M]$-operators, we only needed atomic negation and conjunction. Note also that for each formula of the form $[\bigvee_{l\leq n}\delta_l]\varphi$ we have for all models $\mathfrak{B}$ and points w:

$$\langle \mathfrak{B}, w\rangle \models [\bigvee_{l\leq n}\delta_l]\varphi \iff \langle \mathfrak{B}, w\rangle \models \bigwedge_{l\leq n}[\delta_l]\varphi.$$

Hence, as far as expressivity goes, we can do with atomic negation and conjunction in the definition of Boolean distance operators by bringing δ in disjunctive normal form (DNF) and by replacing the disjunction in δ in favour of conjunctions of formulae. (This procedure, however, can result in an exponential blow up of formulae length and is thus not inert with respect to complexity issues, cf. Lutz and Sattler [2002])

We can now show that the languages $\mathcal{LO}_F$ and $\mathcal{LOB}$ are equally expressive over the class of all models based on distance spaces. The central argument is, in essence, the same as in Theorem 1.17.

THEOREM 1.21 (EXPRESSIVE COMPLETENESS OF $\mathcal{LOB}$ FOR $\mathcal{LO}_F$).
The languages $\mathcal{LO}_F[M]$ and $\mathcal{LOB}[M]$ are equally expressive over the class of all models based on distance spaces.

PROOF. We have already shown that all the distance operators of $\mathcal{LO}_F$ as well as the universal modality are definable in $\mathcal{LOB}$, so $\mathcal{LOB}$ is as expressive as $\mathcal{LO}_F$. It remains to show that there is a translation $.^\dagger : \mathcal{LOB} \longrightarrow \mathcal{LO}_F$ such that for all formulae $\varphi \in \mathcal{LOB}$, models $\mathfrak{B}$, and points w in $\mathfrak{B}$, we have:

$$\langle \mathfrak{B}, w\rangle \models \varphi \iff \langle \mathfrak{B}, w\rangle \models \varphi^\dagger.$$

As sketched above, we can assume without loss of generality that all Boolean distance operators are defined by conjunctions of literals of modal parameters. The translation is defined inductively. As usual we define

- $p_i^\dagger = p_i$;
- $i_k^\dagger = i_k$;

- $(\psi \wedge \chi)^\dagger = \psi^\dagger \wedge \chi^\dagger$;
- $(\neg\psi)^\dagger = \neg\psi^\dagger$.

The remaining case of $\psi = [\delta]\chi$, δ a conjunction of literals, is more difficult. We distinguish several cases.

First we check whether δ is satisfiable. If it is not we have $\|\delta\|^w = \emptyset$ for any model $\mathfrak{B}$ and point w, so we can set:

- $([\delta]\varphi)^\dagger = \top$, if δ is inconsistent;

For the remaining cases we assume δ is satisfiable. If $\approx_a \in \delta$ for some a, then for any model $\mathfrak{B}$ and point $w \in W$ we have $\|\delta\|^w = \| \approx_a \|^w$. Hence we can define

- $([\delta]\varphi)^\dagger = \mathsf{E}^{=a}\varphi^\dagger$, if $\approx_a \in \delta$ for some a and δ is consistent.

It remains to consider the case where

$$\delta = \neg \approx_{a_1} \wedge \ldots \wedge \neg \approx_{a_n} \wedge \neg \prec_{b_1} \wedge \ldots \wedge \neg \prec_{b_m} \wedge \prec_{c_1} \wedge \ldots \wedge \prec_{c_k},$$

with $n, m, k \geq 0$ and $a_1 < \ldots < a_n$, $b_1 < \ldots < b_m$ and $c_1 < \ldots < c_k$. Assume first that $m = k = 0$. Then we can translate

- $([\delta]\varphi)^\dagger = \mathsf{A}^{<a_1}\varphi^\dagger \wedge \mathsf{A}^{>a_1}_{<a_2}\varphi^\dagger \wedge \ldots \wedge \mathsf{A}^{>a_{n-1}}_{<a_n}\varphi^\dagger \wedge \mathsf{A}^{>a_n}\varphi^\dagger$.

We can without loss of generality assume that $m, k \leq 1$. Namely, if $m, k > 0$ set

$$b := max(\{b_i \mid \neg \prec_{b_i} \in \delta\})$$

and

$$c := min(\{c_i \mid \prec_{c_i} \in \delta\}).$$

Then δ is equivalent to δ', where

$$\delta' = \neg \approx_{a_1} \wedge \ldots \wedge \neg \approx_{a_n} \wedge \neg \prec_b \wedge \prec_c .$$

Moreover, since δ is satisfiable, we can assume that the parameters are ordered as follows

$$b < a_1 < a_2 < \ldots < a_n \leq c.$$

It should be clear now how we have to translate the remaining cases. For simplicity, we use the operators $\mathsf{A}^{\geq a}_{<b}$ etc. that are definable in $\mathcal{LO}_F$:

- $([\delta]\varphi)^\dagger = \mathsf{A}^{\geq b}_{<a_1}\varphi^\dagger \wedge \mathsf{A}^{>a_1}_{<a_2}\varphi^\dagger \wedge \ldots \wedge \mathsf{A}^{>a_{n-1}}_{<a_n}\varphi^\dagger \wedge \mathsf{A}^{>a_n}_{<c}\varphi^\dagger$, if $n, m, k > 0$;
- $([\delta]\varphi)^\dagger = \mathsf{A}^{\geq b}_{<a_1}\varphi^\dagger \wedge \mathsf{A}^{>a_1}_{<a_2}\varphi^\dagger \wedge \ldots \wedge \mathsf{A}^{>a_{n-1}}_{<a_n}\varphi^\dagger \wedge \mathsf{A}^{>a_n}\varphi^\dagger$, if $n, m > 0, k = 0$;
- $([\delta]\varphi)^\dagger = \mathsf{A}^{<a_1}\varphi^\dagger \wedge \mathsf{A}^{>a_1}_{<a_2}\varphi^\dagger \wedge \ldots \wedge \mathsf{A}^{>a_{n-1}}_{<a_n}\varphi^\dagger \wedge \mathsf{A}^{>a_n}_{<c}\varphi^\dagger$, if $n, k > 0, m = 0$;
- $([\delta]\varphi)^\dagger = \mathsf{A}^{\geq b}_{<c}\varphi^\dagger$, if $m, k > 0, n = 0$;
- $([\delta]\varphi)^\dagger = \mathsf{A}^{<c}\varphi^\dagger$, if $n, m = 0, k > 0$;
- $([\delta]\varphi)^\dagger = \mathsf{A}^{\geq b}\varphi^\dagger$, if $n, k = 0, m > 0$;

$\square$

Similar to Proposition 1.16 we could as well show that $\mathcal{LOB}$ is structurally expressively complete over the class of all distance spaces for a language $\mathcal{LOB}^-$ that is like $\mathcal{LOB}$, but without nominals or the universal modality. Also, by Theorem 1.17, $\mathcal{LOB}$ is as expressive as the 2-variable first-order language $\mathcal{LF}_2$ over symmetric models. To obtain languages $\mathcal{LO}_F^+$ and $\mathcal{LOB}^+$ that are as expressive as $\mathcal{LF}_2$ over models based on arbitrary distance spaces, we had to add **inverse distance operators** like $\mathsf{A}_-^{\leq a}$ etc., defined by

$$\langle \mathfrak{B}, w\rangle \models \mathsf{E}_-^{\leq a}\varphi \iff \langle \mathfrak{B}, u\rangle \models \varphi \text{ for all } u \in W \text{ with } d(u,w) < a,$$

to the language $\mathcal{LO}_F$, and **inverse Boolean operators**

$$\langle \mathfrak{B}, w\rangle \models [\delta]_-\varphi \iff \text{ for all } v \in \|\delta\|_-^w \text{ we have } \langle \mathfrak{B}, v\rangle \models \varphi,$$

where $\|\delta\|_-^w := \{v \in W \mid \langle v, w\rangle \models \delta\}$, to the language $\mathcal{LOB}$, similarly to what has been done in Lutz et al. [2001b], but we leave the details of these enrichments to the reader.

To close this chapter, we present an overview of the results on expressiveness obtained. An entry in Table 1.5 for languages $\mathcal{L}_1$ and $\mathcal{L}_2$ indicates the respective strongest property we were able to proof for these languages. We use the abbreviations EC^i

	$\mathcal{LF}_2, \mathcal{LO}_F^+, \mathcal{LOB}^+$	$\mathcal{LO}, \mathcal{LO}_F, \mathcal{LOB}$	$\mathcal{L}_F, \mathcal{LOB}^-$	$\mathcal{LO}_D$	$\mathcal{L}_D$
$\mathcal{LF}_2, \mathcal{LO}_F^+, \mathcal{LOB}^+$	–	EC^s	SEC^s	AE^d	AE^d
$\mathcal{LO}, \mathcal{LO}_F, \mathcal{LOB}$	EC^s	–	SEC^d	AE^d	AE^d
$\mathcal{L}_F, \mathcal{LOB}^-$	SEC^s	SEC^d	–	SAE^d	AE^d
$\mathcal{LO}_D$	–	–	–	–	SEC^d
$\mathcal{L}_D$	–	–	–	SEC^d	–

Table 1.2: Expressiveness for distance logics.

($i \in \{d,s,t,m\}$) for expressive completeness with respect to the class of models based on spaces from $\mathcal{D}^i$, SEC^i ($i \in \{d,s,t,m\}$) for structural expressive completeness with respect to the class $\mathcal{D}^i$, AE^i ($i \in \{d,s,t,m\}$) for $\mathcal{L}_j$ being as expressive as $\mathcal{L}_k$ over the class of all models based on spaces from $\mathcal{D}^i$, and SAE^i ($i \in \{d,s,t,m\}$) for $\mathcal{L}_j$ being structurally as expressive as $\mathcal{L}_k$ over the class $\mathcal{D}^i$, where j is the row, and k is the column. Those languages for which we were able to prove expressive completeness with respect to the class of all models based on distance spaces are identified, thus the (empty) diagonal might be read as EC^d.

Some of the results mentioned in the table have not explicitly been proved, but are easy corollaries. For instance, notice that if $\mathcal{L}_1$ is structurally expressively complete for $\mathcal{L}_2$ over $\mathcal{D}^d$ and $\mathcal{L}_2$ is expressively complete for $\mathcal{L}_3$ over symmetric models, then $\mathcal{L}_1$ is structurally expressively complete for $\mathcal{L}_3$ over $\mathcal{D}^s$.

CHAPTER 2

Computational Properties of Distance Logics

2.1. Undecidable First-Order Distance Logics

From the computational point of view, first-order distance logic $\mathcal{LF}$ as well as its two-variable fragment $\mathcal{LF}_2$ turn out to be too expressive, at least when we are primarily interested in metric spaces. We have the following undecidability result.

THEOREM 2.1 (UNDECIDABILITY IN $\mathcal{LF}$ AND $\mathcal{LF}_2$).
(i) *Let $\boldsymbol{K}$ be any class of distance spaces containing $\mathcal{MS}$. Then the satisfiability problem for $\mathcal{LF}[\mathbb{N}]$-formulae in (models based on spaces from) $\boldsymbol{K}$ is undecidable.*
(ii) *The satisfiability problem for $\mathcal{LF}_2[\mathbb{N}]$-formulae in any class $\boldsymbol{K}$ of distance spaces such that $\boldsymbol{K} \subseteq \mathcal{D}^t$ and $\langle \mathbb{R}^2, d_2 \rangle \in \boldsymbol{K}$ is undecidable as well. In particular, $\mathcal{LF}_2[\mathbb{N}]$ satisfiability is undecidable in the classes $\mathcal{D}^t$ and $\mathcal{MS}$.*

PROOF. To prove the former claim, it suffices to observe that $\mathcal{LF}[\mathbb{N}]$ is powerful enough to interpret the theory of graphs (i.e., the theory of structures $\langle W, R \rangle$, where R is a symmetric and reflexive binary relation on W), which is known to be hereditarily undecidable[1] [Rabin, 1965]. Indeed, let $\varphi(x, y)$ be the formula

$$\delta(x, y) = 1 \vee \delta(x, y) = 0.$$

Given a graph $\langle W, R \rangle$, we can define a metric space $\langle W, d \rangle$ by taking, for all $a, b \in W$,

$$d(a, b) = \begin{cases} 0, & \text{if } a = b, \\ 1, & \text{if } a \neq b \text{ and } aRb, \\ 2, & \text{if not } aRb. \end{cases}$$

We then clearly have $\langle W, d \rangle \models \varphi[a, b]$ if and only if aRb. For a formula γ in the signature of graph theory, denote by $\gamma^\bullet$ the result of replacing every occurrence of an atom $R(x, y)$ in γ by $\varphi(x, y)$. Obviously, $\gamma^\bullet$ is an $\mathcal{LF}[\mathbb{N}]$-formula and, for every graph $\langle W, R \rangle$, the formula γ is satisfiable in $\langle W, R \rangle$ if and only if $\gamma^\bullet$ is satisfiable in $\langle W, d \rangle$. Now consider the set Γ of formulae γ in the signature of graph theory such that $\gamma^\bullet$ is true in all $\mathcal{LF}[\mathbb{N}]$-models based on distance spaces in $\boldsymbol{K}$. Clearly, every $\gamma \in \Gamma$ is a theorem of graph theory, so Γ is a subtheory of graph theory. By the result of Rabin [1965] mentioned above, the theory Γ is undecidable, which yields (i).

[1] This means that every subtheory of graph theory is undecidable.

(ii) follows from Theorems 1.17 (i) and 2.2 to be proved below. □

2.2. An Undecidable Modal Distance Logic: $\mathcal{MS}_F$

We now show that the satisfiability problem for $\mathcal{L}_F[\mathbb{N}]$ is undecidable in natural classes of spaces satisfying the triangular inequality. The proof below can be carried out with a number of variations of the language $\mathcal{L}_F$, all containing distance operators like $\mathsf{E}^{>0}_{\leq a}$; details of this may be found in Kutz et al. [2003b].

THEOREM 2.2 (UNDECIDABILITY IN $\mathcal{L}_F$).
Let $\boldsymbol{K} \subseteq \mathcal{D}^t$ contain $\langle \mathbb{R}^2, d_2 \rangle$. Then the satisfiability problem for $\mathcal{L}_F[\mathbb{N}]$-formulae in (models based on spaces from) $\boldsymbol{K}$ is undecidable. In particular, $\mathcal{L}_F[\mathbb{N}]$ satisfiability is undecidable in the classes $\mathcal{D}^t$ and $\mathcal{MS}$, and $\mathcal{L}_F[\mathbb{N}]$ does not have the finite model property.

PROOF. The proof is by reduction to the undecidable $\mathbb{N} \times \mathbb{N}$ tiling problem (see van Emde Boas [1997], Börger et al. [1997] and references therein). We remind the reader that the tiling problem for $\mathbb{N} \times \mathbb{N}$ is formulated as follows: given a finite set of tile types $\mathcal{T} = \{T_1, \ldots, T_l\}$, i.e., squares T_i with colours $left(T_i)$, $right(T_i)$, $up(T_i)$, and $down(T_i)$ on their edges, decide whether the grid $\mathbb{N} \times \mathbb{N}$ can be covered with tiles, each of a type from $\mathcal{T}$, in such a way that the colours of adjacent edges on adjacent tiles match, or, more precisely, whether there exists a function $\tau : \mathbb{N} \times \mathbb{N} \to \mathcal{T}$ such that, for all $n, m \in \mathbb{N}$, we have

$$right(\tau(n,m)) = left(\tau(n+1,m)),$$
$$up(\tau(n,m)) = down(\tau(n,m+1)).$$

So suppose a set of tile types $\mathcal{T} = \{T_1, \ldots, T_l\}$ is given. Our aim is to construct an $\mathcal{L}_F[\mathbb{N}]$-formula which is satisfiable in $\boldsymbol{K}$ if and only if $\mathcal{T}$ can tile $\mathbb{N} \times \mathbb{N}$.

For convenience we use again operators like $\mathsf{A}^{\leq a}$ etc. that are definable in $\mathcal{L}_F[\mathbb{N}]$. Take propositional variables $p_1, \ldots, p_l, q_0, \ldots, q_4$ and $r_0, \ldots, r_4$. Let

$$\chi_{i,j} = \mathsf{A}^{\leq 9}(p_i \wedge q_j), \quad \text{for } i, j \leq 4,$$

and let Γ be the set of the following formulae, where $i, j \leq 4$ and $k \leq l$:

$$p_i \wedge q_j \to \mathsf{E}^{\leq 9}\chi_{i,j}, \;\; \chi_{i,j} \to \mathsf{A}^{>0}_{\leq 80}\neg\chi_{i,j}, \;\; \chi_{i,j} \to \neg\chi_{m,n} \; ((i,j) \neq (m,n)); \tag{4}$$

$$\chi_{i,j} \to \bigvee_{k \leq l} \mathsf{A}^{\leq 9} r_k, \;\; r_m \to \neg r_n \; (n \neq m); \tag{5}$$

$$\chi_{i,j} \wedge r_k \wedge \mathsf{E}^{\leq 20}(\mathsf{E}^{\leq 20}\chi_{i,j} \wedge \chi_{i+_5 1,j} \wedge \bigvee_{right(T_k)=left(T_m)} r_m); \tag{6}$$

$$\chi_{i,j} \wedge r_k \to \mathsf{E}^{\leq 20}(\mathsf{E}^{\leq 20}\chi_{i,j} \wedge \chi_{i,j+_5 1} \wedge \bigvee_{up(T_k)=down(T_m)} r_m); \tag{7}$$

where $+_5$ denotes addition modulo 5.[2] The first formula in (4) is satisfied in a model $\mathfrak{B}$ if and only if the truth-set $(p_i \wedge q_j)^{\mathfrak{B}}$ is the union of a set of spheres of radius 9. The second one is satisfied in $\mathfrak{B}$ if and only if the distance between any two distinct centres of spheres of radius 9, all points in which belong to $(p_i \wedge q_j)^{\mathfrak{B}}$, is more than 80, while the third formula guarantees that the sets $\chi_{i,j}^{\mathfrak{B}}$ are pairwise disjoint. We think of the truth-sets $\chi_{i,j}^{\mathfrak{B}}$, for $i, j \leq 4$, as a finite family of infinite sets making up the grid for the tiling (see Fig. 2.1). The formulae in (5) ensure that every point of the grid is covered by some tile and that different tiles never cover the same point. Finally, formulae (6) and (7) ensure the tiling conditions in the horizontal and vertical directions, respectively.

Note that if $\langle \mathfrak{B}, w \rangle \models \chi_{i,j}$ then, in view of (6), there exist points $u, v \in W$ with $\langle \mathfrak{B}, u \rangle \models \chi_{i+_5 1,j}$ and $\langle \mathfrak{B}, v \rangle \models \chi_{i,j}$ for which $d(w,u) \leq 20$ and $d(u,v) \leq 20$. But then, by the triangular inequality, $d(w,v) \leq 40$, and so, by the second formula in (4), $w = v$. The situation in the vertical direction is similar.

We are going to show that the conjunction of formulae in $\{\neg(\chi_{0,0} \rightarrow \bot)\} \cup \Gamma$ is satisfiable in $\boldsymbol{K}$ if and only if $\mathfrak{T}$ can tile $\mathbb{N} \times \mathbb{N}$. This will be done in two steps.

LEMMA 2.3. *If $\mathfrak{T}$ tiles $\mathbb{N} \times \mathbb{N}$, then $\{\neg(\chi_{0,0} \rightarrow \bot)\} \cup \Gamma$ is satisfiable in the 2-dimensional Euclidean space $\langle \mathbb{R}^2, d_2 \rangle$.*

PROOF. Suppose $\tau : \mathbb{N} \times \mathbb{N} \rightarrow \mathfrak{T}$ is a tiling. For $r \in \mathbb{R}^2$, put

$$S(r) = \{y \in \mathbb{R}^2 : d_2(r,y) \leq 9\}.$$

Define a model $\mathfrak{B}$ on $\langle \mathbb{R}^2, d_2 \rangle$ by taking, for $i, j \leq 4$ and $k \leq l$,

$$p_i^{\mathfrak{B}} = \bigcup_{m,n \in \mathbb{N}} S(20(5n+i), 20m),$$

$$q_j^{\mathfrak{B}} = \bigcup_{m,n \in \mathbb{N}} S(20n, 20(5m+j)),$$

$$r_k^{\mathfrak{B}} = \bigcup_{\tau(n,m) = T_k} S(20n, 20m).$$

It is not difficult to see that this model satisfies $\{\neg(\chi_{0,0} \rightarrow \bot)\} \cup \Gamma$; see Figure 2.1. □

LEMMA 2.4. *Suppose that a model $\mathfrak{B}$ based on a space $\langle W, d \rangle \in \mathfrak{D}^t$ satisfies the conjunction of $\{\neg(\chi_{0,0} \rightarrow \bot)\} \cup \Gamma$. Then there exists a function $f : \mathbb{N} \times \mathbb{N} \rightarrow W$ such that, for all $i, j \leq 4$ and $k_1, k_2 \in \mathbb{N}$,*

(a) $\langle \mathfrak{B}, f(5k_1 + i, 5k_2 + j) \rangle \models \chi_{i,j}$;

(b) $d(f(k_1,k_2), f(k_1+1,k_2)) \leq 20$ *and* $d(f(k_1+1,k_2), f(k_1,k_2)) \leq 20$;

(c) $d(f(k_1,k_2), f(k_1,k_2+1)) \leq 20$ *and* $d(f(k_1,k_2+1), f(k_1,k_2)) \leq 20$.

[2]The first conjunct in the right hand sides of (6) and (7) is redundant if $\boldsymbol{K}$ consists of symmetric spaces only.

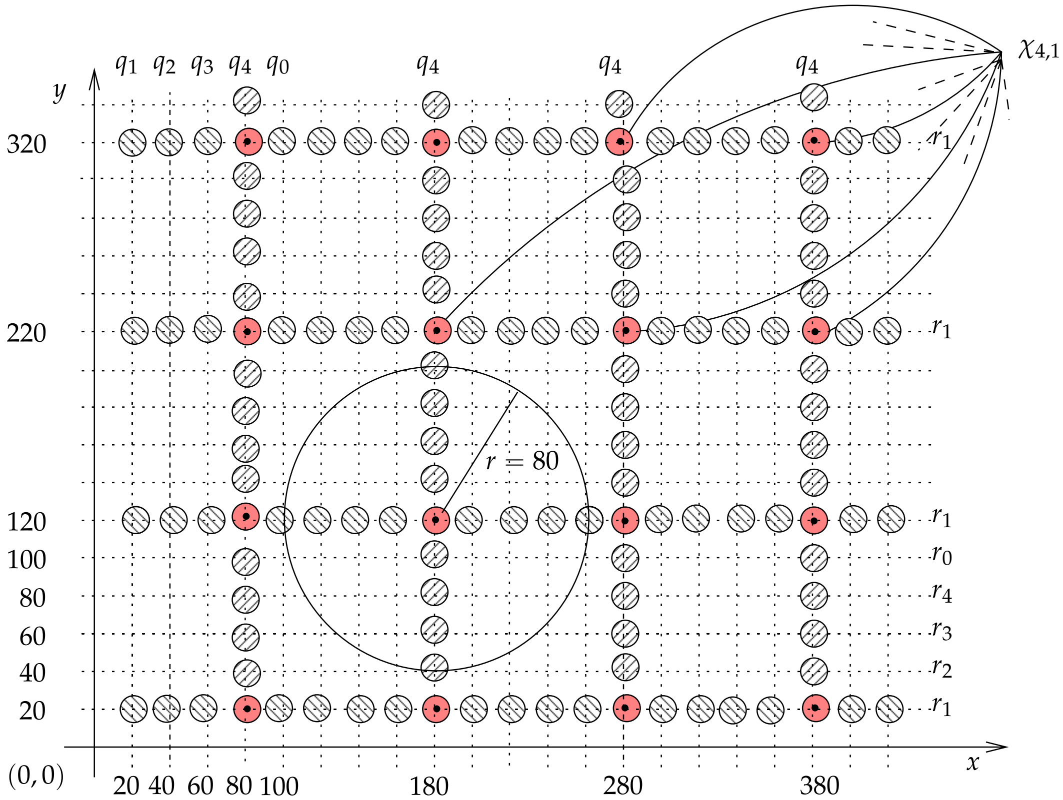


Figure 2.1: Building the grid.

The map $\tau : \mathbb{N} \times \mathbb{N} \to \mathfrak{T}$ *defined by taking*

$$\tau(n, m) = T_k :\Longleftrightarrow \langle \mathfrak{B}, f(n, m)\rangle \models r_k, \textit{ for all } k \leq l \textit{ and all } n, m \in \mathbb{N},$$

is a tiling.

PROOF. We define f inductively. Pick some $f(0,0)$ with $\langle \mathfrak{B} f(0,0)\rangle \models \chi_{0,0}$. By (6), we can then find a sequence $w_n \in W$, for $n \in \mathbb{N}$, such that

- $w_0 = f(0,0)$,
- $\langle \mathfrak{B}, w_{5k+i}\rangle \models \chi_{i,0}$ for all $i \leq 4$ and $k \in \mathbb{N}$,
- $d(w_n, w_{n+1}) \leq 20$ and $d(w_{n+1}, w_n) \leq 20$.

We put $f(n,0) = w_n$ for all $n \in \mathbb{N}$. Similarly, by (7), we find a sequence v_n, for $n \in \mathbb{N}$, such that

- $v_0 = f(0,0)$,
- $\langle \mathfrak{B}, v_{5k+j}\rangle \models \chi_{0,j}$ for all $j \leq 4$ and $k \in \mathbb{N}$,
- $d(v_n, v_{n+1}) \leq 20$ and $d(v_{n+1}, v_n) \leq 20$.

Put $f(0, m) = v_m$ for all $m \in \mathbb{N}$. Suppose now that f has been defined for all (m', n') with $m' + n' < m + n$ so that it satisfies conditions (a)–(c). Without loss of generality we can assume that $n = 5k_1$, $m = 5k_2 + 1$, for some $k_1, k_2 \in \mathbb{N}$. Then $\langle \mathfrak{B}, f(n, m-1)\rangle \models \chi_{0,0}$, and hence $\langle \mathfrak{B}, f(n, m-1)\rangle \models \mathsf{E}^{\leq 20}\chi_{0,1}$. So we can find a $w' \in W$ with $\langle \mathfrak{B}, w'\rangle \models \chi_{0,1}$ and such that $d(f(n, m-1), w') \leq 20$ and $d(w', f(n, m-1)) \leq 20$. We then put $f(n, m) = w'$. It remains to prove that f still satisfies (a)–(c). To this end it suffices to show that $d(f(n-1, m), w') \leq 20$ and $d(w', f(n-1, m)) \leq 20$. We have $\langle \mathfrak{B}, f(n-1, m)\rangle \models \chi_{4,1}$, and so there exists a $w'' \in W$ with $\langle \mathfrak{B}, w''\rangle \models \chi_{0,1}$ and such that $d(f(n-1, m), w'') \leq 20$ and $d(w'', f(n-1, m)) \leq 20$. Thus it is enough to show that $w' = w''$. Suppose otherwise. Then we have

- $d(w'', f(n-1, m)) \leq 20$;
- $d(f(n-1), m), f(n-1, m-1)) \leq 20$;
- $d(f(n-1, m-1), f(n, m-1)) \leq 20$;
- $d(f(n, m-1), w') \leq 20$.

By the triangular inequality, it follows that $d(w'', w') \leq 80$, contrary to the second formula in (4). It is readily seen now that τ is a tiling. □

This completes the proof of Theorem 2.2. □

2.3. A Decidable Logic of Metric Spaces: $\mathcal{MS}_D$

Our main concern in this section is to establish that, when interpreted in the class $\mathcal{MS}$ of all metric spaces, $\mathcal{LO}_D[\mathbb{R}^+]$ has the finite model property (FMP), i.e., every satisfiable formula of $\mathcal{LO}_D[\mathbb{R}^+]$ is satisfiable in a finite metric space, and that satisfiability in $\mathcal{LO}_D[\mathbb{Q}^+]$ is decidable over the class $\mathcal{MS}$. However, in Proposition 1.16 (i) we have shown that $\mathcal{LO}_D[\mathbb{Q}^+]$ and $\mathcal{L}_D[\mathbb{Q}^+]$ are structurally equally expressive over the class of all distance spaces, i.e., for every formula φ of $\mathcal{LO}_D$ there is a formula $\varphi^\dagger$ of $\mathcal{L}_D$ such that φ is satisfiable in a point w of a full model based on a distance space $\langle W, d\rangle$ if and only if $\varphi^\dagger$ is satisfiable in w in a nominal-free model based on $\langle W, d\rangle$. We will therefore, for purely technical reasons, work with the language $\mathcal{L}_D$, i.e., assume that no nominals i_k or the universal modality occur in formulae—thus proving the results first for the language $\mathcal{L}_D$ and then transfer them to the language $\mathcal{LO}_D$ via the following simple corollary.

COROLLARY 2.5. *For $i \in \{d, s, t, m\}$, the following hold:*

(1) *$\mathcal{LO}_D[\mathbb{R}^+]$ has the* FMP *over the class $\mathcal{D}^i$ if and only if $\mathcal{L}_D[\mathbb{R}^+]$ has the* FMP *over the class $\mathcal{D}^i$.*

(2) *Satisfiability of $\mathcal{LO}_D[\mathbb{Q}^+]$-formulae in $\mathcal{D}^i$ is decidable if and only if satisfiability of $\mathcal{L}_D[\mathbb{Q}^+]$-formulae in $\mathcal{D}^i$ is decidable.*

PROOF. By Proposition 1.16 (i). □

But before turning to the details of the proof, we give a relational representation of metric spaces that enables us to use tools and techniques from standard modal logic.

2.3.1. Frame Representation for $\mathcal{L}_D[M]$. As we have two kinds of distance operators in the language $\mathcal{L}_D[\mathbb{R}^+]$, namely, $\mathsf{A}^{\leq a}$ and $\mathsf{A}^{>a}$, the frame-companions of distance spaces $S = \langle W, d \rangle$, as defined in Definition 1.11, are structures of the form

$$\mathfrak{f}_D(S) = \langle W, (R_{\leq a})_{a \in M}, (R_{>a})_{a \in M} \rangle, \tag{8}$$

where $(R_{\leq a})_{a \in \mathbb{R}^+}$ and $(R_{>a})_{a \in \mathbb{R}^+}$ are two families of binary relations on W defined by $uR_{\leq a}v \iff d(u,v) \leq a$ and $uR_{>a}v \iff d(u,v) > a$ for all $a \in \mathbb{R}^+$ and $u, v \in W$. Moreover, in Proposition 1.12 we have shown that for every distance space model $\mathfrak{B} = \langle W, d, p_0^{\mathfrak{B}}, \dots \rangle$ and its frame-companion model $\mathfrak{M} = \langle \mathfrak{f}_D(S), p_0^{\mathfrak{B}}, \dots \rangle$ we have, for all formulae φ in $\mathcal{L}_D[\mathbb{R}^+]$ and points $w \in W$:

$$\langle \mathfrak{B}, w \rangle \models \varphi \iff \langle \mathfrak{f}_D(S), w \rangle \models \varphi. \tag{9}$$

We can, therefore, understand the relations $uR_{\leq a}v$ and $uR_{>a}v$ as stating that 'v is at most a (units) away from u' and 'v is more than a (units) away from u', respectively.

Our next aim is to describe in a more analytic way a suitable class of Kripke frames comprising all frame-companions of metric spaces for the language $\mathcal{L}_D$, where by 'suitable' we mean a class of frames that suffices to characterise theoremhood in the language $\mathcal{L}_D$. Let us impose a number of restrictions on the accessibility relations and say that an M-frame $\mathfrak{f}$ of the form

$$\mathfrak{f} = \langle W, (R_{\leq a})_{a \in M}, (R_{>a})_{a \in M} \rangle \tag{10}$$

is D-**metric**, if the following conditions hold for all $a, b \in M$ and $w, u, v \in W$:

- (D1) $R_{\leq a} \cup R_{>a} = W \times W$;
- (D2) $R_{\leq a} \cap R_{>a} = \varnothing$;
- (D3) If $uR_{\leq a}v$ and $a \leq b$, then $uR_{\leq b}v$;
- (D4) $uR_{\leq 0}v \iff u = v$;
- (D5) $uR_{\leq a}v \iff vR_{\leq a}u$;
- (D6) If $uR_{\leq a}v$ and $vR_{\leq b}w$, then $uR_{\leq a+b}w$, whenever $a + b \in M$.

Properties (D4), (D5) and (D6) reflect Axioms (1)–(2) of metric spaces. Thus, we call an M-frame $\mathfrak{f}$ D-**standard**, if it satisfies (D1)–(D4); we call it D-**symmetric**, if it satisfies (D1)–(D5); and we call it D-**triangular**, if it satisfies (D1)–(D4) and (D6).

Note that all D-metric frames satisfy the additional properties:

(D7) If $uR_{\leq a}v$ and $uR_{>a+b}w$ then $vR_{>b}w$, whenever $a + b \in M$;

(D8) If $uR_{>a}v$ and $a \geq b$, then $uR_{>b}v$;

(D9) $uR_{>a}v \iff vR_{>a}u$;

(D10) $uR_{\leq a}v$ and $uR_{>a}w$, then $vR_{>0}w$,

where (D7) is a consequence of (D1), (D2) and (D6); (D8) follows from (D1), (D2) and (D3), (D9) from (D1), (D2) and (D5), and (D10) from (D1), (D2) and (D4). Thus, all D-standard frames also satisfy (D8) and (D10), all D-triangular frames also satisfy (D7), (D8) and (D10), and all D-symmetric frames also satisfy (D8), (D9) and (D10). Note also that condition (D10) is the special case of condition (D7) for $b = 0$.

We shall denote by $\mathcal{F}^d[M]$, $\mathcal{F}^s[M]$, $\mathcal{F}^t[M]$ and $\mathcal{F}^m[M]$ the **classes** of all D-standard, D-symmetric, D-triangular, and D-metric M-frames, respectively.

The reason why we will sometimes assume these 'redundant' conditions is that condition (D2) is not definable in the language $\mathcal{L}_D$ when interpreted on Kripke-frames, namely we have the following:

PROPOSITION 2.6 (UNDEFINABILITY IN $\mathcal{L}_D$).
There is no set Φ of $\mathcal{L}_D[M]$-formulae such that, for all M-frames $\mathfrak{f}$, we would have

$$\mathfrak{f} \models \Phi \quad \iff \quad \mathfrak{f} \textit{ satisfies condition } \text{(D2)}.$$

PROOF. By the Goldblatt-Thomason Theorem, a first-order definable class of frames is modally definable, that is, characterised by the validity of some set Φ of modal formulae, if and only if it is closed under taking p-morphic images, generated subframes, disjoint unions and reflects ultrafilter extensions, cf. Goldblatt and Thomason [1974] or Goldblatt [1993].

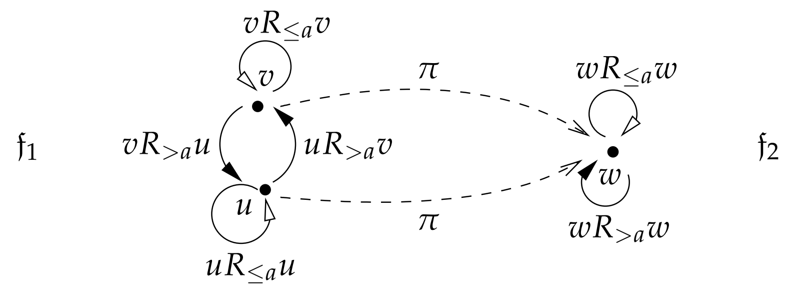


Figure 2.2: (D2) is not definable on frames.

Hence, to prove the lemma, it suffices to give a frame $\mathfrak{f}_1$ satisfying (D2) and a surjective p-morphism $\pi : \mathfrak{f}_1 \longrightarrow \mathfrak{f}_2$ such that (D2) does not hold in $\mathfrak{f}_2$.

Figure 2.2 illustrates two such frames. The M-frame $\mathfrak{f}_1$ in Figure 2.2 (where a ranges over M) clearly satisfies (D2). The depicted map π is obviously a p-morphism from $\mathfrak{f}_1$ onto the M-frame $\mathfrak{f}_2$, but $\mathfrak{f}_2$ does not satisfy (D2). □

In fact, $\mathfrak{f}_1$ satisfies all the properties (D1)–(D10), which means that none of the classes $\mathcal{F}^d$, $\mathcal{F}^s$, $\mathcal{F}^t$, and $\mathcal{F}^m$ is $\mathcal{L}_D[M]$-definable. Basically, this is due to the fact that we treat all our distance operators as standard modal operators. If we would interpret $\mathsf{E}^{>0}$ as a logical modality, namely as the difference operator, then we could simulate nominals and define the universal modality, and thus define (D2). This would, however, not simplify the proof of the finite model property, as will be clear from the proof below.

It is easily checked that, for every metric space $S = \langle W, d\rangle$ (distance space, symmetric space, triangular space) and parameter set M, the frame-companion $\mathfrak{f}_D(S)$ is a D-metric M-frame (D-standard, D-symmetric, D-triangular). Moreover, we have the following representation result.

THEOREM 2.7 (REPRESENTATION OF METRIC SPACES FOR $\mathcal{L}_D$).
(i) For every finite parameter set M and D-metric M-frame $\mathfrak{f}$ (D-standard, D-symmetric, D-triangular) there is a metric space S (distance space, symmetric space, triangular space) such that $\mathfrak{f}$ is its frame-companion, i.e., $\mathfrak{f} = \mathfrak{f}_{D,M}(S)$. In particular, if $\mathfrak{f}$ is finite, so is S.

(ii) For an arbitrary parameter set M we have: An $\mathcal{L}_D[M]$-formula φ is satisfiable in a metric space model (distance space, symmetric space, triangular space) based on a set W if and only if it is satisfiable in a model based on a D-metric M-frame (D-standard, D-symmetric, D-triangular) based on W.

PROOF. Let us first prove (i). Let $M = \{0, a_1, \ldots, a_{n-2}, \gamma\}$ be a finite parameter set with $\gamma = max(M)$ and let

$$\mathfrak{f} = \langle W, (R_{\leq a})_{a\in M}, (R_{>a})_{a\in M}\rangle .$$

Let

$$D := \{a_i + a_j - \gamma \mid a_i + a_j > \gamma \text{ and } a_i, a_j \in M\}$$

and set $\varepsilon := \min(D \cup \{1\}) > 0$. The definition of ε guarantees the following property:

(11) $$\text{If } a, b \in M \text{ and } a + b < \gamma + \varepsilon, \text{ then } a + b \leq \gamma.$$

Thus, by (11), $a, b \in M$ and $a + b < \gamma + \varepsilon$ implies $a + b \in M$, since M is a parameter set. Define a function $d : W \times W \longrightarrow \mathbb{R}^+$ by setting for all $u, v \in W$:

$$d(u, v) = \min(\{\gamma + \varepsilon\} \cup \{a \in M \mid uR_{\leq a}v\}).$$

As M is finite, d is well-defined. Next, we show that if $\mathfrak{f}$ is D-standard then d is a distance function. If $\mathfrak{f}$ is D-symmetric, then d satisfies symmetry, and if $\mathfrak{f}$ is D-triangular, then d satisfies the triangular inequality. Thus, if $\mathfrak{f}$ is D-metric, then, indeed, d is a metric. Clearly, the range of d is $M \cup \{\gamma + \varepsilon\}$.

(a): $d(u, v) = 0$ if and only if $uR_{\leq 0}v$ if and only if $u = v$, by condition (D4).

(b): We assume condition (D5) of D-symmetric frames: suppose $d(u, v) = a$. If we have $a = \gamma + \varepsilon$ then $\neg uR_{\leq b}v$ for all $b \in M$, and so, by condition (D5), $\neg vR_{\leq b}u$ for all $b \in M$, hence $d(v, u) = \gamma + \varepsilon$. Similarly, if $a = \min(\{b \in M \mid uR_{\leq b}v\})$ and so $d(u, v) = a$, then $vR_{\leq a}u$ by (D5), and $\neg vR_{\leq c}u$ for all $c < a$, so $d(v, u) = a$.

(c): We assume condition (D6) of D-triangular frames: suppose $d(u, v) = a$ and $d(v, w) = b$. We have to show that $d(u, w) \leq a + b$. Suppose first that $a + b \geq \gamma + \varepsilon$. Then, since the range of d is $M \cup \{\gamma + \varepsilon\}$, we have $d(u, w) \leq a + b$.

So we may assume that $d(u, v) = a \leq \gamma$, $d(v, w) = b \leq \gamma$, and $a + b < \gamma + \varepsilon$. By (11) we have $a + b \in M$. By condition (D6) it follows that $uR_{\leq a+b}w$. Thus, by the definition of d, we have $d(u, w) \leq a + b$.

Thus we can define a distance space $S := \langle W, d\rangle$. It remains to prove that, indeed, $\mathfrak{f}_{D,M}(S) = \mathfrak{f}$. To this end, we have to show that

(A) $d(u, v) \leq a \iff uR_{\leq a}v$, for all $a \in M$, and
(B) $d(u, v) > a \iff uR_{>a}v$, for all $a \in M$.

Let us first prove (A): Suppose $d(u, v) \leq a$. Then, by the definition of d, there is a $b \in M$ with $b \leq a$ such that $uR_{\leq b}v$. By condition (D3) we obtain $uR_{\leq a}v$. Conversely, if $uR_{\leq a}v$, then $d(u, v) \leq a$ by definition of d.

To prove (B), assume first that $d(u, v) > a$. Then we have $\neg uR_{\leq a}v$ by definition of d. By condition (D1) we obtain $uR_{>a}v$. Conversely, if $uR_{>a}v$ then $\neg uR_{\leq a}v$ by condition (D2). Then, by condition (D3), we have $\neg uR_{\leq b}v$ for all $b \leq a$. Thus, $d(u, v) > a$.

Note that the proof does not depend on whether or not the set W is finite, but only on the finiteness of M.

We can now proceed to prove (ii). One implication is immediate. Suppose that φ is satisfied in the distance space model $\mathfrak{B} = \left\langle W, d, p_0^{\mathfrak{B}}, p_1^{\mathfrak{B}}, \ldots, i_0^{\mathfrak{B}}, i_1^{\mathfrak{B}}, \ldots\right\rangle$ based on the distance space $S = \langle W, d\rangle$, i.e., that $\langle \mathfrak{B}, w\rangle \models \varphi$ for some point $w \in W$. By Proposition 1.12, we have that φ is satisfied in the frame-companion model $\mathfrak{M}_D(\mathfrak{B})$. Moreover, it is easily checked that the frame $\mathfrak{f}_D(S)$ underlying $\mathfrak{M}_D(\mathfrak{B})$ is D-metric, i.e. satisfies properties (D1)–(D6), if S was a metric space, that it is D-symmetric, if S was a symmetric space, etc.

Conversely, assume φ is satisfiable in the D-standard M-frame $\mathfrak{f}$. We first define a finite parameter set $M(\varphi)$ and a D-standard $M(\varphi)$-frame $\mathfrak{f}^{\dagger}$, such that φ is satisfiable in $\mathfrak{f}$ if and only if it is satisfiable in $\mathfrak{f}^{\dagger}$.

Let

$$Par(\varphi) := \{a \in M \mid a \text{ occurs in } \varphi\}$$

and let

$$\gamma := max(Par(\varphi)) + 1.$$

Then define $M(\varphi)$ as follows:

$$M(\varphi) := \{a \in M \mid a = a_1 + \cdots + a_n < \gamma,\ a_1, \ldots, a_n \in Par(\varphi),\ n < \omega\}.$$

Clearly, $M(\varphi)$ is a parameter set. For $0 \in M(\varphi)$ and if $a, b, c \in M(\varphi)$ and $a + b < c < \gamma$, then $a = a_1 + \ldots + a_k$ and $b = b_1 + \cdots + b_l$ with $a_i, b_j \in Par(\varphi)$, $a + b \in M$, and so $a + b \in M(\varphi)$. Further, $M(\varphi)$ is finite. For if μ is the smallest positive number in $Par(\varphi)$, then the number k of summands in a sum $a_1 + \ldots + a_k$ is bounded by $k \cdot \mu < \gamma$. Thus $|M(\varphi)| \leq 2 + |Par(\varphi)|^{\nu}$, where ν is the smallest natural number greater than $\frac{\gamma}{\mu}$.

Now we define the frame $\mathfrak{f}^{\dagger}$ as the frame-reduct of $\mathfrak{f}$ with respect to $M(\varphi)$, that is

$$\mathfrak{f}^{\dagger} := \mathfrak{f} \downharpoonright_{(D,M(\varphi))} .$$

As remarked on Page 21, since $\varphi \in \mathcal{L}_D[M(\varphi)]$, φ is satisfiable in $\mathfrak{f}$ if and only if it is satisfiable in $\mathfrak{f} \downharpoonright_{(D,M(\varphi))}$. By (i) there is a distance space S such that $\mathfrak{f}^{\dagger}$ is its frame-companion, i.e., $\mathfrak{f}_{D,M(\varphi)}(S) = \mathfrak{f} \downharpoonright_{(D,M(\varphi))}$. By Proposition 1.12, φ is satisfiable in S, which had to be shown.

□

In the next section we are going to proof that every satisfiable $\mathcal{L}_D[M]$ formula φ is satisfiable in a finite D-metric M'-frame, where $M' \subseteq M$ is finite, thus proving the finite frame property with respect to D-metric frames. Note that in the presence of the difference operator the finite frame property is, in general, not equivalent to the finite model property [de Rijke, 1992]. The finite model property with respect to metric spaces, however, is an immediate corollary by Theorem 2.7. Further, we can regard the class of all D-metric (D-triangular etc.) M-frames as a relational representation of metric (triangular etc.) spaces in the sense that a formula is satisfiable in a metric (triangular etc.) space model if and only if it is satisfiable in a D-metric (D-triangular etc.) M-frame. However, this correspondence does not carry over to infinite sets of formulae, which will be discussed in more detail in Section 3.3, where we investigate the compactness property. Note also that the technique used in the proof of Theorem 2.7 (i) does not apply to models with an infinite number of relations, that is, where the parameter set M is infinite.

2.3.2. The Finite Model Property. In this section we prove the following theorem:

THEOREM 2.8 (FINITE MODEL PROPERTY OF $\mathcal{L}_D$ IN METRIC SPACES).
An $\mathcal{L}_D[\mathbb{R}^+]$-formula φ is satisfiable in a metric space model if and only if it is satisfiable in a finite metric space model of size $f(|\varphi|)$, where $f(|\varphi|)$ is effectively computable.

PROOF. The theorem follows from Proposition 1.12, Theorem 2.7 and the finite frame property of $\mathcal{L}_D[\mathbb{R}^+]$ with respect to D-metric $\mathbb{R}^+$-frames, Theorem 2.9 to be proved below. □

THEOREM 2.9 (FINITE FRAME PROPERTY). *An $\mathcal{L}_D[\mathbb{R}^+]$-formula φ is satisfiable in a D-metric $\mathbb{R}^+$-frame if and only if it is satisfiable in a finite D-metric $\mathbb{R}^+$-frame. In particular, φ is satisfiable in a finite D-metric $M(\varphi)$-frame, where $M(\varphi) \subset \mathbb{R}^+$ is finite, whenever φ is satisfiable.*

PROOF. We first outline the idea of the proof which consists of three steps. Suppose $\varphi \in \mathcal{L}_D[\mathbb{R}^+]$ is satisfiable in some model $\mathfrak{M}$ based on a D-metric $\mathbb{R}^+$-frame $\mathfrak{f}$.

Step 1. We replace the D-metric $\mathbb{R}^+$-frame $\mathfrak{f}$ by a D-metric $M(\varphi)$-frame $\mathfrak{f}^\dagger$ such that φ is satisfiable in a model $\mathfrak{M}^\dagger$ based on $\mathfrak{f}^\dagger$ and $M(\varphi)$ is a finite parameter set.

Step 2. The next step is to filtrate the model $\mathfrak{M}^\dagger$ through some suitable set $cl(\varphi)$ of formulae of $\mathcal{L}_D[M(\varphi)]$ (see, e.g., Chagrov and Zakharyaschev [1997]). The set $cl(\varphi)$ is a closure of the set $SF(\varphi)$ of subformulae of φ under rules similar to those of the Fischer–Ladner closure for PDL-formulae (cf. Harel [1984]). As a result of the filtration we get a *finite* model $\mathfrak{M}^f$ in which φ is satisfiable and which is based on a frame $\mathfrak{f}^f$ that satisfies all properties of D-metric frames except possibly (D2), and additionally (D7)–(D9).

Step 3. Since $\mathfrak{f}^f$ does not necessarily satisfy condition (D2), there may exist a $v \in W^f$ such that $wR^f_{\leq a}v$ and $wR^f_{>a}v$, for some $w \in W^f$ and $a \in M(\varphi)$. To 'cure' these defects, we make copies of such 'bad' points v and modify the relations $R^f_{\leq a}$ and $R^f_{>a}$ in $\mathfrak{f}^f$ obtaining a finite D-metric $M(\varphi)$-frame $\mathfrak{f}^*$ in which φ is satisfiable. Here, we need the additional property (D7) satisfied by the frame $\mathfrak{f}^f$ to establish the triangular inequality for $\mathfrak{f}^*$, i.e. (D6). (The 'copying-method' was developed by the Bulgarian school of modal logic; see Gargov et al. [1988], Vakarelov [1991]. Our technique follows Goranko [1990a].) Finally, we turn the frame $\mathfrak{f}^*$ into a D-metric $\mathbb{R}^+$-frame $\mathfrak{f}^\ddagger$, in which φ is satisfiable.

Let us now turn to technical details. Suppose φ is satisfiable in a model $\mathfrak{M}$ based on the D-metric $\mathbb{R}^+$-frame $\mathfrak{f} = \langle W, (R_{\leq a})_{a\in\mathbb{R}^+}, (R_{>a})_{a\in\mathbb{R}^+}\rangle$, i.e. $\langle\mathfrak{M}, w\rangle \models \varphi$ for some point $w \in W$.

Step 1. Define $M(\varphi)$ as in the proof of Theorem 2.7 (ii). $M(\varphi)$ is a finite parameter set, γ is the greatest number in $M(\varphi)$, μ is the smallest positive number in $M(\varphi)$, and $M(\varphi)$ is bounded by $|M(\varphi)| \leq 2 + |Par(\varphi)|^{\nu}$, where ν is the smallest natural number greater than $\frac{\gamma}{\mu}$.

Further, define $\mathfrak{f}^{\dagger} := \mathfrak{f} \upharpoonright_{(D,M(\varphi))}$ as the frame-reduct of $\mathfrak{f}$ with respect to $M(\varphi)$. Then for all $\psi \in \mathcal{L}_D[M(\varphi)]$ and points $u \in W$ we have

$$\left\langle \mathfrak{f}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \ldots, u \right\rangle \models \psi \iff \left\langle \mathfrak{f}^{\dagger}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \ldots, u \right\rangle \models \psi.$$

In particular, we have $\langle \mathfrak{f}^{\dagger}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \ldots, w \rangle \models \varphi$. Denote the model $\langle \mathfrak{f}^{\dagger}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \ldots \rangle$ by $\mathfrak{M}^{\dagger}$.

Step 2. Let $SF(\varphi)$ be the set of all subformulae of φ. Define the **closure** $cl(\varphi)$ of $SF(\varphi)$ as the smallest set T of formulae of $\mathcal{L}_D[M(\varphi)]$ such that $SF(\varphi) \subseteq T$ and

(C1) T is closed under subformulae.
(C2) If $\psi \in T$, then $\mathsf{A}^{\leq 0}\psi \in T$ whenever ψ is not of the form $\mathsf{A}^{\leq 0}\chi$.
(C3) If $\mathsf{A}^{\leq a}\psi \in T$ and $a \geq a_1 + \cdots + a_n$, $a_i \in M(\varphi) - \{0\}$, then $\mathsf{A}^{\leq a_1} \ldots \mathsf{A}^{\leq a_n}\psi \in T$.
(C4) If $\mathsf{A}^{>a}\psi \in T$ and $b \in M(\varphi)$, then $\neg\mathsf{A}^{\leq b}\neg\mathsf{A}^{>a}\psi \in T$.
(C5) If $\mathsf{A}^{>a}\psi \in T$ and $b > a$ ($b \in M(\varphi)$), then $\mathsf{A}^{>b}\psi \in T$ and if $a + b \in M(\varphi)$ ($b \in M(\varphi) - \{0\}$), then $\neg\mathsf{A}^{>a+b}\neg\mathsf{A}^{>a}\psi \in T$.

Next, we show that $cl(\varphi)$ is a finite set.

LEMMA 2.10. $|cl(\varphi)| \leq S(\varphi) = 2^{\nu+3} \cdot |SF(\varphi)| \cdot |M(\varphi)|^{2\nu+1}$.

PROOF. Observe that $cl(\varphi)$ can be obtained from $SF(\varphi)$ step-by-step as follows:

First, take the closure of $SF(\varphi)$ under subformulae and (C5) and denote the result by $cl_1(\varphi)$. Second, take the closure of $cl_1(\varphi)$ under subformulae and (C4) and denote the result by $cl_2(\varphi)$, which is still closed under (C5). Third, take the closure of $cl_2(\varphi)$ under subformulae and (C3), denote the result by $cl_3(\varphi)$ and notice that $cl_3(\varphi)$ is closed under (C4) and (C5). Finally, take the closure of $cl_3(\varphi)$ under (C2). This is closed under (C1)–(C5).

The following is now readily checked:

- $|cl_1(\varphi)|$ is bounded by $|SF(\varphi)| \cdot 2^{\nu} \cdot |M(\varphi)|^{\nu}$, because the introduced formulae are of the form $(\neg)\mathsf{A}^{>a_1}(\neg)\mathsf{A}^{>a_2}(\neg)\ldots(\neg)\mathsf{A}^{>a_k}t$, with $a_i - a_{i+1} \geq \mu$ and $(\neg)$ marking a possible occurrence of $\neg$. The length k of such sequences of parameters a_i is bounded by ν, because $a_1 \leq \gamma$ and so $k \cdot \mu \leq \gamma$.
- $|cl_2(\varphi)|$ is bounded by $4 \cdot |cl_1(\varphi)| \cdot |M(\varphi)|$.
- $|cl_3(\varphi)|$ is bounded by $|cl_2(\varphi)| \cdot |M(\varphi)|^{\nu}$ because, as shown in the proof of Theorem 2.7 (ii), no chains $a_1 + \ldots + a_n \leq \gamma$ of length $> \nu$ exist in $M(\varphi)$, so no chain $\mathsf{A}^{\leq a_1} \cdots \mathsf{A}^{\leq a_n}$ of length $> \nu$ is introduced when taking the closure under (C3).

- $|cl(\varphi)|$ is bounded by $2 \cdot |cl_3(\varphi)|$.

So we obtain that $|cl(\varphi)|$ is bounded by $S(\varphi) = 2^{\nu+3} \cdot |SF(\varphi)| \cdot |M(\varphi)|^{2\nu+1}$. □

We are now going to filtrate $\mathfrak{M}^\dagger$ through $\Theta = cl(\varphi)$. Define an equivalence relation $\equiv$ on W by taking

$$u \equiv v :\iff \left\langle \mathfrak{M}^\dagger, u \right\rangle \models \psi \text{ iff } \left\langle \mathfrak{M}^\dagger, v \right\rangle \models \psi \text{ for all } \psi \in \Theta.$$

Let $[u] = \{v \in W : u \equiv v\}$. Construct a filtration $\mathfrak{M}^f = \left\langle \mathfrak{f}^f, p_0^{\mathfrak{M}^f}, p_1^{\mathfrak{M}^f}, \dots \right\rangle$ with $\mathfrak{f}^f = \left\langle W^f, (R^f_{\leq a})_{a \in M(\varphi)}, (R^f_{>a})_{a \in M(\varphi)} \right\rangle$ of $\mathfrak{M}^\dagger$ through Θ by taking

- $W^f = \{[u] : u \in W\}$;
- $p_i^{\mathfrak{M}^f} = \{[u] : u \in p_i^{\mathfrak{M}^\dagger}\}$ for $i < \omega$;
- $[u]R^f_{\leq a}[v] :\iff$ for all formulae $\mathsf{A}^{\leq a}\chi \in \Theta$:
  - $\langle \mathfrak{M}^\dagger, u \rangle \models \mathsf{A}^{\leq a}\chi$ implies $\langle \mathfrak{M}^\dagger, v \rangle \models \chi$ and
  - $\langle \mathfrak{M}^\dagger, v \rangle \models \mathsf{A}^{\leq a}\chi$ implies $\langle \mathfrak{M}^\dagger, u \rangle \models \chi$;
- $[u]R^f_{>a}[v] :\iff$ for all formulae $\mathsf{A}^{>a}\chi \in \Theta$:
  - $\langle \mathfrak{M}^\dagger, u \rangle \models \mathsf{A}^{>a}\chi$ implies $\langle \mathfrak{M}^\dagger, v \rangle \models \chi$ and
  - $\langle \mathfrak{M}^\dagger, v \rangle \models \mathsf{A}^{>a}\chi$ implies $\langle \mathfrak{M}^\dagger, u \rangle \models \chi$.

Since Θ is finite, W^f is finite as well. We summarise the properties of $\mathfrak{M}^f$ in the following lemma:

LEMMA 2.11. (1) *For every $\psi \in \Theta$ and every $u \in W$ we have:*

$$\left\langle \mathfrak{M}^\dagger, u \right\rangle \models \psi \iff \left\langle \mathfrak{M}^f, [u] \right\rangle \models \psi.$$

(2) $\mathfrak{f}^f$ *satisfies conditions* (D1) *and* (D3)–(D9) *from Section 2.3.1.*

(3) $\mathfrak{f}^f$ *is finite and* $|W^f| \leq 2^{S(\varphi)}$.

PROOF. Claim (3) follows immediately from the definition of $\mathfrak{M}^f$ as a filtration through $cl(\varphi)$ and the bound for the size of $cl(\varphi)$ proved in Lemma 2.10.

Claim (1) is proved by an easy induction on the construction of ψ.

To prove (2), let us check conditions (D1) and (D3)–(D9).

(D1): We have to show that $R^f_{\leq a} \cup R^f_{>a} = W^f \times W^f$. Let $\neg[u]R^f_{\leq a}[v]$. Then $\neg uR_{\leq a}v$, and so $uR_{>a}v$, since $\mathfrak{f}^\dagger$ satisfies (i). Thus $[u]R^f_{>a}[v]$.

(D3): If $[u]R^f_{\leq a}[v]$ and $a \leq b$ then $[u]R^f_{\leq b}[v]$. Let $[u]R^f_{\leq a}[v]$ and $a < b$, for $b \in M(\varphi)$. Suppose $\langle \mathfrak{M}^\dagger, u \rangle \models \mathsf{A}^{\leq b}\chi$. By condition (C3) in the definition of $\Theta = cl(\varphi)$, $\mathsf{A}^{\leq a}\chi \in \Theta$, and so $\langle \mathfrak{M}^\dagger, u \rangle \models \mathsf{A}^{\leq a}\chi$. Hence $\langle \mathfrak{M}^\dagger, v \rangle \models \chi$. In the same way we can show that $\langle \mathfrak{M}^\dagger, v \rangle \models \mathsf{A}^{\leq b}\chi$ implies $\langle \mathfrak{M}^\dagger, u \rangle \models \chi$.

(D4): $[u]R^f_{\leq 0}[v] \iff [u] = [v]$. The implication ($\Leftarrow$) is obvious. So suppose $[u]R^f_{\leq 0}[v]$. Take some $\psi \in \Theta$ with $\langle \mathfrak{M}^\dagger, u \rangle \models \psi$. Without loss of generality we may assume that ψ is not of the form $\mathsf{A}^{\leq 0}\chi$. Then, by closure condition (C2), $\mathsf{A}^{\leq 0}\psi \in \Theta$ and $\langle \mathfrak{M}^\dagger, u\rangle \models \mathsf{A}^{\leq 0}\psi$. Hence $\langle \mathfrak{M}^\dagger, v\rangle \models \psi$. In precisely the same way one can show that for all $\psi \in \Theta$, $\langle \mathfrak{M}^\dagger, v\rangle \models \psi$ implies $\langle \mathfrak{M}^\dagger, u\rangle \models \psi$. Therefore, $[u] = [v]$.

(D5) and (D9): $[w]R^f_{\leq a}[u] \iff [u]R^f_{\leq a}[w]$ and $[w]R^f_{>a}[u] \iff [u]R^f_{>a}[w]$ hold by definition.

(D6): If $[u]R^f_{\leq a}[v]$ and $[v]R^f_{\leq b}[w]$, then $[u]R^f_{\leq a+b}[w]$, for $(a+b) \in M(\varphi)$. Suppose $\langle \mathfrak{M}^\dagger, u\rangle \models \mathsf{A}^{\leq a+b}\chi$. Then $\mathsf{A}^{\leq a}\mathsf{A}^{\leq b}\chi \in \Theta$ by (C3) and $\langle \mathfrak{M}^\dagger, u\rangle \models \mathsf{A}^{\leq a}\mathsf{A}^{\leq b}\chi$. It follows that $\langle \mathfrak{M}^\dagger, v\rangle \models \mathsf{A}^{\leq b}\chi$, whence $\langle \mathfrak{M}^\dagger, w\rangle \models \chi$. Now suppose that $\langle \mathfrak{M}^\dagger, w\rangle \models \mathsf{A}^{\leq a+b}\chi$. Again, we have $\mathsf{A}^{\leq b}\mathsf{A}^{\leq a}\chi \in \Theta$ by (C3) and $\langle \mathfrak{M}^\dagger, w\rangle \models \mathsf{A}^{\leq b}\mathsf{A}^{\leq a}\chi$. Then $\langle \mathfrak{M}^\dagger, v\rangle \models \mathsf{A}^{\leq a}\chi$, whence $\langle \mathfrak{M}^\dagger, u\rangle \models \chi$.

(D7): If $[u]R^f_{\leq a}[v]$ and $[u]R^f_{>a+b}[w]$, then $[v]R^f_{>b}[w]$, for $a+b \in M(\varphi)$. Suppose that $\langle \mathfrak{M}^\dagger, v\rangle \models \mathsf{A}^{>b}\chi$. Then $\neg\mathsf{A}^{\leq a}\neg\mathsf{A}^{>b}\chi \in \Theta$ by closure condition (C4) and hence $\langle \mathfrak{M}^\dagger, u\rangle \models \neg\mathsf{A}^{\leq a}\neg\mathsf{A}^{>b}\chi$. It follows that $\langle \mathfrak{M}^\dagger, u\rangle \models \mathsf{A}^{>a+b}\chi$ and so $\langle \mathfrak{M}^\dagger, w\rangle \models \chi$. For the other direction suppose $\langle \mathfrak{M}^\dagger, w\rangle \models \mathsf{A}^{>b}\chi$. Then $\langle \mathfrak{M}^\dagger, u\rangle \models \neg\mathsf{A}^{>a+b}\neg\mathsf{A}^{>b}\chi$ and $\neg\mathsf{A}^{>a+b}\neg\mathsf{A}^{>b}\chi \in \Theta$ by (C5). Hence $\langle \mathfrak{M}^\dagger, u\rangle \models \mathsf{A}^{\leq a}\chi$ and so $\langle \mathfrak{M}^\dagger, v\rangle \models \chi$.

(D8): If $[u]R^f_{>a}[v]$ and $a \geq b$ then $[u]R^f_{>b}[v]$. Let $[u]R^f_{>a}[v]$ and $a > b$. Suppose $\langle \mathfrak{M}^\dagger, u\rangle \models \mathsf{A}^{>b}\chi$. Then $\mathsf{A}^{>a}\chi \in \Theta$ by (C5), $\langle \mathfrak{M}^\dagger, u\rangle \models \mathsf{A}^{>a}\chi$, and so $\langle \mathfrak{M}^\dagger, v\rangle \models \chi$. Again, the other direction is treated analogously.

□

Step 3. Unfortunately, $\mathfrak{f}^f$ does not necessarily satisfy condition (D2) which is required to construct a metric space from a relational model. It may happen that for some points $[u]$, $[v]$ in W^f and $a \in M(\varphi)$, we have both $[u]R^f_{\leq a}[v]$ and $[u]R^f_{>a}[v]$. We therefore have to perform some surgery on the model $\mathfrak{M}^f$. The defects form the set

$$D(W^f) = \{v \in W^f \mid \exists a \in M(\varphi)\ \exists u \in W^f\ (uR^f_{\leq a}v \ \&\ uR^f_{>a}v)\}.$$

Let

$$W^* = \{\langle v, i\rangle \mid v \in D(W^f), i \in \{0,1\}\} \cup \{\langle u, 0\rangle : u \in W^f - D(W^f)\}.$$

So for each $v \in D(W^f)$ we now have two copies $\langle v, 0\rangle$ and $\langle v, 1\rangle$. Define a new model $\mathfrak{M}^* = \langle \mathfrak{f}^*, p_0^{\mathfrak{M}^*}, p_1^{\mathfrak{M}^*}, \dots, \rangle$ with $\mathfrak{f}^* = \langle W^*, (R^*_{\leq a})_{a\in M(\varphi)}, (R^*_{>a})_{a\in M(\varphi)}\rangle$ by setting

- $p_i^{\mathfrak{M}^*} = \{\langle u, i\rangle \in W^* \mid u \in p_i^{\mathfrak{M}^f}\}$, for all $i < \omega$,

and by defining accessibility relations $R^*_{\leq a}$ and $R^*_{>a}$ as follows:

- if $a > 0$ then

$$\langle u, i\rangle\ R^*_{\leq a}\ \langle v, j\rangle :\iff \text{ either } (uR^f_{\leq a}v \ \&\ \neg uR^f_{>a}v), \text{ or } (uR^f_{\leq a}v \ \&\ i = j);$$

- if $a = 0$ then
$$\langle u,i\rangle \, R^*_{\le a} \, \langle v,j\rangle :\iff \langle u,i\rangle = \langle v,j\rangle\,;$$
- $R^*_{>a}$ is defined as the complement of $R^*_{\le a}$, i.e.,
$$\langle u,i\rangle \, R^*_{>a} \, \langle v,j\rangle :\iff \neg \langle u,i\rangle \, R^*_{\le a} \, \langle v,j\rangle\,.$$

LEMMA 2.12. *$\mathfrak{f}^* = \left\langle W^*, (R^*_{\le a})_{a\in M(\varphi)}, (R^*_{>a})_{a\in M(\varphi)} \right\rangle$ is a finite D-metric $M(\varphi)$-frame and $|W^*| \le 2\cdot 2^{S(\varphi)}$.*

PROOF. The upper bound for W^* is obvious. That $\mathfrak{f}^*$ satisfies (D1), (D2), and (D4) follows immediately from the definition. Let us check the remaining conditions.

(D3) Suppose that $\langle u,i\rangle \, R^*_{\le a} \, \langle v,j\rangle$ and $b \in M(\varphi)$ is such that $a < b$. If $i = j$ then clearly $\langle u,i\rangle \, R^*_{\le b} \, \langle v,j\rangle$. So assume $i \ne j$. Then, by definition, $uR^f_{\le a}v$ and $\neg uR^f_{>a}v$. Since $\mathfrak{f}^f$ satisfies (D3) and (D8), we obtain $uR^f_{\le b}v$ and $\neg uR^f_{>b}v$. Thus $\langle u,i\rangle \, R^*_{\le b} \, \langle v,j\rangle$.

(D5) follows from the symmetry of $R^f_{\le a}$ and $R^f_{>a}$, conditions (D5) and (D9).

(D6) Suppose $\langle u,i\rangle \, R^*_{\le a} \, \langle v,j\rangle$, $\langle v,j\rangle \, R^*_{\le b} \, \langle w,k\rangle$ and $a + b \in M(\varphi)$. Then $uR^f_{\le a}v$ and $vR^f_{\le b}w$. As $\mathfrak{f}^f$ satisfies (D6), we have $uR^f_{\le a+b}w$. If $i = k$ then clearly $\langle u,i\rangle \, R^*_{\le a+b} \, \langle w,k\rangle$. So assume $i \ne k$. If $i = j \ne k$ then, using (D7) for $\mathfrak{f}^f$, $\neg uR^f_{>a+b}w$, since $uR^f_{\le a}v$ and $\neg vR^f_{>b}w$. The case $i \ne j = k$ is considered analogously using the fact that the relations in $\mathfrak{f}^f$ are symmetric.

□

LEMMA 2.13. *For all $\langle v,i\rangle \in W^*$ and $\psi \in \Theta$, we have:*
$$\langle \mathfrak{M}^*, \langle v,i\rangle\rangle \models \psi \iff \left\langle \mathfrak{M}^f, v\right\rangle \models \psi.$$
In particular, φ is satisfiable in $\mathfrak{M}^*$, i.e., $\langle \mathfrak{M}^*, \langle w,i\rangle\rangle \models \varphi$, $i = 0,1$.

PROOF. The proof is by induction on ψ. The basis of induction and the case of Booleans are trivial. The cases $\psi = \mathsf{A}^{\le a}\chi$ and $\psi = \mathsf{A}^{>a}\chi$ are consequences of the following claims:

Claim 1: If $uR^f_{\le a}v$ and $\langle u,i\rangle \in W^*$ ($i \in \{0,1\}$), then there exists a j such that $\langle u,i\rangle \, R^*_{\le a} \, \langle v,j\rangle$.

Indeed, this is clear for $i = 0$. Suppose $i = 1$. If v was duplicated, then $\langle v,1\rangle$ is as required. If v was not duplicated, then $\neg uR^f_{>a}v$, and so $\langle v,0\rangle$ is as required.

Claim 2: If $\langle u,i\rangle \, R^*_{\le a} \, \langle v,j\rangle$, then $uR^f_{\le a}v$.

This should be obvious.

Claim 3: If $uR^f_{>a}v$ and $\langle u,i\rangle \in W^*$ ($i \in \{0,1\}$), then there exists a j such that $\langle u,i\rangle \, R^*_{>a} \, \langle v,j\rangle$.

Suppose $i = 0$. If v was not duplicated, then $\neg uR^f_{\leq a}v$. Hence $\neg \langle u, 0\rangle\, R^*_{\leq a}\, \langle v, 0\rangle$. If v was duplicated, then $\neg \langle u, 0\rangle\, R^*_{\leq a}\, \langle v, 1\rangle$. In the case of $i = 1$ we have $\neg \langle u, 1\rangle\, R^*_{\leq a}\, \langle v, 0\rangle$, i.e., $\langle u, 1\rangle\, R^*_{>a}\, \langle v, 0\rangle$.

Claim 4: If $\langle u, i\rangle\, R^*_{>a}\, \langle v, j\rangle$, then $uR^f_{>a}v$.
Indeed, if $i = j$ then $\neg uR^f_{\leq a}v$ and so $uR^f_{>a}v$. And if $i \neq j$, then $uR^f_{>a}v$. □

Thus, we have shown that φ is satisfiable in the model $\mathfrak{M}^*$ based on the finite D-metric $M(\varphi)$-frame $\mathfrak{f}^*$. It remains to transform $\mathfrak{f}^*$ into a D-metric $\mathbb{R}^+$-frame $\mathfrak{f}^\ddagger$ in which φ remains satisfiable. By Theorem 2.7 (i), there is a metric space S based on the same set W^* such that $\mathfrak{f}^*$ is its $M(\varphi)$-frame companion, i.e. $\mathfrak{f}_{D,M(\varphi)}(S) = \mathfrak{f}^*$. By Proposition 1.12, φ is satisfiable in S. Take the $\mathbb{R}^+$-frame companion of S, $\mathfrak{f}^\ddagger = \mathfrak{f}_{D,\mathbb{R}^+}(S)$. Then $\mathfrak{f}^\ddagger$ is a D-metric $\mathbb{R}^+$-frame and φ is satisfiable in it.

This completes the proof of Theorem 2.9. □

2.3.3. Decidability. We can now derive the decidability of the satisfiability problem of formulae from the language $\mathcal{L}_D[\mathbb{Q}^+]$ in the class of metric spaces.

THEOREM 2.14 (DECIDABILITY OF $\mathcal{L}_D$ IN METRIC SPACES).
The satisfiability problem for $\mathcal{L}_D[\mathbb{Q}^+]$-formulae in the class $\mathcal{MS}$ of metric spaces is decidable.

PROOF. Let φ be some $\mathcal{L}_D[\mathbb{Q}^+]$-formula. By Theorems 2.9 and 2.7, φ is satisfiable in a metric space if and only if φ is satisfiable in some finite and D-metric $M(\varphi)$-frame $\mathfrak{f}$, where the size of $\mathfrak{f}$ is at most $f(|\varphi|)$ and $M(\varphi)$ is finite. Thus, to decide whether φ is satisfiable in the class $\mathcal{MS}$, we first enumerate all D-metric $M(\varphi)$-frames of size at most $f(|\varphi|)$ (clearly, it is decidable whether a finite frame of size n is a D-metric frame), of which there are only finitely many, and then check whether φ is satisfiable in one of them. □

2.4. Decidable Logics of Non-Metric Distance Spaces

Let us now consider the satisfiability problem in the class $\mathcal{D}^d$ of arbitrary distance spaces and its subclasses $\mathcal{D}^s$ and $\mathcal{D}^t$. For $\mathcal{D}^d$ and $\mathcal{D}^s$ we can prove decidability even for the language $\mathcal{LF}_2[\mathbb{Q}^+]$. For $\mathcal{D}^t$ we will consider the language $\mathcal{L}_D[\mathbb{Q}^+]$.

THEOREM 2.15 (DECIDABILITY OF $\mathcal{LF}_2$ IN NON-METRIC SPACES).
The satisfiability problem for $\mathcal{LF}_2[\mathbb{Q}^+]$-formulae in $\mathcal{D}^d$ and $\mathcal{D}^s$ is decidable. Moreover, both problems are NEXPTIME*-complete (for binary encoding of parameters), and, in both cases, any satisfiable formula is satisfiable in a finite model.*

PROOF. The proof is based on a simple reduction to the satisfiability problem for the two-variable fragment of first-order logic. Recall that atomic formulae $\delta(x, y) < a$

and $\delta(x,y) = a$ can be regarded as binary predicates $P_{<a}(x,y)$ and $P_{=a}(x,y)$. Denote by φ^+ the result of replacing all subformulae in φ of the form $\delta(x,y) < a$ and $\delta(x,y) = a$ by $P_{<a}(x,y)$ and $P_{=a}(x,y)$, respectively. Let

$$0 = a_0 < a_1 < \cdots < a_n$$

be the list of rational numbers that occur in φ, together with 0, and let Γ be the set of the following formulae, for $i \leq n$:

$$\forall x,y\big(P_{=a_i}(x,y) \to \bigwedge_{0\leq j\leq i} \neg P_{<a_j}(x,y) \wedge \bigwedge_{i\neq j} \neg P_{=a_j}(x,y) \wedge \bigwedge_{n\geq j>i} P_{<a_j}(x,y)\big);$$

$$\forall x,y\big(P_{<a_i}(x,y) \to \bigwedge_{i<j\leq n} P_{<a_j}(x,y)\big);$$

$$\forall x,y \neg P_{<0}(x,y);$$

$$\forall x,y\big(P_{=0}(x,y) \leftrightarrow x = y\big).$$

We claim that the set $\Gamma \cup \{\varphi^+\}$ (which is of size polynomial in the size of φ for binary encoding of parameters) is satisfiable in a first-order structure

$$\mathfrak{A} = \left\langle W, P^{\mathfrak{A}}_{=a_0}, \dots, P^{\mathfrak{A}}_{<a_0}, \dots, P^{\mathfrak{A}}_1, \dots, c^{\mathfrak{A}}_1, \dots \right\rangle$$

if and only if φ is satisfiable in a distance space model.

The direction ($\Leftarrow$) is clear. So suppose that $\mathfrak{A}$ satisfies $\Gamma \cup \{\varphi^+\}$. Define a distance space structure

$$\mathfrak{B} = \left\langle W, d, P^{\mathfrak{A}}_1, \dots, c^{\mathfrak{A}}_1, \dots \right\rangle$$

by taking, for $a, b \in W$:

$$d(a,b) := \begin{cases} a_i & \text{if } \mathfrak{A} \models P_{=a_i}(a,b); \\ \frac{a_i + a_{i+1}}{2} & \text{if } \mathfrak{A} \models \neg P_{<a_i}(a,b) \wedge P_{<a_{i+1}}(a,b) \wedge \neg P_{=a_i}(a,b); \\ 2 \cdot a_n & \text{if } \mathfrak{A} \models \neg P_{<a_n}(a,b) \wedge \neg P_{=a_n}(a,b). \end{cases}$$

It is not difficult to see that $\mathfrak{B}$ satisfies φ, we just need to show that d is a well-defined distance function and satisfies, for all $a, b \in W$:

$$d(a,b) < a_i \iff \mathfrak{A} \models P_{<a_i}(a,b) \quad \text{and} \quad d(a,b) = a_i \iff \mathfrak{A} \models P_{=a_i}(a,b).$$

But this is exactly what is guaranteed by $\mathfrak{A} \models \Gamma$.

Hence, to decide whether φ is satisfiable in a distance space model, it suffices to check whether $\Gamma \cup \{\varphi^+\}$ is satisfiable in a first-order structure. This proves the decidability of satisfiability in $\mathcal{D}^d$.

For $\mathcal{D}^s$, we take the set Γ_s which is

$$\Gamma \cup \{\forall x,y\big(P_{<a_i}(x,y) \leftrightarrow P_{<a_i}(y,x)\big),\ \forall x,y\big(P_{=a_i}(x,y) \leftrightarrow P_{=a_i}(y,x)\big) \ \mid\ i \leq n\}.$$

It is readily checked that φ is satisfiable in $\mathcal{D}^s$ if and only if $\Gamma_s \cup \{\varphi^+\}$ is satisfiable.

Since the satisfiability problem for the two-variable fragment with unary predicates only is already NEXPTIME-complete, the satisfiability problem for $\mathcal{LF}_2[\mathbb{Q}^+]$ is clearly NEXPTIME-hard. The remaining claims follow immediately from the NEXPTIME-completeness of the two-variable fragment of first-order logic (with binary relations) and its finite model property, compare Mortimer [1975], Fürer [1984] and Grädel et al. [1997]. □

In Theorem 1.17, we have shown that the first-order language $\mathcal{LF}_2[\mathbb{Q}^+]$ is expressively complete for the modal language $\mathcal{LO}_F[\mathbb{Q}^+]$ over symmetric structures and that $\mathcal{LF}_2[\mathbb{Q}^+]$ is as expressive as $\mathcal{LO}_F[\mathbb{Q}^+]$ over arbitrary distance spaces. We can now use this correspondence to easily derive corresponding upper complexity bounds for $\mathcal{LO}_F$ over symmetric and arbitrary distance spaces.

COROLLARY 2.16. *The satisfiability problem for $\mathcal{LO}_F[\mathbb{Q}^+]$-formulae (and all its sublanguages) is decidable in* NEXPTIME *(for binary encoding of parameters) in the classes D^s and D^d. $\mathcal{LO}_F[\mathbb{Q}^+]$ has the finite model property in the classes D^s and D^d.*

PROOF. Since the translation from $\mathcal{LO}_F[\mathbb{Q}^+]$-formulae to $\mathcal{LF}_2[\mathbb{Q}^+]$-formulae given in Theorem 1.17 is polynomial and satisfiability preserving, $\mathcal{LO}_F[\mathbb{Q}^+]$ satisfiability is decidable in NEXPTIME. Further, $\mathcal{LO}_F[\mathbb{Q}^+]$ has the finite model property over both, arbitrary and symmetric distance spaces, since $\mathcal{LF}_2[\mathbb{Q}^+]$ has the finite model property over these classes by Theorem 2.15. □

We leave it as an open problem whether the complexity of the satisfiability problem for $\mathcal{LO}_F[\mathbb{Q}^+]$-formulae in symmetric and arbitrary distance spaces is NEXPTIME-complete. Similarly to the situation in Boolean modal logic enriched with converse operators and the difference operator, compare Lutz et al. [2001b], it might be the case that satisfiability in $\mathcal{LO}_F[M]$ is EXPTIME-complete for finite M, but NEXPTIME-complete for M infinite, i.e., $M = \mathbb{N}, \mathbb{Q}^+$.

Let us now consider the satisfiability problem in the class $\mathcal{D}^t$ of spaces satisfying the triangular inequality.

THEOREM 2.17 (DECIDABILITY AND FMP OF $\mathcal{L}_D$ IN TRIANGULAR SPACES).
(i) *The satisfiability problem for $\mathcal{L}_D[\mathbb{Q}^+]$-formulae in $\mathcal{D}^t$ is decidable.*

(ii) *Any $\mathcal{L}_D[\mathbb{Q}^+]$-formula φ satisfiable in $\mathcal{D}^t$ is satisfiable in a finite member of $\mathcal{D}^t$ of size at most $g(|\varphi|)$, where $g(|\varphi|)$ is computable.*

PROOF. The proof of (ii) is essentially the same as the proof of Theorem 2.8. Given a formula φ satisfiable in a triangular space, we find a D-triangular $\mathbb{R}^+$-frame in which φ is satisfiable by Proposition 1.12. By Theorem 2.18 to be proved below, $\mathcal{L}_D[\mathbb{R}^+]$ has the finite frame property with respect to D-triangular $\mathbb{R}^+$-frames. In particular, we find a D-triangular $M(\varphi)$-frame in which φ is satisfiable and such that $M(\varphi)$ is a finite parameter set. Finally, by Theorem 2.7 we find a finite triangular space S in which φ is satisfiable.

Claim (i) follows, as in the proof of Theorem 2.14, from the strong finite model property together with the Representation Theorem 2.7. □

Next, we prove the finite frame property of $\mathcal{L}_D[\mathbb{R}^+]$ with respect to D-triangular $\mathbb{R}^+$-frames.

THEOREM 2.18 (FINITE FRAME PROPERTY OF $\mathcal{L}_D$ IN $\mathfrak{F}^t$).
A $\mathcal{L}_D[\mathbb{R}^+]$-formula φ is satisfiable in a D-triangular $\mathbb{R}^+$-frame if and only if it is satisfiable in a finite D-triangular $\mathbb{R}^+$-frame. In particular, φ is satisfiable in a finite D-triangular $M(\varphi)$-frame, where $M(\varphi) \subset \mathbb{R}^+$ is a finite parameter set, whenever φ is satisfiable.

PROOF. The proof is quite similar to that of Theorem 2.9 and proceeds in three steps. Steps 1 is as before. So we may assume that we have a model $\mathfrak{M}$ based on a D-triangular $M(\varphi)$-frame $\mathfrak{f}$ such that $\left\langle \mathfrak{f}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \dots, w \right\rangle \models \varphi$ and $M(\varphi)$ is a finite parameter set.

However, since models are now not necessarily symmetric, two important modifications are required in Steps 2 and 3. One concerns the filtration, another the copying technique:

Step 2. Let $SF(\varphi)$ be the set of all subformulae of φ. Define the **closure** $cl(\varphi)$ of $SF(\varphi)$ as the smallest set T of formulae of $\mathcal{L}_D[M(\varphi)]$ such that $SF(\varphi) \subseteq T$ and

(C1) T is closed under subformulae.
(C3) If $\mathsf{A}^{\leq a}\psi \in T$ and $a \geq a_1 + \cdots + a_n$, $a_i \in M(\varphi) - \{0\}$, then $\mathsf{A}^{\leq a_1} \dots \mathsf{A}^{\leq a_n}\psi \in T$.
(C4) If $\mathsf{A}^{>a}\psi \in T$ and $b \in M(\varphi)$, then $\neg\mathsf{A}^{\leq b}\neg\mathsf{A}^{>a}\psi \in T$.
(C6) If $\mathsf{A}^{>a}\psi \in T$ and $b > a$, for $b \in M(\varphi)$, then $\mathsf{A}^{>b}\psi \in T$.

The closure $cl(\varphi)$ of $SF(\varphi)$ is defined in almost the same way as on page 48; the only differences are that the last condition (C5) has been replaced with the condition (C6), and (C2) is now not longer necessary since we define the relation $R^f_{\leq 0}$ differently:

The filtration of $\mathfrak{M}$ through $\Theta = cl(\Phi)$ is modified in the following way. Define an equivalence relation $\equiv$ on W by taking

$$u \equiv v :\iff \text{ for all } \psi \in \Theta \text{ we have } \langle \mathfrak{M}, u \rangle \models \psi \text{ iff } \langle \mathfrak{M}, v \rangle \models \psi.$$

Let $[u] = \{v \in W : u \equiv v\}$. Construct a filtration $\mathfrak{M}^f = \left\langle \mathfrak{f}^f, p_0^{\mathfrak{M}^f}, \dots \right\rangle$ with the frame $\mathfrak{f}^f = \left\langle W^f, (R^f_{\leq a})_{a \in M(\varphi)}, (R^f_{>a})_{a \in M(\varphi)} \right\rangle$ of $\mathfrak{M}$ through Θ by taking

- $W^f = \{[u] : u \in W\}$;
- $p_i^{\mathfrak{M}^f} = \{[u] : u \in p_i^{\mathfrak{M}}\}$ for $i < \omega$;

- $a > 0$: $[u]R^f_{\leq a}[v] \iff \mathsf{A}^{\leq a}\psi \in \Theta$ and $\langle\mathfrak{M}, u\rangle \models \mathsf{A}^{\leq a}\psi$ implies $\langle\mathfrak{M}, v\rangle \models \psi$;
- $a = 0$: $[u]R^f_{\leq a}[v] \iff [u] = [v]$;
- $a \geq 0$: $[u]R^f_{>a}[v] \iff \mathsf{A}^{>a}\psi \in \Theta$ and $\langle\mathfrak{M}, u\rangle \models \mathsf{A}^{>a}\psi$ implies $\langle\mathfrak{M}, v\rangle \models \psi$.

Since Θ is finite, W^f is finite as well. Note also that we have that $uR_{\leq a}v$ implies $[u]R^f_{\leq a}[v]$, and $uR_{>a}v$ implies $[u]R^f_{>a}[v]$.

LEMMA 2.19.
(1) *For every* $\psi \in \Theta$ *and every* $u \in W$: $\langle\mathfrak{M}, u\rangle \models \psi \iff \langle\mathfrak{M}^f, [u]\rangle \models \psi$.
(2) $\mathfrak{f}^f$ *satisfies (D1), (D3)–(D4) and (D6)–(D8) from Section* 2.3.1, *Page* 42.

PROOF. (1) is proved by an easy induction; To prove (2), we have to check conditions (D1), (D3)–(D4) and (D6)–(D8). The proof of (D1), i.e., $R^f_{\leq a} \cup R^f_{>a} = W^f \times W^f$, is as in Lemma 2.11.

(D3): if $[u]R^f_{\leq a}[v]$ and $a \leq b$ then $[u]R^f_{\leq b}[v]$. Let $[u]R^f_{\leq a}[v]$ and $a < b$ for some $b \in M(\varphi)$. Suppose $\langle\mathfrak{M}, u\rangle \models \mathsf{A}^{\leq b}\psi$. By the definition of Θ, (C3), $\mathsf{A}^{\leq a}\psi \in \Theta$. Thus, since $a < b$, $\langle\mathfrak{M}, u\rangle \models \mathsf{A}^{\leq a}\psi$. Then $[u]R^f_{\leq a}[v]$ implies $\langle\mathfrak{M}, v\rangle \models \psi$, and $[u]R^f_{\leq b}[v]$ follows.

(D4): $[u]R^f_{\leq 0}[v] \iff [u] = [v]$ holds by the definition of $R^f_{\leq 0}$.

(D6): if $[u]R^f_{\leq a}[v]$ and $[v]R^f_{\leq b}[w]$, then $[u]R^f_{\leq a+b}[w]$, for $a + b \in M(\varphi)$. Suppose that we have $\langle\mathfrak{M}, u\rangle \models \mathsf{A}^{\leq a+b}\psi$. Then $\mathsf{A}^{\leq a}\mathsf{A}^{\leq b}\psi \in \Theta$ by (C3) and $\langle\mathfrak{M}, u\rangle \models \mathsf{A}^{\leq a}\mathsf{A}^{\leq b}\psi$. So $\langle\mathfrak{M}, v\rangle \models \mathsf{A}^{\leq b}\psi$, whence $\langle\mathfrak{M}, w\rangle \models \psi$.

(D7): if $[u]R^f_{\leq a}[v]$ and $[u]R^f_{>a+b}[w]$ then $[v]R^f_{>b}[w]$, for $a + b \in M(\varphi)$. Let $\langle\mathfrak{M}, v\rangle \models \mathsf{A}^{>b}\psi$ and $\mathsf{A}^{>b}\psi \in \Theta$. Then we have $\neg\mathsf{A}^{\leq a}\neg\mathsf{A}^{>b}\psi \in \Theta$ by (C4) and $\langle\mathfrak{M}, u\rangle \models \neg\mathsf{A}^{\leq a}\neg\mathsf{A}^{>b}\psi$, for otherwise (since Θ is closed under subterms) $\langle\mathfrak{M}, u\rangle \models \mathsf{A}^{\leq a}\neg\mathsf{A}^{>b}\psi$ together with $[u]R^f_{\leq a}[v]$ would imply $\langle\mathfrak{M}, v\rangle \models \neg\mathsf{A}^{>b}\psi$, which is a contradiction. Suppose that $uR_{>a+b}x$ for some point $x \in W$. Since $\langle\mathfrak{M}, u\rangle \models \neg\mathsf{A}^{\leq a}\neg\mathsf{A}^{>b}\psi$, there is a point $y \in W$ such that $uR_{\leq a}y$ and $\langle\mathfrak{M}, y\rangle \models \mathsf{A}^{>b}\psi$. As $\mathfrak{M}$ satisfies (D7), it follows that $yR_{>b}x$, and so $\langle\mathfrak{M}, x\rangle \models \psi$. Hence $\langle\mathfrak{M}, u\rangle \models \mathsf{A}^{>(a+b)}\psi$, which implies $\langle\mathfrak{M}, w\rangle \models \psi$.

(D8): if $[u]R^f_{>a}[v]$ and $a \geq b$ then $[u]R^f_{>b}[v]$. Let $[u]R^f_{>a}[v]$ and $a > b$ for some $b \in M(\varphi)$. Suppose $\langle\mathfrak{M}, u\rangle \models \mathsf{A}^{>b}\psi$. By the definition of Θ, (C6), $\mathsf{A}^{>a}\psi \in \Theta$. Thus, since $a > b$, $\langle\mathfrak{M}, u\rangle \models \mathsf{A}^{>a}\psi$. Then $[u]R^f_{>a}[v]$ implies $\langle\mathfrak{M}, v\rangle \models \psi$, and $[u]R^f_{>b}[v]$ follows.

□

Step 3. We are now again facing the problem that $\mathfrak{f}^f$ may not satisfy condition (D2). To avoid this problematic case—the situation where for some points $[u]$, $[v]$ in W^f and $a \in M(\varphi)$ both $[u]R^f_{\leq a}[v]$ and $[u]R^f_{>a}[v]$ hold—we modify the copying technique in the following way. The problematic points form the set

$$D(W^f) = \{v \in W^f \mid \exists a \in M(\varphi)\ \exists u \in W^f\ (uR^f_{\leq a}v \ \&\ uR^f_{>a}v)\}.$$

Let

$$W^* = \{\langle v, i\rangle \mid v \in D(W^f),\ i \in \{0,1,2\}\} \cup \{\langle u, 0\rangle \mid u \in W^f - D(W^f)\}.$$

So for each $v \in D(W^f)$ we have now three copies $\langle v, 0\rangle$, $\langle v, 1\rangle$ and $\langle v, 2\rangle$. Define an new model $\mathfrak{M}^* = \left\langle \mathfrak{f}^*, p_0^{\mathfrak{M}^*}, \dots \right\rangle$ with $\mathfrak{f}^* = \left\langle W^*, (R^*_{\leq a})_{a \in M(\varphi)}, (R^*_{>a})_{a \in M(\varphi)} \right\rangle$ by setting

$$p_i^{\mathfrak{M}^*} = \{\langle u, i\rangle \in W^* \mid u \in p_i^{\mathfrak{M}^f}\}, \text{ for all } i < \omega,$$

and by defining accessibility relations $R^*_{\leq a}$ and $R^*_{>a}$ as follows:

- If $a > 0$, then $\langle u, i\rangle\, R^*_{\leq a}\, \langle v, j\rangle :\Longleftrightarrow$ either
  - $uR^f_{\leq a}v$ and $\neg uR^f_{>a}v$, or
  - $uR^f_{\leq a}v$ and $j = 0$, or
  - $\langle u, i\rangle = \langle v, j\rangle$ (then also $uR^f_{\leq a}v$).
- If $a = 0$, then $\langle u, i\rangle\, R^*_{\leq a}\, \langle v, j\rangle :\Longleftrightarrow \langle u, i\rangle = \langle v, j\rangle$.
- $R^*_{>a}$ is defined as the complement of $R^*_{\leq a}$, i.e.,

$$\langle u, i\rangle\, R^*_{>a}\, \langle v, j\rangle :\Longleftrightarrow \neg \langle u, i\rangle\, R^*_{\leq a}\, \langle v, j\rangle .$$

LEMMA 2.20. *The frame $\mathfrak{f}^*$ is a D-triangular $M(\varphi)$-frame, i.e., satisfies conditions (D6) and (D1)–(D4) .*

PROOF. That $\mathfrak{f}^*$ satisfies (D1), (D2), and (D4), follows immediately from the definitions of $R^*_{\leq a}$ and $R^*_{>a}$. Let us check the remaining conditions.

(D3) Suppose $\langle u, i\rangle\, R^*_{\leq a}\, \langle v, j\rangle$ and $a < b$, for $b \in M(\varphi)$. If $\langle u, i\rangle = \langle v, j\rangle$, then $\langle u, i\rangle\, R^*_{\leq b}\, \langle v, j\rangle$ follows immediately from the definition. So assume $\langle u, i\rangle \neq \langle v, j\rangle$. By definition we have $uR^f_{\leq a}v$, and since $\mathfrak{f}^f$ satisfies (D3), $uR^f_{\leq b}v$ holds as well. If $\neg uR^f_{>b}v$, then clearly $\langle u, i\rangle\, R^*_{\leq b}\, \langle v, j\rangle$. So suppose $uR^f_{>b}v$. Since $\mathfrak{f}^f$ satisfies (D8), we then have $uR^f_{>a}v$, whence $j = 0$ and so $\langle u, i\rangle\, R^*_{\leq b}\, \langle v, j\rangle$.

(D6) Suppose $\langle u, i\rangle\, R^*_{\leq a}\, \langle v, j\rangle$ and $\langle v, j\rangle\, R^*_{\leq b}\, \langle w, k\rangle$, for $a, b, a + b \in M(\varphi)$. We have to show that $\langle u, i\rangle\, R^*_{\leq a+b}\, \langle w, k\rangle$. First, if $\langle u, i\rangle = \langle v, j\rangle$ or $\langle v, j\rangle = \langle w, k\rangle$, then $\langle u, i\rangle\, R^*_{\leq a+b}\, \langle w, k\rangle$ follows immediately from (D3), since $a, b \leq a + b$. So we may assume that $\langle u, i\rangle \neq \langle v, j\rangle$ and $\langle v, j\rangle \neq \langle w, k\rangle$. Then by definition, $uR^f_{\leq a}v$ and $vR^f_{\leq b}w$, whence $uR^f_{\leq a+b}w$, because $\mathfrak{f}^f$ satisfies (D6). If $\neg uR^f_{>a+b}w$, then $\langle u, i\rangle\, R^*_{\leq a+b}\, \langle w, k\rangle$ follows from the definition. So assume $uR^f_{>a+b}w$ holds in $\mathfrak{f}^f$ as well. From $uR^f_{\leq a}v$ and (D7) we obtain $vR^f_{>b}w$, and so $k = 0$. But then again, $\langle u, i\rangle\, R^*_{\leq a+b}\, \langle w, k\rangle$ follows from the definition. □

LEMMA 2.21. *For all $\langle u, i\rangle \in W^*$, $i \in \{0, 1, 2\}$ and all $\psi \in \Theta$, we have*

$$\langle \mathfrak{M}^*, \langle u, i\rangle\rangle \models \psi \quad\Longleftrightarrow\quad \left\langle \mathfrak{M}^f, u \right\rangle \models \psi.$$

PROOF. The proof is by induction on ψ. The basis of induction follows from the definition and the case of Booleans is trivial. The cases of $\psi = (\mathsf{A}^{\leq a}\chi)$ and $\psi = (\mathsf{A}^{>a}\chi)$ are consequences of the following claims.

Claim 1: If $uR^f_{\leq a}v$ and $\langle u,i\rangle \in W^*$, then there is j such that $\langle u,i\rangle\, R^*_{\leq a}\, \langle v,j\rangle$.

Indeed, if $a > 0$, we put $j = 0$, and $\langle u,i\rangle\, R^*_{\leq a}\, \langle v,j\rangle$ follows from the definition. If $a = 0$, then $u = v$; so we can take $i = j$.

Claim 2: If $\langle u,i\rangle\, R^*_{\leq a}\, \langle v,j\rangle$, then $uR^f_{\leq a}v$.

This follows immediately from the definition of $R^*_{\leq a}$.

Claim 3: If $uR^f_{>a}v$ and $\langle u,i\rangle \in W^*$, then there exists j such that $\langle u,i\rangle\, R^*_{>a}\, \langle v,j\rangle$.

Fix some $uR^f_{>a}v$ and $\langle u,i\rangle \in W^*$. Suppose first that $a = 0$. If $\neg uR^f_{\leq 0}v$ we then have $u \neq v$, since $R^f_{\leq 0}$ satisfies (D4), and so we can choose $j = 0$. If $uR^f_{\leq 0}v$ then v has been copied, so we can choose $j = i + 1(mod\ 2)$ and $\langle u,i\rangle \neq \langle v,j\rangle$, from which $\langle u,i\rangle\, R^*_{>a}\, \langle v,j\rangle$. Suppose now that $a > 0$. Consider two cases.

Case 1: $uR^f_{\leq a}v$. Then v has been copied, i.e., W^* contains $\langle v,0\rangle$, $\langle v,1\rangle$ and $\langle v,2\rangle$. Then put $j \neq 0, i$ which is always possible, because we have three copies of v. But then all three defining properties of $\langle u,i\rangle\, R^*_{\leq a}\, \langle v,j\rangle$ are violated, which means $\langle u,i\rangle\, R^*_{>a}\, \langle v,j\rangle$.

Case 2: $\neg uR^f_{\leq a}v$. Then $u \neq v$. So we can put $j = 0$, and again all three defining properties are violated.

Claim 4: If $\langle u,i\rangle\, R^*_{>a}\, \langle v,j\rangle$ then $uR^f_{>a}v$.

There are again two cases.

Case 1: $a > 0$. If $j = 0$ then $\neg uR^f_{\leq a}v$, and thus $uR^f_{>a}v$ by (D1). If $j \neq 0$, then, since the first defining property of $R^*_{\leq a}$ is violated, $uR^f_{\leq a}v$ follows again by (D1).

Case 2: $a = 0$. Then $\langle u,i\rangle \neq \langle v,j\rangle$. If $u \neq v$, then $\neg uR^f_{\leq 0}v$ and hence $uR^f_{>0}v$ as required. If $u = v$ and $i \neq j$, then u has been copied. So there are $w \in W^f$ and $b \in M$ such that $wR^f_{\leq b}u$ and $wR^f_{>b}u$. Since the latter can be written as $wR^f_{>b+0}u$, condition (D7) yields $uR^f_{>0}u$, as required.

Now, consider the induction step for $\psi = \mathsf{A}^{\leq a}\chi$. Suppose $\langle \mathfrak{M}^*, \langle u,i\rangle\rangle \models \mathsf{A}^{\leq a}\chi$ and pick some v such that $uR^f_{\leq a}v$. By Claim 1, there exists $j \in \{0,1,2\}$ such that $\langle u,i\rangle\, R^*_{\leq a}\, \langle v,j\rangle$. Then $\langle \mathfrak{M}^*, \langle v,j\rangle\rangle \models \chi$ and, by the induction hypotheses, it follows that $\langle \mathfrak{M}^f, v\rangle \models \chi$. Hence $\langle \mathfrak{M}^f, u\rangle \models \mathsf{A}^{\leq a}\chi$. Conversely, if $\langle \mathfrak{M}^f, u\rangle \models \mathsf{A}^{\leq a}\chi$ and $\langle v,j\rangle$ is such that $\langle u,i\rangle\, R^*_{\leq a}\, \langle v,j\rangle$, then by Claim 2, $uR^f_{\leq a}v$ and $\langle \mathfrak{M}^f, v\rangle \models \chi$, and so by the induction hypotheses, $\langle \mathfrak{M}^*, \langle v,j\rangle\rangle \models \chi$, i.e., $\langle \mathfrak{M}^*, \langle u,i\rangle\rangle \models \mathsf{A}^{\leq a}\chi$.

The case of $\psi = A^{>a}\chi$ follows analogously from Claims 3 and 4. ☐

Thus, we have shown that the model $\mathfrak{M}^*$ satisfies φ, i.e., $\langle \mathfrak{M}^*, \langle w, 0 \rangle \rangle \models \varphi$, and that the frame $\mathfrak{f}^*$ it is based on is a D-triangular $M(\varphi)$-frame. Again, it remains to show that this frame can be transformed in a D-triangular $\mathbb{R}^+$-frame in which φ is satisfiable, which can be done in precisely the same way as in Theorem 2.9, i.e., with the help of Theorem 2.7 (i) and Proposition 1.12.

□

CHAPTER 3

Logical Properties of Distance Logics

This chapter studies logical properties of distance logics. We have introduced logics of distance spaces in Definition 1.9 semantically, that is, as the sets of formulae of some language valid in some class of distance spaces. Let us begin our investigations by stating the obvious:

PROPOSITION 3.1. *All the logics* $\mathcal{MS}^i_O[M]$ *(*$\mathcal{MSO}^i_O[M]$*), where* $O \subseteq \mathfrak{O}(M)$ *is some operator set and* $i \in \{d, s, t, m\}$*, are normal multi-modal (hybrid) logics.*

PROOF. Let $\mathcal{D}^i$, $i \in \{d, s, t, m\}$, be a class of distance spaces. It easily follows from the definition of the truth-relation that

(1) all propositional tautologies are valid in $\mathcal{D}^i$,

(2) the **K**-Axioms

$$\bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi),$$

for $\bigcirc \in \{\blacksquare\} \cup \mathfrak{O}(M)$ are valid in $\mathcal{D}^i$, and

(3) that the rules of (sorted) substitution, modus ponens and necessitation, i.e.

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ \text{(MP)} \qquad \frac{\varphi}{\bigcirc\varphi}\ \text{(RN)} \quad \bigcirc \in \{\blacksquare\} \cup \mathfrak{O}(M),$$

preserve validity. □

In the next section, we present Hilbert-style axiomatisations of the logics $\mathcal{MS}^i_D[M]$, $i \in \{d, s, t, m\}$, in the languages $\mathcal{L}_D[M]$ not containing nominals. To prove completeness in Theorem 3.3, we employ the relational representation for these languages given in Section 2.3 and use Sahlqvist completeness theory as well as variants of the filtration and 'frame-repair' techniques used to prove the decidability of the satisfiability problem for these languages in Chapter 2.

In Section 3.2, we draw our attention to the modal distance logic $\mathcal{MSO}_F[M]$, whose language we showed in Section 1.4 to be expressively complete for the two-variable fragment $\mathcal{LF}_2[M]$ interpreted in metric spaces. We show that even for this (undecidable) language, an elementary relational representation of metric spaces can be given that captures theoremhood. To axiomatise the corresponding class of frames in the language $\mathcal{LO}_F[M]$, we will use some rather general completeness results from hybrid completeness theory

Finally, in Sections 3.3 and 3.4, we discuss several themes related to the (frame) representation theorems for languages $\mathcal{LO}_D[M]$ and $\mathcal{LO}_F[M]$. We first show, in Section 3.3, that while F-metric frames can capture theoremhood in $\mathcal{LO}_F[M]$, the local consequence relations with respect to metric spaces and F-metric frames differ in that the latter is compact, while the former is not. We then use the frame representations, in Section 3.4, to derive corresponding representation theorems for a variety of sublanguages of the full modal distance language $\mathcal{LO}[M]$, and provide sound and complete axiom systems for the respective logics. We close our investigations on distance logics by deriving a few results on Craig interpolation.

3.1. Axiomatising $\mathcal{MS}_D$

In this section, we present Hilbert-style axiomatisations of the logics $\mathcal{MS}_D^i$, for $i \in \{d, s, t, m\}$. As noted after the proof of Proposition 1.16, we have an effective translation $.^\sharp$ such that

$$\varphi \in \mathcal{MSO}_D^i \iff \varphi^\sharp \in \mathcal{MS}_D^i, \text{ for } i \in \{d, s, t, m\}.$$

Thus, the axiomatisations given can be understood as giving axiomatisations for the logics including nominals, as well. Note again that, as proved in Proposition 1.16 (i), the language $\mathcal{L}_D$ 'contains' standard modal operators like

- the universal modality: $\Box_a \varphi = \mathsf{A}^{\leq a}\varphi \wedge \mathsf{A}^{>a}\varphi$ (φ holds 'everywhere'),
- its dual: $\Diamond_a \varphi = \mathsf{E}^{\leq a}\varphi \vee \mathsf{E}^{>a}\varphi)$ (φ holds 'somewhere'), and
- the difference operator $D\varphi = \mathsf{E}^{>0}\varphi$ (φ holds 'somewhere else').

as *definable operators* in the sense of Definition 1.13.

Since $\mathsf{A}^{\leq a}$ and $\mathsf{A}^{>a}$ are both normal modal operators, the operator $\Box_a$ is normal as well, for any a in a given parameter set M.

3.1.1. The Axiomatic Systems for $\mathcal{MS}_D^i$. We start by presenting the Hilbert-style calculi for the logics $\mathcal{MS}_D^d[M]$, $\mathcal{MS}_D^s[M]$, $\mathcal{MS}_D^t[M]$, and $\mathcal{MS}_D^m[M]$, where the choice of the parameter set $M \subseteq \mathbb{R}^+$ is arbitrary. The corresponding axiomatic systems will be denoted by $\mathsf{MS}_D^d[M]$, $\mathsf{MS}_D^s[M]$, $\mathsf{MS}_D^t[M]$, and $\mathsf{MS}_D^m[M]$.

As usual, given a logic L and an axiom schema Θ, we denote by $L' = L \oplus \Theta$ the smallest normal modal logic containing L and Θ. Let MS_D^d be the axiomatic system with the axiom schemata and inference rules listed in Table 3.1.

Let us make a few comments on the choice of those axioms: The schema $(\mathsf{K}_\bigcirc)$ reflects that we are dealing with standard normal modal operators, while the axiom schemata $(\mathsf{Mo}_{\mathsf{A}^\leq})$ and $(\mathsf{Mo}_{\mathsf{A}^>})$ are sound, since d is a function and the values $d(u, v)$ are compared by the usual ordering on $\mathbb{R}^+$. The Axiom $(\mathsf{T}_{\mathsf{A}^{\leq 0}})$ codifies the assumption that the modality $\mathsf{A}^{\leq 0}$ satisfies reflexivity. Intuitively, it states that the distance from

AXIOM SCHEMATA FOR $\mathsf{MS}^d_D[M]$

(CL)	Axioms of propositional calculus;	
($\mathsf{K}_{\bigcirc}$)	$\bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi)$, where $\bigcirc \in \{\mathsf{A}^{\leq a}, \mathsf{A}^{>a} \mid a, b \in M\}$	
($\mathsf{Mo}_{\mathsf{A}^{\leq}}$)	$\mathsf{A}^{\leq a}\varphi \to \mathsf{A}^{\leq b}\varphi$	$(a, b \in M,\ a \geq b)$
($\mathsf{Mo}_{\mathsf{A}^{>}}$)	$\mathsf{A}^{>a}\varphi \to \mathsf{A}^{>b}\varphi$	$(a, b \in M,\ a \leq b)$
($\mathsf{T}_{\mathsf{A}^{\leq 0}}$)	$\mathsf{A}^{\leq 0}\varphi \to \varphi$	
($\mathsf{T}^c_{\mathsf{A}^{\leq 0}}$)	$\varphi \to \mathsf{A}^{\leq 0}\varphi$	
(Diff)	$\mathsf{E}^{\leq a}\mathsf{A}^{>0}\varphi \to \mathsf{A}^{>a}\varphi$	$(a \in M)$
(U1)	$\Box_0\varphi \to \Box_a\varphi$	$(a \in M)$
(U2)	$\Box_a\varphi \to \Box_0\varphi$	$(a \in M)$
($4_\Box$)	$\Box_a\varphi \to \Box_a\Box_a\varphi$	$(a \in M)$
($\mathsf{B}_\Box$)	$\varphi \to \Box_a\Diamond_a\varphi$	$(a \in M)$

INFERENCE RULES:

The inference rules are **modus ponens** and **necessitation** for both $\mathsf{A}^{\leq a}$ and $\mathsf{A}^{>a}$:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ \text{(MP)} \qquad \frac{\varphi}{\mathsf{A}^{\leq a}\varphi}\ \text{(RN1)} \qquad \frac{\varphi}{\mathsf{A}^{>a}\varphi}\ \text{(RN2)} \quad (a \in M)$$

Table 3.1: The axiomatic system $\mathsf{MS}^d_D[M]$.

a point to itself is zero. Conversely, Axiom ($\mathsf{T}^c_{\mathsf{A}^{\leq 0}}$) states that this is never the case for distinct points. Axiom (Diff) asserts a weak form of the triangular inequality, namely, that if a point u is at most a far away, and a point v is more than a far away, then the distance from u to v should be greater than 0. Axiom (U1) says that if φ holds everywhere, then φ holds at all points whose distance is less or equal to a or greater than a. Axiom (U2) is the converse of (U1). Finally, the last two Axioms plus item (4) from Lemma 3.2 state that the modality $\Box_a$ is an **S5** modal operator. Here, to ensure the soundness, we basically need that every pair of reals is assigned a distance.

The following lemma gives two simple theorems of $\mathsf{MS}^d_D[M]$ which will be used in the proof of Theorem 3.5.

LEMMA 3.2. *For any $\varphi \in \mathcal{L}_D[M]$ we have:*

($\mathsf{Eq}_\Box$):	$\vdash_{\mathsf{MS}^d_D} \Box_a\varphi \leftrightarrow \Box_b\varphi$	$(a, b \in M)$
($\mathsf{T}_\Box$):	$\vdash_{\mathsf{MS}^d_D} \Box_a\varphi \to \varphi$	$(a \in M)$

PROOF. *(ad 1).* This follows easily from (U1) and (U2) and some propositional reasoning. *(ad 2).* By ($\mathsf{Eq}_\Box$) it suffices to show that $\vdash_{\mathsf{MS}^d_D} \Box_0\varphi \to \varphi$. But this is a weakening of Axiom ($\mathsf{T}_{\mathsf{A}^{\leq 0}}$). $\Box$

ADDITIONAL AXIOM SCHEMATA FOR $\mathsf{MS}^s_D[M]$, $\mathsf{MS}^t_D[M]$, AND $\mathsf{MS}^m_D[M]$.

$(\mathsf{B}_{\mathsf{A}^\leq})$	$\varphi \to \mathsf{A}^{\leq a}\mathsf{E}^{\leq a}\varphi$	$(a \in M)$
$(\mathsf{B}_{\mathsf{A}^>})$	$\varphi \to \mathsf{A}^{>a}\mathsf{E}^{>a}\varphi$	$(a \in M)$
(Tr1)	$\mathsf{A}^{\leq a+b}\varphi \to \mathsf{A}^{\leq a}\mathsf{A}^{\leq b}\varphi$	$(a, b \in M)$
(Tr2)	$\mathsf{E}^{\leq a}\mathsf{A}^{>b}\varphi \to \mathsf{A}^{>a+b}\varphi$	$(a, b \in M)$

THE AXIOMATIC SYSTEMS $\mathsf{MS}^s_D[M]$, $\mathsf{MS}^t_D[M]$, AND $\mathsf{MS}^m_D[M]$.

$$\begin{array}{lcl} \mathsf{MS}^s_D[M] & = & \mathsf{MS}^d_D[M] \oplus (\mathsf{B}_{\mathsf{A}^\leq}) \oplus (\mathsf{B}_{\mathsf{A}^>}); \\ \mathsf{MS}^t_D[M] & = & \mathsf{MS}^d_D \oplus (\mathsf{Tr1}) \oplus (\mathsf{Tr2}); \\ \mathsf{MS}^m_D[M] & = & \mathsf{MS}^d_D \oplus (\mathsf{B}_{\mathsf{A}^\leq}) \oplus (\mathsf{B}_{\mathsf{A}^>}) \oplus (\mathsf{Tr1}) \oplus (\mathsf{Tr2}). \end{array}$$

Table 3.2: The axiomatic systems $\mathsf{MS}^s_D[M]$, $\mathsf{MS}^t_D[M]$, and $\mathsf{MS}_D[M]$.

To axiomatise $\mathcal{MS}^s_D[M]$, $\mathcal{MS}^t_D[M]$, and $\mathcal{MS}_D[M]$ ($= \mathcal{MS}^m_D[M]$), we require four extra axiom schemata as specified in Table 3.2.

For an $\mathcal{L}_D[M]$-formula φ we write $\vdash_{\mathsf{MS}^m_D[M]} \varphi$, $\vdash_{\mathsf{MS}^s_D[M]} \varphi$ etc. if φ is a theorem of $\mathsf{MS}^m_D[M]$, $\mathsf{MS}^s_D[M]$ etc. To simplify notation, we will usually omit M and write MS^m_D, MS^s_D, $\vdash_{\mathsf{MS}^m_D} \varphi$, $\vdash_{\mathsf{MS}^s_D} \varphi$, etc.

The main result of this section is the following:

THEOREM 3.3 (WEAK COMPLETENESS). *Let MS^i_D, $i \in \{d, s, t, m\}$, be any of the axiomatic systems and $\mathcal{MS}^i_D$ the corresponding logic. Then for all $\mathcal{L}_D[M]$-formula φ:*

$$\vdash_{\mathsf{MS}^i_D} \varphi \quad \Longleftrightarrow \quad \varphi \in \mathcal{MS}^i_D.$$

We begin the proof of this theorem by establishing the soundness of the axiomatic systems.

LEMMA 3.4 (SOUNDNESS). *Let MS^i_D, $i \in \{d, s, t, m\}$, be any of the axiomatic systems and $\mathcal{MS}^i_D$ the corresponding logic. Then for every $\mathcal{L}_D[M]$-formula φ:*

$$\vdash_{\mathsf{MS}^i_D} \varphi \quad \textit{implies} \quad \varphi \in \mathcal{MS}^i_D.$$

PROOF. (a) Let us start with the system MS^d_D and the class of all distance spaces as intended models. The validity of the **K**-schemata follows from the semantic definition of the modal operators. The validity of the schemata $(\mathsf{Mo}_{\mathsf{A}^\leq})$ and $(\mathsf{Mo}_{\mathsf{A}^>})$ follows from the **K**-Axioms (in the case of $a = b$) and the definition of distance spaces. For suppose that $a > b$ and

$$\langle \mathfrak{B}, w \rangle \models \mathsf{A}^{\leq a}\varphi \wedge \mathsf{E}^{\leq b}\neg\varphi.$$

Then there exists a $u \in W$ such that $d(w,u) \leq b$ and $\langle \mathfrak{B}, u \rangle \not\models \varphi$. But since $\leq$ is the usual linear order on $\mathbb{R}$, we have $d(w,u) \leq a$, and hence $\langle \mathfrak{B}, u \rangle \models \varphi$, which is a contradiction. The Axiom schema ($\mathsf{Mo}_{\mathsf{A}^>}$) is considered analogously.

Consider now the Axiom (Diff). Suppose that $\langle \mathfrak{B}, u \rangle \models \mathsf{E}^{\leq a}\mathsf{A}^{>0}\varphi$. Then there is a point $v \in W$ with $d(u,v) \leq a$ such that for all points w with $d(v,w) > 0$ (i.e., $w \neq v$) we have $\langle \mathfrak{B}, w \rangle \models \varphi$. Now take any point w' with $d(u,w') > a$. Then clearly $v \neq w'$, and hence $\langle \mathfrak{B}, w' \rangle \models \varphi$, from which $\langle \mathfrak{B}, u \rangle \models \mathsf{A}^{>a}\varphi$.

The validity of the remaining axioms follows immediately from the definitions (note that $\langle \mathfrak{B}, w \rangle \models \Box_a \varphi$ means that φ is valid in $\mathfrak{B}$), and it should be clear that validity is preserved under the inference rules.

(b) Now assume that the distance function d is symmetric and consider Axiom ($\mathsf{B}_{\mathsf{A}^{\leq}}$). Suppose that $\langle \mathfrak{B}, w \rangle \not\models \varphi \to \mathsf{A}^{\leq a}\mathsf{E}^{\leq a}\varphi$. Then $\langle \mathfrak{B}, w \rangle \models \varphi$ and there is a point u with $d(w,u) \leq a$ such that $\langle \mathfrak{B}, u \rangle \models \mathsf{A}^{\leq a}\neg\varphi$. Since d is symmetric, we have $d(u,w) \leq a$, and hence $\langle \mathfrak{B}, w \rangle \models \neg\varphi$, which is a contradiction. The validity of Axiom ($\mathsf{B}_{\mathsf{A}^{>}}$) in symmetric distance spaces is shown in a similar manner.

(c) Suppose that the distance function d satisfies the triangular inequality (3) and $\langle \mathfrak{B}, w \rangle \models \mathsf{A}^{\leq a+b}\varphi$. Take any points u, v such that $d(w,u) \leq a$ and $d(u,v) \leq b$. By (3), we have $d(w,v) \leq d(w,u) + d(u,v) \leq a + b$. Therefore, $\langle \mathfrak{B}, v \rangle \models \varphi$, and so we obtain $\langle \mathfrak{B}, w \rangle \models \mathsf{A}^{\leq a}\mathsf{A}^{\leq b}\varphi$, which shows the validity of (Tr1) in triangular spaces. To show the validity of (Tr2), assume that $\langle \mathfrak{B}, w \rangle \models \mathsf{E}^{\leq a}\mathsf{A}^{>b}\varphi$, i.e., that there is a u with $d(w,u) \leq a$ such that $\langle \mathfrak{B}, u \rangle \models \mathsf{A}^{>b}\varphi$. Take any point v such that $d(w,v) > a + b$. We then have $a + d(u,v) \geq d(w,u) + d(u,v) \geq d(w,v) > a + b$, from which $d(u,v) > b$, and hence $\langle \mathfrak{B}, w \rangle \models \mathsf{A}^{>a+b}\varphi$.

(d) The case of metric spaces is a consequence of (a), (b) and (c). $\Box$

To prove completeness, we will use the representation of distance spaces by the respective standard classes of M-frames, as given in Theorem 2.7.

3.1.2. Frame Completeness and Finite Frame Property. To proceed with the proof of Theorem 3.3, we first show that our axiomatic systems are sound and complete with respect to the classes (of finite frames in) $\mathcal{F}^d$, $\mathcal{F}^s$, $\mathcal{F}^t$, and $\mathcal{F}^m$, respectively.

THEOREM 3.5 (FRAME COMPLETENESS). *Let MS^i_D, $i \in \{d, s, t, m\}$, be any of the axiomatic systems and $\mathcal{F}^i$ the corresponding class of standard frames. Then, for every formula φ of $\mathcal{L}_D[M]$, we have:*

$$\vdash_{\mathsf{MS}^i_D} \varphi \quad \iff \quad \mathfrak{f} \models \varphi, \text{ for all finite } \mathfrak{f} \in \mathcal{F}^i.$$

PROOF. ($\Longrightarrow$) Suppose $\vdash_{\mathsf{MS}^i_D} \varphi^i$, for $i \in \{d,s,t,m\}$. Let $M' \supset Par(\varphi^i)$ be a finite parameter set containing the parameters occurring in φ^i. Then, given any (finite) D-standard M-frame $\mathfrak{f}^i \in \mathcal{F}^i$, φ^i is valid in $\mathfrak{f}^i$ if and only if φ^i is valid in the frame-reduct $\mathfrak{f}^i \downharpoonleft_{(D,M'))} := \langle W, (R_{\leq a})_{a \in M'}, (R_{>a})_{a\in M'}\rangle$. By Proposition 2.7 (i), $\mathfrak{f}^i \downharpoonleft_{(D,M')}$ is the frame-companion of some distance space S, where S is a metric space if $\mathfrak{f}^i \downharpoonleft_{(D,M')}$ is a D-metric M'-frame, etc. By Soundness with respect to the respective classes of distance spaces, Lemma 3.4, φ^i is valid in the class $\mathcal{D}^i$, thus φ^i is valid in the respective class $\mathcal{F}^i$ of (finite) M-frames.

($\Longleftarrow$) Let MS^i_D, $i \in \{d,s,t,m\}$, be any of the axiomatic systems and $\mathfrak{M}^i$ its canonical model based on the canonical frame $\mathfrak{f}^i$. As all axioms of MS^i_D are Sahlqvist formulae, for $i \in \{d,s,t,m\}$, by Sahlqvist's Theorem we have $\mathfrak{f}^i \models \mathsf{MS}^i_D$. It is not hard to see that $\mathfrak{f}^i$ satisfies all the frame properties corresponding to $\mathcal{F}^i$, except perhaps (D1) and (D2). (For instance, conditions (D3) and (D8) are first-order equivalents of ($\mathsf{Mo}_{\mathsf{A}^{\leq}}$) and ($\mathsf{Mo}_{\mathsf{A}^{>}}$).)

As an example we consider condition (D7). Suppose that $uR^d_{\leq a}v$, $uR^d_{>a}w$ and $\mathsf{A}^{>0}\varphi \in v$. Then $\mathsf{E}^{\leq a}\mathsf{A}^{>0}\varphi \in u$, for otherwise $\mathsf{A}^{\leq a}\neg\mathsf{A}^{>0}\varphi \in u$ (since u is a maximal consistent set of formulae), and so $\neg\mathsf{A}^{>0}\varphi \in v$ by the definition of $R^d_{\leq a}$, contrary to v being consistent. By Axiom (Diff) we then have $\mathsf{A}^{>a}\varphi \in u$, whence $\varphi \in w$ by the definition of $R^d_{>a}$, and so $vR^d_{>0}w$.

Suppose now that $\nvdash_{\mathsf{MS}^i_D} \varphi^i$. Then there exists a point w_i in $\mathfrak{f}^i$ such that we have $\langle \mathfrak{M}^i, w_i\rangle \not\models \varphi^i$. Take the submodel

$$\mathfrak{M}^i_{w_i} = \left\langle W^i, (R^i_{\leq a})_{a\in M}, (R^i_{>a})_{a\in M}, \mathfrak{b}^i \right\rangle$$

of $\mathfrak{M}^i$ generated by w_i, where $\mathfrak{b}^i$ is the respective canonical valuation. Then, clearly, $\langle \mathfrak{M}^i_{w_i}, w_i\rangle \not\models \varphi^i$ and the underlying frame $\mathfrak{f}^i_{w_i}$ satisfies all the properties mentioned above. We claim that, for $i \in \{d,s,t,m\}$, $\mathfrak{f}^i_{w_i}$ satisfies (D1) as well. Indeed, by ($4_\Box$), ($\mathsf{B}_\Box$) and ($\mathsf{T}_\Box$), for every $a \in M$, $\Box_a$ is an **S5**-box interpreted by the relation $R^i_{\leq a} \cup R^i_{>a}$. It follows that the $R^i_{\leq a} \cup R^i_{>a}$ are equivalence relations on W^i. By ($\mathsf{Eq}_\Box$), we also have

$$R^i_{\leq a} \cup R^i_{>a} = R^i_{\leq b} \cup R^i_{>b}$$

for all $a, b \in M$. And since $\mathfrak{f}^i_{w_i}$ is rooted, we can conclude that $R^i_{\leq a} \cup R^i_{>a}$ is the universal relation on W^i, i.e., $R^i_{\leq a} \cup R^i_{>a} = W^i \times W^i$, as required.

It remains to transform $\mathfrak{M}^i_{w_i}$ into a finite model $\mathfrak{M}^\ddagger_i$ which still refutes φ^i and has all the properties corresponding to $\mathcal{F}^i$, including (D2). For the cases of D-metric and D-triangular frames we have already shown how models of type $\mathfrak{M}^m_{w_m}$ and $\mathfrak{M}^t_{w_t}$ can be transformed in this way in Theorems 2.9 and 2.18. It remains to consider the simpler cases of D-symmetric and D-standard frames. As before, the required construction involves finite filtrations of the models $\mathfrak{M}^s_{w_s}$ and $\mathfrak{M}^d_{w_d}$ that are manipulated by duplicating certain points to obtain finite models satisfying (D2), extending similar techniques developed in Gargov et al. [1988] and Goranko [1990a].

We will treat the two cases simultaneously. Again, as condition (D2) is not definable, we will have to assume 'redundant' frame conditions in the middle of the construction, this time, condition (D10). So suppose that $\langle \mathfrak{M}_d, w_d \rangle \not\models \varphi^d$, where $\mathsf{MS}^d_D \nvdash \varphi^d$, and that $\langle \mathfrak{M}_s, w_s \rangle \not\models \varphi^s$, where $\mathsf{MS}^s_D \nvdash \varphi^s$.

Without loss of generality we may assume that the models $\mathfrak{M}_d$ and $\mathfrak{M}_s$ are based on $M(\varphi^i)$-frames $\mathfrak{f}_d$ and $\mathfrak{f}_s$ as defined in the proof of Theorem 2.9 Step 1, with $M(\varphi^d)$, $M(\varphi^s)$ a finite parameter set. Set $M_d := M(\varphi^d)$ and $M_s := M(\varphi^s)$. Then the frame $\mathfrak{f}_d$ underlying $\mathfrak{M}_d$ is of the form

$$\mathfrak{f}_d = \left\langle W_d, (D_{\leq a})_{a \in M_d}, (D_{>a})_{a \in M_d} \right\rangle$$

and satisfies (D1), (D3)–(D4), (D8), and (D10), whereas the frame $\mathfrak{f}_s$ underlying $\mathfrak{M}_s$ is of the form

$$\mathfrak{f}_s = \langle W_s, (S_{\leq a})_{a \in M_s}, (S_{>a})_{a \in M_s} \rangle$$

and satisfies (D1), (D3)–(D5), and (D8)–(D10).

We want to convert the models $\mathfrak{M}_d$ and $\mathfrak{M}_s$ into finite models $\mathfrak{M}^{\ddagger}_d$ and $\mathfrak{M}^{\ddagger}_s$ which also satisfy (D2) and still refute φ^d and φ^s, respectively.

We continue with the construction of the filtrations. Define the **closures** $cl(\varphi^d)$ and $cl(\varphi^s)$ of φ^d and φ^s to be the smallest sets T^d and T^s of $\mathcal{L}_D[M]$-formulae such that $\varphi^d \in T^d$, $\varphi^s \in T^s$, and, for $i \in \{d, s\}$

- (C1) T^i is closed under subformulae.
- (C6) If $\mathsf{A}^{>a}\psi \in T^i$ and $b > a$, then $\mathsf{A}^{>b}\psi \in T^i$ ($b \in M_i$).
- (C7) if $\mathsf{A}^{\leq a}\psi \in T^i$ and $a > b$, then $\mathsf{A}^{\leq b}\psi \in T^i$ ($b \in M_i$).
- (C8) If $\mathsf{A}^{>0}\psi \in T^i$, then $\neg\mathsf{A}^{\leq b}\neg\mathsf{A}^{>0}\psi \in T^i$ ($b \in M_i$).
- (C9) If $\mathsf{A}^{>0}\psi \in T^s$, then $\neg\mathsf{A}^{>b}\neg\mathsf{A}^{>0}\psi \in T^s$ ($b \in M_s$).

Note that T^d is closed only under (C1) and (C6)–(C8), while T^s is also closed under (C9). By slightly modifying the proof of Lemma 2.10, it is easily seen that the closures of φ^d and φ^s are finite sets.

Now we filtrate $\mathfrak{M}_d$ and $\mathfrak{M}_s$ through $\Theta^d = cl(\varphi^d)$ and $\Theta^s = cl(\varphi^s)$, respectively. Define equivalence relations $\equiv_d$ and $\equiv_s$ on W_d and W_s by taking, for $i \in \{d, s\}$ and $u, v \in W_i$:

$$u \equiv_i v :\Longleftrightarrow \langle \mathfrak{M}_i, u \rangle \models \psi \text{ iff } \langle \mathfrak{M}_i, v \rangle \models \psi \quad \text{for all } \psi \in \Theta^i.$$

Let $[u]_i = \{v \in W_i \mid u \equiv_i v\}$ and construct filtrations

$$\mathfrak{M}_d^f = \left\langle W_d^f, (D_{\leq a}^f)_{a \in M_d}, (D_{>a}^f)_{a \in M_d}, p_0^{\mathfrak{M}_d^f}, p_1^{\mathfrak{M}_d^f}, \dots \right\rangle$$

and

$$\mathfrak{M}_s^f = \left\langle W_s^f, (S_{\leq a}^f)_{a \in M_s}, (S_{>a}^f)_{a \in M_s}, p_0^{\mathfrak{M}_s^f}, p_1^{\mathfrak{M}_s^f}, \dots \right\rangle$$

of $\mathfrak{M}_d$, $\mathfrak{M}_s$ through Θ^d, Θ^s, by taking, for $i \in \{d, s\}$:

- $W_i^f = \{[u]_i \mid u \in W_i\}$;
- $p_k^{\mathfrak{M}_i^f} = \{[u]_i \mid u \in p_k^{\mathfrak{M}_i}\}$ for $k < \omega$;

and by defining the accessibility relations as

- $a > 0:$ $[u]_d D_{\leq a}^f [v]_d :\iff$ for all $\mathsf{A}^{\leq a}\psi \in \Theta^d$
  - $\langle \mathfrak{M}_d, u\rangle \models \mathsf{A}^{\leq a}\psi$ implies $\langle \mathfrak{M}_d, v\rangle \models \psi$;
- $a = 0:$ $[u]_d D_{\leq 0}^f [v]_d :\iff [u]_d = [v]_d$;
- $a \geq 0:$ $[u]_d D_{>a}^f [v]_d :\iff$ for all $\mathsf{A}^{>a}\psi \in \Theta^d$
  - $\langle \mathfrak{M}_d, u\rangle \models \mathsf{A}^{>a}\psi$ implies $\langle \mathfrak{M}_d, v\rangle \models \psi$,

for the frame $\mathfrak{f}_d^f$, and

- $a > 0:$ $[u]_s S_{\leq a}^f [v]_s :\iff$ for all $\mathsf{A}^{\leq a}\psi \in \Theta^s$
  - $\langle \mathfrak{M}_s, u\rangle \models \mathsf{A}^{\leq a}\psi$ implies $\langle \mathfrak{M}_s, v\rangle \models \psi$;
  - $\langle \mathfrak{M}_s, v\rangle \models \mathsf{A}^{\leq a}\psi$ implies $\langle \mathfrak{M}_s, u\rangle \models \psi$;
- $a = 0:$ $[u]_s S_{\leq 0}^f [v]_s :\iff [u]_s = [v]_s$;
- $a \geq 0:$ $[u]_s S_{>a}^f [v]_s :\iff$ for all $\mathsf{A}^{>a}\psi \in \Theta^s$
  - $\langle \mathfrak{M}_s, u\rangle \models \mathsf{A}^{>a}\psi$ implies $\langle \mathfrak{M}_s, v\rangle \models \psi$;
  - $\langle \mathfrak{M}_s, v\rangle \models \mathsf{A}^{>a}\psi$ implies $\langle \mathfrak{M}_s, u\rangle \models \psi$,

for the frame $\mathfrak{f}_s^f$. Since Θ^d and Θ^s are finite, W_d^f and W_s^f are finite as well.

LEMMA 3.6. (i) *For every $\psi^i \in \Theta^i$ and every $u_i \in W_i$, $i \in \{d, s\}$:*

$$\langle \mathfrak{M}_i, u_i\rangle \models \psi^i \iff \left\langle \mathfrak{M}_i^f, [u]_i \right\rangle \models \psi^i.$$

In particular, $\left\langle \mathfrak{M}_d^f, [w_d]_d \right\rangle \not\models \varphi^d$ and $\left\langle \mathfrak{M}_s^f, [w_s]_s \right\rangle \not\models \varphi^s$.
(ii) *The frame $\mathfrak{f}_d^f$ underlying $\mathfrak{M}_d^f$ satisfies* (D1), (D3)–(D4), (D8) *and* (D10).
(iii) *The frame $\mathfrak{f}_s^f$ underlying $\mathfrak{M}_s^f$ satisfies* (D1), (D3)–(D5) *and* (D8)–(D10).

PROOF. Claim (i) is proved by an easy induction on the construction of ψ^d, ψ^s, respectively.

Let us prove (ii) and (iii): Note that we have $uD_{\leq a}v$ implies $[u]_d D_{\leq a}^f [v]_d$ and $uD_{>a}v$ implies $[u]_d D_{>a}^f [v]_d$. The same holds true for the relations $S_{\leq a}^f$ and $S_{>a}^f$. For $a = 0$ and the relations $D_{\leq 0}^f$ and $S_{\leq 0}^f$ this is true by virtue of condition (D4) which holds for both,

$\mathfrak{f}_d$ and $\mathfrak{f}_s$. For the relations $S^f_{\leq a}$ and $S^f_{>a}$ this holds since $\mathfrak{f}_s$ satisfies conditions (D3) and (D8).

(D1): For $\mathfrak{f}^f_d$ we have to show that $D^f_{\leq a} \cup D^f_{>a} = W^f_d \times W^f_d$. Suppose that $\neg[u]_d D^f_{\leq a}[v]$. Then $\neg u D_{\leq a} v$, and so $u D_{>a} v$, since $\mathfrak{f}_d$ satisfies (D1). Thus $[u]_d D^f_{>a}[v]_d$. The proof for $\mathfrak{f}_s$ is the same.

(D3): We check that if $[u]_d D^f_{\leq a}[v]_d$ and $a \leq b$ then $[u]_d D^f_{\leq b}[v]_d$. Suppose $[u]_d D^f_{\leq a}[v]_d$ and $a < b$, for $b \in M_d$. Assume further that $\langle \mathfrak{M}_d, u\rangle \models \mathsf{A}^{\leq b}\psi$ and $\mathsf{A}^{\leq b}\psi \in \Theta^d$. By the definition of Θ^d, condition (C7), $\mathsf{A}^{\leq a}\psi \in \Theta^d$. Since (D3) holds in $\mathfrak{f}_d$, $\langle \mathfrak{M}_d, u\rangle \models \mathsf{A}^{\leq a}\psi$ and thus $\langle \mathfrak{M}_d, v\rangle \models \psi$. The proof for $\mathfrak{f}_s$ is similar.

(D4): $[u]_d D^f_{\leq 0}[v]_d \iff [u]_d = [v]_d$ holds by the definition of $D^f_{\leq 0}$. The same holds true for $\mathfrak{f}_s$.

(D8): If $[u]_d D^f_{>a}[v]_d$ and $a \geq b$ then $[u]_d D^f_{>b}[v]_d$. As above, let us assume that $[u]_d D^f_{>a}[v]_d$ and $a > b$, for $b \in M_d$. Suppose that $\langle \mathfrak{M}_d, u\rangle \models \mathsf{A}^{>b}\psi$. Then $\mathsf{A}^{>a}\psi \in \Theta^d$ by (C6), $\langle \mathfrak{M}_d, u\rangle \models \mathsf{A}^{>a}\psi$, and so $\langle \mathfrak{M}_d, v\rangle \models \psi$. Again, the proof for $\mathfrak{f}_s$ is similar.

(D10): Suppose that $[u]_d D^f_{\leq a}[v]_d$, $[u]_d D^f_{>a}[w]_d$ and $\langle \mathfrak{M}_d, v\rangle \models \mathsf{A}^{>0}\psi$, where $\mathsf{A}^{>0}\psi \in \Theta^d$. Then, by the definition of Θ^d, condition (C8), $\mathsf{E}^{\leq a}\mathsf{A}^{>0}\psi \in \Theta^d$ (so $\mathsf{A}^{\leq a}\neg\mathsf{A}^{>0}\psi \in \Theta^d$ as well) and $\langle \mathfrak{M}_d, u\rangle \models \mathsf{E}^{\leq a}\mathsf{A}^{>0}\psi$ (for otherwise we would have $\langle \mathfrak{M}_d, u\rangle \models \mathsf{A}^{\leq a}\neg\mathsf{A}^{>0}\psi$ and then $\langle \mathfrak{M}_d, v\rangle \models \neg\mathsf{A}^{>0}\psi$). So there is a $w' \in W_d$ such that $u D_{\leq a} w'$ and it holds that $\langle \mathfrak{M}_d, w'\rangle \models \mathsf{A}^{>0}\psi$. So, for any w'' with $u D_{>a} w''$, by condition (D10) which holds for $\mathfrak{f}_d$, we have $w' D_{>0} w''$ and thus $\langle \mathfrak{M}_d, w''\rangle \models \psi$. It follows that $\langle \mathfrak{M}_d, u\rangle \models \mathsf{A}^{>a}\psi$. By definition of Θ^d, $\mathsf{A}^{>a}\psi \in \Theta^d$ since $a \geq 0$ and $\mathsf{A}^{>0}\psi \in \Theta^d$. So $[u]_d D^f_{>a}[w]_d$ implies $\langle \mathfrak{M}_d, w\rangle \models \psi$, from which $[v]_d D^f_{>0}[w]_d$.

Next, to prove (D10) for $\mathfrak{f}_s$ we can first repeat the above proof to show that $\langle \mathfrak{M}_s, v\rangle \models \mathsf{A}^{>0}\psi$, and $\mathsf{A}^{>0}\psi \in \Theta^s$ implies $\langle \mathfrak{M}_s, w\rangle \models \psi$. Further, assume that $\langle \mathfrak{M}_s, w\rangle \models \mathsf{A}^{>0}\psi$, and $\mathsf{A}^{>0}\psi \in \Theta^s$. Then $\mathsf{E}^{>a}\mathsf{A}^{>0}\psi \in \Theta^s$ by (C9) and we also have $\langle \mathfrak{M}_s, u\rangle \models \mathsf{E}^{>a}\mathsf{A}^{>0}\psi$ (for otherwise, we would have $\langle \mathfrak{M}_s, u\rangle \models \mathsf{A}^{>a}\neg\mathsf{A}^{>0}\psi$ and then $\langle \mathfrak{M}_s, w\rangle \models \neg\mathsf{A}^{>0}\psi$). So there is a $w' \in W_s$ such that $u S_{>a} w'$ and $\langle \mathfrak{M}_s, w'\rangle \models \mathsf{A}^{>0}\psi$. Let w'' be such that $u S_{\leq a} w''$. By condition (D10) for $\mathfrak{f}_s$ we obtain $w'' S_{>0} w'$ and by condition (D9) $w' S_{>0} w''$. Thus $\langle \mathfrak{M}_s, w''\rangle \models \psi$ and so $\langle \mathfrak{M}_s, u\rangle \models \mathsf{A}^{\leq a}\psi$, which implies that $\langle \mathfrak{M}_s, v\rangle \models \psi$. This shows that $[v]_s S^f_{>0}[w]_s$.

(D5) and (D9): These conditions follow immediately from the definitions of $S^f_{\leq a}$ and $S^f_{>a}$.

□

Next we perform the copying technique to guarantee that $\mathfrak{f}^f_d$ and $\mathfrak{f}^f_s$ also satisfy condition (D2).

The possible defects form the sets

$$D(W_d^f) = \{v \in W_d^f \mid \exists a \in M_d\ \exists u \in W_d^f\ (uD_{\leq a}^f v\ \&\ uD_{>a}^f v)\},$$

and

$$D(W_s^f) = \{v \in W_s^f \mid \exists a \in M_s\ \exists u \in W_s^f\ (uS_{\leq a}^f v\ \&\ uS_{>a}^f v)\}.$$

Let, for $i \in \{d, s\}$:

$$W_i^* = \{\langle v, j\rangle \mid v \in D(W_i^f), j \in \{0, 1\}\} \cup \{\langle u, 0\rangle \mid u \in W_i^f - D(W_i^f)\}.$$

So, for each $v \in D(W_i^f)$, $i \in \{d, s\}$, we now have two copies $\langle v, 0\rangle$ and $\langle v, 1\rangle$. Define new models $\mathfrak{M}_d^*$ and $\mathfrak{M}_s^*$ based on frames

$$\mathfrak{f}_d^* = \left\langle W_d^*, (D_{\leq a}^*)_{a\in M_d}, (D_{>a}^*)_{a\in M_d}\right\rangle$$

and

$$\mathfrak{f}_s^* = \left\langle W_s^*, (S_{\leq a}^*)_{a\in M_s}, (S_{>a}^*)_{a\in M_s}\right\rangle,$$

by taking, for $i \in \{d, s\}$, $k < \omega$, and $j \in \{0, 1\}$

$$p_k^{\mathfrak{M}_i^*} = \{\langle u, j\rangle \in W_i^* \mid u \in p^{\mathfrak{M}_i^f}\},$$

and by defining accessibility relations $D_{\leq a}^*$, $D_{>a}^*$, $S_{\leq a}^*$ and $S_{>a}^*$ as follows:

- $a > 0$: $\langle u, i\rangle\, D_{\leq a}^*\, \langle v, j\rangle :\Longleftrightarrow$
  - $i \neq j$ and $uD_{\leq a}^f v$ and $\neg uD_{>a}^f v$, or
  - $i = j$ and $uD_{\leq a}^f v$;
- $a = 0$: $\langle u, i\rangle\, D_{\leq a}^*\, \langle v, j\rangle :\Longleftrightarrow \langle u, i\rangle = \langle v, j\rangle$;
- $D_{>a}^*$ is defined as the complement of $D_{\leq a}^*$, i.e.,

$$\langle u, i\rangle\, D_{>a}^*\, \langle v, j\rangle :\Longleftrightarrow \neg\, \langle u, i\rangle\, D_{\leq a}^*\, \langle v, j\rangle.$$

- The relations $S_{\leq a}^*$ and $S_{>a}^*$ are defined in exactly the same way.

The new models $\mathfrak{M}_d^*$ $\mathfrak{M}_s^*$ now have all the required properties, namely, we have:

LEMMA 3.7. (i) *The model $\mathfrak{M}_d^*$ is D-standard.*
(ii) *The model $\mathfrak{M}_s^*$ is D-symmetric.*

PROOF. That both, $\mathfrak{M}_d^*$ and $\mathfrak{M}_s^*$, satisfy (D1), (D2) and (D4) follows immediately from the definition. Hence (D10) is an immediate consequence. Let us check the remaining conditions.

(D3) Suppose that $\langle u, i\rangle\, D_{\leq a}^*\, \langle v, j\rangle$ and $a < b \in M_d$. If $i = j$ then clearly $\langle u, i\rangle\, D_{\leq b}^*\, \langle v, j\rangle$ by (D3) for $\mathfrak{f}_d^f$. So assume $i \neq j$. Then, by definition, $uD_{\leq a}^f v$ and $\neg uD_{>a}^f v$. Since $\mathfrak{M}_d^f$ satisfies (D3) and (D8), we obtain $uD_{\leq b}^f v$ and $\neg uD_{>b}^f v$. Thus $\langle u, i\rangle\, D_{\leq b}^*\, \langle v, j\rangle$. The proof for $\mathfrak{M}_s^*$ is the same. (D8) is a consequence of (D1), (D2) and (D3).

(D5) Suppose that $\langle u,i\rangle\, S^*_{\leq a}\, \langle v,j\rangle$. If $i=j$, then $uS^f_{\leq a}v$ and so $vS^f_{\leq a}u$ by (D5) for $\mathfrak{F}^f_s$, hence $\langle v,j\rangle\, S^*_{\leq a}\, \langle u,i\rangle$. Similarly, if $i\neq j$, then $uS^f_{\leq a}v$ and $\neg uS^f_{>a}v$, whence, by (D5) and (D9) for $\mathfrak{F}^f_s$, $vS^f_{\leq a}u$ and $\neg vS^f_{>a}u$, so $\langle v,j\rangle\, S^*_{\leq a}\, \langle u,i\rangle$. (D9) is a consequence of (D1), (D2) and (D5).

□

LEMMA 3.8. *For $i\in\{d,s\}$ and all $\langle v,j\rangle\in W^*_i$ and all $\psi\in\Theta^i$, we have*

$$\langle \mathfrak{M}^*_i, \langle v,j\rangle\rangle \models \psi \quad\iff\quad \left\langle \mathfrak{M}^f_i, v\right\rangle \models \psi.$$

PROOF. We prove the lemma for the case $i=d$, i.e., for the case of distance spaces. The proof for the symmetric case is identical. We proceed by induction on ψ. The basis of induction and the case of Booleans are trivial. The cases $\psi=(\mathsf{A}^{\leq a}\chi)$ and $\psi=(\mathsf{A}^{>a}\chi)$ are consequences of the following claims:

Claim 1. If $uD^f_{\leq a}v$ and $\langle u,j\rangle\in W^*_d$ ($j\in\{0,1\}$), then there exists a k such that $\langle u,j\rangle\, D^*_{\leq a}\, \langle v,k\rangle$.

$a>0$: Indeed, this is clear for $j=0$. Suppose $j=1$. If v has been duplicated, then $\langle v,1\rangle$ is as required. If v has not been duplicated, then $\neg uR^f_{>a}v$, and so $\langle v,0\rangle$ is as required.

$a=0$: Then $u=v$. If $j=0$ pick $k=0$. If $j=1$, then u has been duplicated and we can pick $k=1$.

Claim 2. If $\langle u,j\rangle\, D^*_{\leq a}\, \langle v,k\rangle$, then $uD^f_{\leq a}v$.

This should be obvious.

Claim 3. If $uD^f_{>a}v$ and $\langle u,j\rangle\in W^*_d$ ($j\in\{0,1\}$), then there exists a k such that $\langle u,j\rangle\, D^*_{>a}\, \langle v,k\rangle$.

$a>0$: Suppose $j=0$. If v has not been duplicated, then $\neg uD^f_{\leq a}v$. Hence $\neg\langle u,0\rangle\, D^*_{\leq a}\, \langle v,0\rangle$ by Claim 2, and so $\langle u,0\rangle\, D^*_{>a}\, \langle v,0\rangle$ by definition. If v has been duplicated, then $\neg\langle u,0\rangle\, D^*_{\leq a}\, \langle v,1\rangle$. In the case of $j=1$ we have $\neg\langle u,1\rangle\, D^*_{\leq a}\, \langle v,0\rangle$, i.e., $\langle u,1\rangle\, D^*_{>a}\, \langle v,0\rangle$.

$a=0$: If $\neg uD^f_{\leq 0}v$, then $u\neq v$, since $D^f_{\leq 0}$ satisfies (D4), and so we can choose $k=0$. If $uD^f_{\leq 0}v$ then v has been copied, so we can choose $k=j+1(\mathit{mod}\ 2)$ and $\langle u,j\rangle\neq\langle v,k\rangle$, from which $\langle u,j\rangle\, D^*_{>a}\, \langle v,k\rangle$.

Claim 4. If $\langle u,j\rangle\, D^*_{>a}\, \langle v,k\rangle$, then $uD^f_{>a}v$.

$a>0$: Indeed, if $j=k$ then $\neg uD^f_{\leq a}v$ and so $uD^f_{>a}v$ by (D1). And if $j\neq k$, then $\neg uD^f_{\leq a}v$ or $uD^f_{>a}v$ by definition, so $uD^f_{>a}v$ by (D1), as well.

$a=0$: Then $\langle u,j\rangle\neq\langle v,k\rangle$. If $u\neq v$, then $\neg uD^f_{\leq 0}v$ and hence $uD^f_{>0}v$ as required. If $u=v$ and $j\neq k$, then u has been copied. So there are $w\in W^f_d$ and $b\in M_d$ such that $wD^f_{\leq b}u$ and $wD^f_{>b}u$. Now condition (D10) yields $uD^f_{>0}u$, as required.

Now consider the induction step for $\psi = \mathsf{A}^{\leq a}\chi$. Suppose $\langle \mathfrak{M}_d^*, \langle u, j\rangle\rangle \models \mathsf{A}^{\leq a}\chi$ and pick some v such that $uD^f_{\leq a}v$. By Claim 1, there exists $k \in \{0,1\}$ such that $\langle u, j\rangle\, D^*_{\leq a}\, \langle v, k\rangle$. Then $\langle \mathfrak{M}_d^*, \langle v, k\rangle\rangle \models \chi$ and, by the induction hypotheses, it follows that $\left\langle \mathfrak{M}_d^f, v\right\rangle \models \chi$. Hence $\left\langle \mathfrak{M}_d^f, u\right\rangle \models \mathsf{A}^{\leq a}\chi$. Conversely, if $\left\langle \mathfrak{M}_d^f, u\right\rangle \models \mathsf{A}^{\leq a}\chi$ and $\langle v, k\rangle$ is such that $\langle u, j\rangle\, D^*_{\leq a}\, \langle v, k\rangle$, then by Claim 2, $uD^f_{\leq a}v$ and $\left\langle \mathfrak{M}_d^f, v\right\rangle \models \chi$, from which, by the induction hypotheses, $\langle \mathfrak{M}_d^*, \langle v, k\rangle\rangle \models \chi$, i.e., $\langle \mathfrak{M}_d^*, \langle u, j\rangle\rangle \models \mathsf{A}^{\leq a}\chi$.

The case of $\psi = (A^{>a}\chi)$ follows analogously from Claims 3 and 4. □

To complete the proof of Theorem 3.5, we transform the frames $\mathfrak{f}_d^*$, $\mathfrak{f}_s^*$ underlying the models $\mathfrak{M}_d^*$, $\mathfrak{M}_s^*$ into D-standard and D-symmetric M-frames refuting φ^d, φ^s, respectively. This can be done as before, i.e., since M_d and M_s are finite, we find a distance space S_d and a symmetric distance space S_s whose M_d, M_s-frame-companions are $\mathfrak{f}_d^*$ and $\mathfrak{f}_s^*$, respectively. Then, take the full M-frame companions $\mathfrak{f}_d^{\ddagger}$ and $\mathfrak{f}_s^{\ddagger}$. These are, respectively, finite D-standard and D-symmetric M-frames, and they still refute φ^d, φ^s, respectively.

This completes the proof of Theorem 3.5. □

3.1.3. Completeness. We are in a position now to prove Theorem 3.3, i.e., derive the completeness of the axiomatic systems introduced.

THEOREM 3.9 (COMPLETENESS). *Let* MS^i_D, $i \in \{d, s, t, m\}$, *be any of the axiomatic systems and let* $\mathcal{MS}^i_D$ *be the corresponding logic. Then, for every* $\mathcal{L}_D[M]$*-formula* φ, *we have:*

$$\varphi \in \mathcal{MS}^i_D \quad \textit{implies} \quad \vdash_{\mathsf{MS}^i_D} \varphi.$$

PROOF. Suppose $\nvdash_{\mathsf{MS}^i_D} \varphi$. By Theorem 3.5, we then have a model refuting φ which is based on a finite M-frame $\mathfrak{f} \in \mathcal{F}^i$ of the corresponding standard frame class. It remains to transform $\mathfrak{f}$ into an appropriate distance space for $\mathcal{MS}^i_D$ which also refutes φ. That this can always be done was shown in Theorem 2.7 (ii). □

Furthermore, although this follows already from Corollary 2.16, we have reproved, as an immediate consequence of Theorem 3.5, that the logics $\mathcal{MS}^i_D[\mathbb{Q}^+]$, $i \in \{d, s, t, m\}$, are decidable and have the finite model property.

COROLLARY 3.10. *All the logics* $\mathcal{MS}^i_D[\mathbb{Q}^+]$, $i \in \{d, s, t, m\}$, *are decidable and have the finite model property.*

PROOF. It suffices to observe that all these logics are recursively axiomatisable and complete with respect to their finite standard frames with an effectively computable upper bound $f(|\varphi|)$ for models satisfying a formula φ, and use Harrop's Theorem (see, e.g., Chagrov and Zakharyaschev [1997]). □

3.2. Axiomatising $\mathcal{MSO}_F$

In this section, we will present an axiomatisation of the logic $\mathcal{MSO}_F[M]$ (for some fixed parameter set $M \subseteq \mathbb{R}^+$)—thus axiomatising the two-variable logic $\mathcal{FM}_2[M]$ (via translation)—and show it to be weakly complete with respect to metric spaces.

In the previous section, we gave 'orthodox' axiomatisations for the logics $\mathcal{MS}_D^i[M]$, for $i \in \{d, s, t, m\}$, in the sense that the axiomatic systems given are standard (modal) Hilbert calculi comprising as rules of proof just *modus ponens* and *necessitation*. We proceeded by applying finite filtrations to the canonical models and by 'repairing' the resulting models to obtain standard models still refuting a given formula.

Unfortunately, as concerns the distance logic $\mathcal{MS}_F[M]$, this proof technique is rather difficult to apply. First, while in the case of D-metric frames we had to deal only with one condition which was not definable in the language, namely (D2), an adequate relational representation of metric spaces for the language $\mathcal{L}_F[M]$ requires, as we will see below, several frame conditions that are not definable in $\mathcal{L}_F[M]$. Second, notice that, by Theorem 2.2, the language $\mathcal{L}_F[M]$ does not have the finite model property, and so we cannot expect to be able to apply a filtration technique similar to the one employed in Section 3.1.

However, we can axiomatise the logic $\mathcal{MSO}_F[M]$ by making use of its *hybrid* character, i.e., the presence of both, nominals and the universal modality, and by using general results from hybrid completeness theory involving the use of non-standard rules of inference, namely (a simplified version of) the *covering rule* (COV) used, e.g., in Goranko [1998].

The main technical tool for showing completeness is a 'finitary' and elementary relational representation of metric spaces—whose frame conditions are 'suggested' by the semantics of the Boolean variant $\mathcal{LOB}[M]$ of the language—that captures theoremhood in metric spaces, given in Theorem 3.12. The representation theorem implies that, to axiomatise $\mathcal{MSO}_F[M]$, it suffices to axiomatise the class $\mathfrak{F}_F[M]$ of F-metric M-frames, defined below in Definition 3.11. More specifically, it allows us to transfer *weak completeness* from F-metric M-frames to metric spaces. That this is all we can hope for in general follows from the fact that *strong completeness* implies compactness, and the non-compactness of $\mathcal{MSO}_F[M]$ for infinite (unbounded) M, which we will discuss in Section 3.3.

3.2.1. Frame Representation for $\mathcal{LO}_F[M]$. Let $M \subseteq \mathbb{R}^+$ be a parameter set. An M-**frame** for the language $\mathcal{LO}_F[M]$ is a structure of the form

$$\mathfrak{f} = \left\langle W, \{R_{<a}, R_{>a}, R_{=a}, R_{<b}^{>a}\}_{a,b \in M} \right\rangle$$

which consists of a set W and families $\{R_{<a}, R_{>a}, R_{=a}, R^{>a}_{<b}\}$ of binary relations on W, depending on M. The intended meaning of, e.g., $uR^{>a}_{<b}v$ will be "the distance from u to v is between a and b", etc.

A model based on a frame is of the form

$$\mathfrak{M} = \left\langle \mathfrak{f}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \dots i_0^{\mathfrak{M}}, i_1^{\mathfrak{M}}, \dots \right\rangle,$$

where the $p_n^{\mathfrak{M}}$ are subsets of W and the $i_m^{\mathfrak{M}}$ singleton subsets. The notions of truth (in a pointed model) and validity in M-models and M-frames are again the usual Kripkean ones, with the addition that nominals are interpreted as singleton sets. For instance,

$$\langle \mathfrak{M}, w \rangle \models \mathsf{A}^{>a}_{<b}\varphi \iff \langle \mathfrak{M}, u \rangle \models \varphi \text{ for all } u \in W \text{ such that } wR^{>a}_{<b}u;$$
$$\langle \mathfrak{M}, w \rangle \models \blacksquare\varphi \iff \langle \mathfrak{M}, u \rangle \models \varphi \text{ for all } u \in W;$$
$$\langle \mathfrak{M}, w \rangle \models i \iff i^{\mathfrak{M}} = \{w\}.$$

Similarly for the other operators.

The following definition of an *F-metric frame* singles out those M-frames that reflect properties of metric spaces. In particular, notice that the conditions (F1)–(F12) correspond rather directly to validities (in metric spaces) of the Boolean modal language $\mathcal{LOB}[M]$ from Section 1.5.

DEFINITION 3.11 (F-METRIC FRAMES). *An M-frame $\mathfrak{f}$ is called F-**metric**, if it meets the following requirements for all $u, v, w \in W$:*

(F1)	$R_{>a} = \overline{R_{<a}} \cap \overline{R_{=a}}$	$(a \in M)$
(F2)	$R^{>a}_{<b} = \overline{R_{<a}} \cap \overline{R_{=a}} \cap R_{<b}$	$(a, b \in M)$
(F3)	$R_{=0} = \{\langle w, w\rangle \mid w \in W\}$	
(F4)	$R_{<0} = \varnothing$	
(F5)	$R_{=a} \cap R_{<b} = \varnothing$	$(a \geq b)$
(F6)	$R_{=a} \cap R_{=b} = \varnothing$	$(a \neq b)$
(F7)	$R_{<a} \subseteq R_{<b}$	$(a \leq b)$
(F8)	$R_{=a} \subseteq R_{<b}$	$(a < b)$
(F9)	$R_{=a}$ *and* $R_{<a}$ *are symmetric*	$(a \in M)$
(F10)	$(uR_{=a}v \wedge vR_{=b}w) \implies (uR_{=a+b}w \vee uR_{<a+b}w)$	$(a + b \in M)$
(F11)	$(uR_{<a}v \wedge vR_{<b}w) \implies uR_{<a+b}w$	$(a + b \in M)$
(F12)	$(uR_{=a}v \wedge vR_{<b}w) \implies uR_{<a+b}w$	$(a + b \in M)$

We denote the class of all F-metric M-frames by $\mathfrak{F}_F[M]$.

Note that, for $a \neq 0$, (F3) and (F6) imply that the relation $R_{=a}$ is irreflexive, and (F3) and (F8) imply that $R_{<a}$ is reflexive. Thus, in all F-metric M-frames we have additionally.

(F13) $R_{=a}$ is irreflexive $(a \neq 0)$
(F14) $R_{<a}$ is reflexive $(a \neq 0)$

We are now in a position to prove a representation theorem in the spirit of Theorem 2.7 which shows that the notion of an F-metric M-frame is sufficient to capture validity in metric spaces. This representation is 'finitary' in the sense that, given a formula $\varphi \in \mathcal{LO}_F[M]$ satisfiable in some F-metric M-frame $\mathfrak{f}$, we construct a finite parameter set $M(\varphi)$ such that φ is satisfiable in a possibly infinite F-metric $M(\varphi)$-frame $\mathfrak{g}$, but which is based on the finitely many relations induced by $M(\varphi)$, and from which we can construct an 'equivalent' metric space, i.e., one whose frame-companion is $\mathfrak{g}$.

THEOREM 3.12 (REPRESENTATION OF METRIC SPACES FOR $\mathcal{LO}_F$).
(i) For every finite parameter set M and F-metric M-frame $\mathfrak{f}$ there is a metric space S such that $\mathfrak{f}$ is its frame-companion, i.e., $\mathfrak{f} = \mathfrak{f}_{F,M}(S)$. In particular, if $\mathfrak{f}$ is finite, so is S.

(ii) For an arbitrary parameter set M we have: an $\mathcal{LO}_F[M]$-formula φ is satisfiable in a metric space model based on a set W if and only if it is satisfiable in a model based on an F-metric M-frame based on W if and only if it is satisfiable in a model based on an F-metric $M(\varphi)$-frame based on W, with $M(\varphi)$ finite.

PROOF. We first prove (i). The proof is similar to the proof of Theorem 2.7. However, this time the definition of an appropriate metric requires much more care since, for instance, we obviously cannot choose as the value of $d(v, w)$ the minimum of the parameters a such that $vR_{<a}w$, but have to choose a slightly smaller value.

Let M be a finite parameter set and

$$\mathfrak{f} = \langle W, \{R_{<a}, R_{>a}, R_{=a}, R^{>a}_{<b}\}_{a,b \in M} \rangle$$

an F-metric M-frame. Enumerate the N elements of M as

$$M = \langle 0 = a_0, a_1, \ldots, a_{N-2}, a_{N-1} = \gamma \rangle \text{ with } a_i < a_j, \text{ if } i < j.$$

Thus, $\gamma = \max(M)$. If $a = a_i \in M$, we refer to the position i of a in the enumeration also by i_a. Now for the definition of the metric. Let

$$D := \{a_i + a_j - \gamma \mid a_i + a_j > \gamma, a_i, a_j \in M\} \cup \{a_i - a_j \mid a_i > a_j, a_i, a_j \in M\},$$

and let $\mu := \min(D \cup \{1\})$. Next, we choose some $\varepsilon > 0$ satisfying

$$\varepsilon < \frac{\mu}{2^N + 1}.$$

Before we proceed to define a metric, let us summarise some properties of ε that we will need later on:

LEMMA 3.13. *The following hold:*

(1) $a_j < a_i$ *if and only if* $a_j < a_i - 2^i \cdot \varepsilon$.

(2) $a_i - 2^i \cdot \varepsilon > 0$ *for all* $a_i \in M - \{0\}$.

(3) $a_i + a_j < \gamma + (2^i + 2^j + 1) \cdot \varepsilon$ *implies* $a_i + a_j \leq \gamma$.

(4) $a_i + a_j < \gamma + (2^i + 1) \cdot \varepsilon$ *implies* $a_i + a_j \leq \gamma$.

(5) $a_i + a_j < \gamma + \varepsilon$ *implies* $a_i + a_j \leq \gamma$.

PROOF. (1): $a_i > a_j$ if and only if $\mu \leq a_i - a_j$ and thus

$$\varepsilon < \frac{\mu}{2^N + 1} \leq \frac{a_i - a_j}{2^N + 1} < \frac{a_i - a_j}{2^i},$$

i.e., $a_j < a_i - 2^i \cdot \varepsilon$.

(2): By the definitions of ε and μ we have

$$a_i - 2^i \cdot \varepsilon > a_i - 2^i \cdot \frac{\mu}{2^N + 1} > a_i - \mu > 0,$$

for $0 \neq a_i \in M$.

(3): First, note that for all $i, j < \omega$:

$$2^i + 2^j \leq 2 \cdot 2^{\max(i,j)} = 2^{\max(i,j)+1}.$$

Thus we obtain:

$$a_i + a_j < \gamma + (2^i + 2^j + 1) \cdot \varepsilon \leq \gamma + (2^{\max(i,j)+1} + 1) \cdot \varepsilon \leq \gamma + (2^N + 1) \cdot \varepsilon.$$

So, since by definition of ε we have $(2^N + 1) \cdot \varepsilon < \mu$, we have

$$a_i + a_j < \gamma + \mu,$$

which implies $a_i + a_j \leq \gamma$. For, if $a + b > \gamma$ then $0 < a + b - \gamma \geq \mu$ by definition of μ, and so $a + b \geq \gamma + \mu$.

(4) and (5) are a consequence of (3). □

Now, define a function d by setting:

$$d(v, w) := \begin{cases} \gamma + \varepsilon & \text{if } vR_{>a}w \text{ for all } a \in M; \\ a & \text{if } vR_{=a}w \text{ for some } a \in M; \\ a_i - 2^i \cdot \varepsilon & \text{if } i = \min\{j < N \mid vR_{<a_j}w\} \text{ and } \forall a \in M : \neg vR_{=a}w. \end{cases}$$

We first show that the function d is well-defined and total. Note that in the case $d(v, w) = a_i - 2^i \cdot \varepsilon$, $a_i = 0$ cannot occur, because of condition (F4), $R_{<0} = \varnothing$. This, together with (2) of Lemma 3.13 shows that $d(v, w) \geq 0$ whenever d is defined.

Moreover, the three cases in the definition of $d(v, w)$ are mutually exclusive, but exhaustive. If for all $a \in M$ we have $vR_{>a}w$ then, for all $a \in M$, $\neg vR_{<a}w$ and $\neg vR_{=a}w$ by Property (F1). If $vR_{=a}w$ for some $a \in M$, then, again by (F1), $\neg vR_{>a}w$. And if for all

$a \in M$ we have $\neg vR_{=a}w$ and there is a $b \in M$ such that $\neg vR_{>b}w$, then, by (F1), $vR_{<b}w$. So d is always defined. Lastly, by Property (F6), we cannot have $vR_{=a}w$ and $vR_{=b}w$ for $a \neq b$, which shows that d is well-defined.

Next, we show that d is indeed a metric:

(a): $d(v,w) = 0$ iff $v = w$.

By (2), $a_i - 2^i \cdot \varepsilon > 0$ for all $i > 0$. Hence $d(v,w) = 0$ iff $vR_{=0}w$ iff $v = w$, according to Property (F3) of F-metric frames.

(b): $d(v,w) = d(w,v)$.

If $d(v,w) = \gamma + \varepsilon$, then $vR_{>a}w$ for all $a \in M$, i.e., by Property (F1), $\neg vR_{=a}w$ and $\neg vR_{<a}w$ for all $a \in M$. By Property (F9), $R_{=a}$ and $R_{<a}$ are symmetric, thus $wR_{>a}v$ for all $a \in M$ by (F1), and so $d(w,v) = \gamma + \varepsilon$.

Suppose $d(v,w) = a$ for some $a \in M$. By (1) this is the case if and only if $vR_{=a}w$, and so $wR_{=a}v$ by the symmetry of $R_{=a}$, thus $d(w,v) = a$. The case of $d(v,w) = a_i - 2^i \cdot \varepsilon$ follows similarly from the symmetry of $R_{<a}$, Property (F9).

(c): $d(u,v) + d(v,w) \geq d(u,w)$.

First, we can assume without loss of generality that $d(u,v), d(v,w) \neq 0$. Otherwise, if e.g. $d(u,v) = 0$, we have $u = v$ by (a) and the inequality obtains.

Case (i): If $d(u,v) + d(v,w) \geq \gamma + \varepsilon$, the inequality obtains.

Case (ii): Suppose $d(u,v) = a$ and $d(v,w) = b$ because of $uR_{=a}v$ and $vR_{=b}w$, with $a, b \in M$, and $a + b < \gamma + \varepsilon$. By (5) we then have $a + b \leq \gamma$ and thus $a + b \in M$, since M is a parameter set. By Property (F10), we have either (ii.i) $uR_{=a+b}w$, or (ii.ii) $uR_{<a+b}w$. In Case (ii.i) we have $d(u,w) = a + b$, and the inequality obtains. In Case (ii.ii) we have $d(u,w) \leq a + b - 2^{i_{a+b}} \cdot \varepsilon < a + b$.

Case (iii): $d(u,v) = a_i$, $d(v,w) = a_j - 2^j \cdot \varepsilon$ with $a_i, a_j \in M$, and, by assumption, $a_i + a_j - 2^j \cdot \varepsilon < \gamma + \varepsilon$. By definition of d, $uR_{=a_i}v$, $vR_{<a_j}w$ and $\neg vR_{<a_k}w$ for all $k < j$. Further, by (4), $a_i + a_j \leq \gamma$ and so $a_i + a_j \in M$.

By Property (F12) we obtain $uR_{<a_i+a_j}w$ and so

$$d(u,w) \leq a_i + a_j - 2^{i_{a_i+a_j}} \cdot \varepsilon \leq a_i + a_j - 2^j \cdot \varepsilon = d(u,v) + d(v,w),$$

since $i_{a_i+a_j} \geq \max(i,j) \geq j$.

Case (iv): $d(u,v) = a_i - 2^i \cdot \varepsilon$ and $d(u,v) = a_j$ with $a_i, a_j \in M$. This is similar to case (iii). We use again (F14) and additionally symmetry.

Case (v): $d(u,v) = a_i - 2^i \cdot \varepsilon$ and $d(v,w) = a_j - 2^j \cdot \varepsilon$, with $a_i, a_j \in M$. By definition of d, $uR_{\leq a_i}v$ and $vR_{\leq a_j}w$. By assumption, $a_i + a_j - 2^i \cdot \varepsilon - 2^j \cdot \varepsilon < \gamma + \varepsilon$. By (3) of

Lemma 3.13 we obtain $a_i + a_j \leq \gamma$ and thus $a_i + a_j \in M$. By Property (F11) we have $uR_{<a_i+a_j}w$. Thus

$$d(u,w) \leq a_i + a_j - 2^{i_{a_i+a_j}} \cdot \varepsilon \leq a_i + a_j - 2^{\max(i,j)+1} \leq a_i + a_j - 2^i \cdot \varepsilon - 2^j \cdot \varepsilon,$$

since we can assume $i, j \neq 0$ and so $2^{i_{a_i+a_j}} > 2^{\max(i,j)+1} = 2 \cdot 2^{\max(i,j)} \geq 2^i + 2^j$. Hence the inequality follows.

Now that we have established that d is a metric, we can define the metric space $S = \langle W, d \rangle$ and show that $\mathfrak{f}$ is its M-frame companion, i.e., that $\mathfrak{f}_{F,M}(S) = \mathfrak{f}$. To this end, we have to show that:

(A) $d(u,v) = a \iff uR_{=a}v$, for all $a \in M$;
(B) $d(u,v) < a \iff uR_{<a}v$, for all $a \in M$;
(C) $d(u,v) > a \iff uR_{>a}v$, for all $a \in M$;
(D) $a < d(u,v) < b \iff uR^{>a}_{<b}v$, for all $a, b \in M$;

(A): To prove (A), note that by (1) we have $\gamma + \varepsilon, a_i - 2^i \cdot \varepsilon \neq a$ for all $a \in M$. Thus we immediately obtain from the definition of d that $uR_{=a}v$ for some $a \in M$ if and only if $d(u,v) = a$.

(B): Suppose first that $d(u,v) < a$, then either (i) $d(u,v) = b < a$ and $wR_{=b}v$ for some $b \in M$, or, (ii) $d(u,v) = a_i - 2^i \cdot \varepsilon < a$. In Case (i) we obtain $uR_{<a}v$ by condition (F8). In Case (ii) we have $uR_{<a_i}v$ with a_i minimal with this property. By (1) of Lemma 3.13 we have $a_i - 2^i \cdot \varepsilon < a$ implies $a_i < a$. Hence, by condition (F7), $uR_{<a}v$.

Conversely, suppose that $uR_{<a}v$. Then, by (F5), $\neg uR_{=b}v$ for all $b \geq a$. We again have to distinguish two cases. Case (i): There exists a $b < a$ with $uR_{=b}v$. But then we have $d(u,v) = b < a$. In Case (ii), we have $\neg uR_{=b}v$ for all $b \in M$ and hence $d(u,v) = a_i - 2^i \cdot \varepsilon$ with $a_i \leq a$. Hence $d(u,v) < a$.

(C): Suppose first that $d(u,v) > a$. There are three cases to consider:
Case (i): $d(u,v) = \gamma + \varepsilon$. Then $uR_{>b}v$ for all $b \in M$. Hence, in particular, $uR_{>a}v$. Case (ii): $d(u,v) = b > a$ for some $b \in M$ and $uR_{=b}v$. By (F5) we have $\neg uR_{<a}v$ and by (F6) $\neg uR_{=a}v$. Hence, (F1) implies $uR_{>a}v$. Case (iii): $d(u,v) = a_i - 2^i \cdot \varepsilon > a$. Then $\neg uR_{=a}v$ by definition of d and $\neg uR_{<a}v$ since otherwise $d(u,v) < a$. Hence, by (F1), $uR_{>a}v$.

Conversely, suppose $uR_{>a}v$. There are again three cases. But note first that we cannot have $uR_{=b}v$ for $b \leq a$ due to (F1) and (F8). Case (i): For all $b \in M$ we have $uR_{>b}v$. Then $d(u,v) = \gamma + \varepsilon > a$ by definition of d. Case (ii): There is some $b > a$ with $uR_{=b}v$. Then $d(u,v) = b > a$. Case (iii): There is some $b > a$ with $uR_{<b}v$. Then $d(u,v) = a_i - 2^i \cdot \varepsilon$ with $a < a_i \leq b$. But by (1), $a_i - 2^i \cdot \varepsilon > a$, as required.

(D): Suppose first that $a < d(u,v) < b$. Then, clearly, $d(u,v) < \gamma$, so we cannot have $uR_{>c}v$ for all $c \in M$. There are two cases. Case (i): $d(u,v) = c$ with $a < c < b$, and $uR_{=c}v$. By (F8) we then have $uR_{<b}v$, by (F6) $\neg uR_{=a}v$, and by (F5) $\neg uR_{<a}v$. Hence, (F2) gives $uR_{<b}^{>a}v$. Case (ii): $d(u,v) = a_i - 2^i \cdot \varepsilon$ with $uR_{<a_i}v$, a_i minimal with this property, and $a < a_i - 2^i \cdot \varepsilon < b$. By (1) we have $a < a_i$, so $\neg uR_{<a}v$. Because we also have $\neg uR_{=a}v$ we get $uR_{>a}v$ by (F1). Further, we have $a_i \leq b$, so $uR_{<b}v$ by (F7). Now, (F2) implies $uR_{<b}^{>a}v$, as required.

Conversely, assume $uR_{<b}^{>a}v$. By (F1) and (F2) we get $uR_{>a}v$, $uR_{<b}v$, $\neg uR_{<a}v$ and $\neg uR_{=a}v$. We further distinguish two cases. Case (i): $d(u,v) = c$ and $uR_{=c}v$, $c \in M$. Then $b > c$ by (F5), $c = a$ is impossible by (F1), and $c < a$ would imply $uR_{<a}v$ by (F8). Hence $a < c < b$. Case (ii): $d(u,v) = a_i - 2^i \cdot \varepsilon$, a_i minimal with $uR_{<a_i}v$. Then, since a_i is minimal, we have $a_i \leq b$, hence $d(u,v) < b$. Finally, by (F7) we cannot have $a \geq a_i$, so $a < a_i$ and hence $a < d(u,v) < b$ by (1).

We can now prove (ii). Suppose that φ is satisfied in the metric space model

$$\mathfrak{B} = \left\langle W, d, p_0^{\mathfrak{B}}, p_1^{\mathfrak{B}}, \ldots, i_0^{\mathfrak{B}}, i_1^{\mathfrak{B}}, \ldots \right\rangle$$

based on the metric space $S = \langle W, d \rangle$, i.e., that $\langle \mathfrak{B}, w \rangle \models \varphi$ for some point $w \in W$. By Proposition 1.12, φ is satisfied in the frame-companion model $\mathfrak{M}_F(\mathfrak{B})$. It is easily checked that the relations of the frame companion $\mathfrak{f}_F(S)$ satisfy properties (F1)–(F14). Thus, $\mathfrak{f}_F(S)$ is an F-metric M-frame in which φ is satisfiable.

Conversely, suppose that φ is satisfied in an F-metric frame model

$$\mathfrak{M} = \left\langle \mathfrak{f}, p_0^{\mathfrak{M}}, p_1^{\mathfrak{M}}, \ldots, i_0^{\mathfrak{M}}, i_1^{\mathfrak{M}}, \ldots \right\rangle$$

based on the F-metric M-frame $\mathfrak{f} = \left\langle W, \{R_{<a}, R_{>a}, R_{=a}, R_{<b}^{>a}\}_{a,b \in M} \right\rangle$. We first define a finite parameter set $M(\varphi)$ and an F-metric $M(\varphi)$-frame $\mathfrak{f}^\dagger$ such that φ is satisfiable in $\mathfrak{f}$ if and only if it is satisfiable in $\mathfrak{f}^\dagger$.

Just as in the proof of Theorem 2.7, let

$$Par(\varphi) := \{a \in M \mid a \text{ occurs in } \varphi\}$$

and let $\gamma_\varphi := \max(Par(\varphi)) + 1$ and define $M(\varphi)$ as follows:

$$M(\varphi) := \{b \in M : \gamma_\varphi > b = b_1 + \ldots + b_n, b_i \in Par(\varphi), n < \omega\}.$$

As before, it is easily seen that $M(\varphi)$ is a finite parameter set.

Now we define the frame $\mathfrak{f}^\dagger$ as the frame-reduct of $\mathfrak{f}$ with respect to $M(\varphi)$, that is

$$\mathfrak{f}^\dagger := \mathfrak{f} \restriction_{(F, M(\varphi))} .$$

As remarked on Page 21, since $\varphi \in \mathcal{LO}_F[M(\varphi)]$, φ is satisfiable in $\mathfrak{f}$ if and only if it is satisfiable in $\mathfrak{f} \restriction_{(F,M(\varphi))}$. By (i) there is a metric space S such that $\mathfrak{f}^\dagger$ is its frame-companion, i.e., $\mathfrak{f}_{F,M(\varphi)}(S) = \mathfrak{f} \restriction_{(F,M(\varphi))}$. By Proposition 1.12, φ is satisfiable in S, which had to be shown.

□

3.2.2. Completeness. Theorem 3.12 implies that, to axiomatise the logic $\mathcal{MSO}_F[M]$ of metric spaces, it suffices to axiomatise the class $\mathfrak{F}_F[M]$ of F-metric M-frames. More specifically, the set of validities in metric spaces of the language $\mathcal{LO}_F[M]$ coincides with its validities over the class $\mathfrak{F}_F[M]$.

By using general completeness results from hybrid logic, however, the axiomatic system $\mathsf{MSO}_F[M]$, given below in Table 3.3, is in fact **strongly complete** with respect to the class of F-metric M-frames. Let us discuss and summarise the relevant parts of hybrid completeness theory.

The (second-order) **standard translation** from formulae of standard modal logic to the second-order frame correspondence language can be straightforwardly extended to cover nominals and the universal modality, and similarly to formulae from the *Sahlqvist fragment*, **pure formulae**, that is, formulae containing *only* nominals (rather than propositional variables), define frame classes that are always first-order definable (see, e.g., Gargov and Goranko [1993, Proposition 3.1]):

PROPOSITION 3.14. *Pure formulae define first-order definable classes of frames.*

As early as in Bull [1970], it was realised that axiomatisations with pure formulae give rise to 'easy' completeness proofs. Very roughly, completeness proofs for languages involving nominals and universal modality or the @-operator proceed by combining the techniques of canonical models from modal logic and a Henkin construction as in first-order logic. The main task is to show that we can construct **named models** from the canonical models, i.e., models which consist only of maximally consistent sets that contain a nominal, and such that we can still prove the usual truth lemma.

In Blackburn and Tzakova [1999], we find such a completeness proof for languages containing nominals and @-operator, and in Gargov and Goranko [1993] we find a completeness proof for languages with nominals and the universal modality. Although many first-order conditions that are modally or Sahlqvist definable are definable by pure formulae, e.g. reflexivity to name one of the simplest examples, this is not true in general. The Church-Rosser property

$$\forall u \forall v \forall w (uRv \wedge uRw \rightarrow \exists x (vRx \wedge wRx))$$

is modally definable by the Sahlqvist formula

$$\Diamond\Box\varphi \to \Box\Diamond\varphi,$$

which is not equivalent to any pure formula [Goranko and Vakarelov, 2001].

Now, it is known that any extension with pure axioms of the basic logic of the universal modality enriched with nominals is strongly complete with respect to the class of frames defined by the pure formulae when using the additional *covering rule* of inference (COV) [Goranko and Vakarelov, 2001]. To state this result properly, let us discuss the rule (COV) first.

(COV) was introduced in Passy and Tinchev [1991] to axiomatise **PDL** with nominals, and further examined most notably in Gargov et al. [1987] and Gargov and Goranko [1993]. Usually, (COV) takes on a rather complicated form, being formulated with the help of *universal forms* $u(\sharp)$ (compare Goldblatt [1982] and Gargov et al. [1987]), which are defined as follows:

DEFINITION 3.15 (UNIVERSAL FORMS). *Let $\sharp$ be a symbol not appearing in the language $\mathcal{LO}_F[M]$. The set $UF(\sharp)$ of* ***universal forms*** *of $\sharp$ is the smallest set that is closed under the following conditions:*

- *$\sharp$ is a universal form of $\sharp$, i.e. $\sharp \in UF(\sharp)$;*
- *If $u(\sharp) \in UF(\sharp)$ and $\varphi \in \mathcal{LO}_F[M]$ a formula, then $\varphi \to u(\sharp) \in UF(\sharp)$;*
- *If $u(\sharp) \in UF(\sharp)$ and $\bigcirc \in \{\mathsf{A}^{>a}, \mathsf{A}^{<a}, \mathsf{A}^{=a}, \mathsf{A}^{>a}_{<b}, \blacksquare \mid a, b \in M\}$, then $\bigcirc\, u(\sharp) \in UF(\sharp)$;*

Every universal form $u(\sharp)$ has precisely one occurrence of the symbol $\sharp$, so we may denote by $u(\varphi)$ the formula that results from the universal form $u(\sharp)$ by substituting the formula φ for the symbol $\sharp$. Furthermore, up to propositional equivalence, every universal form can be rewritten as

$$\varphi_0 \to \triangledown_1^{k_1}(\varphi_1 \to \ldots \to \triangledown_{n-1}^{k_{n-1}}(\varphi_{n-1} \to \triangledown_n^{k_n}(\varphi_n \to \sharp)) \ldots)$$

where the $\triangledown_i^{k_i}$ are sequences of k_i 'universal' distance operators from the list $\{\mathsf{A}^{>a}, \mathsf{A}^{<a}, \mathsf{A}^{=a}, \mathsf{A}^{>a}_{<b} \mid a, b \in M\}$ or the universal modality $\blacksquare$, and some of the φ_i may be $\top$, when necessary. The number n is called the **depth** of $u(\sharp)$. Given a sequence $\triangledown_i^{k_i}$, we denote by $\triangle_i^{k_i}$ the sequence of 'existential' distance operators, including the existential $\blacklozenge$, that result from $\triangledown_i^{k_i}$ by replacing every $\mathsf{A}^{>a}$ by $\mathsf{E}^{>a}$ etc. Then, the negation $\neg u(\sharp)$ of a universal form is called an **existential form** and is, again modulo propositional equivalence, of the form

$$\varphi_0 \wedge \triangle_1^{k_1}(\varphi_1 \wedge \ldots \wedge \triangle_n^{k_n}(\varphi_n \wedge \sharp) \ldots).$$

The standard (COV) rule—which is needed in general for pure extensions—states that if $u(\neg i)$ is derivable for some universal form $u(\sharp)$ and nominal i not appearing in $u(\sharp)$, then infer $u(\bot)$. To understand this rule a bit better, it is probably best to look at a soundness proof for it, given here with respect to the class of all distance spaces:

LEMMA 3.16 (SOUNDNESS OF (COV)). *The rule* (COV) *preserves validity in the class* $\mathcal{D}$ *of all distance spaces.*

PROOF. We show something slightly stronger than the usual soundness, namely that (COV) preserves validity in each *fixed* distance space $\langle W, d\rangle$. Suppose $u(\sharp)$ is a universal form such that $\langle W, d\rangle \models u(\neg i)$, i.e., that $u(\neg i)$ is valid in on $\langle W, d\rangle$. We may suppose without loss of generality that $u(\sharp)$ is of the form

$$u(\sharp) = \varphi_0 \to \triangledown_1^{k_1}(\varphi_1 \to \ldots \to \triangledown_{n-1}^{k_{n-1}}(\varphi_{n-1} \to \triangledown_n^{k_n}(\varphi_n \to \sharp))\ldots),$$

where the depth of $u(\sharp)$ is n, some of the φ_i might be $\top$, and the nominal i does not occur in $u(\sharp)$. Assume that there is a full model $\mathfrak{M}$ based on some distance space $S = \langle W, d\rangle$ and a point v in W such that $\langle \mathfrak{M}, v\rangle \models \neg u(\bot)$, i.e., that

$$\langle \mathfrak{M}, v\rangle \models \varphi_0 \wedge \triangle_1^{k_1}(\varphi_1 \wedge \ldots \triangle_n^{k_n}(\varphi_n \wedge \top)\ldots).$$

Then there is a sequence $\langle v, w_1, w_2, \ldots, w_n\rangle$ in W that witnesses the truth of $\neg u(\bot)$, i.e., $\langle \mathfrak{M}, w_i\rangle \models \varphi_i$, for $1 \le i \le n$. Since i does not occur in $u(\sharp)$, we can define an i-variant $\mathfrak{M}'$ of $\mathfrak{M}$ by setting $i^{\mathfrak{M}'} = \{w_n\}$ and $j^{\mathfrak{M}'} = j^{\mathfrak{M}}$ for every nominal $j \neq i$ and $p^{\mathfrak{M}'} = p^{\mathfrak{M}}$ for every propositional variable p, such that we still have $\langle \mathfrak{M}', w_i\rangle \models \varphi_i$, for $1 \le i \le n$. But $\langle W, d\rangle \models u(\neg i)$ implies $\langle \mathfrak{M}', v\rangle \models u(\neg i)$ and thus $\langle \mathfrak{M}', w_n\rangle \models \varphi_n \to \neg i$. Since $\{w_n\} = i^{\mathfrak{M}'}$, this implies $\langle \mathfrak{M}', w_n\rangle \models \neg\varphi_n$, which is a contradiction. □

We can now state the general completeness result mentioned above. By $\mathbf{K}_{\blacksquare}[M]$ we shall denote the minimal multi-modal logic in the language $\mathcal{LO}_F[M]$, comprising the **S5** Axioms for $\blacksquare$, the Axioms (Nom_1), (Nom_2) and ($\mathsf{Inc}_{\bigcirc}$) from Table 3.3, the covering rule (COV), modus ponens, generalisation for $\blacksquare$, and **sorted substitution**, i.e., nominals may be substituted for nominals and arbitrary formulae for propositional variables.

Further, by $L \oplus_C \Gamma$ we denote the smallest hybrid logic obtained by adding the formulae in Γ as axioms to the logic L, and which is closed under modus ponens, generalisation, sorted substitution, and (COV).

THEOREM 3.17 (PURE COMPLETENESS). *Let* M *be a parameter set and* Π *a set of pure formulae in language* $\mathcal{LO}_F[M]$. *Then the multi-modal (hybrid) logic* $\mathbf{K}_{\blacksquare}[M] \oplus_C \Pi$ *that has* (COV) *as additional rule of inference is strongly sound and complete with respect to the class of frames that* Π *defines.*

The role that the (COV) rule plays in the proof of this result is to ensure that, given the canonical model for the logic $\mathbf{K}_{\blacksquare}[M] \oplus_C \Pi$, we can select from any *definable* subset of the model, i.e., every set of points at which some formula is true, a named member, i.e., a point that contains a nominal [Gargov and Goranko, 1993]. Thus, there are, as it were, *enough* named maximally consistent sets to still prove a truth lemma.

Note that, as opposed to the case of the logic $\mathcal{MS}_D[M]$ where nominals were not available, we can, in a way, *'internalise'* the distance function in the modal language $\mathcal{LO}_F[M]$ with nominals. For, if $\mathfrak{M}$ is any full model based on a distance space $\langle W, d\rangle$ and i, j are nominals, then, for any point u of W we have

$$\langle \mathfrak{M}, u\rangle \models \blacksquare(i \to \mathsf{E}^{=a} j) \iff d(i^{\mathfrak{M}}, j^{\mathfrak{M}}) = a,$$

and so on for the cases $d(i^{\mathfrak{M}}, j^{\mathfrak{M}}) < a$, $d(i^{\mathfrak{M}}, j^{\mathfrak{M}}) > a$ and $a < d(i^{\mathfrak{M}}, j^{\mathfrak{M}}) < b$.

Now, to axiomatise the class $\mathfrak{F}_F$ using the language $\mathcal{LO}_F[M]$, we have a number of options. First, we can check that, indeed, all the frame conditions from Definition 3.11 are definable by pure formulae. But this is an immediate corollary to the next lemma. First, recall that the **first-order correspondence language** $\mathcal{L}^1_{F,M}$ for a language of type $\mathcal{LO}_F[M]$ comprises a countably infinite set $\{x_i \mid x < \omega\}$ of variables, a countably infinite set $\{c_i \mid i < \omega\}$ of constants (one for each nominal), binary relation $R_{<a}, R_{>a}, R_{=a}$ and $R^{>a}_{<b}$ for any $a, b \in M$, as well as equality $\doteq$. Then we have:

LEMMA 3.18. *Every class of frames defined by universal first-order formulae of the first-order correspondence language $\mathcal{L}^1_{F,M}$ of $\mathcal{LO}_F[M]$ can be defined by pure formulae of $\mathcal{LO}_F[M]$, where M is arbitrary.*

PROOF. We define a translation $.^\dagger$ from first-order formulae of $\mathcal{L}^1_{F,M}$ to pure formulae such that, for any M-frame $\mathfrak{f}$ and universal first-order formula φ of $\mathcal{L}^1_{F,M}$, we have

$$(\star) \qquad \mathfrak{f} \models \varphi \iff \mathfrak{f} \models \varphi^\dagger.$$

Clearly, every universal formula of $\mathcal{L}^1_{F,M}$ can be written, without loss of generality, in the form

$$\varphi(\bar{x}) = \forall x_1 \ldots \forall x_n B\big(x_1, \ldots, x_n, c_1, \ldots, c_m\big),$$

where $B\big(x_1, \ldots, x_n, c_1, \ldots, c_m\big)$ is a Boolean combination of atoms of the form $R_{<a}(s_a, t_a)$, etc., and equalities $s \doteq t$, where the terms s_a, t_a, s, t are either variables from $\{x_1, \ldots, x_n\}$ or constants from $\{c_1, \ldots c_m\}$, and $a \in M$.

Define the translation $.^\dagger$ by, given φ, associating with every variable x_l a nominal $(x_l)^\dagger = i_l$ and with every constant c_k a nominal $(c_k)^\dagger = j_k$ such that

$$\{i_1, \ldots, i_n\} \cap \{j_1, \ldots j_m\} = \varnothing,$$

removing the quantifiers $\forall x_1 \ldots \forall x_n$, and by simultaneously replacing

- equalities $s \doteq t$ with $\blacksquare(s^\dagger \leftrightarrow t^\dagger)$;
- binary atoms

– $R_{<a}(s,t)$ with $\blacksquare(s^\dagger \to \mathsf{E}^{<a} t^\dagger)$;
– $R_{>a}(s,t)$ with $\blacksquare(s^\dagger \to \mathsf{E}^{>a} t^\dagger)$;
– $R_{=a}(s,t)$ with $\blacksquare(s^\dagger \to \mathsf{E}^{=a} t^\dagger)$;
– $R^{>a}_{<b}(s,t)$ with $\blacksquare(s^\dagger \to \mathsf{E}^{>a}_{<b} t^\dagger)$.

Now, an easy induction shows $(\star)$. □

This immediately gives us the following completeness result:

THEOREM 3.19. *$\mathcal{MSO}_F[M]$ is strongly sound and complete with respect to a pure extension of* $\mathbf{K}_\blacksquare[M]$.

PROOF. Obviously, conditions (F1)–(F12) are universal first-order conditions formulated in $\mathcal{L}^1_{F,M}$. Thus, Lemma 3.18 together with Theorem 3.17 proves the Theorem. □

Yet, we can do a bit better. Notice that all the modalities in the set F are symmetric by conditions (F1), (F2) and (F9). Fortunately, symmetric modalities are a special case of the *versatile similarity types* of Venema [1993], or the *reversive languages* of Goranko [1998]. Moreover, it is clear that the languages $\mathcal{LO}_F[M]$ (having nominals and the universal modality) and $\mathcal{L}_F[M]$ (having the difference operator $\mathsf{E}^{>0}$) have the same expressive power when it comes to frame definability [de Rijke, 1992]. Thus, we can choose between giving an axiomatisation over the language $\mathcal{L}_F[M]$—using the general completeness theorems available for languages employing the difference operator and Sahlqvist axioms [Venema, 1993]—and a (mixed) axiomatisation using pure formulae and Sahlqvist schemes, using the (generalised) Sahlqvist completeness theorem from Goranko and Vakarelov [2001].

To be precise about the results that we will be using in the completeness proof below, we give simplified versions of some of the definitions and results of Goranko and Vakarelov [2001]. For a **polyadic modal** language $\mathcal{L}$, the definition of **reversiveness** of the language requires that, for every n-ary modal operator $\triangledown$ interpreted by an $n+1$ ary relation R, there are n-ary modal operators $\triangledown^{-i}$, $i = 1, \dots, n$, interpreted by the **inverse relations** R^{-i} as follows:

$$xR^{-i}y_1 \dots y_i \dots y_n \iff y_i R y_1 \dots x \dots y_n.$$

Obviously, for unary modalities, this requirement boils down to having a tense similarity type, i.e, for every modality $\Box$ interpreted by a binary relation R there is an inverse modality $\Box^{-1}$ interpreted by R^{-1}, where

$$xR^{-1}y \iff yRx.$$

Also, modalities that are interpreted by **symmetric** binary relations are therefore their own inverses in this sense.

Now, by $\mathbf{K}^r_{\blacksquare}[M]$ we shall denote the logic in the language $\mathcal{LO}_F[M]$ that is defined just like $\mathbf{K}_{\blacksquare}[M]$, with the exception that the rule (COV) is replaced by the rule (COV_0), that is, infer φ from $i \to \varphi$, $(i \notin \varphi)$, and it additionally contains the axioms

$$(\mathsf{B}_{\bigcirc}) \qquad \varphi \to \bigcirc\neg\bigcirc\neg\varphi \quad \bigcirc \in \{\mathsf{A}^{=a}, \mathsf{A}^{<a}, \mathsf{A}^{>a}, \mathsf{A}^{>a}_{<b}\} \quad (a, b \in M).$$

Further, we use $L \oplus_{C_0} \Gamma$ to denote the smallest hybrid logic obtained by adding the formulae in Γ as axioms to the logic L and which is closed under modus ponens, generalisation, sorted substitution, and (COV_0).

Note that, in the case of symmetric modalities, and more generally in the case of reversive languages, universal forms are no longer necessary: given the rule (COV_0), the rule (COV) becomes derivable. A universal form of type

$$\varphi_0 \to \triangledown_1^{k_1}(\varphi_1 \to \ldots \to \triangledown_{n-1}^{k_{n-1}}(\varphi_{n-1} \to \triangledown_n^{k_n}(\varphi_n \to \sharp))\ldots)$$

is provable in $\mathbf{K}^r_{\blacksquare}[M]$ if and only if its 'converse'

$$\neg\sharp \to (\varphi_n \to \triangledown_n^{k_n}(\varphi_{n-1} \to \ldots \to \triangledown_1^{k_1}\neg\varphi_0)\ldots),$$

is provable in $\mathbf{K}^r_{\blacksquare}[M]$, compare Gabbay and Hodkinson [1990] and Gargov and Goranko [1993]. Thus, if $u(\neg i)$ is provable for some universal form, so is

$$i \to (\varphi_n \to \triangledown_n^{k_n}(\varphi_{n-1} \to \ldots \to \triangledown_1^{k_1}\neg\varphi_0)\ldots)$$

and so, by (COV_0), we obtain

$$\varphi_n \to \triangledown_n^{k_n}(\varphi_{n-1} \to \ldots \to \triangledown_1^{k_1}\neg\varphi_0)\ldots),$$

which is equivalent to $u(\bot)$.

Assume now **Sahlqvist formulae** in the language $\mathcal{LO}_F[M]$ are defined as usual.[1] We say that two formulae φ and ψ are **axiomatically equivalent** over the base logic $\mathbf{K}^r_{\blacksquare}[M]$, if $\varphi \vdash_{\mathbf{K}^r_{\blacksquare}[M]} \psi$ and $\psi \vdash_{\mathbf{K}^r_{\blacksquare}[M]} \varphi$.

THEOREM 3.20 (Goranko & Vakarelov). *Over the base logic $\mathbf{K}^r_{\blacksquare}[M]$, every Sahlqvist formula is axiomatically equivalent to a pure formula.*

The following lemma is immediate from the fact that the inference rules preserve validity on frames.

LEMMA 3.21. *Axiomatically equivalent formulae define the same classes of frames.*

Thus, the following theorem is an almost immediate consequence of Theorems 3.17 and 3.20.

[1] In Goranko and Vakarelov [2001], the (standard) class of Sahlqvist formulae is substantially extended, and the results we mention here hold for this larger class as well.

THEOREM 3.22 (SAHLQVIST COMPLETENESS). *Every extension* $\mathbf{K}^r_{\blacksquare}[M] \oplus_{C_0} \Sigma \oplus_{C_0} \Pi$ *of* $\mathbf{K}^r_{\blacksquare}[M]$ *in the language* $\mathcal{LO}_F[M]$ *by a set* Σ *of Sahlqvist axioms and a set* Π *of pure axioms is complete with respect to the class of symmetric M-frames defined by the first-order conditions defined by* $\Sigma \cup \Pi$.

We now give the axiomatisation of the class $\mathfrak{F}_F[M]$ for the language with nominals, using both, pure and Sahlqvist axioms, which we feel to be the most elegant formulation. The axiomatic system, which is listed in Table 3.3, will be denoted by $\mathsf{MSO}_F[M]$.

Some comments on the choice of axioms might be in order. First, we have the standard **S5** and 'inclusion' axioms for the universal modality, as well as the Axioms (Nom_1) and (Nom_2) which suffice to axiomatise nominals in the presence of the universal modality, compare Gargov and Goranko [1993]. The remaining axioms are basically derived from the Representation Theorem, Theorem 3.12. They precisely define the first-order conditions given for F-metric M-frames, i.e., those that are needed to construct an appropriate metric space from a frame.

The inference rules of the system $\mathsf{MSO}_F[M]$ are **sorted substitution** (SSUB), **modus ponens**, **necessitation** for $\blacksquare$, as well as (COV_0).

Note that, in the presence of the Inclusion Axioms $(\mathsf{Inc}_{\bigcirc})$, all of the rules of necessitation

$$\frac{\varphi}{\bigcirc\varphi}\ (\mathrm{RN}\bigcirc), \quad \bigcirc \in \{\mathsf{A}^{>a}, \mathsf{A}^{<a}, \mathsf{A}^{=a}, \mathsf{A}^{>a}_{<b} \mid a, b \in M\}$$

are derivable in MSO_F.

Thus, the details of the proof of the following theorem are easily spelled out:

THEOREM 3.23 (STRONG FRAME COMPLETENESS).
For every $\mathcal{LO}_F[M]$*-formula* φ *and set of formulae* Γ*:*

$$\Gamma \vdash_{\mathsf{MSO}_F} \varphi \quad \Longleftrightarrow \quad \Gamma \models_{\mathfrak{F}_F[M]} \varphi.$$

PROOF. We first prove the soundness part.

The validity of the axioms for the universal modality, the Inclusion Axioms $(\mathsf{Inc}_<)$ and $(\mathsf{Inc}_=)$, and the **S5**-Axioms $(4_\blacksquare)$, $(\mathsf{B}_\blacksquare)$ and $(\mathsf{T}_\blacksquare)$, is immediate. Similarly, the validity of the axioms for nominals, (Nom_1) and (Nom_2), is proved as usual.

Checking the validity of the remaining axiom schemata is a routine matter: they directly correspond to properties of metrics as specified by the F-metric M-frame conditions (F1)–(F12). To illustrate this, let us check the validity of the schemata $(\mathsf{Def}^>_<)$ and (Tra_1). Let $\mathfrak{M}$ be a full model based on an F-metric M-frame $\mathfrak{f}$, $u \in W$, and $a, b, a+b \in M$.

$(\mathsf{Def}^>_<)$: We have

$$\langle \mathfrak{M}, u \rangle \models \mathsf{E}^{>a}_{<b} i$$

Axiom schemata for $\mathsf{MSO}_F[M]$

(CL)	Axioms of propositional calculus	
$(\mathsf{K}_{\bigcirc})$	$\bigcirc(\varphi \to \psi) \to (\bigcirc\varphi \to \bigcirc\psi)$, where $\bigcirc \in \{\blacksquare, \mathsf{A}^{=a}, \mathsf{A}^{>a}, \mathsf{A}^{<a}, \mathsf{A}^{>a}_{<b}\}$	$(a, b \in M)$
$(\mathsf{Def}^{>})$	$\mathsf{E}^{>a} i \leftrightarrow (\mathsf{A}^{<a}\neg i \wedge \mathsf{A}^{=a}\neg i)$	$(a \in M)$
$(\mathsf{Def}^{>}_{<})$	$\mathsf{E}^{>a}_{<b} i \leftrightarrow (\mathsf{A}^{<a}\neg i \wedge \mathsf{A}^{=a}\neg i \wedge \mathsf{A}^{>b}\neg i \wedge \mathsf{A}^{=b}\neg i)$	$(a, b \in M)$
$(\mathsf{Dis}^{\leq}_{=})$	$\mathsf{E}^{=a} i \to \mathsf{A}^{<b}\neg i$	$(a \geq b)$
$(\mathsf{Dis}_{=})$	$\mathsf{E}^{=a} i \to \mathsf{A}^{=b}\neg i$	$(a \neq b)$
$(\mathsf{T}_{=0})$	$\mathsf{A}^{=0}\varphi \leftrightarrow \varphi$	$(0 \in M)$
$(\mathsf{Bot}_{<0})$	$\mathsf{A}^{<0}\bot$	$(0 \in M)$
$(\mathsf{Mon}_{<})$	$\mathsf{A}^{<a}\varphi \to \mathsf{A}^{<b}\varphi$	$(a \geq b)$
$(\mathsf{Mon}^{\leq}_{=})$	$\mathsf{A}^{<a}\varphi \to \mathsf{A}^{=b}\varphi$	$(a > b)$
$(\mathsf{B}_{\bigcirc})$	$\varphi \to \bigcirc\neg\bigcirc\neg\varphi$, $\quad \bigcirc \in \{\mathsf{A}^{=a}, \mathsf{A}^{>a}, \mathsf{A}^{<a}, \mathsf{A}^{>a}_{<b}\}$	$(a, b \in M)$
(Tra_1)	$(\mathsf{A}^{<a+b}\varphi \wedge \mathsf{A}^{=a+b}\varphi) \to \mathsf{A}^{=a}\mathsf{A}^{=b}\varphi$	$(a + b \in M)$
(Tra_2)	$\mathsf{A}^{<a+b}\varphi \to \mathsf{A}^{<a}\mathsf{A}^{<b}\varphi$	$(a + b \in M)$
(Tra_3)	$\mathsf{A}^{<a+b}\varphi \to \mathsf{A}^{<a}\mathsf{A}^{=b}\varphi$	$(a + b \in M)$
$(\mathsf{Inc}_{\bigcirc})$	$\blacksquare\varphi \to \bigcirc\varphi$, $\quad \bigcirc \in \{\mathsf{A}^{=a}, \mathsf{A}^{>a}, \mathsf{A}^{<a}, \mathsf{A}^{>a}_{<b}\}$	$(a, b \in M)$
$(4_{\blacksquare})$	$\blacksquare\varphi \to \blacksquare\blacksquare\varphi$	
$(\mathsf{B}_{\blacksquare})$	$\varphi \to \blacksquare\blacklozenge\varphi$	
$(\mathsf{T}_{\blacksquare})$	$\blacksquare\varphi \to \varphi$	
(Nom_1)	$\blacklozenge i$	
(Nom_2)	$\blacklozenge(i \wedge \varphi) \to \blacksquare(i \to \varphi)$	

Inference rules

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ (\mathrm{MP}) \qquad \frac{\varphi}{\blacksquare\varphi}\ (\mathrm{RN}) \qquad \frac{i \to \varphi}{\varphi}, i \notin \varphi\ (\mathrm{COV}_0) \qquad (\mathrm{SSUB})$$

Table 3.3: The axiomatic system $\mathsf{MSO}_F[M]$.

if and only if $uR^{>a}_{<b}i^{\mathfrak{M}}$ if and only if $\neg uR_{=a}i^{\mathfrak{M}}$ and $\neg uR_{<a}i^{\mathfrak{M}}$ and $uR_{<b}i^{\mathfrak{M}}$ (by condition (F2)) if and only if $\neg uR_{=a}i^{\mathfrak{M}}$ and $\neg uR_{<a}i^{\mathfrak{M}}$ and $\neg uR_{>b}i^{\mathfrak{M}}$ and $\neg uR_{=b}i^{\mathfrak{M}}$ (by conditions (F1) and (F5)) if and only if

$$\langle \mathfrak{M}, u \rangle \models \mathsf{A}^{<a}\neg i \wedge \mathsf{A}^{=a}\neg i \wedge \mathsf{A}^{>b}\neg i \wedge \mathsf{A}^{=b}\neg i.$$

(Tra_1): Suppose $\langle \mathfrak{M}, u\rangle \models \mathsf{E}^{=a}\mathsf{E}^{=b}\neg\varphi$. Then there are points $v, w \in W$ such that $uR_{=a}v$, $vR_{=b}w$, and $\langle \mathfrak{M}, w\rangle \models \neg\varphi$. By condition (F10), we obtain (i) $uR_{=a+b}w$, or (ii) $uR_{<a+b}w$. Thus $\langle \mathfrak{M}, u\rangle \models \mathsf{E}^{<a+b}\neg\varphi \vee \mathsf{E}^{=a+b}\neg\varphi$.

The soundness of the remaining schemata is checked in a similar way.

That the rules of modus ponens and necessitation preserve validity is clear. So let us check the soundness of the rule (COV_0), which just amounts to a much simplified version of the proof of Lemma 3.16. Fix some F-metric M-frame $\mathfrak{f}$ such that $\mathfrak{f} \models i \rightarrow \varphi$, where the nominal i does not appear in φ. Assume that there is a full model $\mathfrak{M}$ based on $\mathfrak{f}$ and a point v in W such that $\langle \mathfrak{M}, v\rangle \models \neg\varphi$. Since i does not occur in φ, we can define an i-variant $\mathfrak{M}'$ of $\mathfrak{M}$ by setting $i^{\mathfrak{M}'} = \{v\}$ and $j^{\mathfrak{M}'} = j^{\mathfrak{M}}$ for every nominal $j \neq i$ and $p^{\mathfrak{M}'} = p^{\mathfrak{M}}$ for every propositional variable p, such that $\langle \mathfrak{M}', v\rangle \models i \wedge \neg\varphi$, which is a contradiction.

For the completeness part it suffices to show, according to Theorem 3.22, that the axioms given in Table 3.3 indeed define the frame conditions (F1)–(F12). We show this only for the condition (F10), and leave the remaining cases to the reader. We have shown above that Axiom (Tra_1) is valid in the class of M-frames satisfying condition (F10). Now assume that condition (F10) is violated, i.e., take an M-frame $\mathfrak{f}$ such that there are points $u, v, w \in W$ with $uR_{=a}v$, $vR_{=b}w$, $\neg uR_{=a+b}w$, and $\neg uR_{<a+b}w$. Define a model $\mathfrak{M}$ based on $\mathfrak{f}$ by setting $p^{\mathfrak{M}} = W \setminus \{w\}$, and $q^{\mathfrak{M}}$ arbitrary for propositional variables $q \neq p$. Then

$$\langle \mathfrak{M}, u\rangle \models \mathsf{A}^{=a+b}p \wedge \mathsf{A}^{<a+b}p \wedge \mathsf{E}^{=a}\mathsf{E}^{=b}\neg p,$$

from which it follows that (Tra_1) is not valid in M-frames violating (F10). □

Next, as a corollary to Theorem 3.12 and Theorem 3.23, we obtain:

THEOREM 3.24 (WEAK METRIC COMPLETENESS). *For every $\mathcal{LO}_F[M]$-formula φ:*

$$\vdash_{\mathsf{MSO}_F} \varphi \quad \Longleftrightarrow \quad \varphi \in \mathcal{MSO}_F.$$

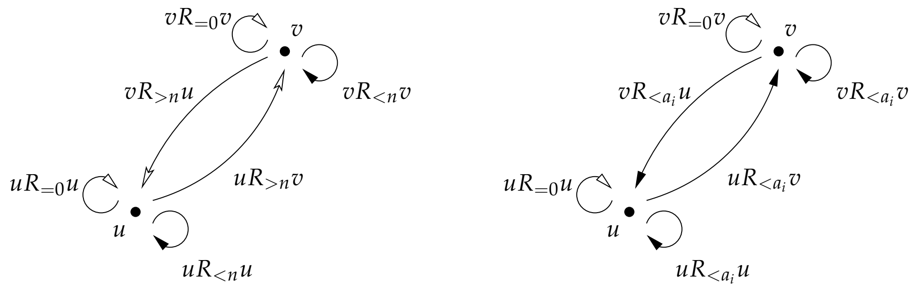
3.3. Compactness

To clarify the relationship between Theorems 3.23 and 3.24, let us briefly discuss the compactness property in metric spaces. We have shown in Theorem 3.12 (i) that, for finite parameter sets M, there is a precise correspondence between F-metric M-frames and metric spaces, that is, there are maps

$$\mathcal{MS} \xrightarrow{h} \mathfrak{F}_F[M], \text{ with } h(S) := \mathfrak{f}_{F,M}(S)$$

and

$$\mathfrak{F}_F[M] \xrightarrow{g} \mathcal{MS}, \text{ with } h \circ g = \mathsf{id}_{\mathfrak{F}_F[M]},$$



¬(†) ¬(‡)

Figure 3.1: Non-standard frames with 'points at infinity' and 'infinitesimal points'.

where $\mathrm{id}_{\mathfrak{F}_F[M]}$ is the identity function on the class $\mathfrak{F}_F[M]$ (but note that, in general, $g \circ h(S) \neq S$). This means that, for finite M, the local consequence relations $\models_{\mathcal{MS}}$ and $\models_{\mathfrak{F}_F[M]}$ coincide and are, by strong completeness with respect to F-metric M-frames, both compact. This picture changes radically, however, when we move to infinite parameter sets. For instance, let $M = \mathbb{N}$ and consider the set

$$\Gamma = \{\neg \mathsf{E}^{<n} p \mid n < \omega\} \cup \{\blacklozenge p\}.$$

Then every finite subset Γ_0 of Γ is satisfiable in some metric space, but Γ is not. This shows that the local consequence relation $\models_{\mathcal{MS}}$ is not compact for the language $\mathcal{LO}_F[\mathbb{N}]$. Since the local consequence relation $\models_{\mathfrak{F}_F[M]}$ is compact independently of M, Γ is satisfiable in some F-metric M-frame. In fact, define a 'non-standard' F-metric M-frame $\mathfrak{f}$ that is based on the set $W = \{u, v\}$ by setting $uR_{>n}v$, for all $n \in \mathbb{N}$, $uR_{=0}u$, $vR_{=0}v$, $uR_{<n}u$ and $vR_{<n}v$, for all $n > 0$, and $R^{>n}_{<m} = \varnothing$, for all n, m, see Figure 3.1.

Then $\mathfrak{f}$ is an F-metric $[\mathbb{N}]$-frame and Γ is satisfiable in u. But $\mathfrak{f}$ is not the frame-companion of any metric space S, since any such frame-companion satisfies the additional infinitary condition

$$(\dagger) \qquad \bigcup_{n<\omega} R_{<n} = W \times W.$$

This condition holds generally in any frame-companion of some metric space, whenever the parameter set M is **unbounded**: since the distance between any two points is functionally determined by the metric, the **Archimedean axiom** for the real numbers

$$\forall a, b \in \mathbb{R}^+ \exists n \in \mathbb{N} : n \cdot a > b$$

guarantees that, eventually, we will find some $n \in \mathbb{N}$ such that $d(u, v) < n$, which rules out frames as the one defined above.

Similarly, suppose M is a **dense** subset of $\mathbb{R}^+$, e.g. $M = \mathbb{Q}^+$. Suppose $\mathfrak{a} = (a_n)_{n<\omega}$ is some strictly decreasing sequence of numbers from M, i.e., $a_{n+1} < a_n$ for all $n < \omega$, with $\inf(\mathfrak{a}) = 0$. Then every frame-companion of some metric space $\langle W, d\rangle$ satisfies the following condition:

$$(\ddagger) \qquad \forall v, w \in W : v \neq w \implies \inf(\{a_i \mid a_i \in \mathfrak{a} \text{ and } vR_{<a_i}w\}) > 0.$$

On the other hand, there are F-metric $\mathbb{Q}^+$-frames that violate it, and which are hence not the frame-companion of a metric space, compare, again, Figure 3.1.

It should be now rather clear that the concept of 'metric space' is not first-order definable on frames: an adequate stronger relational representation of metric spaces also has to 'represent' the theory of real numbers. In particular, given an arbitrary F-metric M-frame $\mathfrak{f}$ with an infinite parameter set M, it is in general not possible to find an equivalent metric space S, i.e, such that $\mathsf{Th}(\mathfrak{f}) = \mathsf{Th}(S)$. At this point, we can proceed in different ways. One possibility is to enrich the language $\mathcal{LO}_F[M]$ by numerical variables $x, y, \dots$ that range over M and can take the place of parameters, and to allow explicit quantification over these variables, with the obvious semantic interpretation.[2] Then, for instance, the formula

$$\mathsf{E}^{>0}\varphi \to \mathsf{E}^{\exists x.<x}\varphi$$

taken as an extra axiom corresponds to the frame-condition

$$\forall u, v.\exists a \in M.uR_{<a}v,$$

thus expressing (†).

3.4. More Axioms and Interpolation

In this section, we discuss several themes related to the frame representations of metric spaces given above. Using the fact that by algebraically manipulating the frame conditions given for F-metric M-frames (or D-metric M-frames) into equivalent forms taking other operator sets and the corresponding relations as *primitive* (in the sense that the relations $R_{=a}$ and $R_{<a}$ were used to *define* the relations $R^{>a}_{<b}$ in Definition 3.11), the frame representation for the language $\mathcal{LO}_F[M]$ can be used to derive corresponding representation theorems for various sublanguages $\mathcal{LO}_O[M]$, $O \subseteq \mathfrak{O}$, and to give respective sound and complete axiom systems. Finally, we can use these relational representations to derive various positive and negative results about Craig interpolation for languages without nominals.

[2]A similar extension with variables but not allowing quantification was considered for weaker decidable logics of distance in Wolter and Zakharyaschev [2003], and it was shown that adding variables ranging over parameters and linear inequalities as constraints preserves decidability.

The fragments we are interested in here, besides the operator sets F and D defined earlier, are the following:

DEFINITION 3.25 (OPERATOR SETS). *Given some parameter set M, let:*

(1) $L_1[M] := \{\mathsf{A}^{\leq a}, \mathsf{A}^{<a} \mid a \in M\}$;
(2) $L_2[M] := \{\mathsf{A}^{\leq a} \mid a \in M\}$;
(3) $L_3[M] := \{\mathsf{A}^{<a} \mid a \in M\}$;
(4) $G_1[M] := \{\mathsf{A}^{\geq a}, \mathsf{A}^{>a} \mid a \in M\}$;
(5) $G_2[M] := \{\mathsf{A}^{\geq a} \mid a \in M\}$;
(6) $G_3[M] := \{\mathsf{A}^{>a} \mid a \in M\}$;

This defines languages $\mathcal{LO}_{L_i}[M]$, $\mathcal{L}_{L_i}[M]$, etc. Again, we will usually suppress the dependence on the parameter set M and will talk about operator sets L_1, G_2, etc.

As concerns the interpolation property, in modal logic, one usually distinguishes a *weak* and a *strong interpolation property*. Strong interpolation is also referred to as *arrow interpolation* or *Craig interpolation*, and weak interpolation as *turnstile interpolation*. Furthermore, weak interpolation also depends on the consequence relation being used, i.e., depends on whether we use the local or the global consequence. Let us define these notions properly. If φ is a formula in $\mathcal{LO}[M]$, we shall denote by $\mathcal{V}(\varphi)$ the set of propositional variables and nominals appearing in φ and by $\mathcal{P}(\varphi)$ the set of propositional variables appearing in φ. As specified in Definition 1.7, we distinguish the *local* and *global* consequence relations $\models^i_l$ and $\models^i_g$ with respect to the classes $\mathcal{D}^i$ of distance spaces. Further, as we have discussed in the last section, the local consequence relations with respect to metric spaces and the respective classes of standard frames coincide, whenever the parameter sets M are finite, and can thus be identified whenever convenient. Indeed, as we will explain in a moment, it is enough to investigate the interpolation properties with respect to the relational semantics. If $\boldsymbol{F}$ is a class of frames, let us denote the local consequence relation with respect to $\boldsymbol{F}$ by $\models^{\boldsymbol{F}}_l$, and the global consequence relation by $\models^{\boldsymbol{F}}_g$.

DEFINITION 3.26 (INTERPOLATION PROPERTIES). *Let $\mathcal{L}_O[M]$ be some language (without nominals), $\boldsymbol{F}$ a class of frames for $\mathcal{L}_O[M]$, $L_{O,\boldsymbol{F},M}$ the logic of the class $\boldsymbol{F}$ in language $\mathcal{L}_O[M]$, and $k \in \{l, g\}$. We distinguish the following interpolation properties:*

(AIP) *$L_{O,\boldsymbol{F},M}$ has the **arrow interpolation property** if, for all formulae φ and ψ such that $\varphi \to \psi \in L_{O,\boldsymbol{F},M}$, there is a formula θ with $\mathcal{P}(\theta) \subseteq \mathcal{P}(\varphi) \cap \mathcal{P}(\psi)$ and such that $\varphi \to \theta \in L_{O,\boldsymbol{F},M}$ and $\theta \to \psi \in L_{O,\boldsymbol{F},M}$.*

(TIP$_k$) *$L_{O,\boldsymbol{F},M}$ has the **k-turnstile interpolation property** if, for all formulae φ, ψ such that $\varphi \models^{\boldsymbol{F}}_k \psi$, there is a formula θ with $\mathcal{P}(\theta) \subseteq \mathcal{P}(\varphi) \cap \mathcal{P}(\psi)$ and such that $\varphi \models^{\boldsymbol{F}}_k \theta$ and $\theta \models^{\boldsymbol{F}}_k \psi$.*

(SIP$_k$) $L_{O,\boldsymbol{F},M}$ *has the* k***-splitting interpolation property*** *if, for all formulae* $\varphi_0, \varphi_1, \psi$ *such that* $\varphi_0 \wedge \varphi_1 \models_k^{\boldsymbol{F}} \psi$, *there is a formula* θ *with* $\mathcal{P}(\theta) \subseteq \mathcal{P}(\varphi_0) \cap (\mathcal{P}(\varphi_1) \cup \mathcal{P}(\psi))$ *and such that* $\varphi_0 \models_k^{\boldsymbol{F}} \theta$ *and* $\varphi_1 \wedge \theta \models_k^{\boldsymbol{F}} \psi$.

The interpolation properties just defined are interrelated as follows, compare e.g. Areces and Marx [1998].

LEMMA 3.27. *Let* $\mathcal{L}_O[M]$ *be a language and* $\boldsymbol{F}$ *a class of frames for* $\mathcal{L}_O[M]$.

(i) *(AIP), (TIP$_l$) and (SIP$_l$) are equivalent for* $L_{O,\boldsymbol{F},M}$.

(ii) *If the local consequence relation* $\models_l^{\boldsymbol{F}}$ *is compact and* $\boldsymbol{F}$ *is closed under taking point-generated subframes, then (AIP) implies (TIP$_g$), and (TIP$_g$) and (SIP$_g$) are equivalent.*

PROOF. To prove (i), just notice that with the local consequence relation we have the standard deduction theorem available, and thus $\models^{\boldsymbol{F}} \varphi \rightarrow \psi$ if and only if $\varphi \models_l^{\boldsymbol{F}} \psi$.

(ii) follows from the fact that we can switch from the global consequence to the local consequence using the deduction theorem for the global consequence, the assumption that the class $\boldsymbol{F}$ of frames is closed under taking point-generated subframes, and compactness. □

Now, suppose the class $\boldsymbol{F}$ of frames, closed under taking point-generated subframes, is an elementary relational representation of a class $\mathcal{D}^i$ of distance spaces, i.e., the logics $L_{O,\boldsymbol{F},M}$ and $\mathcal{MS}_O^i[M]$ coincide. Then, clearly, if we are able to prove (AIP) for $L_{O,\boldsymbol{F},M}$, then (AIP) holds for $\mathcal{MS}_O^i[M]$ as well. Further, the local consequence relation $\models_l^{\boldsymbol{F}}$ is compact by using the standard translation and the compactness of first-order logic, and thus, if we are able to disprove (TIP$_g$) for $L_{O,\boldsymbol{F},M}$, then also (AIP) fails for $L_{O,\boldsymbol{F},M}$ by (ii), and so we have disproved (AIP) for $\mathcal{MS}_O^i[M]$.

As mentioned above, we will concentrate here on (the failure of) the interpolation property for the languages $\mathcal{L}_O[M]$ without nominals and the universal modality. If nominals and the universal modality are available in a language, Craig interpolation usually fails. For instance, as we show below, the language $\mathcal{L}_{L_2}[M]$ does have Craig interpolation. Now consider the language $\mathcal{LO}_{L_2}[M]$ and let $0 \neq a \in M$. Then the formula

$$\mathsf{E}^{\leq a} p \wedge \mathsf{E}^{\leq a} \neg p \rightarrow (\blacksquare(q \rightarrow i) \rightarrow \blacklozenge \neg q)$$

is a tautology with respect to any class of distance spaces: suppose the formulae $\mathsf{E}^{\leq a} p \wedge \mathsf{E}^{\leq a} \neg p$ and $\blacksquare(q \rightarrow i)$ are true at some point w of a model $\mathfrak{B}$ based on a distance space $\langle W, d \rangle$. Then there are at least two 'p-worlds' and at most one 'q-world', whence $\blacklozenge \neg q$ is true at w as well. Possible interpolants for this implication have to be build from $\top$ and $\bot$, which is impossible in this language. If we treat nominals on a par with modalities, i.e., allowing them to appear freely in the interpolant, we may restore interpolation. For example, the formula $\blacklozenge\neg i$ can serve as an interpolant for the implication above. Sometimes, a more restricted version of Craig interpolation is investigated, where interpolants not only have to be build from shared propositional variables, but also from shared modal operators. This, though, makes no sense in this general form in the context of distance logics. Consider, for instance, the formula

$$\mathsf{A}^{\leq 2}\varphi \to \mathsf{A}^{\leq 1}\mathsf{A}^{\leq 1}\varphi.$$

While the language containing the operators of type $\mathsf{A}^{\leq a}$ have Craig interpolation in all classes of distance spaces, the above formula, being tautological in triangular spaces, has clearly no interpolant using no distance operators at all. The question, on the other hand, exactly which parameters from a parameter set M are needed in an interpolant, given the parameters appearing in the antecedent and consequent of some valid implication, is a non-trivial and interesting question.

3.4.1. The Logics $\mathcal{MS}_{L_i}$ and $\mathcal{MSO}_{L_i}$, $i = 1, 2, 3$. Let us now continue by analysing the languages $\mathcal{LO}_{L_1}[M]$ and its sublanguages $\mathcal{LO}_{L_2}[M]$ and $\mathcal{LO}_{L_3}[M]$.

Call an M-frame $\mathfrak{f}$ of the form

$$\mathfrak{f} = \langle W, (R_{\leq a})_{a \in M} \rangle$$

L_2-**metric**, if the following conditions hold for all $a, b \in M$ and $w, u, v \in W$:

- (L1) If $uR_{\leq a}v$ and $a \leq b$, then $uR_{\leq b}v$;
- (L2) $uR_{\leq 0}v \iff u = v$;
- (L3) $uR_{\leq a}v \iff vR_{\leq a}u$;
- (L4) If $uR_{\leq a}v$ and $vR_{\leq b}w$, then $uR_{\leq a+b}w$, whenever $a + b \in M$.

These are the same frame conditions as used for D-metric frames given on Page 42, save conditions (D1) and (D2). An inspection of the proof of the Representation Theorem 2.7 shows that the logic $\mathcal{MS}_{L_2}$ coincides with the logic of the class of L_2-metric frames. By a standard Sahlqvist argument, this class is easily seen to be axiomatised by the axiomatic system MS_{L_2} given in Table 3.4.

Similarly, call an M-frame $\mathfrak{f}$ of the form

$$\mathfrak{f} = \langle W, (R_{<a})_{a \in M} \rangle$$

L_3-**metric**, if the following conditions hold for all $a, b \in M$ and $w, u, v \in W$:

- (L5) If $uR_{<a}v$ and $a \leq b$, then $uR_{<b}v$;
- (L6) $uR_{<a}u$, whenever $a \neq 0$;
- (L7) $R_{<0} = \varnothing$;
- (L8) If $uR_{<a}v$ and $vR_{<b}w$, then $uR_{<a+b}w$, whenever $a + b \in M$.

AXIOM SCHEMATA FOR $\mathsf{MS}_{L_2}[M]$

(CL)	Axioms of propositional calculus	
$(\mathsf{K}_{\leq})$	$\mathsf{A}^{\leq a}(\varphi \to \psi) \to (\mathsf{A}^{\leq a}\varphi \to \mathsf{A}^{\leq a}\psi)$	$(a \in M\}$
$(\mathsf{Mo}_{\leq})$	$\mathsf{A}^{\leq a}\varphi \to \mathsf{A}^{\leq b}\varphi$	$(a, b \in M,\ a \geq b)$
$(\mathsf{T}_{\leq 0})$	$\mathsf{A}^{\leq 0}\varphi \to \varphi$	
$(\mathsf{T}^c_{\leq 0})$	$\varphi \to \mathsf{A}^{\leq 0}\varphi$	
$(\mathsf{B}_{\leq})$	$\varphi \to \mathsf{A}^{\leq a}\mathsf{E}^{\leq a}\varphi$	$(a \in M)$
$(\mathsf{Tr}_{\leq})$	$\mathsf{A}^{\leq a+b}\varphi \to \mathsf{A}^{\leq a}\mathsf{A}^{\leq b}\varphi$	$(a + b \in M)$

INFERENCE RULES:

The inference rules are **modus ponens** and **necessitation** for $\mathsf{A}^{\leq a}$:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ \text{(MP)} \qquad \frac{\varphi}{\mathsf{A}^{\leq a}\varphi}\ \text{(RN)} \qquad (a \in M)$$

Table 3.4: The axiomatic system $\mathsf{MS}_{L_2}[M]$.

AXIOM SCHEMATA FOR $\mathsf{MS}_{L_3}[M]$

(CL)	Axioms of propositional calculus	
$(\mathsf{K}_{<})$	$\mathsf{A}^{<a}(\varphi \to \psi) \to (\mathsf{A}^{<a}\varphi \to \mathsf{A}^{<a}\psi)$	$(a \in M)$
$(\mathsf{Mo}_{\mathsf{A}<})$	$\mathsf{A}^{<a}\varphi \to \mathsf{A}^{<b}\varphi$	$(a, b \in M,\ a \geq b)$
$(\mathsf{T}_{\mathsf{A}<})$	$\mathsf{A}^{<a}\varphi \to \varphi$	$(a \neq 0)$
$(\mathsf{Bot}_{\mathsf{A}<0})$	$\mathsf{A}^{<0}\bot$	
$(\mathsf{B}_{\mathsf{A}<})$	$\varphi \to \mathsf{A}^{<a}\mathsf{E}^{<a}\varphi$	$(a \in M)$
$(\mathsf{Tr}_{<})$	$\mathsf{A}^{<a+b}\varphi \to \mathsf{A}^{<a}\mathsf{A}^{<b}\varphi$	$(a + b \in M)$

INFERENCE RULES:

The inference rules are **modus ponens** and **necessitation** for $\mathsf{A}^{<a}$:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ \text{(MP)} \qquad \frac{\varphi}{\mathsf{A}^{<a}\varphi}\ \text{(RN)} \qquad (a \in M)$$

Table 3.5: The axiomatic system $\mathsf{MS}_{L_3}[M]$.

Again, it is easily seen that the logic $\mathcal{MS}_{L_3}$ coincides with the logic of the class of L_3-metric frames. The axiomatic system MS_{L_3} listed in Table 3.5 axiomatises this class.

Further, a representation theorem for the language $\mathcal{L}_{L_1}$ is obtained by adding to the properties (L1)–(L8) 'interaction' conditions. Call an M-frame $\mathfrak{f}$ of the form

$$\mathfrak{f} = \langle W, (R_{<a})_{a \in M}, (R_{\leq a})_{a \in M} \rangle$$

L_1-**metric**, if the following conditions hold for all $a, b \in M$ and $w, u, v \in W$:

(L1)–(L8);

(L9) If $uR_{<a}v$, then $uR_{\leq a}v$;

(L10) If $uR_{\leq a}v$ and $a < b$, then $uR_{<b}v$;

(L11) If $uR_{\leq a}v$ and $vR_{<b}w$, then $uR_{<a+b}w$, whenever $a + b \in M$;

That the logic $\mathcal{MS}_{L_1}$ coincides with the logic of the class of L_1-metric frames follows from the Representation Theorem 3.12 and has independently been shown in Lutz et al. [2003]. The axiomatic system MS_{L_1} listed in Table 3.6 axiomatises this class.

AXIOM SCHEMATA FOR $\mathsf{MS}_{L_1}[M]$

$\mathsf{MS}_{L_2}[M] \oplus \mathsf{MS}_{L_3}[M] \oplus$

$(\mathsf{Inc}^{\leq}_{<})$	$\mathsf{A}^{\leq a}\varphi \to \mathsf{A}^{<a}\varphi$	$(a \in M)$
$(\mathsf{Mo}^{\leq}_{<})$	$\mathsf{A}^{<a}\varphi \to \mathsf{A}^{\leq b}\varphi$	$(b < a)$
$(\mathsf{Tr}^{\leq}_{<})$	$\mathsf{A}^{<a+b}\varphi \to \mathsf{A}^{\leq a}\mathsf{A}^{<b}\varphi$	$(a + b \in M)$

INFERENCE RULES:

The inference rules are **modus ponens** and **necessitation** for $\mathsf{A}^{<a}$:

$$\frac{\varphi \quad \varphi \to \psi}{\psi} \text{ (MP)} \qquad \frac{\varphi}{\mathsf{A}^{<a}\varphi} \text{ (RN)} \qquad (a \in M)$$

Table 3.6: The axiomatic system $\mathsf{MS}_{L_1}[M]$.

As concerns the axiomatisations of the logics $\mathcal{MS}_{L_2}$ and $\mathcal{MS}_{L_3}$, we run across a peculiar phenomenon. Both modalities, $\mathsf{A}^{<0}$ and $\mathsf{A}^{\leq 0}$, are trivial in the sense that $\mathsf{A}^{\leq 0}\varphi$ can always be replaced by φ, and $\mathsf{A}^{<0}\varphi$ can always be replaced by $\top$. So when considering languages with parameter sets that do not contain 0, or when performing a trivial translation, the tautologies of $\mathcal{L}_{L_2}$ and $\mathcal{L}_{L_3}$ completely coincide (when replacing $\mathsf{A}^{\leq}$ by $\mathsf{A}^{<}$). On the other hand, when it comes to expressivity on models based on distance spaces, the languages $\mathcal{L}_{L_2}$ and $\mathcal{L}_{L_3}$ are incomparable. More specifically, since $\mathsf{A}^{<a}\varphi$ is true in a point w of a model $\mathfrak{M}$ based on a metric space S if and only if φ is true in an 'open ball' around w, there is no translation $.^\sharp$, taking formulae of $\mathcal{L}_{L_2}$ to formulae of $\mathcal{L}_{L_3}$, such that for any metric space S and point w we would have

$$\langle S, w\rangle \models \mathsf{A}^{<a}\varphi \iff \langle S, w\rangle \models (\mathsf{A}^{<a}\varphi)^\sharp.$$

We do not give a formal prove of this here, but remark that it follows, e.g., from considering topological extensions to these languages with interior and closure operators, as investigated in Wolter and Zakharyaschev [2004].[3]

Next, we look at interpolation for these languages. Recall that a **universal Horn formula** is a universally quantified conjunction of disjunctions of literals, where in each disjunction at most one literal appears positively. Since all of the axioms given for the logics $\mathcal{MS}_{L_i}$, $i = 1, 2, 3$, are of Sahlqvist type and correspond to universal Horn conditions, i.e., conditions (L1)–(L11) can be rewritten to be universal Horn, we can make use of the following theorem, a proof of which can be found in Marx and Venema [1997].

THEOREM 3.28 (Marx & Venema). *Every Sahlqvist axiomatisable modal logic whose axioms correspond to universal Horn formulae enjoys the arrow interpolation property.*

Thus, we immediately obtain:

THEOREM 3.29 (INTERPOLATION FOR $\mathcal{MS}^i_{L_j}$).
The logics $\mathcal{MS}^i_{L_j}[M]$ have Craig interpolation, for $j = 1, 2, 3$ and $i \in \{d, s, t, m\}$.

Failure of Craig interpolation, though, is the norm for distance logics, as we will see in the next section.

3.4.2. The Logics $\mathcal{MS}_{G_i}$ and $\mathcal{MSO}_{G_i}$, $i = 1, 2, 3$. Call an M-frame $\mathfrak{f}$ of the form

$$\mathfrak{f} = \langle W, (R_{>a})_{a \in M} \rangle$$

G_3-**metric** if the following conditions hold for all $a, b \in M$ and $w, u, v \in W$:

(G1) If $uR_{>a}v$ and $b \leq a$, then $uR_{>b}v$;

(G2) $uR_{>a}v \iff vR_{>a}u$;

(G3) $uR_{>0}v \iff u \neq v$;

(G4) If $uR_{>a+b}w$, then $uR_{>a}v$ or $wR_{>b}v$, whenever $a + b \in M$.

Again, it is easily seen that the logic $\mathcal{MS}_{G_3}$ coincides with the logic of the class of G_3-metric frames. This follows immediately from the Representation Theorem 2.7 by replacing the condition for $R_{\leq}$ by their 'converses' for $R_{>}$—e.g., replace

(D3) If $uR_{\leq a}v$ and $a \leq b$, then $uR_{\leq b}v$;

by (G1)—and by noticing that the proof of Theorem 2.7 does not depend on whether we specify frame-conditions for $R_{\leq a}$ or $R_{>a}$ due to the conditions (D1) and (D2) that make the relations $R_{\leq a}$ and $R_{>a}$ mutually exclusive but exhaustive.

[3]Note, however, that we can find such a translation for every fixed finite model, since then the topologies are trivial.

Conditions (G1) and (G2) are defined, respectively, by the Sahlqvist axioms ($\mathsf{Mo}_{\mathsf{A}^>}$) and ($\mathsf{B}_{\mathsf{A}^>}$) from Table 3.7. Condition (G3) tells us that the operator $\mathsf{E}^{>0}$ has to be interpreted as the **difference operator**, that is, by the inequality relation. Clearly, condition (G4) is definable by a pure axiom, e.g.

$$\mathsf{E}^{>a+b}i \to \mathsf{E}^{>a}j \vee \blacksquare(j \to \mathsf{E}^{>b}i).$$

Moreover, if we assume that $\mathsf{E}^{>0}$ acts as the difference operator on a class of frames, we can in fact show that the Axiom ($\mathsf{Tr}_>$)

$$(\mathsf{Tr}_>) \quad (\varphi \vee \mathsf{E}^{>0}\varphi) \wedge \mathsf{E}^{>a+b}\mathsf{A}^{>b}\neg\varphi \to \mathsf{E}^{>a}\varphi \qquad (a+b \in M)$$

defines the class of frames satisfying condition (G4):

Suppose $\mathfrak{f} = \langle W, (R_{>a})_{a\in M}\rangle$ satisfies conditions (G3) and (G4). Let $u \in W$, $\mathfrak{M}$ be a model based on $\mathfrak{f}$, and assume

$$\langle \mathfrak{M}, u\rangle \models (\varphi \vee \mathsf{E}^{>0}\varphi) \wedge \mathsf{E}^{>a+b}\mathsf{A}^{>b}\neg\varphi.$$

Then there is some point $v \in W$ (possibly $v = u$) such that $\langle \mathfrak{M}, v\rangle \models \varphi$ and a point $w \in W$ such that $\langle \mathfrak{M}, w\rangle \models \mathsf{A}^{>b}\neg\varphi$ and $uR_{>a+b}w$. It follows that $\neg wR_{>b}v$ and hence, by condition (G4), $uR_{>a}v$, from which $\langle \mathfrak{M}, u\rangle \models \mathsf{E}^{>a}\varphi$. This shows that ($\mathsf{Tr}_>$) is valid on the class of frames satisfying (G4) and on which $\mathsf{E}^{>0}$ acts as the difference operator.

Conversely, pick a frame $\mathfrak{f} = \langle W, (R_{>a})_{a\in M}\rangle$ that violates condition (G4), i.e., there are $u, v, w \in W$ such that $uR_{>a+b}w$, $\neg uR_{>a}v$, and $\neg wR_{>b}v$. We have to show that ($\mathsf{Tr}_>$) can be refuted on $\mathfrak{f}$. Define a model $\mathfrak{M}$ on $\mathfrak{f}$ by setting $p^{\mathfrak{M}} = \{v\}$ and $q^{\mathfrak{M}} = \varnothing$ for all $q \neq p$. Then

$$\langle \mathfrak{M}, u\rangle \models \mathsf{E}^{>0}p \wedge \mathsf{E}^{>a+b}\mathsf{A}^{>b}\neg p,$$

since $wR_{>b}v'$ implies $v' \neq v$. But $\langle \mathfrak{M}, u\rangle \not\models \mathsf{E}^{>a}p$, since $\neg uR_{>a}v$ and, by definition of $\mathfrak{M}$, $p^{\mathfrak{M}} = \{v\}$.

The Axiom ($\mathsf{Tr}_>$) is also a Sahlqvist formula, for $\mathsf{E}^{>a}p$ is **positive**, that is, the propositional variable p appears in the scope of an even number of negations, and the formula $(p \vee \mathsf{E}^{>0}p) \wedge \mathsf{E}^{>a+b}\mathsf{A}^{>b}\neg p$ is a **Sahlqvist antecedent**, for it is build from $\top$, $\bot$, **boxed atoms** (i.e. p) and **negative** formulae (i.e. $\mathsf{E}^{>a+b}\mathsf{A}^{>b}\neg p$), using $\wedge$, $\vee$, and existential modal operators only.[4] In particular, note that if we treat $\mathsf{E}^{>0}$ just as a normal modal operator rather than the difference operator, the first-order correspondent of ($\mathsf{Tr}_>$) is weaker than $(G4)$, namely, for $a + b \in M$:

(G4)′ If $uR_{>a+b}w$ and $u = v \vee uR_{>0}v$, then $uR_{>a}v$ or $wR_{>b}v$.

That the axiomatic system MS_{G_3} listed in Table 3.7 axiomatises this class follows from the *D-Sahlqvist Theorem*, which is proved in Venema [1993, Theorem 7.7]. Similarly to

[4]We use here the definition of **Sahlqvist formula** as defined in Blackburn et al. [2001].

AXIOM SCHEMATA FOR $\mathsf{MS}_{G_3}[M]$

(CL)	the axiom schemata of classical propositional calculus	
($\mathsf{K}_>$)	$\mathsf{A}^{>a}(\varphi \to \psi) \to (\mathsf{A}^{>a}\varphi \to \mathsf{A}^{>a}\psi)$	$(a \in M)$
($\mathsf{Mo}_>$)	$\mathsf{A}^{>a}\varphi \to \mathsf{A}^{>b}\varphi$	$(a \leq b \in M)$
($\mathsf{B}_>$)	$\varphi \to \mathsf{A}^{>a}\mathsf{E}^{>a}\varphi$	$(a \in M)$
($\mathsf{Tr}_>$)	$(\varphi \vee \mathsf{E}^{>0}\varphi) \wedge \mathsf{E}^{>a+b}\mathsf{A}^{>b}\neg\varphi \to \mathsf{E}^{>a}\varphi$	$(a+b \in M)$
(Diff)	$\mathsf{E}^{>0}\mathsf{E}^{>0}\varphi \to (\varphi \vee \mathsf{E}^{>0}\varphi)$	$(0 \in M)$

INFERENCE RULES:

The inference rules are **modus ponens**, substitution, **necessitation** for $\mathsf{A}^{>a}$, and the **irreflexivity rule** for $\mathsf{E}^{>0}$, $(\mathrm{IR})_{\mathsf{E}^{>0}}$:

$$\frac{\varphi \quad \varphi \to \psi}{\psi}\ (\mathrm{MP}) \qquad \frac{\varphi}{\mathsf{A}^{<a}\varphi}\ (\mathrm{RN}) \qquad \frac{p \wedge \mathsf{A}^{>0}\neg p \to \varphi}{\varphi}\,(p \notin \varphi)\quad (\mathrm{IR})_{\mathsf{E}^{>0}} \quad (a \in M)$$

Table 3.7: The axiomatic system $\mathsf{MS}_{G_3}[M]$.

the condition of *reversiveness* discussed earlier, the condition of **versatility** used in the formulation of the D-Sahlqvist Theorem below takes on a very simple form for the case of unary modal operators, in particular, temporal or symmetric modalities are a special case of versatile similarity types.

THEOREM 3.30 (D-SAHLQVIST THEOREM, Venema). *Let Σ be a collection of Sahlqvist formulae in a versatile similarity type containing the difference operator. Then the minimal D-logic enriched with the axioms in Σ is strongly sound and complete with respect to the class of frames defined by the first-order frame correspondents of the axioms in Σ and where D is interpreted as inequality.*

Call an M-frame $\mathfrak{f}$ of the form

$$\mathfrak{f} = \langle W, (R_{\geq a})_{a \in M} \rangle$$

G_2-**metric**, if the following conditions hold for all $a, b \in M$ and $w, u, v \in W$:

(G5) If $uR_{\geq a}v$ and $b \leq a$, then $uR_{\geq b}v$;

(G6) $uR_{\geq a}v \iff vR_{\geq a}u$;

(G7) $uR_{\geq 0}v = W \times W$;

(G8) If $uR_{\geq a+b}w$, then $uR_{\geq a}v$ or $wR_{\geq b}v$, whenever $a + b \in M$.

Again, by modifying Theorem 2.7, it is easily seen that the logic $\mathcal{MS}_{G_2}$ coincides with the logic of the class of G_2-metric frames.

Note that, while the operators $\mathsf{A}^{\leq 0}$ and $\mathsf{A}^{<0}$ where both trivial, the operators $\mathsf{A}^{\geq 0}$ and $\mathsf{A}^{>0}$ are both rather powerful, but $\mathsf{A}^{>0}$ is strictly more expressive than $\mathsf{A}^{\geq 0}$ over models, i.e., as we have discussed in Chapter 1, the operator $\mathsf{E}^{>0}$ (the difference operator) allows for the 'simulation of nominals', while the operator $\mathsf{E}^{\geq 0}$ (the universal modality) does not, compare also de de Rijke [1992]. Furthermore, we cannot give an axiomatisation for this logic in the same way as for $\mathcal{MS}_{G_3}$, since the language does not contain the difference operator, and so the D-Sahlqvist Theorem is not applicable. We could, however, obviously axiomatise the class of G_2-metric frames in the stronger language $\mathcal{LO}_{G_2}$ including nominals. Also, the logic $\mathcal{MS}_{G_1}[M]$ is easily characterised as the logic of a class of frames comprising the conditions (G1)–(G8) and additionally 'interaction' conditions for the modalities $\mathsf{E}^{>a}$ and $\mathsf{E}^{\geq a}$, similarly to what we have done in the last section.

As shown in ten Cate [2004], any language that has Craig interpolation over a class of frames and extends the basic logic of the difference operator is at least as expressive as the first-order correspondence language. Thus, all distance logics containing the operator $\mathsf{E}^{>0}$ fail to have Craig interpolation.

THEOREM 3.31 (FAILURE OF INTERPOLATION IN $\mathcal{L}_{G_1}, \mathcal{L}_{G_3}$).
The logics $\mathcal{MS}_{G_i}[M]$, $i = 1, 3$, fail to have Craig interpolation.

Even worse, if we consider the languages $\mathcal{L}_D[M \setminus \{0\}]$ comprising the operators $\mathsf{A}^{\leq a}$ and $\mathsf{A}^{>a}$, but leaving out the difference operator $\mathsf{E}^{>0}$, we can still construct a counterexample for Craig interpolation following the lines of the proof for failure of interpolation in Humberstone's inaccessibility logic, which comprises modal operators for a binary relation and its complement [Areces and Marx, 1998]. This will be discussed below in detail.

3.4.3. The Logics $\mathcal{MS}^i_D, i \in \{d, s, t, m\}$. In Theorem 3.5 we have shown that the logics $\mathcal{MS}^i_D, i \in \{d, s, t, m\}$, are complete with respect to the finite members in their standard frame classes $\mathcal{F}^i$ as defined on Page 42. Using the results from the last sections, we can now without further ado give a hybrid variant of the axiomatic system $\mathsf{MS}^m_D[M]$. We just have to add to the base logic $\mathbf{K}^r_{\blacksquare}[M]$ in language $\mathcal{LO}_D$ pure axioms that define the conditions (D1)–(D6) of D-metric M-frames. The corresponding axiomatic system is listed in Table 3.8.

We next show that the logics $\mathcal{MS}^i_D[M]$ fail to have interpolation for finite parameter sets M, even if we leave out the operators $\mathsf{E}^{\leq 0}$ and $\mathsf{E}^{>0}$ (that is, the difference operator), and even in the weak form of the splitting interpolation property.

To prove this, we will use a technique from Areces and Marx [1998], where failure of interpolation was shown for a variety of combinations of logics, like certain fusions, products, and Humberstone's inaccessibility logic. But let us first introduce

AXIOM SCHEMATA FOR $\mathsf{MSO}_D[M]$

$\mathbf{K}^r_{\blacksquare}[M]\oplus_{\mathsf{C}_0}$		
$(\mathsf{Def}^{>}_{\leq})$	$\mathsf{E}^{>a}j \leftrightarrow \mathsf{A}^{\leq a}\neg j$	$(a \in M)$
$(\mathsf{Mon}^h_{\leq})$	$\mathsf{E}^{\leq a}i \rightarrow \mathsf{E}^{\leq b}i$	$(a \leq b)$
$(\mathsf{T}^h_{\leq 0})$	$\mathsf{E}^{\leq 0}i \leftrightarrow i$	$(0 \in M)$
(Tra^h_D)	$\mathsf{E}^{\leq a}\mathsf{E}^{\leq b}j \rightarrow \mathsf{E}^{\leq a+b}j$	$(a, b \in M)$

INFERENCE RULES

$$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}\ (\mathrm{MP}) \qquad \frac{\varphi}{\blacksquare\varphi}\ (\mathrm{RN}) \qquad \frac{i \rightarrow \varphi}{\varphi}, i \notin \varphi,\ (\mathrm{COV}_0)$$

Table 3.8: The axiomatic system $\mathsf{MSO}_D[M]$.

some terminology needed. Given two frames, $\mathfrak{f} = \langle W, (R_{\leq a})_{a\in M}, (R_{>a})_{a\in M}\rangle$ and $\mathfrak{g} = \langle V, (S_{\leq a})_{a\in M}, (S_{>a})_{a\in M}\rangle$, a **p-morphism** from $\mathfrak{f}$ to $\mathfrak{g}$ is a function $f : \mathfrak{f} \rightarrow \mathfrak{g}$ such that the following two conditions hold for all $a \in M$:

- If $w_1 R_{\leq a} w_2$, then $f(w_1) S_{\leq a} f(w_2)$; the same for $R_{>a}$;
- If $f(w_1) S_{\leq a} v$, then there is a $w_2 \in W$ such that $f(w_2) = v$ and $w_1 R_{\leq a} w_2$; the same for $S_{>a}$.

If the function f is surjective, we call $\mathfrak{g}$ the **p-morphic image** of $\mathfrak{f}$ under f and then write $f : \mathfrak{f} \twoheadrightarrow \mathfrak{g}$ or $\mathfrak{f} \overset{f}{\twoheadrightarrow} \mathfrak{g}$. The notions of a frame **generated by** a set X and generated subframes are the usual ones from modal logic, compare, e.g., [Kracht, 1999]. In particular, if $\mathfrak{f}$ is generated by a single point $w \in W$, it is called **rooted** and w is called the **root** of $\mathfrak{f}$.

We can now state the criterion for failure of interpolation mentioned above.

LEMMA 3.32 (Marx and Areces). *Let* $\boldsymbol{F}$ *be a class of frames for a poly-modal language with finitely many modal operators. Suppose there are finite frames* $\mathfrak{g}, \mathfrak{h} \in \boldsymbol{F}$ *and a frame* $\mathfrak{f}$ *such that:*

(1) *There are surjective p-morphisms* m, n*, such that* $\mathfrak{g} \overset{m}{\twoheadrightarrow} \mathfrak{f} \overset{n}{\twoheadleftarrow} \mathfrak{h}$;
(2) $\mathfrak{f}$ *is point-generated by the root* w*, every* $u \in m^{-1}(w)$ *generates* $\mathfrak{g}$*, every* $v \in n^{-1}(w)$ *generates* $\mathfrak{h}$;
(3) *There is no frame* $\mathfrak{j} \in \boldsymbol{F}$ *with commuting surjective p-morphisms* g *and* h *from* $\mathfrak{j}$ *onto* $\mathfrak{g}$ *and* $\mathfrak{h}$*, i.e., such that* $\mathfrak{g} \overset{g}{\twoheadleftarrow} \mathfrak{j} \overset{h}{\twoheadrightarrow} \mathfrak{h}$ *and* $m \circ g = n \circ h$.

Then an explicit counterexample for the splitting interpolation property (SIP) can be constructed from the frames and functions $\mathfrak{g} \overset{m}{\twoheadrightarrow} \mathfrak{f} \overset{n}{\twoheadleftarrow} \mathfrak{h}$.

The proof of this criterion depends on the fact that finite frames can be characterised syntactically up to bisimulation [Fine, 1974] and that a counterexample for (SIP) can be explicitly constructed from the formulae describing the frames and functions $\mathfrak{g}$, $\mathfrak{h}$, m and n.

For the case of distance logics we are interested in here, though, this criterion can be simplified and strengthened. Since in D-standard M-frames the relations $R_{>a}$ are the complement of $R_{\leq a}$, for all $a \in M$, D-standard frames are point-generated by every point. Moreover, the classes $\mathcal{F}^i$, $i \in \{d,s,t,m\}$, are elementary and obviously closed under point-generated subframes. More precisely, D-standard frames are **simple**, i.e., every generated subframe of a D-standard frame is the frame itself. We thus have the following stronger variant of Lemma 3.32, compare [Areces and Marx, 1998, Lemma 2.5]:

LEMMA 3.33. *Let $\boldsymbol{F}$ be one of $\mathcal{F}^i$, $i \in \{d,s,t,m\}$. Suppose there are finite frames $\mathfrak{g}, \mathfrak{h} \in \boldsymbol{F}$ and a frame $\mathfrak{f}$ for $\mathcal{L}_D$ containing just one world such that*

(1) *There are surjective p-morphisms m, n such that $\mathfrak{g} \overset{m}{\twoheadrightarrow} \mathfrak{f} \overset{n}{\twoheadleftarrow} \mathfrak{h}$;*
(2) *There is no frame $\mathfrak{j} \in \boldsymbol{F}$ with commuting surjective p-morphisms g and h from $\mathfrak{j}$ onto $\mathfrak{g}$ and $\mathfrak{h}$, i.e., such that $\mathfrak{g} \overset{g}{\twoheadleftarrow} \mathfrak{j} \overset{h}{\twoheadrightarrow} \mathfrak{h}$ and $m \circ g = n \circ h$.*

Then all interpolation properties fail including the relevance property, i.e., there are formulae φ and ψ with $\mathcal{P}(\varphi) \cap \mathcal{P}(\psi) = \varnothing$, such that $\varphi \wedge \psi \models_{\boldsymbol{F}} \bot$ and there is no splitting interpolant.

THEOREM 3.34 (FAILURE OF INTERPOLATION IN $\mathcal{L}_D$). *The logics $\mathcal{MS}^i_D[M]$ and $\mathcal{MS}^i_D[M \setminus \{0\}]$, for $i \in \{d,s,t,m\}$ and $|M| < \omega$ finite, do not have the relevance property, and thus fail to have Craig interpolation.*

PROOF. We will use Lemma 3.33 and show the failure of the relevance property for the logic $\mathcal{MS}^i_D[M \setminus \{0\}]$ over some 'non-standard' parameter set excluding 0. First, let us define two finite D-metric frames $\mathfrak{g}$ and $\mathfrak{h}$. Define

$$\mathfrak{g} = \left\langle U, (R_{\leq a})_{a \in M\setminus\{0\}}, (R_{>a})_{a \in M\setminus\{0\}} \right\rangle \text{ and } \mathfrak{h} = \left\langle V, (S_{\leq a})_{a \in M\setminus\{0\}}, (S_{>a})_{a \in M\setminus\{0\}} \right\rangle$$

by letting

$$U := \{u, v, w\}, R_{\leq a} := \{\langle u,u\rangle, \langle v,v\rangle, \langle w,w\rangle\} \text{ and } R_{>a} := (U \times U) \setminus R_{\leq a},$$

and

$$V := \{u', v'\}, S_{\leq a} := \{\langle u,u\rangle, \langle v,v\rangle\} \text{ and } S_{>a} := (V \times V) \setminus R_{\leq a},$$

for every $a \in M \setminus \{0\}$. Note that by D-metric frames for parameter sets $M \setminus \{0\}$ we still mean frames satisfying all of the conditions of D-metric frames even if the relation $R_{>0}$ is not explicitly present in the frame. Thus, for instance, the relations $R_{>a}$ are assumed to be irreflexive. Further, define the one-point frame $\mathfrak{f}$ based on the set $W = \{x\}$ by setting $T_{\leq a} = T_{>a} = \{\langle x,x\rangle\}$, for every $a \in M \setminus \{0\}$. Finally, define functions m, n with $\mathfrak{g} \overset{m}{\twoheadrightarrow} \mathfrak{f}$ and $\mathfrak{h} \overset{n}{\twoheadrightarrow} \mathfrak{f}$ by setting $m(u) = m(v) = m(w) = x$ and

$n(u') = n(v') = x$. Obviously, m and n are surjective p-morphisms. In Figure 3.2 below, white arrows represent the relation $R_{\leq a}$, black arrows represent $R_{>a}$ (holding for all a), and the functions m and n are shown as dotted lines.

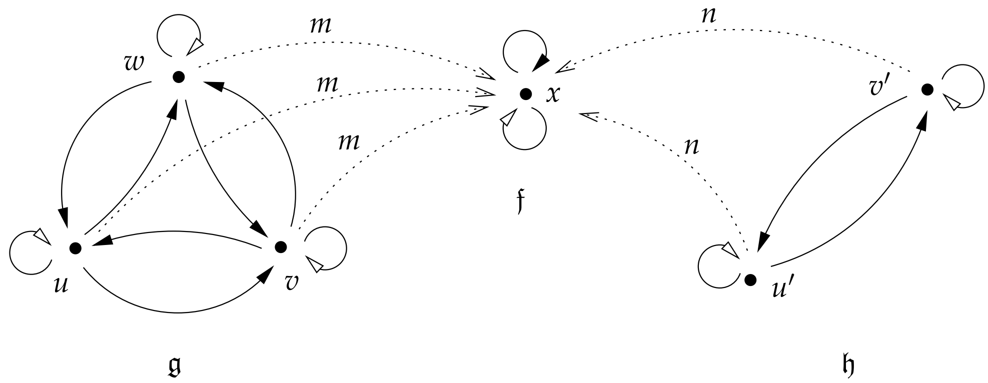


Figure 3.2: Counterexample for interpolation in $\mathcal{L}_D[M \setminus \{0\}]$.

Now, suppose that there is a D-metric frame

$$\mathfrak{j} = \langle J, (P_{\leq a})_{a \in M \setminus \{0\}}, (P_{>a})_{a \in M \setminus \{0\}} \rangle$$

with commuting surjective p-morphisms g and h from $\mathfrak{j}$ onto $\mathfrak{g}$ and $\mathfrak{h}$, i.e., such that $\mathfrak{g} \overset{g}{\twoheadleftarrow} \mathfrak{j} \overset{h}{\twoheadrightarrow} \mathfrak{h}$ and $m \circ g = n \circ h$. Pick some y_1 in $\mathfrak{j}$ such that $g(y_1) = u$. This exists by surjectivity of g. Then we have, for some a, $g(y_1)R_{>a}vR_{>a}w$, and so there is a y_2 in $\mathfrak{j}$ with $g(y_2) = v$ and $y_1P_{>a}y_2$. Since $\mathfrak{j}$ is assumed to be D-metric, $P_{>a}$ is irreflexive and so $y_1 \neq y_2$. Repeating this argument shows that there is a y_3 in $\mathfrak{j}$ with $y_2P_{>a}y_3$, $g(y_3) = w$, and $y_2 \neq y_3$. Since $g(y_1) = u \neq w = g(y_3)$, we also have $y_1 \neq y_3$. Note that, at this point, we directly obtain a contradiction if $\mathsf{E}^{>0}$ is in the signature: y_1, y_2, y_3 are all distinct, which implies that $y_1P_{>0}y_2P_{>0}y_3P_{>0}y_1$, and since h is a p-morphism, we obtain

$$h(y_1)S_{>0}h(y_2)S_{>0}h(y_3)S_{>0}h(y_1),$$

which implies that $\mathfrak{h}$ has at least 3 points, which is a contradiction.

Let us now continue with the case of $M \setminus \{0\}$. Without loss of generality, we may assume $h(y_1) = u'$. Then $h(y_2) = v'$, for $y_1P_{>a}y_2$ implies $h(y_1)S_{>a}h(y_2)$ and so, by irreflexivity, $h(y_1) \neq h(y_2)$. Similarly, it follows that $h(y_3) \neq h(y_2)$, and so $h(y_3) = u'$. It follows that $h(y_1)S_{\leq a}h(y_3)$ and so $y_1P_{\leq a}y_3$, for otherwise we would have $y_1P_{>a}y_3$ by condition (D1) of D-standard frames, and thus $h(y_1)S_{>a}h(y_3)$ contrary to the definition of $\mathfrak{h}$. But $y_1P_{\leq a}y_3$ implies $g(y_1)R_{\leq a}g(y_3)$, i.e., $uR_{\leq a}w$, contradicting condition (D2) of D-metric frames.

□

Note that the proof of the last theorem can be readily adapted to prove failure of interpolation for the language $\mathcal{L}_F[M]$. Simply notice that the modality $\mathsf{E}^{\leq a}$ is definable in $\mathcal{L}_F[M]$ and that the modalities $\mathsf{E}^{=a}$ and $\mathsf{E}^{>a}_{<b}$ can be 'trivialised' in the frames used in the proof: just set $R^{>a}_{<b} = S^{>a}_{<b} = \emptyset$, $R_{=a} = S_{=a} = \emptyset$ for $a \neq 0$, and $R_{=0} = S_{=0} = \mathsf{id}$.

It is open, however, whether Craig interpolation still fails over metric spaces, if we only have 'weak' difference operators in the form of $\mathsf{E}^{>a}$ saying 'somewhere at least a far away', but not their complements (Craig interpolation is obtained for this language if we leave out the triangular inequality, since then all frame conditions are again universal Horn).

Part 2

$\mathcal{E}$-CONNECTIONS

CHAPTER 4

$\mathcal{E}$-Connections and Abstract Description Systems

4.1. Introducing $\mathcal{E}$-Connections

Logic-based formalisms play a prominent role in modern Artificial Intelligence (AI) research. The numerous logical systems employed in various applications can roughly be divided into three categories:

(1) very expressive but undecidable logics, typically variants of first- or higher-order logics;
(2) quantifier-free formalisms of low computational complexity (typically P- or NP-complete), such as (fragments of) classical propositional logic and its non-monotonic variants;
(3) decidable logics with restricted quantification located 'between' propositional and first-order logics; typical examples are modal, description and propositional temporal logics.

The use of formalisms of the third kind is motivated by the fact that logics of category (2) are often not sufficiently expressive, e.g., for terminological, spatial, and temporal reasoning, while logics of the first kind are usually too complex to be used for efficient reasoning in realistic application domains.

Thus, the trade-off between expressiveness and effectiveness is the main design problem in the third approach, with decidability being an important indicator that the computational complexity of the language devised *might* be sufficiently low for successful applications. Over the last few years, an enormous progress has been made in the design and implementation of special purpose languages in this area—witness surprisingly fast representation and reasoning systems of description and temporal logics.[1] In contrast to first-order and propositional logics, however, these systems are useful only for very specific tasks, say, pure temporal, spatial, or terminological reasoning.

Since usually realistic application domains comprise various aspects of the world, the next target within this third approach is the design of suitable *combinations* of formalisms modelling each of these aspects. Following the underlying idea that to devise useful languages one has to search for a compromise between expressiveness

[1] Cf. Schwendimann [1998], Hustadt and Konev [2002], Möller and Haarslev [2003], Horrocks [2003].

and effectiveness, the problem then is to find combination methodologies which are sufficiently robust in the sense that the computational behaviour of the resulting hybrids should not be much worse than that of the combined components. The need for such methodologies has been clearly recognised by the AI community,[2] and various approaches to combining logics have been proposed, e.g., description logics with **concrete domains** in Lutz [2003], multi-dimensional **spatio-temporal logics** in Wolter and Zakharyaschev [2000, 2002], **independent fusions** in Kracht and Wolter [1991], Fine and Schurz [1996], Baader et al. [2002], **fibring** in Gabbay [1999], **temporalised logics** in Finger and Gabbay [1992], **temporal epistemic logic** in Fagin et al. [1995], or more general **logics of rational agency** in Rao and Georgeff [1998], van der Hoek and Wooldridge [2003].

In this part of the thesis, we introduce and investigate a novel combination method with a wide range of applications and a very robust computational behaviour in the sense that the combination is decidable whenever all of its components are decidable.

This combination method can be applied in the following setting. Suppose that we have n mutually disjoint domains $D_1, \ldots, D_n$ together with appropriate languages $L_1, \ldots, L_n$ for speaking about them. Although the domains are disjoint, they can represent different aspects of the same objects (say, a concrete house as an instance of a general concept house, its spatial extension and life span). So we can assume that we have a set $\mathcal{E} = \{E_j \mid j \in J\}$ of links establishing certain relations $E_j \subseteq D_1 \times \cdots \times D_n$ among objects of the domains.

Now, we form a new language L containing all of the L_i, $1 \leq i \leq n$, which is supposed to talk about the union $\bigcup_{i=1}^{n} D_i$, where the D_i are connected by the links in $\mathcal{E}$. The fragments L_i of L can still talk about each of the D_i, but the super-language L contains extra $(n-1)$-ary operators $\langle E_j \rangle^i$, $1 \leq i \leq n$, $j \in J$, which, given an input $(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$, for $X_\ell \subseteq D_\ell$, return

$$\{x \in D_i \mid \forall \ell \neq i \; \exists x_\ell \in X_\ell \; (x_1, \ldots, x_{i-1}, x, x_{i+1}, x_n) \in E_j\}.$$

In other words, the value of $\langle E_j \rangle^i (X_1, \ldots, X_{i-1}, X_{i+1}, \ldots X_n)$ is the i-th factor of

$$(X_1 \times \cdots \times X_{i-1} \times D_i \times X_{i+1} \times \cdots \times X_n) \cap E_j.$$

For instance, if $i = 2$ then, for all $X_1 \subseteq D_1$ and $X_2 \subseteq D_2$, we have

$$x_1 \in \langle E_j \rangle^1 (X_2) \iff \exists x_2 \in X_2. \, (x_1, x_2) \in E_j,$$
$$x_2 \in \langle E_j \rangle^2 (X_1) \iff \exists x_1 \in X_1. \, (x_1, x_2) \in E_j.$$

[2] It suffices to mention the workshop series ‘Frontiers of Combining Systems’ (FroCoS’96–02) and subsequent volumes [Baader and Schulz, 1996, Gabbay and de Rijke, 2000, Kirchner and Ringeissen, 2000, Armando, 2002]

We call the new system L the **basic $\mathcal{E}$-connection**[3] of $L_1, \ldots, L_n$. The operators $\langle E_j \rangle^i$ correspond to the exists-restrictions of standard description logics [Baader et al., 2003], or, in terms of first-order logic, to an E_j-guarded quantification over the members of foreign domains [Andréka et al., 1998].

Here are four simple examples of $\mathcal{E}$-connections; in more detail they will be considered in Section 4.5.

Description Logic–Spatial Logic. A description logic L_1 (say, $\mathcal{ALC}$ or $\mathcal{SHIQ}$ [Horrocks et al., 1999]) talks about a domain D_1 of certain 'abstract' objects. A spatial logic L_2 (say, qualitative $\mathbf{S4_u}$[4] or quantitative $\mathcal{MSO}_D$ from Chapter 1) talks about some spatial domain D_2. An obvious $\mathcal{E}$-connection is given by the relation $E \subseteq D_1 \times D_2$ defined by taking $(x, y) \in E$ if and only if y belongs to the spatial extension of x—whenever x occupies some space. Then, given an L_1-concept, say, river, the operator $\langle E \rangle^2$ (river) provides us with the spatial extension of all rivers. Conversely, given a spatial region of L_2, say, the Alps, $\langle E \rangle^1$ (Alps) provides the concept comprising all objects whose spatial extension has a non-empty intersection with the Alps. So the concept country $\sqcap \langle E \rangle^1$ (Alps) will then denote the set of all alpine countries.

Description Logic–Temporal Logic. Next, let L_3 be a temporal logic (say, point-based **PTL** [Gabbay et al., 1994] or Halpern-Shoham's logic of intervals **HS** [Halpern and Shoham, 1991]) and let D_3 be a set of time points or, respectively, time intervals interpreting L_3. In this case, a natural link relation $E \subseteq D_1 \times D_3$ is given by taking $(x, y) \in E$ if and only if y belongs to the life-span of x.

Description Logic–Description Logic. Besides the description logic L_1 talking about the domain D_1, another description logic L_4 may be given that is used to formalise knowledge about a domain D_4 closely related to D_1. For instance, if L_1 talks about countries and companies, while L_4 talks about people, we may have two relations $W, L \subseteq D_1 \times D_4$, where $(x, y) \in W$ if and only if y works in x (for x a company) and $(x, y) \in L$ if and only if y lives in x (for x a country). Typically, L_1 and L_4 will also use different sets of concept constructors.

Similar combinations, called **distributed description logics**, have been constructed by Borgida and Serafini [2002], whose motivation was the integration of and logical reasoning in loosely federated information systems. In more detail, the relationship between $\mathcal{E}$-connections of description logics and distributed description logics

[3] Basic $\mathcal{E}$-connections were first introduced in Kutz et al. [2001] and further investigated in Kutz et al. [2002b]. The construction of $\mathcal{E}$-connections was in part inspired by the author's work on counterpart theoretic semantics for modal predicate logics, where we have a set of relations interconnecting the objects of different first-order domains, compare [Kracht and Kutz, 2002, Kutz, 2003, Kracht and Kutz, 2004].

[4] Cf. Tarski [1938], Bennett [1996], Shehtman [1999], Gabbay et al. [2003].

will be analysed in Section 6.1, where we will show that distributed description logics can be thought of as special instances of $\mathcal{E}$-connections.

Description Logic–Spatial Logic–Temporal Logic. Further, we can combine the three logics L_1, L_2, L_3 above into a single formalism by defining a ternary relation $E \subseteq D_1 \times D_2 \times D_3$ such that $(x, y, z) \in E$ if and only if y belongs to the spatial extension of x at moment (interval) z.

So far, we have given only the rough idea of how we intend to construct $\mathcal{E}$-connections. To make it more precise, and to provide evidence for the claim that this combination technique is computationally robust, we will use the framework of **abstract description systems** (ADSs, for short) introduced in Baader et al. [2002]. Basically, all description, modal, temporal, epistemic and similar logics (in particular, modal logics of space) can be represented in the form of ADSs with the same computational behaviour as the original formalisms. For this reason, ADSs appear to be a good level of abstraction for investigating $\mathcal{E}$-connections.

The next question is how one might 'prove' that the formation of $\mathcal{E}$-connections is a computationally robust operation, for, obviously, 'computational robustness' is a rather vague term. Here, we adopt the idea that a proof of the decidability of the main reasoning tasks provided by a formalism is an important first step towards 'good' computational behaviour. Of course, only experiments will show whether a *particular* $\mathcal{E}$-connection is of sufficiently low complexity to be useful in practice; this is, however, left for future research.

Thus, our aim is to prove transfer results of the following form:

(1) *if a certain reasoning task for each of the component* ADSs *of an $\mathcal{E}$-connection is decidable, then this reasoning task for the $\mathcal{E}$-connection itself is decidable as well.*

On the other hand, to show that our results are in a sense optimal, and that, indeed, we have found—at least on the theoretical level—a good compromise between expressivity and effectiveness, we provide examples which demonstrate that

(2) *the transfer results in* (1) *do not hold if we take more expressive $\mathcal{E}$-connections.*

All 'positive' decidability transfer theorems come with the following complexity result:

(3) *the time complexity of a reasoning task for an $\mathcal{E}$-connection is at most one non-deterministic exponential higher than the maximal time complexity of its components; in some cases this upper bound is optimal.*

The increase of the worst-case time complexity by one exponential shows that in general $\mathcal{E}$-connections are not given 'for free', that is, they are so to speak more expressive than the 'sum of their parts'. On the other hand, this result also shows that the formation of $\mathcal{E}$-connections is a 'relatively cheap' combination methodology compared, for instance, with the multi-dimensional approach (see Section 6.4). We hope the ideas underlying the proofs of the decidability transfer theorems do not only indicate that 'practical algorithms' may exist for some particular cases, but may also help the designer of such algorithms.

The structure of this part of the thesis is as follows:

Section 4.2 introduces abstract description systems, describes four important logic-based knowledge representation formalisms—being used in examples of $\mathcal{E}$-connections given in Section 4.5—and their translations into ADSs. Section 4.2 also discusses the treatment of the nominal constructor within the framework of ADSs. In Section 4.3, we discuss two important technical aspects of ADSs: *number tolerance*, a property of ADSs, and *singleton satisfiability*, a new reasoning problem for ADSs. Both notions will play an important role in Chapters 5 and 6 when proving (negative) transfer results for $\mathcal{E}$-connections involving number restrictions on links or constructors motivated by DDLs. In Section 4.4, we formally introduce basic $\mathcal{E}$-connections, and in Section 4.5 we give a number of examples illustrating the new combination technique.

Chapter 5 is concerned with an analysis of the computational behaviour of various kinds of $\mathcal{E}$-connections. After a discussion of basic $\mathcal{E}$-connections in Section 5.1, we consider extensions of $\mathcal{E}$-connections which allow more interaction between the combined formalisms, namely, link operators on object variables (Section 5.2), Boolean operators on link relations (Section 5.3), and number restrictions on link relations (Section 5.4). Decidability transfer results as well as counterexamples to the transfer of decidability describe the trade-off between expressive power and computational behaviour.

In Chapter 6, we compare the methodology of $\mathcal{E}$-connections with related combination techniques and analyse the expressive power of basic $\mathcal{E}$-connections. We start, in Section 6.1, by considering the relation between $\mathcal{E}$-connections and distributed description logics as introduced in Borgida and Serafini [2002]. To fully capture the constructors employed in DDLs, we have to define a new variant of $\mathcal{E}$-connections. We give a number of decidability transfer results concerning DDLs as well as show where general transfer fails. In Section 6.2, we analyse the expressivity of basic $\mathcal{E}$-connections by lifting the concept of bisimulations to $\mathcal{E}$-connections and by providing a number of undefinable properties. In Section 6.3, we then show how basic $\mathcal{E}$-connections can be extended by means of assertions stating certain first-order constraints on models of $\mathcal{E}$-connections such that those undefinable properties can be expressed and we still preserve decidability. Finally, in Section 6.4, we briefly discuss $\mathcal{E}$-connections in the light of other combination methodologies such as multi-dimensional products of logics, independent fusions, fibrings, and description logics with concrete domains.

4.2. Abstract Description Systems

4.2.1. Basic Definitions. Abstract description systems (ADSs) have been proposed in Baader et al. [2002] as a common generalisation of description logics, modal logics, temporal logics, and some other formalisms. Our presentation of ADSs in this section will be brief, yet self-contained. As illustrating examples, we describe several logics that have been proposed in the literature for knowledge representation and reasoning, and show how these logics can be viewed as abstract description systems. For more details about ADSs, the reader is referred to Baader et al. [2002].

An abstract description system consists of an abstract description language and a class of admissible models specifying the intended semantics.

DEFINITION 4.1 (ADL). *An **abstract description language** (ADL) $\mathcal{L}$ is determined by a countably infinite set $\mathcal{V}$ of **set variables**, a countably infinite set $\mathcal{X}$ of **object variables**, a countable set $\mathcal{R}$ of **relation symbols** R of arity m_R, and a countable set $\mathcal{F}$ of **function symbols** f of arity n_f such that $\neg, \wedge \notin \mathcal{F}$. The **terms** t_j of $\mathcal{L}$ are built in the following way:*

$$t_j \quad ::= \quad x \quad | \quad \neg t_1 \quad | \quad t_1 \wedge t_2 \quad | \quad f(t_1, \ldots, t_{n_f}),$$

*where $x \in \mathcal{V}$ and $f \in \mathcal{F}$. The **term assertions** of $\mathcal{L}$ are of the form*

$$t_1 \sqsubseteq t_2,$$

*where t_1 and t_2 are terms, and the **object assertions** are*

- *$R(a_1, \ldots, a_{m_R})$, for $a_1, \ldots, a_{m_R} \in \mathcal{X}$ and $R \in \mathcal{R}$;*
- *$a : t$, for $a \in \mathcal{X}$ and t a term.*

*The sets of term and object assertions together form the set of $\mathcal{L}$**-assertions**. We will write $t_1 = t_2$ as an abbreviation for the two assertions $t_1 \sqsubseteq t_2$, $t_2 \sqsubseteq t_1$.*

The semantics of ADLs is defined via abstract description models.

DEFINITION 4.2 (ADM). *Given an* ADL $\mathcal{L} = \langle \mathcal{V}, \mathcal{X}, \mathcal{R}, \mathcal{F} \rangle$, *an **abstract description model** (ADM) for $\mathcal{L}$ is a structure of the form*

$$\mathfrak{W} = \left\langle W, \mathcal{V}^{\mathfrak{W}} = (x^{\mathfrak{W}})_{x \in \mathcal{V}}, \mathcal{X}^{\mathfrak{W}} = (a^{\mathfrak{W}})_{a \in \mathcal{X}}, \mathcal{F}^{\mathfrak{W}} = (f^{\mathfrak{W}})_{f \in \mathcal{F}}, \mathcal{R}^{\mathfrak{W}} = (R^{\mathfrak{W}})_{R \in \mathcal{R}} \right\rangle,$$

where W is a non-empty set, $x^{\mathfrak{W}} \subseteq W$, $a^{\mathfrak{W}} \in W$, each $f^{\mathfrak{W}}$ is a function mapping n_f-tuples $\left\langle X_1, \ldots, X_{n_f} \right\rangle$ of subsets of W to a subset of W, and the $R^{\mathfrak{W}}$ are m_R-ary relations on W.

*The **value** $t^{\mathfrak{W}} \subseteq W$ of an $\mathcal{L}$-term t in $\mathfrak{W}$ is defined inductively by taking*

- *$(\neg t)^{\mathfrak{W}} = W \setminus (t)^{\mathfrak{W}}$, $(t_1 \wedge t_2)^{\mathfrak{W}} = t_1^{\mathfrak{W}} \cap t_2^{\mathfrak{W}}$,*
- *$(f(t_1, \ldots, t_{m_f}))^{\mathfrak{W}} = f^{\mathfrak{W}}(t_1^{\mathfrak{W}}, \ldots, t_{m_f}^{\mathfrak{W}})$.*

*The **truth-relation** $\mathfrak{W} \models \varphi$ for an $\mathcal{L}$-assertion φ is defined in the obvious way:*

- $\mathfrak{W} \models R(a_1, \ldots, a_{m_R}) \iff R^{\mathfrak{W}}(a_1^{\mathfrak{W}}, \ldots, a_{m_R}^{\mathfrak{W}})$,
- $\mathfrak{W} \models a : t \iff a^{\mathfrak{W}} \in t^{\mathfrak{W}}$,
- $\mathfrak{W} \models t_1 \sqsubseteq t_2 \iff t_1^{\mathfrak{W}} \subseteq t_2^{\mathfrak{W}}$.

If $\mathfrak{W} \models \varphi$ holds, we say that φ is ***satisfied*** *in $\mathfrak{W}$. For sets Γ of assertions, we write $\mathfrak{W} \models \Gamma$ if $\mathfrak{W} \models \varphi$ holds for all $\varphi \in \Gamma$.*

ADSs become a powerful tool by providing a choice of an appropriate class of ADMs in which the ADL is interpreted. In this way, we can, e.g., ensure that a function symbol has the desired semantics, and that relation symbols are interpreted as relations having desired properties, say, transitivity.

DEFINITION 4.3 (ADS). *An* ***abstract description system*** (ADS) *is a pair* $(\mathcal{L}, \mathcal{M})$, *where $\mathcal{L}$ is an* ADL *and $\mathcal{M}$ is a class of* ADMs *for $\mathcal{L}$ that is closed under the following operations:*

(i) *if $\mathfrak{W} = \langle W, \mathcal{V}^{\mathfrak{W}}, \mathcal{X}^{\mathfrak{W}}, \mathcal{F}^{\mathfrak{W}}, \mathcal{R}^{\mathfrak{W}} \rangle$ is in $\mathcal{M}$ and $\mathcal{V}^{\mathfrak{W}'} = (x^{\mathfrak{W}'})_{x \in \mathcal{V}}$ is a new assignment of set variables in W, then $\mathfrak{W}' = \left\langle W, \mathcal{V}^{\mathfrak{W}'}, \mathcal{X}^{\mathfrak{W}}, \mathcal{F}^{\mathfrak{W}}, \mathcal{R}^{\mathfrak{W}} \right\rangle$ is in $\mathcal{M}$ as well;*

(ii) *for every finite $\mathcal{G} \subseteq \mathcal{F}$, there exists a finite set $\mathcal{X}_{\mathcal{G}} \subseteq \mathcal{X}$ such that, for every model $\mathfrak{W} = \langle W, \mathcal{V}^{\mathfrak{W}}, \mathcal{X}^{\mathfrak{W}}, \mathcal{F}^{\mathfrak{W}}, \mathcal{R}^{\mathfrak{W}} \rangle$ from $\mathcal{M}$ and every assignment $\mathcal{X}^{\mathfrak{W}'} = (a^{\mathfrak{W}'})_{a \in \mathcal{X}}$ of object variables in W such that $a^{\mathfrak{W}} = a^{\mathfrak{W}'}$ for all $a \in \mathcal{X}_{\mathcal{G}}$, there is an interpretation $\mathcal{F}^{\mathfrak{W}'} = (f^{\mathfrak{W}'})_{f \in \mathcal{F}}$ of the function symbols such that $f^{\mathfrak{W}'} = f^{\mathfrak{W}}$ for all $f \in \mathcal{G}$ and $\mathfrak{W}' = \left\langle W, \mathcal{V}^{\mathfrak{W}}, \mathcal{X}^{\mathfrak{W}'}, \mathcal{F}^{\mathfrak{W}'}, \mathcal{R}^{\mathfrak{W}} \right\rangle$ is in $\mathcal{M}$.*

The first closure condition imposed on the class of models $\mathcal{M}$ means that set variables are treated as *variables* in any ADS, i.e., their values are not fixed. Closure condition (ii) deals with object variables and is slightly weaker: it states that object variables behave like variables, except that the interpretation of a finite number of function symbols may determine the assignments of a finite number of object variables. This weakening is required to enable the representation of the important 'nominal-constructor' from modal and description logic (which associates with any object variable a nullary function symbol; see below for more details) in abstract description systems. Mostly, however, the example ADSs we are going to discuss satisfy the stronger condition:

(ii′) if $\mathfrak{W} = \langle W, \mathcal{V}^{\mathfrak{W}}, \mathcal{X}^{\mathfrak{W}}, \mathcal{F}^{\mathfrak{W}}, \mathcal{R}^{\mathfrak{W}} \rangle \in \mathcal{M}$ and $\mathcal{X}^{\mathfrak{W}'} = (a^{\mathfrak{W}'})_{a \in \mathcal{X}}$ is a new assignment of object variables in W, then $\mathfrak{W}' = \left\langle W, \mathcal{V}^{\mathfrak{W}}, \mathcal{X}^{\mathfrak{W}'}, \mathcal{F}^{\mathfrak{W}}, \mathcal{R}^{\mathfrak{W}} \right\rangle$ is in $\mathcal{M}$ as well.

The main reasoning tasks for an ADS $\mathcal{S}$ we will be concerned with are the *satisfiability problem* for finite sets of assertions, and the satisfiability problem restricted to sets Γ of *object assertions*, which we will call the *A-satisfiability problem* for $\mathcal{S}$ (here, the 'A' stands for ABox; see below).

DEFINITION 4.4 (SATISFIABILITY IN ADS). *Let $\mathcal{S} = (\mathcal{L}, \mathcal{M})$ be an* ADS. *A finite set Γ of $\mathcal{L}$-assertions is called **satisfiable** in $\mathcal{S}$ if there exists an* ADM $\mathfrak{W} \in \mathcal{M}$ *such that $\mathfrak{W} \models \Gamma$. The problem of deciding, given any finite set Γ of object assertions from $\mathcal{L}$, whether Γ is satisfiable, is called the **A-satisfiability** problem in $\mathcal{S}$.*

Note that the **entailment** of term assertions and, e.g., object assertions of the form $a : t$—to decide, given such an object assertion φ and a finite set of assertions Γ, whether $\mathfrak{W} \models \Gamma$ implies $\mathfrak{W} \models \varphi$ for all models $\mathfrak{W}$—is clearly reducible to the satisfiability problem. For example, Γ entails $a : t$ if and only if $\Gamma \cup \{a : \neg t\}$ is not satisfiable.

We now introduce a number of logics that have been proposed for knowledge representation and reasoning in AI, and show how these logics can be viewed as ADSs. Again, our presentation will be brief, but self-contained. We put these formalisms to work in Section 4.5, where we give several examples illustrating $\mathcal{E}$-connections. Moreover, we will discuss the treatment of nominals in the framework of ADS.

4.2.2. Description Logics. Description logics (DLs) are formalisms devised for the representation of and reasoning about conceptual knowledge. Such knowledge is represented in terms of compound concepts which are composed from atomic concepts (unary predicates) and roles (binary predicates) using the concept and role constructors provided by the given DL. Description logic knowledge bases consist of

- a TBox containing concept inclusion statements of the form $C_1 \sqsubseteq C_2$, where both C_1 and C_2 are concepts, and
- an ABox containing assertions of the form $a : C$ and $(a, b) : R$, where a, b are object names, C is a concept, and R is a role.

Description logics have found applications in various fields of Artificial Intelligence, for example, as languages for describing ontologies in the context of the semantic web.[5] In Baader et al. [2002], it has been shown that almost all description logics can be regarded as ADSs. Here, we briefly describe three members of this family of logics and their translations into ADSs. We start with the basic description logic, $\mathcal{ALC}$.

The alphabet of $\mathcal{ALC}$ comprises concept names $A_1, A_2, \ldots$, role names $R_1, R_2, \ldots$, object names $a_1, a_2, \ldots$, the Boolean constructors $\neg$ and $\sqcap$, and the existential and universal restrictions $\exists$ and $\forall$, respectively. $\mathcal{ALC}$-**concepts** C_i are built according to the following rule:

$$C_i \quad ::= \quad A_i \quad | \quad \neg C_1 \quad | \quad C_1 \sqcap C_2 \quad | \quad \exists R.C \quad | \quad \forall R.C.$$

[5] More information on DLs can be found in the handbook [Baader et al., 2003].

As usual, we use $C_1 \sqcup C_2$ as an abbreviation for $\neg(\neg C_1 \sqcap \neg C_2)$, and $\exists R.C$ as an abbreviation for $\neg\forall R.\neg C$. An $\mathcal{ALC}$-**model** is a structure of the form

$$\mathfrak{I} = \left\langle \Delta, A_1^{\mathfrak{I}}, \ldots, R_1^{\mathfrak{I}}, \ldots, a_1^{\mathfrak{I}}, \ldots \right\rangle,$$

where Δ is a non-empty set, the $A_i^{\mathfrak{I}}$ are subsets of Δ, the $R_i^{\mathfrak{I}}$ are binary relations on Δ, and the $a_i^{\mathfrak{I}}$ are elements of Δ. The interpretation of complex concepts is defined by setting

$$(\neg C)^{\mathfrak{I}} = \Delta \setminus C^{\mathfrak{I}}, \qquad (C \sqcap D)^{\mathfrak{I}} = C^{\mathfrak{I}} \cap D^{\mathfrak{I}},$$
$$(\exists R.C)^{\mathfrak{I}} = \{w \in \Delta \mid \exists v \in \Delta((w,v) \in R^{\mathfrak{I}} \wedge v \in C^{\mathfrak{I}})\},$$
$$(\forall R.C)^{\mathfrak{I}} = \{w \in \Delta \mid \forall v \in \Delta((w,v) \in R^{\mathfrak{I}} \to v \in C^{\mathfrak{I}})\}.$$

The concepts of $\mathcal{ALC}$ can be regarded as terms $C^\sharp$ of an ADS $\mathcal{ALC}^\sharp$. Indeed, we can associate with each concept name A_i a set variable $A_i^\sharp$, with each role name R_i two unary function symbols $f_{\forall R_i}$ and $f_{\exists R_i}$, and then set inductively:

$$(\neg C)^\sharp = \neg C^\sharp, \qquad (C \sqcap D)^\sharp = C^\sharp \wedge D^\sharp,$$
$$(\exists R_i.C)^\sharp = f_{\exists R_i}(C^\sharp), \qquad (\forall R_i.C)^\sharp = f_{\forall R_i}(C^\sharp).$$

The object names of $\mathcal{ALC}$ are treated as object variables of $\mathcal{ALC}^\sharp$ and the role names as its binary relations. Thus, $\mathcal{ALC}^\sharp$-term assertions correspond to concept inclusion statements, while object assertions correspond to ABox assertions. The class $\mathcal{M}$ of ADMs for $\mathcal{ALC}^\sharp$ is defined as follows. For every $\mathcal{ALC}$-model $\mathfrak{I} = \left\langle \Delta, A_1^{\mathfrak{I}}, \ldots, R_1^{\mathfrak{I}}, \ldots, a_1^{\mathfrak{I}}, \ldots \right\rangle$, the class $\mathcal{M}$ contains the model

$$\mathfrak{M} = \left\langle \Delta, \mathcal{V}^{\mathfrak{M}}, \mathcal{X}^{\mathfrak{M}}, \mathcal{F}^{\mathfrak{M}}, \mathcal{R}^{\mathfrak{M}} \right\rangle,$$

where, for every concept name A, role name R, and every object name a,

$$(A^\sharp)^{\mathfrak{M}} = A^{\mathfrak{I}}, \qquad R^{\mathfrak{M}} = R^{\mathfrak{I}}, \qquad a^{\mathfrak{M}} = a^{\mathfrak{I}},$$
$$f_{\exists R}^{\mathfrak{M}}(X) = \{w \in \Delta \mid \exists v\ ((w,v) \in R^{\mathfrak{I}} \wedge v \in X)\},$$
$$f_{\forall R}^{\mathfrak{M}}(X) = \{w \in \Delta \mid \forall v\ ((w,v) \in R^{\mathfrak{I}} \to v \in X)\}.$$

Observe that the semantics of the function symbols $f_{\exists R}$ and $f_{\forall R}$ is obtained in a straightforward way from the semantics of the DL constructors $\exists R.C$ and $\forall R.C$. Since the interpretations of concept and object names can be changed arbitrarily, $\mathcal{M}$ satisfies the closure conditions (i) and (ii′) (and therefore (ii) as well). Now, considering this translation, it is easily seen that

- the satisfiability problem of $\mathcal{ALC}^\sharp$ corresponds to the problem of whether an $\mathcal{ALC}$-ABox is satisfiable with respect to a TBox;[6]

[6]Note that in the literature the TBoxes we are concerned with are usually called **general** TBoxes.

- the A-satisfiability problem of $\mathcal{ALC}^\sharp$ corresponds to the problem of whether an $\mathcal{ALC}$-ABox is satisfiable without any reference to TBoxes.

Our second description logic, $\mathcal{SHIQ}$, extends $\mathcal{ALC}$ by various additional constructors. For brevity, we define here only those that will be used in the examples later on, viz., inverse roles and qualified number restrictions.[7] The **inverse roles** allow us to use roles of the form R^{-1} (where R is a role name) in place of role names, and the **qualified number restrictions** are concept constructors of the form $(\geq nR.C)$ and $(\leq nR.C)$; their semantics is almost obvious:

$$\begin{aligned}
(R^{-1})^{\mathcal{I}} &= \{(w,v) \mid (v,w) \in R^{\mathcal{I}}\},\\
(\geq nR.C)^{\mathcal{I}} &= \{w \in \Delta \mid \left|\{v \in \Delta \mid (w,v) \in R^{\mathcal{I}} \wedge v \in C^{\mathcal{I}}\}\right| \geq n\},\\
(\leq nR.C)^{\mathcal{I}} &= \{w \in \Delta \mid \left|\{v \in \Delta \mid (w,v) \in R^{\mathcal{I}} \wedge v \in C^{\mathcal{I}}\}\right| \leq n\}.
\end{aligned}$$

More details on $\mathcal{SHIQ}$ can be found in Horrocks et al. [1999, 2000]. By extending the translation $.^\sharp$ of $\mathcal{ALC}$ above in a straightforward way, one can transform $\mathcal{SHIQ}$ into the corresponding ADS $\mathcal{SHIQ}^\sharp$. Details of this translation can be found in Baader et al. [2002].

The third description logic we deal with is called $\mathcal{ALCO}$; it extends $\mathcal{ALC}$ with the nominal constructor $\{a\}$, where a is an object name; cf. Schaerf [1994], Horrocks and Sattler [2001]. The semantics of the concepts $\{a\}$ is as expected: $\{a\}^{\mathcal{I}} = \{a^{\mathcal{I}}\}$. Thus, the difference between $\mathcal{ALC}$ and $\mathcal{ALCO}$ is that $\mathcal{ALCO}$ allows the use of object names *in concepts* rather than only in ABox assertions. The corresponding ADS, $\mathcal{ALCO}^\sharp$, is obtained from $\mathcal{ALC}^\sharp$ by introducing, for every object variable a of $\mathcal{ALC}^\sharp$, the nullary function symbol f_a such that, for every model $\mathfrak{M}$, $f_a^{\mathfrak{M}} = \{a^{\mathfrak{M}}\}$, and by setting $\{a\}^\sharp = f_a$. While $\mathcal{ALC}^\sharp$ and $\mathcal{SHIQ}^\sharp$ satisfy the closure condition (ii′) following Definition 4.3—simply observe that there is no interaction between the interpretation of function symbols and object variables—this is obviously not the case for $\mathcal{ALCO}^\sharp$: by changing the assignment of an object variable a we also change the interpretation of the nullary function symbol f_a. However, $\mathcal{ALCO}^\sharp$ does satisfy (ii). Indeed, given a finite set $\mathcal{G}$ of function symbols of $\mathcal{ALCO}^\sharp$, let $\mathcal{X}_\mathcal{G}$ be the set of all object variables a such that $f_a \in \mathcal{G}$. Now, for any new assignment of the variables in $\mathcal{X} \setminus \mathcal{X}_\mathcal{G}$, the new interpretation of the function symbols not occurring in $\mathcal{G}$ is obtained by interpreting every nominal f_a, $a \in \mathcal{X} \setminus \mathcal{X}_\mathcal{G}$, as the singleton set containing the object newly assigned to a. The remaining function symbols are interpreted as before.

To determine the computational complexity of reasoning with the ADSs defined above, let us recall that, for $\mathcal{ALC}$, $\mathcal{SHIQ}$, and $\mathcal{ALCO}$, ABox-satisfiability with respect to TBoxes is EXPTIME-complete, compare De Giacomo and Lenzerini [1996], Tobies

[7] $\mathcal{SHIQ}$ also provides for transitive roles and role hierarchies, and the application of the qualified number restrictions constructor is limited to so-called *simple* roles.

[2001a], and Areces et al. [1999], respectively. It follows immediately that we have the following:

PROPOSITION 4.5. *The satisfiability problem for* $\mathcal{ALC}^\sharp$*,* $\mathcal{SHIQ}^\sharp$*, and* $\mathcal{ALCO}^\sharp$ *is* EXPTIME*-complete.*

In what follows, it will turn out that the difference between $\mathcal{ALC}$ and $\mathcal{ALCO}$ is rather important, also on the level of ADSs.

4.2.3. Nominals in ADS**.** To be precise about the notion of 'nominal' within the framework of ADSs, let us first give a definition.

DEFINITION 4.6 (NOMINALS). *An* ADS $\mathcal{S} = (\mathcal{L}, \mathcal{M})$*, where* $\mathcal{L} = \langle \mathcal{V}, \mathcal{X}, \mathcal{R}, \mathcal{F} \rangle$*, is said to* ***have nominals*** *if* $\mathcal{F}$ *contains a nullary function symbol* f_a*, for each* $a \in \mathcal{X}$*, such that, for every* $\mathfrak{W} = \langle W, \mathcal{V}^{\mathfrak{W}}, \mathcal{X}^{\mathfrak{W}}, F^{\mathfrak{W}}, R^{\mathfrak{W}} \rangle$ *in* $\mathcal{M}$*, we have* $f_a^{\mathfrak{M}} = \{a^{\mathfrak{M}}\}$*. Usually, we will denote the function symbols* f_a *by* $\{a\}$ *and call them* ***nominals****.*

The ADS $\mathcal{ALCO}^\sharp$ obviously has nominals in the sense of this definition, while the ADSs $\mathcal{ALC}^\sharp$ and $\mathcal{SHIQ}^\sharp$ do not.

REMARK 4.7. There is a close connection between nominals and object assertions: for an ADS with nominals, object assertions of the form $a : t$ can be reformulated as $\{a\} \sqsubseteq t$. On the other hand, in general, object assertions of the form $R(a_1, \ldots, a_m)$ cannot be rephrased in this style. Yet, for some ADSs they are equivalent to assertions of the form $\{a_1\} \sqsubseteq f(a_2, \ldots, a_m)$, as will be clear from examples below. We could give a more general definition of 'to have nominals' by replacing nullary function symbols f_a with terms t_a. The results we are going to obtain for ADSs with nominals hold true under this more general definition as well.

In the examples that follow, some expressive means provided by the ADSs have no direct counterparts in the corresponding logics. For instance, none of these logics has explicit term and object assertions. However, we will see that this additional expressivity can be regarded just as 'syntactic sugar'.

4.2.4. A Modal Logic of Topological Spaces. The modal logic $\mathbf{S4_u}$, i.e., Lewis's modal system **S4** enriched with the universal modality, is an important formalism for reasoning about spatial knowledge. The interpretation of the basic **S4** logic (without the universal modality) in topological spaces dates back to Tarski [1938]. Later, the universal box was added in order to allow the representation of and reasoning about the well-known RCC-8 set of relations between two regions in a topological space [Randell et al., 1992].[8] We discuss the encoding of the RCC-8 relations in $\mathbf{S4_u}$ in Section 4.5.2.

[8] Compare also Renz [1998], Bennett [1996], Renz and Nebel [1998], Shehtman [1999], Wolter and Zakharyaschev [2002], Gabbay et al. [2003].

The language of $\mathbf{S4_u}$ is built from **region variables** $X_1, X_2, \dots$ (in the modal context, **propositional variables**), the Boolean operators, the **interior operator** $\boldsymbol{I}$ (the *necessity operator*), and the **universal quantifier** $\blacksquare$ (the *universal modality*). More precisely, **$\mathbf{S4_u}$-formulae** φ_i are defined as follows:

$$\varphi_i \quad ::= \quad X_j \quad | \quad \neg\varphi_1 \quad | \quad \varphi_1 \wedge \varphi_2 \quad | \quad \boldsymbol{I}\varphi_1 \quad | \quad \blacksquare\,\varphi_1.$$

As usual, we use $\varphi_1 \vee \varphi_2$ as an abbreviation for $\neg(\neg\varphi_1 \wedge \neg\varphi_2)$, $\blacklozenge\,\varphi$ (the *universal diamond*) as an abbreviation for $\neg\blacksquare\,\neg\varphi$, and the **closure operator** $\boldsymbol{C}\varphi$ (the *possibility operator*) as an abbreviation for $\neg\boldsymbol{I}\neg\varphi$. A (**topological**) **$\mathbf{S4_u}$-model**

$$\mathfrak{I} = \left\langle T, \mathbb{I}, \mathbb{C}, X_1^{\mathfrak{I}}, X_2^{\mathfrak{I}}, \dots \right\rangle$$

consists of a topological space $\langle T, \mathbb{I}\rangle$, where $\mathbb{I}$ is an **interior operator** mapping subsets X of T to their **interior** $\mathbb{I}(X) \subseteq T$ and satisfying **Kuratowski's Axioms**

$$\mathbb{I}(X \cap Y) = \mathbb{I}(X) \cap \mathbb{I}(Y), \quad \mathbb{I}\mathbb{I}(X) = \mathbb{I}(X), \quad \mathbb{I}(X) \subseteq X,$$

for all $X, Y \subseteq T$, $\mathbb{C}$ is the closure operator defined by $\mathbb{C}(X) = T \setminus \mathbb{I}(T \setminus X)$, and the $X_i^{\mathfrak{I}}$ are subsets of T (interpreting the region variables of $\mathbf{S4_u}$). The **value** $\varphi^{\mathfrak{I}}$ of an $\mathbf{S4_u}$-formula φ in $\mathfrak{I}$ is defined inductively in the natural way:

$$\begin{array}{rclrcl} (\neg\psi)^{\mathfrak{I}} & = & T \setminus \psi^{\mathfrak{I}}, & (\chi \wedge \psi)^{\mathfrak{I}} & = & \chi^{\mathfrak{I}} \cap \psi^{\mathfrak{I}}, \\ (\boldsymbol{I}\psi)^{\mathfrak{I}} & = & \mathbb{I}\psi^{\mathfrak{I}}, & (\blacksquare\,\psi)^{\mathfrak{I}} & = & \begin{cases} \varnothing & \text{if } \psi^{\mathfrak{I}} \neq T, \\ T & \text{if } \psi^{\mathfrak{I}} = T. \end{cases} \end{array}$$

We say that φ is **satisfiable** if there is an $\mathbf{S4_u}$-model $\mathfrak{I}$ such that $\varphi^{\mathfrak{I}} \neq \varnothing$.

Let us see now how $\mathbf{S4_u}$ can be represented as an ADS $\mathbf{S4_u^\sharp}$. The corresponding ADL contains the set variables $X_1^\sharp, X_2^\sharp, \dots$, the unary function symbols $f_{\boldsymbol{I}}$ and $f_{\blacksquare}$, but no relation symbols. Besides, according to the definition, $\mathbf{S4_u^\sharp}$ must contain a countably infinite set of object variables a_i. The translation $.^\sharp$ of $\mathbf{S4_u}$-formulae into $\mathbf{S4_u^\sharp}$-terms is obvious, e.g., $(\Box\varphi)^\sharp = f_\Box(\varphi^\sharp)$, where $\Box \in \{\boldsymbol{I}, \blacksquare\}$.

Define a class $\mathcal{M}$ of ADMs for $\mathbf{S4_u^\sharp}$ by taking, for every $\mathbf{S4_u}$-model $\mathfrak{I}$ as above, the ADMs

$$\mathfrak{M} = \left\langle T, \mathcal{V}^{\mathfrak{M}}, \mathcal{X}^{\mathfrak{M}}, f_{\boldsymbol{I}}^{\mathfrak{M}}, f_{\blacksquare}^{\mathfrak{M}} \right\rangle,$$

where $(X_i^\sharp)^{\mathfrak{M}} = X^{\mathfrak{I}_i}$, $a^{\mathfrak{M}} \in T$, for every $a \in \mathcal{X}$, $f_{\boldsymbol{I}}^{\mathfrak{M}} = \mathbb{I}$, and, for every $Y \subseteq T$,

$$f_{\blacksquare}^{\mathfrak{M}}(Y) = \begin{cases} \varnothing & \text{if } Y \neq T, \\ T & \text{if } Y = T. \end{cases}$$

Obviously, $\mathbf{S4_u^\sharp}$ satisfies the closure conditions (i) and (ii) of Definition 4.3 (it even satisfies (ii′)); so it is an ADS. Unlike $\mathbf{S4_u^\sharp}$, the logic $\mathbf{S4_u}$ does not have assertions of the form $t_1 \sqsubseteq t_2$ or $a : t$. So we have to be careful when relating the computational complexity of $\mathbf{S4}_{\mathbf{u}}^{\sharp}$ to that of $\mathbf{S4}_{\mathbf{u}}$. However, we have the following:

PROPOSITION 4.8. *The satisfiability problem for* $\mathbf{S4}_{\mathbf{u}}^{\sharp}$ *is* PSPACE-*complete.*

PROOF. PSPACE-hardness follows from PSPACE-hardness of the satisfiability problem for **S4** [Ladner, 1977]. We establish the corresponding upper bound by means of a reduction to the satisfiability problem for $\mathbf{S4}_{\mathbf{u}}$ enriched with *nominals* in topological models,[9] which is known to be PSPACE-complete [Areces et al., 2000]. Namely, given a set Γ of $\mathbf{S4}_{\mathbf{u}}^{\sharp}$-assertions we define an $\mathbf{S4}_{\mathbf{u}}$-formula φ_Γ as the conjunction of all formulae in the set

$$\{\blacksquare(\varphi_1 \to \varphi_2) \mid (\varphi_1^{\sharp} \sqsubseteq \varphi_2^{\sharp}) \in \Gamma\} \cup \{\blacksquare(\{a\} \to \psi) \mid (a : \psi^{\sharp}) \in \Gamma\}.$$

Obviously, φ_Γ is satisfiable in some topological model if and only if Γ is satisfiable, which gives us the required PSPACE-upper bound. Reductions of this type are known as **internalisations** of TBoxes by means of the universal modality [Schild, 1991]. □

Note, that $\mathbf{S4}_{\mathbf{u}}^{\sharp}$ does not have nominals.

4.2.5. A Logic of Metric Spaces. Formalisms like $\mathbf{S4}_{\mathbf{u}}$ allow the representation of qualitative spatial knowledge using, e.g., the RCC-8 relations. Motivated by the fact that many spatial AI applications also require representations of *quantitative* information, we investigated logics of distance spaces in Part 1 of this thesis. Here, we consider one member of this family, namely $\mathcal{MSO}_D[\mathbb{Q}^+]$ introduced on Page 20, and define a corresponding ADS.[10]

Recall that formulae in the language $\mathcal{LO}_D$ of $\mathcal{MSO}_D$ are build from propositional variables p_i and nominals i_k using the Booleans, the universal modality $\blacksquare$, as well as the operators $\mathsf{A}^{\leq a}$ and $\mathsf{A}^{>a}$, for $a \in \mathbb{Q}^+$. More precisely, $\mathcal{MSO}_D$-**formulae** φ_i are defined as follows:

$$\varphi_i \ ::= \ p_j \ \mid \ i_k \ \mid \ \neg\varphi_1 \ \mid \ \varphi_1 \wedge \varphi_2 \ \mid \ \blacksquare\varphi_1 \ \mid \ \mathsf{A}^{\leq a}\varphi_1 \ \mid \ \mathsf{A}^{>a}\varphi_1.$$

Intuitively, given a subset p in a metric space, $\mathsf{A}^{\leq a}p$ is the set of all those points u in the space such that all points located at distance $\leq a$ from u belong to p. An $\mathcal{MSO}_D$-**model**

$$\mathfrak{B} = \left\langle W, d, p_1^{\mathfrak{B}}, \ldots, i_1^{\mathfrak{B}}, \ldots \right\rangle$$

consists of a metric space $\langle W, d\rangle$ together with interpretations of propositional variables p_i as subsets $p_i^{\mathfrak{B}}$ of W and nominals i_k as singleton subsets $i_k^{\mathfrak{B}}$ of W. We remind the reader that d is a function from $W \times W$ into the set $\mathbb{R}^+$ (of non-negative real numbers) satisfying Axioms (1)–(3) from Page 7.

[9]Nominals $\{a\}$ are interpreted as singleton sets of topological spaces.

[10]The logic we consider here is called $\mathcal{MS}_2$ in Sturm et al. [2000] and $\mathcal{MS}^{\sharp}$ in Kutz et al. [2003b].

To define a corresponding ADS $\mathcal{MSO}_D^\sharp$, we reserve a set variable $p_i^\sharp$ for each propositional variable p_i, an object variable $i_k^\sharp$ and a nullary function symbol $f_{i_k^\sharp}$ for each nominal i_k, and take unary function symbols $f_{\mathsf{A}^{\leq a}}$ and $f_{\mathsf{A}^{>a}}$, for each $a \in \mathbb{Q}^+$. Again, the set of relation symbols is empty. It should now be clear how to devise a translation $.^\sharp$ of $\mathcal{MSO}_D$-formulae into $\mathcal{MSO}_D^\sharp$-set terms. Moreover, to describe the class of ADMs, similarly to what was done in the preceding two sections, note that the semantics of the function symbols $f_{\mathsf{A}^{\leq a}}$ and $f_{\mathsf{A}^{>a}}$ can be derived from the semantics of the operators $\mathsf{A}^{\leq a}$ and $\mathsf{A}^{>a}$ in a straightforward way. Namely, every $\mathcal{MSO}_D$-model $\mathfrak{B}$ gives rise to an ADM

$$\mathfrak{M} = \left\langle W, \mathcal{V}^{\mathfrak{M}}, \mathcal{X}^{\mathfrak{M}}, \mathcal{F}^{\mathfrak{M}} \right\rangle$$

for $\mathcal{MSO}_D^\sharp$, where $(p_i^\sharp)^{\mathfrak{M}} = p_i^{\mathfrak{I}}$, $(i_k^\sharp)^{\mathfrak{M}} = u \in W$, for $(i_k)^{\mathfrak{I}} = \{u\}$, and $\mathcal{F}^{\mathfrak{M}}$ consists of all nullary functions $f_{i_k^\sharp}$ interpreted as $(f_{i_k^\sharp})^{\mathfrak{M}} = \{{i_k^\sharp}^{\mathfrak{M}}\}$, and $f_{\mathsf{A}^{\leq a}}$, $f_{\mathsf{A}^{>a}}$, for $a \in \mathbb{Q}^+$, where the latter are defined by taking

$$f^{\mathfrak{M}}_{\mathsf{A}^{\leq a}}(Y) = \{w \in W \mid \forall x \in W\ (d(w,x) \leq a \rightarrow x \in Y)\},$$
$$f^{\mathfrak{M}}_{\mathsf{A}^{>a}}(Y) = \{w \in W \mid \forall x \in W\ (\delta(w,x) > a \rightarrow x \in Y)\},$$

for every $Y \subseteq W$.

It is shown in Wolter [2004] that the satisfiability problem for $\mathcal{MSO}_D[\mathbb{Q}^+]$ is EXPTIME-complete if the parameters from $\mathbb{Q}^+$ are encoded in unary. Thus, since the universal modality is available in $\mathcal{MSO}_D$, we can essentially repeat the proof of Proposition 4.8 and obtain:

PROPOSITION 4.9. *The satisfiability problem for $\mathcal{MSO}_D^\sharp$ is* EXPTIME-*complete (for unary encoding of parameters).*

The ADS $\mathcal{MSO}_D^\sharp$ does have nominals.

4.2.6. Propositional Temporal Logic. Finally, we consider the propositional temporal logic **PTL** [Goldblatt, 1987, Gabbay et al., 1994, Fagin et al., 1995] which is a well-known tool for reasoning about time. **PTL-formulae** φ_i are composed from propositional variables p_i by means of the Booleans and the binary **temporal operators** $\mathcal{U}$ ('until') and $\mathcal{S}$ ('since'):

$$\varphi_i \quad ::= \quad p_j \quad | \quad \neg\varphi_1 \quad | \quad \varphi_1 \wedge \varphi_2 \quad | \quad \varphi_1 \mathcal{U} \varphi_2 \quad | \quad \varphi_1 \mathcal{S} \varphi_2.$$

We introduce $\Diamond_F \varphi$ ('eventually φ'), $\Box_F \varphi$ ('always in the future φ'), $\Diamond_P \varphi$ ('sometime in the past φ'), $\Box_P \varphi$ ('always in the past φ') as abbreviations for $\top \mathcal{U} \varphi$, $\neg \Diamond_F \neg \varphi$, $\top \mathcal{S} \varphi$, and $\neg \Diamond_P \neg \varphi$, respectively. A **PTL-model** is a structure of the form

$$\mathfrak{I} = \left\langle \mathbb{N}, <, p_0^{\mathfrak{I}}, p_1^{\mathfrak{I}}, \ldots \right\rangle,$$

where $\langle \mathbb{N}, < \rangle$ is the intended **flow of time**, and $p_i^{\mathfrak{I}} \subseteq \mathbb{N}$. The **temporal extension** $\varphi^{\mathfrak{I}}$ of a **PTL**-formula φ is defined inductively in the standard way, the interesting cases being:

$$(\varphi_1 \mathcal{U} \varphi_2)^{\mathfrak{I}} = \{u \in \mathbb{N} \mid \exists z > u \; (z \in \varphi_2^{\mathfrak{I}} \wedge \forall y \in (u, z) \; y \in \varphi_1^{\mathfrak{I}})\},$$
$$(\varphi_1 \mathcal{S} \varphi_2)^{\mathfrak{I}} = \{u \in \mathbb{N} \mid \exists z < u \; (z \in \varphi_2^{\mathfrak{I}} \wedge \forall y \in (z, u) \; y \in \varphi_1^{\mathfrak{I}})\},$$

where $(u, v) = \{w \in \mathbb{N} \mid u < w < v\}$.

To obtain the corresponding ADS $\mathbf{PTL}^\sharp$, we associate with $\mathcal{U}$ and $\mathcal{S}$ **binary** function symbols $f_{\mathcal{U}}$ and $f_{\mathcal{S}}$. It is not hard now to define a translation $.^\sharp$ from **PTL**-formulae to $\mathbf{PTL}^\sharp$-terms. We represent individual time points and the precedence relation $<$ by adding nominals and the relation symbol $<$ to **PTL**, i.e., the language $\mathbf{PTL}^\sharp$ has the function symbols $f_{\mathcal{U}}$, $f_{\mathcal{S}}$ and $\{a\}$, for any object variable a, and the binary relation symbol $<$ interpreted by the precedence relation on $\mathbb{N}$. Note that although **PTL** itself contains none of these explicitly, nominals $\{a\}$ (and so object variables) can be simulated as **PTL**-formulae $p_a \wedge \neg\Diamond_F p_a \wedge \neg\Diamond_P p_a$, and the assertion $a < b$ can be simulated as

$$(p_a \wedge \neg\Diamond_F p_a \wedge \neg\Diamond_P p_a) \wedge \Diamond_F(p_b \wedge \neg\Diamond_F p_b \wedge \neg\Diamond_P p_b).$$

The definition of the class of ADMs for $\mathbf{PTL}^\sharp$ is now straightforward. Note that the satisfiability problem of $\mathbf{PTL}^\sharp$ is not more complex than the satisfiability problem in **PTL**:

PROPOSITION 4.10. *The satisfiability problem for* $\mathbf{PTL}^\sharp$ *is* PSPACE-*complete.*

PROOF. It it proved in Sistla and Clarke [1985] that the satisfiability problem for **PTL** is PSPACE-complete. As we have already seen above, the nominals and the binary relation $<$ can be simulated in **PTL**. Observe that the universal box $\blacksquare\,\varphi$ can be expressed as well, by using the formula $\Box_F \varphi \wedge \varphi \wedge \Box_P \varphi$. Therefore, we can employ the same internalisation reduction as in the proof of Proposition 4.8 to show that the satisfiability problem for $\mathbf{PTL}^\sharp$ is PSPACE-complete. □

4.3. Number Tolerance and Singleton Satisfiability

In this section, we introduce and discuss an important property of ADSs, *number tolerance*, which will play a key role in the decidability transfer results for $\mathcal{E}$-connections allowing number restrictions on links. Furthermore, we will introduce a new reasoning problem for ADSs, *singleton satisfiability*, that we will employ in Section 5.4 to show that enriching $\mathcal{E}$-connections with number restrictions does, in general, lead to undecidability.

4.3.1. Number Tolerance. Given a finite set Σ of $\mathcal{L}$-assertions, we denote by $term(\Sigma)$ the set of all terms in Σ.

DEFINITION 4.11 (NUMBER TOLERANCE). *An* ADS $\mathcal{S} = (\mathcal{L}, \mathcal{M})$ *is called* ***number tolerant*** *if there is a cardinal κ such that, for every $\kappa' \geq \kappa$ and every satisfiable finite set Σ of assertions, there exists a model $\mathfrak{W} \in \mathcal{M}$ satisfying Σ and such that, for each $d \in W$, there are precisely κ' elements $d' \in W$ for which*

$$\{t \in term(\Sigma) \mid d \in t^{\mathfrak{W}}\} = \{t \in term(\Sigma) \mid d' \in t^{\mathfrak{W}}\}.$$

Intuitively, being number tolerant means that if a knowledge base Σ is satisfiable we can find a model of Σ in which each occurring **type**, that is, set of terms, is satisfied a 'very large' number of times. For example, ADSs of modal logics that are invariant for the formation of disjoint unions of structures are clearly number tolerant. In contrast, ADSs with nominals cannot be number tolerant because nominals are always interpreted as singleton sets. Thus, for instance, the ADSs $\mathcal{ALCO}^\sharp$ and $\mathcal{MSO}_D^\sharp$ are not number tolerant. Indeed, even $\mathbf{PTL}^\sharp$ and $\mathcal{MS}_D^\sharp$ are not number tolerant since they can simulate nominals, as shown in the last section for **PTL** and in Proposition 1.16 for $\mathcal{MS}_D$.

We now use results from Baader et al. [2002] to obtain a straightforward proof that the ADSs for numerous description logics, in particular $\mathcal{ALC}^\sharp$ and $\mathcal{SHIQ}^\sharp$, are number tolerant. The following notion of a *local* ADS was introduced in Baader et al. [2002], where the transfer of decidability from local ADSs to their so-called **fusions** is proved:

DEFINITION 4.12 (LOCALNESS AND DISJOINT UNIONS). *Given a family $(\mathfrak{W}_p)_{p\in P}$ of* ADMs

$$\mathfrak{W}_p = \left\langle W_p, \mathcal{V}^{\mathfrak{W}_p}, \mathcal{X}^{\mathfrak{W}_p}, \mathcal{F}^{\mathfrak{W}_p}, \mathcal{R}^{\mathfrak{W}_p} \right\rangle$$

over pairwise disjoint domains W_p, we say that

$$\mathfrak{W} = \left\langle W, \mathcal{V}^{\mathfrak{W}}, \mathcal{X}^{\mathfrak{W}}, \mathcal{F}^{\mathfrak{W}}, \mathcal{R}^{\mathfrak{W}} \right\rangle$$

is a ***disjoint union*** *of $(\mathfrak{W}_p)_{p\in P}$ if*

- $W = \bigcup_{p\in P} W_p$;
- $f^{\mathfrak{W}}(X_1, \ldots, X_{n_f}) = \bigcup_{p\in P} f^{\mathfrak{W}_p}(X_1 \cap W_p, \ldots, X_{n_f} \cap W_p),$
  for all $X_1, \ldots, X_{n_f} \subseteq W$ and all $f \in \mathcal{F}$;
- $R^{\mathfrak{W}} = \bigcup_{p\in P} R^{\mathfrak{W}_p}$ *for all $R \in \mathcal{R}$.*

An ADS $S = (\mathcal{L}, \mathcal{M})$ *is called* ***local*** *if $\mathcal{M}$ is closed under disjoint unions.*

The following result is easily proved and illustrates the relationship between localness and number tolerance.

PROPOSITION 4.13. *Every local* ADS *is number tolerant.*

PROOF. Suppose that an ADS $(\mathcal{L}, \mathcal{M})$ is local. Let κ be any infinite cardinal such that, for every finite satisfiable Σ, there exists a model $\mathfrak{W} \in \mathcal{M}$ of cardinality $\leq \kappa$ which satisfies Σ. The supremum of all the minimal cardinals needed to satisfy each Σ will do, for instance. We show that κ is as required. Suppose that $\kappa' \geq \kappa$ and that Σ is satisfiable. Take any model

$$\mathfrak{W}_0 = \left\langle W_0, \mathcal{V}^{\mathfrak{W}_0}, \mathcal{X}^{\mathfrak{W}_0}, \mathcal{F}^{\mathfrak{W}_0}, \mathcal{R}^{\mathfrak{W}_0} \right\rangle$$

from $\mathcal{M}$ which satisfies Σ and is of cardinality $\leq \kappa$. Now take the disjoint union $\mathfrak{W}$ of κ' isomorphic copies $\mathfrak{W}_i$, $i < \kappa'$, of $\mathfrak{W}_0$ in which

- $x^{\mathfrak{W}} = \bigcup_{i<\kappa'} x^{\mathfrak{W}_i}$, for $x \in \mathcal{V}$;
- $a^{\mathfrak{W}} = a^{\mathfrak{W}_0}$, for $a \in \mathcal{X}$.

By cardinal arithmetic, the size of $\mathfrak{W}$ is κ', and it is not difficult to show that $\mathfrak{W}$ satisfies all of the conditions we need. □

It is an immediate consequence of Proposition 15 in Baader et al. [2002] that both $\mathcal{ALC}^\sharp$ and $\mathcal{SHIQ}^\sharp$ are local. By applying Proposition 4.13, we thus get:

PROPOSITION 4.14. $\mathcal{ALC}^\sharp$ *and* $\mathcal{SHIQ}^\sharp$ *are number tolerant.*

Note, however, that localness and number tolerance are not the same. For instance, the ADS $\mathsf{S4}_u^\sharp$ is a counterexample: it is number tolerant but not local.

PROPOSITION 4.15. $\mathsf{S4}_u^\sharp$ *is number tolerant.*

PROOF. To prove that $\mathsf{S4}_u^\sharp$ is number tolerant, we show that $\aleph_0$ is the required cardinal number. Suppose that $\kappa' \geq \aleph_0$ and that Σ is satisfiable. Let

$$\mathfrak{W}_0 = \left\langle T_0, \mathcal{V}^{\mathfrak{W}_0}, \mathcal{X}^{\mathfrak{W}_0}, f_{\mathbf{I}}^{\mathfrak{W}_0}, f_{\mathbf{C}}^{\mathfrak{W}_0}, f_{\blacksquare}^{\mathfrak{W}_0} \right\rangle$$

be a countable model satisfying Σ. Take the disjoint union $\mathfrak{W}'$ of κ' isomorphic copies $\mathfrak{W}'_i$, $i < \kappa'$, of the reduct

$$\mathfrak{W}'_0 = \left\langle T_0, \mathcal{V}^{\mathfrak{W}_0}, \mathcal{X}^{\mathfrak{W}_0}, f_{\mathbf{I}}^{\mathfrak{W}_0}, f_{\mathbf{C}}^{\mathfrak{W}_0} \right\rangle$$

of $\mathfrak{W}_0$, leaving out the universal modality, in which

- $x^{\mathfrak{W}'} = \bigcup_{i<\kappa'} x^{\mathfrak{W}_0}$, for $x \in \mathcal{V}$;
- $a^{\mathfrak{W}'} = a^{\mathfrak{W}_0}$, for $a \in \mathcal{X}$.

Now we extend $\mathfrak{W}'$ to a model $\mathfrak{W}$ of the required signature by setting

$$f_{\blacksquare}^{\mathfrak{W}}(Y) = \begin{cases} \varnothing & \text{if } Y \neq \bigcup\limits_{i<\kappa'} T_i, \\ \bigcup\limits_{i<\kappa'} T_i & \text{if } Y = \bigcup\limits_{i<\kappa'} T_i, \end{cases}$$

for every subset Y of $\bigcup_{i<\kappa'} T_i$. It is readily seen that the constructed ADM $\mathfrak{W}$ is as required. □

That $\mathsf{S4}_u^\sharp$ is not local follows from the fact that it is equipped with the universal modality: if we take the disjoint union of two ADMs, then the function symbols for the universal modality 'lose' their universality.

4.3.2. Singleton Satisfiability. The concept of singleton satisfiability discussed in this section is of a rather technical nature. However, the undecidability results we obtain will prove very useful when showing that automatic transfer of decidability fails for certain types of $\mathcal{E}$-connections.

DEFINITION 4.16 (SINGLETON SATISFIABILITY). *Let $\mathcal{S} = (\mathcal{L}, \mathcal{M})$ be an* ADS. *We call an $\mathcal{L}$-term t **singleton satisfiable** if there exists a model $\mathfrak{M} \in \mathcal{M}$ such that $|t^{\mathfrak{M}}| = 1$. The **singleton satisfiability problem** for the* ADS *$\mathcal{S}$ is to decide, given any term $t \in \mathcal{L}$, whether t is singleton satisfiable.*

As the following theorem shows, there exist ADSs that are number tolerant and have decidable satisfiability problems, but for which singleton satisfiability is, nevertheless, undecidable:

THEOREM 4.17 (UNDECIDABLE SINGLETON SATISFIABILITY).
There exist number tolerant ADSs *with decidable satisfiability problems for which singleton satisfiability is undecidable. In particular, there exist number tolerant* ADSs *with decidable satisfiability problems whose extensions with nominals have undecidable satisfiability problems.*

PROOF. Consider the ADS $\mathcal{ALC}^\sharp = (\mathcal{L}, \mathcal{M})$ corresponding to the description logic $\mathcal{ALC}$. It follows from, e.g., Theorem 13.15 of Chagrov and Zakharyaschev [1997]—which proves that there is a continuum of generally Post-complete logics in the lattice NExt**K4**—that there exists an uncountable set $\mathcal{K} = \{\mathcal{S}_i \mid i \in I\}$ of ADSs $\mathcal{S}_i = (\mathcal{L}, \mathcal{M}_i)$ such that $\mathcal{M}_i \subseteq \mathcal{M}$ for $i \in I$ and

(1) for all $i \in I$ and any $\mathcal{L}$-term t, satisfiability of $a : t$ in $\mathcal{S}_i$ implies singleton satisfiability of t in $\mathcal{S}_i$;
(2) for all $i, j \in I$ with $i \neq j$, there exists a constant term t (i.e., a term composed using the Booleans and function symbols from the symbol $\top$) such that $a : t$ is $\mathcal{S}_i$-satisfiable and not $\mathcal{S}_j$-satisfiable or vice versa.

By property (2), $i \neq j$ implies that the set of constant terms satisfiable in $\mathcal{S}_i$ is not identical to the set of constant terms satisfiable in $\mathcal{S}_j$. Since there exist only countably many algorithms (i.e., Turing machines), the fact that $\mathcal{K}$ is uncountable implies that there exists an $i_0 \in I$ such that satisfiability of constant terms in $\mathcal{S}_{i_0}$ is undecidable.

Since for any satisfiable $a : t$ the term t is singleton satisfiable by (1), it is undecidable whether a constant term is singleton satisfiable in $\mathcal{S}_{i_0}$.

Let $\mathcal{M}'$ denote those members of $\mathcal{M}$ which are disjoint unions of at least $\aleph_0$ isomorphic copies of some model in $\mathcal{M}$. By $\mathcal{M}'' \subseteq \mathcal{M}$ we denote the closure of $\mathcal{M}'$ under disjoint unions and arbitrary re-interpretations of object and set variables. The important properties of $\mathcal{M}''$ are as follows:

(a) A knowledge base Γ is satisfiable in $(\mathcal{L}, \mathcal{M}'')$ if and only if it is satisfiable in $\mathcal{ALC}^\sharp$.

(b) If $a : t$ is satisfied in some $\mathfrak{M} \in \mathcal{M}''$ and t is a constant term, then $\left|t^{\mathfrak{M}}\right| \geq \aleph_0$. Hence no satisfiable constant term t is singleton satisfiable in $\mathcal{M}''$.

That property (a) holds should be clear. Property (b) follows from the fact that the extension of constant terms does not depend on the interpretation of set or object variables. Now set $\mathcal{N} = \mathcal{M}_{i_0} \cup \mathcal{M}''$ and $\mathcal{S} = (\mathcal{L}, \mathcal{N})$. We claim that $\mathcal{S}$ is as required. Obviously, $\mathcal{S}$ is number tolerant and singleton satisfiability is undecidable. It remains to observe that the satisfiability problem for $\mathcal{S}$ coincides with the satisfiability problem for $\mathcal{ALC}^\sharp$, which is decidable.

The extension of $\mathcal{S}$ by means of nominals has an undecidable satisfiability problem, since $\{a\} = t$ is satisfiable if and only if t is singleton satisfiable, for any term t. □

4.4. Basic $\mathcal{E}$-Connections of Abstract Description Systems

We are now in a position to formally introduce the **basic $\mathcal{E}$-connection** of abstract description systems. Similarly to the definition of ADSs, an $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is specified by first defining its language, i.e., its alphabet and sets of terms and assertions, and by specifying its semantics.

Let $\mathcal{S}_1, \ldots, \mathcal{S}_n$ be n ADSs, where $\mathcal{S}_i = (\mathcal{L}_i, \mathcal{M}_i)$, for $1 \leq i \leq n$. Without loss of generality we assume that, for $1 \leq i < j \leq n$, the alphabets of the ADSs $\mathcal{S}_i$ and $\mathcal{S}_j$ (i.e., the sets of set variables, object variables, function symbols, and relation symbols) are disjoint apart from the Boolean operators.[11]

To define the **basic $\mathcal{E}$-connection** $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ of n ADSs $\mathcal{S}_1, \ldots, \mathcal{S}_n$, the first ingredient we have to specify is the set $\mathcal{E}$ of links:

DEFINITION 4.18 (LINKS AND LINK OPERATORS). *Let $\mathcal{S}_1, \ldots, \mathcal{S}_n$ be n* ADSs.

(i) *A non-empty set of n-ary relation symbols*

$$\mathcal{E} = \{E_j \mid j \in J\} \neq \varnothing$$

[11]It should be noted that this condition is different from the one required for fusions of ADSs in Baader et al. [2000, 2002]: when forming fusions, we assume that the respective sets of set and object variables of the ADSs to be combined coincide. In the case of $\mathcal{E}$-connections, these sets of symbols should be disjoint, since they are used in the combined system to represent knowledge about disjoint domains.

*is called a **link-set** for $\mathcal{S}_1, \ldots, \mathcal{S}_n$. The elements of $\mathcal{E}$ are called **link relations** or simply **links**.*

(ii) *Given a link-set $\mathcal{E} = \{E_j \mid j \in J\}$, the set of **link operators** generated by $\mathcal{E}$ is the set*

$$\{\langle E_j \rangle^i \mid 1 \leq i \leq n, j \in J\}$$

*of function symbols $\langle E_j \rangle^i$ of arity $n-1$ which are assumed to be distinct from the function symbols of $\mathcal{S}_1, \ldots, \mathcal{S}_n$. The function symbols $\langle E_j \rangle^i$ are called **link operators**.*

Next, we introduce the terms of an $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$. The set of $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$-terms consists, intuitively, of n disjoint sets of i-terms, $1 \leq i \leq n$, which, in turn, consist of the terms of $\mathcal{L}_i$ enriched with the new function symbols $\langle E_j \rangle^i$, for each $j \in J$. Here is a formal inductive definition:

DEFINITION 4.19 (TERMS AND i-TERMS OF $\mathcal{E}$-CONNECTIONS). *Let $\mathcal{S}_1, \ldots, \mathcal{S}_n$ be n* ADSs *in languages $\mathcal{L}_i$, $1 \leq i \leq n$, and $\mathcal{E} = \{E_j \mid j \in J\}$ a link-set. The set of **terms** of $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is the disjoint union of the sets of **i-terms**, $1 \leq i \leq n$, defined inductively as follows:*

- *every set variable of $\mathcal{L}_i$ is an i-term;*
- *the set of i-terms is closed under $\neg$, $\wedge$ and the function symbols of $\mathcal{L}_i$;*
- *if $(t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ is a sequence of k-terms t_k, for $k \neq i$, then*

$$\langle E_j \rangle^i (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$$

  is an i-term, for every $j \in J$.

There are three types of assertions of $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$. The first two types are the term assertions and object assertions of the component ADSs, taking into account the new sets of i-terms. Additionally, to be able to speak about the new ingredients of $\mathcal{E}$-connections, link relations, we require so-called *link assertions*. A formal definition is as follows:

DEFINITION 4.20 (ASSERTIONS IN $\mathcal{E}$-CONNECTIONS). *Assume, for $1 \leq i \leq n$, the sets of i-terms are already defined. Then define*

- *if t_1 and t_2 are i-terms then*

$$t_1 \sqsubseteq t_2$$

  *is an i-**term assertions**;*
- *if a, $a_1, \ldots, a_{m_R}$ are object variables of $\mathcal{L}_i$, t is an i-term, and R is a relation symbol of $\mathcal{L}_i$ of arity m_R, then*

$$a : t \text{ and } R(a_1, \ldots, a_{m_R})$$

  *are i-**object assertions**;*

– *if a_i, $1 \leq i \leq n$, are object variables of $\mathcal{L}_i$ and $j \in J$ then*

$$(a_1, \ldots, a_n) : E_j$$

*is a **link assertions**.*

*Taken together, the sets of all link assertions, i-term assertions, and i-object assertions form the set of **assertions** of the $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$. A finite set of assertions is also called a **knowledge base** of $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$.*

We now introduce the semantics of $\mathcal{E}$-connections.

DEFINITION 4.21 (SEMANTICS OF $\mathcal{E}$-CONNECTIONS). *Let $\mathcal{S}_1, \ldots, \mathcal{S}_n$ be n ADSs, where $\mathcal{S}_i = (\mathcal{L}_i, \mathcal{M}_i)$, for $1 \leq i \leq n$, and let $\mathcal{E} = \{E_j \mid j \in J\}$ be a link-set. A **model** for the $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is a structure of the form*

$$\mathfrak{M} = \left\langle (\mathfrak{W}_i)_{i \leq n}, \mathcal{E}^{\mathfrak{M}} = (E_j^{\mathfrak{M}})_{j \in J}) \right\rangle,$$

where $\mathfrak{W}_i \in \mathcal{M}_i$, for $1 \leq i \leq n$, and

$$E_j^{\mathfrak{M}} \subseteq W_1 \times \cdots \times W_n$$

for each $j \in J$.

*The **extension** $t^{\mathfrak{M}} \subseteq W_i$ of an i-term t is defined by induction. For set and object variables X and a of $\mathcal{L}_i$, we put $X^{\mathfrak{M}} = X^{\mathfrak{W}_i}$ and $a^{\mathfrak{M}} = a^{\mathfrak{W}_i}$. The inductive steps for the Booleans and function symbols of $\mathcal{L}_i$ are the same as in Definition 4.2:*

– $(\neg t_1)^{\mathfrak{M}} = W_i \setminus t_1^{\mathfrak{M}}$, $\quad (t_1 \wedge t_2)^{\mathfrak{M}} = t_1^{\mathfrak{M}} \wedge t_2^{\mathfrak{M}}$,

– $(f(t_1, \ldots, t_{m_f}))^{\mathfrak{M}} = f^{\mathfrak{W}_i}(t_1^{\mathfrak{M}}, \ldots, t_{m_f}^{\mathfrak{M}})$.

Now let $\bar{t}_i = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ be a sequence of j-terms t_j, $j \neq i$. Then set

$$(\langle E_j \rangle^i (\bar{t}_i))^{\mathfrak{M}} = \{x \in W_i \mid \underset{\ell \neq i}{\exists} x_\ell \in t_\ell^{\mathfrak{M}} \ (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) \in E_j^{\mathfrak{M}}\}.$$

*Finally, the **extension** $R^{\mathfrak{M}}$ of a relation symbol R of $\mathcal{L}_i$ is just $R^{\mathfrak{W}_i}$.*

*The **truth-relation** $\models$ between models $\mathfrak{M}$ for the $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ and assertions of $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is defined in the obvious way:*

– $\mathfrak{M} \models t_1 \sqsubseteq t_2 \iff t_1^{\mathfrak{M}} \subseteq t_2^{\mathfrak{M}}$;

– $\mathfrak{M} \models a : t \iff a^{\mathfrak{M}} \in t^{\mathfrak{M}}$;

– $\mathfrak{M} \models R(a_1, \ldots, a_{m_R}) \iff R^{\mathfrak{M}}(a_1^{\mathfrak{M}}, \ldots, a_{m_R}^{\mathfrak{M}})$;

– $\mathfrak{M} \models (a_1, \ldots, a_n) : E_j \iff E_j^{\mathfrak{M}}(a_1^{\mathfrak{M}}, \ldots, a_n^{\mathfrak{M}})$.

As in the case of ADSs, *we say that φ is **satisfied** in $\mathfrak{M}$ if $\mathfrak{M} \models \varphi$. A set Γ of $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$-assertions is **satisfiable** if there exists a model $\mathfrak{M}$ for $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ which satisfies all assertions in Γ. In this case we write $\mathfrak{M} \models \Gamma$. If Γ contains only object assertions then, as before, we use the term **A-satisfiability** instead of satisfiability.*

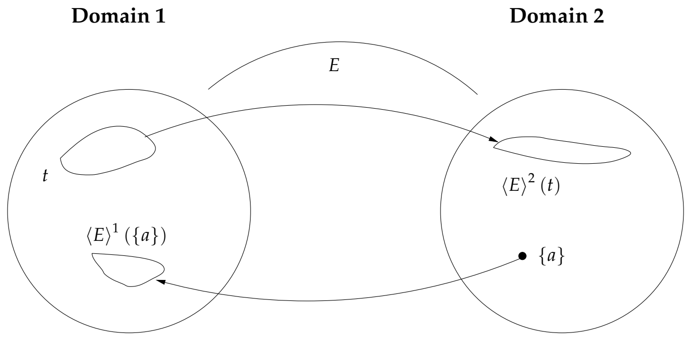


Figure 4.1: A two-dimensional connection.

As in the case of ADSs, the entailment of term assertions and object assertions of the form $a : t$ can be reduced to the satisfiability problem.

Observe that, technically, the $\mathcal{E}$-connection of ADSs is not an ADS itself because the structure of models for $\mathcal{E}$-connections is different from the structure of models for ADSs. This approach was taken on purpose. Since we define the $\mathcal{E}$-connection as an n-ary operation, there is hardly any need to connect $\mathcal{E}$-connections. An alternative would be to extend the definition of ADSs in order to capture $\mathcal{E}$-connections. Although this is not a problem in general, it would further complicate the definition of ADSs and, in turn, also of $\mathcal{E}$-connections.

Several examples of $\mathcal{E}$-connections are given in the next section. For now, we refer the reader to Figure 4.1 for an illustration of the semantics of $\mathcal{E}$-connections: the figure displays the connection of two ADSs by means of a single link relation E, highlighting the extensions of two 1-terms and two 2-terms (one of the latter is a nominal and thus has a singleton extension).

The central result on $\mathcal{E}$-connections, as we have already mentioned, is that they preserve decidability of the satisfiability problem. More precisely, as we prove in the next chapter in Theorem 5.1, given n abstract description systems $\mathcal{S}_1, \ldots, \mathcal{S}_n$ whose satisfiability problems are all decidable, the satisfiability problem for any $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is decidable as well, where the choice of the link-set $\mathcal{E}$ is arbitrary.

4.5. Examples of $\mathcal{E}$-Connections

Our aim in this section is to demonstrate the versatility of the new combination technique. We will give four examples of $\mathcal{E}$-connections using the knowledge representation formalisms introduced in Section 4.2. The first three examples are 'two-dimensional,' while the fourth one connects three ADSs. To simplify notation, we use the syntax of the underlying logical formalism rather than the syntax of the corresponding ADSs. This is justified, since the ADS representation of a logical formalism is always at least as expressive as the formalism itself.

4.5.1. A Metric-Conceptual $\mathcal{E}$-Connection: $\mathcal{C}^{\mathcal{E}}(\mathcal{ALC}^{\sharp}, \mathcal{MSO}_{D}^{\sharp})$. Suppose that you are developing a KR&R system for an estate agency. You imagine yourself to be a customer hunting for a house in Liverpool. The following variation of Example 1.1 from Page 9 could, for instance, be a list of requirements (constraints) that you might have:

(A) The house should not be too far from the Chadwick Tower, not more than 3 miles.
(B) The house should be close to a shop selling newspapers, say, within 0.2 miles.
(C) It should be located in the 'golden area', that is, 1 mile around the house is the city centre, in each direction.
(D) There must be a pub around, say, within 0.1 miles, and moreover, all pubs of the city centre should be reachable on foot, i.e., they should be within, say, 2 miles.
(E) Public transport should be easily accessible: whenever you are within 3 miles of your home, the nearest bus stop should be reachable within 0.1 miles.
(F) The house should have broadband.
(G) The neighbours should be ignorant to extreme noise.

The terminology usually requires some background ontology. In this case, you may also need statements like:

(H) Supermarkets are shops which provide no service and sell newspapers, coffee, etc.
(I) Newsagents are shops which sell magazines and newspapers.

The resulting constraints (A)–(I) contain two kinds of knowledge. (F)–(I) can be classified as conceptual knowledge which is captured by almost any description logic, say, $\mathcal{ALC}$:

(F′) $\textit{house} : \exists \mathsf{has}.\mathsf{Broadband};$
(G′) $\textit{house} : \forall \mathsf{neighbor}.\forall \mathsf{sensitive}.\neg \mathsf{Extreme\_noise};$
(H′) $\mathsf{Supermarket} \sqsubseteq \mathsf{Shop} \sqcap \forall \mathsf{service}.\bot \sqcap \exists \mathsf{sell}.\mathsf{Newspaper} \sqcap \exists \mathsf{sell}.\mathsf{Coffee};$
(I′) $\mathsf{Newsagent} \sqsubseteq \mathsf{Shop} \sqcap \exists \mathsf{sell}.\mathsf{Magazine} \sqcap \exists \mathsf{sell}.\mathsf{Newspaper}.$

(A)–(E) speak about distances and can be represented in the logic $\mathcal{MSO}_D[\mathbb{Q}^+]$ of metric spaces:

(A′) $house \to \mathsf{E}^{\leq 3} Chadwick\_tower$;
(B′) $house \to \mathsf{E}^{\leq 0.2}\mathsf{Newspaper\_shop}$;
(C′) $house \to \mathsf{A}^{\leq 1}\mathsf{City\_center}$;
(D′) $house \to (\mathsf{E}^{\leq 0.1}\mathsf{Pub}) \sqcap (\mathsf{A}^{>2}\neg(\mathsf{City\_center} \sqcap \mathsf{Pub}))$;
(E′) $house \to \mathsf{A}^{\leq 3}\mathsf{E}^{\leq 0.1}\mathsf{Bus\_stop}$.

Note that *house* and *Chadwick_tower* are nominals of $\mathcal{MSO}_D$, i.e., location constants representing single points, while Newspaper_shop, City_centre, etc. are propositional variables representing subsets of a space.

However, we cannot just join these two knowledge bases together without connecting them. They speak about the same things, but from different points of view. For instance, in (H), 'shop' is used as a *concept*, while (B) deals with the *space* occupied by 'shops selling newspapers'. Without connecting these different aspects we cannot deduce from the knowledge base that a supermarket or a news agent within 0.2 miles is sufficient to satisfy constraint (B). Moreover, it is obviously not too natural for the spatial part of the knowledge base to deal with primitive set variables for regions occupied by 'shops selling newspapers'.

The required interaction can be easily captured by an $\mathcal{E}$-connection between $\mathcal{ALC}^\sharp$ and $\mathcal{MSO}_D^\sharp$, where $\mathcal{E} = \{E\}$ and the relation E is intended to relate abstract points of an $\mathcal{ALC}$-model with points in a metric space understood as the abstract point's spatial extension. Indeed, take relations has, neighbour, sensitive, sell, service and set variables Broadband, Supermarket, Shop, City_centre etc. from $\mathcal{ALC}^\sharp$, and the object variable *Chadwick_tower* from $\mathcal{MSO}_D^\sharp$. Now, using the constructors $\langle E\rangle^1$ and $\langle E\rangle^2$ connecting models of $\mathcal{ALC}^\sharp$ and $\mathcal{MSO}_D^\sharp$, we can represent constraints (A)–(I) as the concept Good_house defined by the following knowledge base in $\mathcal{C}^{\mathcal{E}}(\mathcal{ALC}^\sharp, \mathcal{MSO}_D^\sharp)$:

$$\begin{aligned}
\mathsf{Good_house} &= \mathsf{House} \sqcap \mathsf{Well_located} \sqcap \exists \mathsf{has.Broadband} \sqcap \\
&\sqcap\ \forall \mathsf{neighbor}.\forall \mathsf{sensitive}.\neg \mathsf{Extreme_noise};\\
\mathsf{Well_located} &= \langle E\rangle^1 \big(\mathsf{E}^{\leq 3}\{Chadwick_tower\} \sqcap \mathsf{E}^{\leq 0.2}\, \langle E\rangle^2\, (\exists \mathsf{sell.Newspaper}) \sqcap \\
&\sqcap\ \mathsf{A}^{\leq 1}\, \langle E\rangle^2\, (\mathsf{City_center}) \sqcap \mathsf{E}^{\leq 2}\, \langle E\rangle^{0.1}\, (\mathsf{Pub}) \sqcap \\
&\sqcap\ \mathsf{A}^{>2}\neg\, \langle E\rangle^2\, (\mathsf{City_center} \sqcap \mathsf{Pub})\, \sqcap \mathsf{A}^{\leq 3}\mathsf{E}^{\leq 0.1}\, \langle E\rangle^2\, (\mathsf{Bus_stop})\big);\\
\mathsf{Supermarket} &\sqsubseteq \mathsf{Shop} \sqcap \forall \mathsf{service}.\bot \sqcap \exists \mathsf{sell.Newspaper} \sqcap \exists \mathsf{sell.Coffee};\\
\mathsf{Newsagent} &\sqsubseteq \mathsf{Shop} \sqcap \exists \mathsf{sell.Magazine} \sqcap \exists \mathsf{sell.Newspaper}.
\end{aligned}$$

If we also want to specify that the house should be available at a reasonable price, $\mathcal{ALC}$ can be extended with a suitable 'concrete domain' dealing with (natural or rational) numbers such that the resulting description logic is still decidable [Lutz, 2002, 2003]. As shown in Baader et al. [2002], description logics with concrete domains can still be regarded as ADSs and, therefore, the decidability of the $\mathcal{E}$-connection is preserved as well.

As mentioned in Section 4.4, the satisfiability checking algorithms for $\mathcal{ALC}^\sharp$ and $\mathcal{MSO}_D^\sharp$ can be combined to obtain an algorithm for their $\mathcal{E}$-connection. This algorithm can then be used to check whether the formulated requirements are consistent. However, we can go one step further: to answer the query whether such a house really exists *in Liverpool*, we should not perform reasoning with respect to arbitrary metric spaces, but rather take a suitable map of Liverpool as our metric space. This scenario can be represented by an $\mathcal{E}$-connection of $\mathcal{ALC}^\sharp$ with the following ADS. Suppose that our map is a structure

$$\mathfrak{D} = \langle D, \delta, P_1, \ldots, P_n, c_1, \ldots, c_m \rangle ,$$

where D is a finite set, δ a distance function on D, the P_i are subsets of D representing spatial extensions of concepts like House, Pub, etc., and the c_i are elements of D representing objects such as *Chadwick_tower*.[12] We then define an ADS

$$\mathcal{MAP} = (\mathcal{MAP}_l, \mathcal{MAP}_m).$$

Here, the ADL $\mathcal{MAP}_l$ extends the language of $\mathcal{MSO}_D^\sharp$ by 0-ary function symbols $f_{P_1}, \ldots, f_{P_n}$ and $f_{c_1}, \ldots, f_{c_m}$, and $\mathcal{MAP}_m$ contains models of the form

$$\mathfrak{M} = \left\langle D, \mathcal{V}^{\mathfrak{M}}, \mathcal{X}^{\mathfrak{M}}, \mathcal{F}^{\mathfrak{M}}, f_{P_1}^{\mathfrak{M}}, \ldots, f_{P_n}^{\mathfrak{M}}, f_{c_1}^{\mathfrak{M}}, \ldots, f_{c_m}^{\mathfrak{M}} \right\rangle ,$$

where $\langle D, \mathcal{V}^{\mathfrak{M}}, \mathcal{X}^{\mathfrak{M}}, \mathcal{F}^{\mathfrak{M}} \rangle$ is an $\mathcal{MSO}_D^\sharp$-model corresponding to $\langle D, \delta \rangle$ as defined in Section 4.2.5, $f_{P_1}^{\mathfrak{M}} = P_1, \ldots, f_{P_n}^{\mathfrak{M}} = P_n$, and $f_{c_1}^{\mathfrak{M}} = \{c_1\}, \ldots, f_{c_m}^{\mathfrak{M}} = \{c_m\}$. Note that $\mathcal{MAP}_m$ contains more than one model since, according to Definition 4.3, the class of ADMs of any ADS is closed under arbitrary variations of the extensions of set variables. For this reason, we have to take 0-ary function symbols rather than set variables to represent the sets P_i and 0-ary function symbols rather than object variables to represent the constants c_i. However, since all models in $\mathcal{MAP}_m$ agree on $\mathcal{F}^{\mathfrak{M}}$, the $f_{P_i}^{\mathfrak{M}}$, and the $f_{c_i}^{\mathfrak{M}}$, the ADS $\mathcal{MAP}$ uniquely describes a single map.

Now, returning to our example, let us assume that the map $\mathfrak{D}$ contains subsets P_1 = *City_center*, P_2 = *Pub*, P_3 = *Bus_stop*, P_4 = *Supermarket*, P_5 = *Newsagent*, and a point c_1 = *Chadwick_tower* (but no subset marked by *shop*). We can then modify the knowledge base above by replacing *Chadwick_tower* with f_{c_1} and by adding the following equations to the knowledge base in order to fix the spatial extensions of

[12] This representation depends, of course, on the size or granularity of the map.

certain concepts:

$$\langle E \rangle^2 (\mathsf{City\_center}) = f_{P_1}, \qquad \langle E \rangle^2 (\mathsf{Pub}) = f_{P_2},$$
$$\langle E \rangle^2 (\mathsf{Bus\_stop}) = f_{P_3}, \qquad \langle E \rangle^2 (\mathsf{Supermarket}) = f_{P_4},$$
$$\langle E \rangle^2 (\mathsf{Newsagent}) = f_{P_5}.$$

Although shops selling newspapers are not marked in the map, it will follow from the subsumption relations (H) and (I) of the $\mathcal{ALC}^{\sharp}$-part of the knowledge base that any supermarket or shop at distance ≤ 0.2 in the map is sufficient to satisfy the constraint on shops selling newspapers.

Finally, by adding

$$house : \mathsf{Good\_house}$$

to the knowledge base and checking its satisfiability, we can find out whether Liverpool has the house of our dreams.[13]

4.5.2. A Topo-Conceptual $\mathcal{E}$-Connection: $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCO}^{\sharp}, \mathbf{S4}_{\mathbf{u}}^{\sharp})$. Now imagine that you are employed by the EU parliament to develop a geographical information system about Europe. One part of the task is easy. You take the description logic $\mathcal{ALCO}$ and, using concepts Country, Treaty, etc., object names *EU*, *Schengen_treaty*, *Spain*, *Luxembourg*, *UK*, etc., and a role member, write

$$Luxembourg : \exists \mathsf{member}.\{EU\} \sqcap \exists \mathsf{member}.\{Schengen\_treaty\};$$
$$Iceland : \exists \mathsf{member}.\{Schengen\_treaty\} \sqcap \neg \exists \mathsf{member}.\{EU\};$$
$$France : \mathsf{Country};$$
$$Schengen\_treaty : \mathsf{Treaty};$$
$$\exists \mathsf{member}.\{Schengen\_treaty\} \sqsubseteq \mathsf{Country}, \quad \text{etc.}$$

After that you have to say something about the geography of Europe. To this end, you can use the spatial logic $\mathbf{S4}_{\mathbf{u}}$ in which, as we have mentioned already, the topological meaning of the RCC-8 predicates can be encoded as follows, where X, Y are set

[13] In fact, in Liverpool, our constraints are satisfiable.

variables and $\top = Z \vee \neg Z$:

$$
\begin{array}{ll}
\mathsf{DC}(X,Y): & \top = \neg\blacklozenge\,(X \wedge Y);\\
\mathsf{EQ}(X,Y): & \top = (X \leftrightarrow Y);\\
\mathsf{EC}(X,Y): & \top = \blacklozenge\,(X \wedge Y) \wedge \neg\blacklozenge\,(\boldsymbol{I}X \wedge \boldsymbol{I}Y);\\
\mathsf{PO}(X,Y): & \top = \blacklozenge\,(\boldsymbol{I}X \wedge \boldsymbol{I}Y) \wedge \blacklozenge\,(\boldsymbol{I}X \wedge \neg Y) \wedge \blacklozenge\,(\boldsymbol{I}Y \wedge \neg X);\\
\mathsf{TPP}(X,Y): & \top = (\neg X \vee Y) \wedge \blacklozenge\,(X \wedge \neg\boldsymbol{I}Y) \wedge \blacklozenge\,(\neg X \wedge Y);\\
\mathsf{NTPP}(X,Y): & \top = \blacksquare\,(\neg X \vee \boldsymbol{I}Y) \wedge \blacklozenge\,(\neg X \wedge Y);
\end{array}
$$

($\mathsf{TPPi}(X,Y) = \mathsf{TPP}(Y,X)$ and $\mathsf{NTPPi}(X,Y) = \mathsf{NTPP}(Y,X)$). To ensure that RCC-8 predicates are only applied to regular closed sets, one can add the assertions $\boldsymbol{CI}X = X$ and $\boldsymbol{CI}Y = Y$ to the knowledge base.

Now, using an $\mathcal{E}$-connection between $\mathcal{ALCO}^\sharp$ and $\mathbf{S4}_\mathbf{u}^\sharp$, you can continue:

$$
\begin{array}{l}
\mathsf{EQ}(\langle E\rangle^2(\{EU\}), \langle E\rangle^2(\{Portugal\} \sqcup \{Spain\} \sqcup \cdots \sqcup \{UK\});\\
\mathsf{EC}(\langle E\rangle^2(\{France\}), \langle E\rangle^2(\{Luxembourg\}));\\
\mathsf{NTPP}(\langle E\rangle^2(\{Luxembourg\}), \langle E\rangle^2(\exists \mathsf{member}.\{Schengen\_Treaty\}));\\
Austria : \langle E\rangle^1(Alps);
\end{array}
$$

i.e., 'The space occupied by the EU is the space occupied by its members', 'France and Luxembourg have a common border' (see Figure 4.2), 'If you cross the border of Luxembourg you enter a member of the Schengen Treaty', 'Austria is an alpine country' (*Alps* is a set variable of $\mathbf{S4}_\mathbf{u}^\sharp$). You can even say that Germany, Austria and Switzerland meet at one point:

$$
\begin{array}{l}
\blacklozenge\,(\langle E\rangle^2(\{Austria\}) \sqcap \langle E\rangle^2(\{Germany\}) \sqcap \langle E\rangle^2(\{Switzerland\}))\ \wedge\\
\quad \wedge\ \neg\blacklozenge\,(\boldsymbol{I}\,\langle E\rangle^2(\{Austria\}) \sqcap \boldsymbol{I}\,\langle E\rangle^2(\{Germany\}))\ \wedge\\
\quad \wedge\ \neg\blacklozenge\,(\boldsymbol{I}\,\langle E\rangle^2(\{Austria\}) \sqcap \boldsymbol{I}\,\langle E\rangle^2(\{Switzerland\}))\ \wedge\\
\quad \wedge\ \neg\blacklozenge\,(\boldsymbol{I}\,\langle E\rangle^2(\{Switzerland\}) \sqcap \boldsymbol{I}\,\langle E\rangle^2(\{Germany\})).
\end{array}
$$

Of course, to ensure that the spatial extensions of the *EU*, *France*, etc. are not degenerate and to comply with requirements of RCC-8 you should guarantee that all mentioned spatial regions are interpreted by regular closed sets, i.e.,

$$
\begin{array}{c}
\langle E\rangle^2(\{EU\}) = \boldsymbol{CI}\,\langle E\rangle^2(\{EU\});\\
\langle E\rangle^2(\{France\}) = \boldsymbol{CI}\,\langle E\rangle^2(\{France\});\\
\text{etc.}
\end{array}
$$

Suppose now that you want to test your system and ask whether France is a member of the Schengen treaty, i.e., $France : \exists \mathsf{member}.\{\mathsf{Schengen\_treaty}\}$. The answer will be

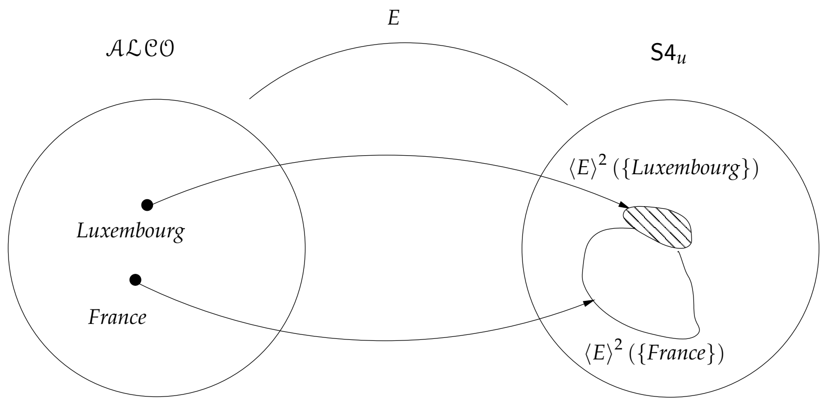


Figure 4.2: France and Luxembourg have a common border.

'Don't know!' because you did not tell your system that the spatial extensions of any two countries do not overlap. If you add, for example,

$$\neg\blacklozenge\,(\boldsymbol{I}(\langle E\rangle^2\,(\mathsf{Country} \sqcap \neg\exists\mathsf{member}.\{\mathit{Schengen\_treaty}\}) \wedge \\ \wedge\ \boldsymbol{I}(\langle E\rangle^2\,(\exists\mathsf{member}.\{\mathit{Schengen\_treaty}\}))$$

('The members of the Schengen treaty do not overlap with the non-Schengen countries') to the knowledge base, then the answer to the query will be 'Yes!'.

Clearly, the representation task is much easier if complete knowledge about the geography of Europe is available. Then you could have taken an existing spatial database describing the RCC-8 relations between the European countries, mountains, etc., and thus use a fixed model of $\mathbf{S4_u}$ with a fixed link relation E. This database can be conceived of as an ADS in the same manner as the map of Liverpool in the previous example.

4.5.3. A Purely Conceptual $\mathcal{E}$-Connection: $\mathcal{C}^{\mathcal{E}}(\mathcal{SHIQ}^\sharp, \mathcal{ALCO}^\sharp)$. Having satisfied your boss in the EU parliament with the constructed GIS, you get a new task: to develop a knowledge base regulating relations between people in the EU (citizenship, jobs, etc.). On the one hand, you already have the $\mathcal{ALCO}$ knowledge base describing countries in the EU from the previous example. But on the other hand, you must also be able to express laws like

(i) 'No citizen of the EU may have more than one spouse';
(ii) 'All children of UK citizens are UK citizens'; or
(iii) 'A person whose residence is the UK either is a child of a person whose residence is the UK, or is a UK citizen or has a work permit in the UK'.

This means, in particular, that you need more constructors than $\mathcal{ALCO}$ can provide, say, qualified number restrictions and inverse roles. It is known, however, that inverse roles, number restrictions, and nominals are difficult to handle algorithmically in one system [Horrocks and Sattler, 2001]. The fusion of $\mathcal{ALCO}$ with, say, the description logic $\mathcal{SHIQ}$ of Horrocks et al. [1999] having the required constructors, does not help either, because transfer results for fusions are available so far only for DLs whose models are closed under disjoint unions which is not the case if nominals are allowed as concept constructors [Baader et al., 2002]. It seems, a perspective way to attack this problem is to *connect* $\mathcal{SHIQ}^\sharp$ with $\mathcal{ALCO}^\sharp$.

Let $\mathcal{E}$ contain three binary relations between the domains of $\mathcal{SHIQ}$ (people, companies, etc.) and $\mathcal{ALCO}$ (countries): xCy means that x is a citizen of y, xRy means that x has residence in y, and xWy means that x has a work permit in y. For example, $\langle R\rangle^1(\{UK\})$ denotes all people having residence in the UK, and $\langle C\rangle^1(\{UK\})$ all UK citizens. The subsumptions below represent the regulations (i)–(iii):

$$\langle C\rangle^1(\{EU\}) \sqsubseteq \neg(\geq 2\mathsf{married}.\top);$$

$$\exists\mathsf{child\_of}.\langle C\rangle^1(\{UK) \sqsubseteq \langle C\rangle^1(\{UK\});$$

$$\langle R\rangle^1(\{UK\}) \sqsubseteq \exists\mathsf{child\_of}^{-1}.\langle R\rangle^1(\{UK\}) \sqcup \langle C\rangle^1(\{UK\}) \sqcup \langle W\rangle^1(\{UK\}).$$

4.5.4. A Concept-Topo-Temporal $\mathcal{E}$-Connection: $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCO}^\sharp, \mathbf{S4}_{\mathbf{u}}^\sharp, \mathbf{PTL}^\sharp)$. 'The EU is developing!', said your boss, 'We are going to have new members by 2004'. So you extend the connection $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCO}^\sharp, \mathbf{S4}_{\mathbf{u}}^\sharp)$ with one more ADS—propositional temporal logic $\mathbf{PTL}^\sharp$. Now, besides object variables *EU*, *Germany*, etc. of $\mathcal{ALCO}^\sharp$, and set variables *Alps*, *Basel*, etc. of $\mathbf{S4}_{\mathbf{u}}^\sharp$, we use the terms $\{0\}, \{1\}, \ldots$ as abbreviations for $(\neg \bigcirc_P^n \top \wedge \bigcirc_P^{n-1}\top)^\sharp$, where $\bigcirc_P \varphi$ stands for $\bot\mathcal{S}\varphi$. We then have $\{n\}^{\mathfrak{W}} = \{n\}$, for any $\mathbf{PTL}^\sharp$-model $\mathfrak{W}$. The ternary relation $E(x,y,z)$ means now that at moment z (from the domain of $\mathbf{PTL}$) point y (in the domain of $\mathbf{S4_u}$) belongs to the spatial region occupied by object x (in the domain of $\mathcal{ALCO}$). Then we can say, for example:

$$\langle E\rangle^2(\{Poland\}, \{2004\}) \sqsubseteq \langle E\rangle^2(\{EU\}, \{2004\});$$

$$\mathsf{PO}(\langle E\rangle^2(\{Austria\}, \{1914)\}, \langle E\rangle^2(\{Italy\}, \{1950\}));$$

$$\Box_F \neg\langle E\rangle^3(\{Basel\}, \{EU\});$$

i.e., 'In 2004, the territory of Poland will belong to the territory occupied by the EU' (see Figure 4.3), 'The territory of Austria in 1914 partially overlaps the territory of Italy in 1950', 'No part of Basel will ever belong to the EU'.[14]

[14]This example was devised before Poland actually became a member of the EU.

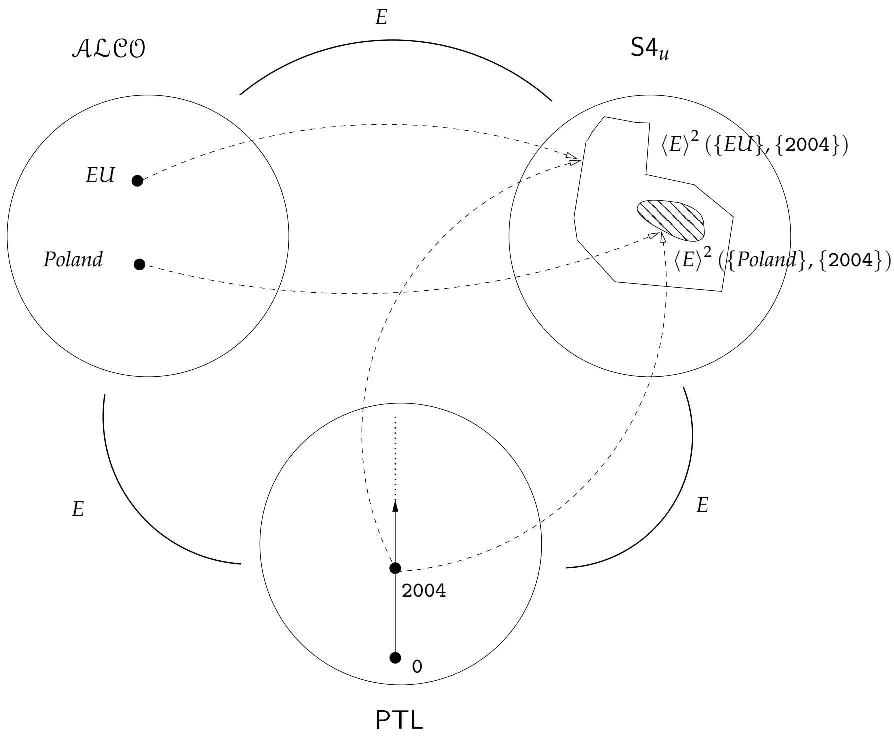


Figure 4.3: In 2004, Poland will be part of the EU.

CHAPTER 5

Computational Properties of $\mathcal{E}$-Connections

This chapter studies the computational behaviour of $\mathcal{E}$-connections. After discussing basic $\mathcal{E}$-connections in the next section, we investigate several extensions that allow a closer interaction of the combined formalisms by extending basic $\mathcal{E}$-connections with more powerful link operators. In Section 5.2, we add link operators that can be applied to object variables even though the connected ADSs do not have nominals. In Section 5.3, we add operators that can 'talk' about Boolean combinations of links, and, in Section 5.4, we add operators that correspond, in description logic terminology, to 'qualified number restrictions' on links—they can be used to express, e.g., that a given link operator is a partial function. We provide (brief) examples illustrating the expressive power of the new constructors and study the computational properties of the resulting formalisms.

5.1. Basic $\mathcal{E}$-Connections

Our central result on basic $\mathcal{E}$-connections is that they preserve decidability of the satisfiability problem. We will not give a proof of this theorem directly, though, since it follows immediately from the extension of basic $\mathcal{E}$-connections with link operators that can be applied to object variables, which we investigate in Section 5.2.

THEOREM 5.1 (TRANSFER OF DECIDABILITY FOR BASIC $\mathcal{E}$-CONNECTIONS). *Let $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ be an $\mathcal{E}$-connection of* ADSs *$\mathcal{S}_1, \ldots, \mathcal{S}_n$. If the satisfiability problem for each of $\mathcal{S}_1, \ldots, \mathcal{S}_n$ is decidable, then it is decidable for $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ as well.*

PROOF. By Theorem 5.7 to be proved in Section 5.2. □

Intuitively, the decision procedure for $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ works as follows (for simplicity, we confine ourselves to the connection of *two* ADSs and a single link relation). To check whether there exists a model $\mathfrak{M} = \langle \mathfrak{W}_1, \mathfrak{W}_2, E \rangle$ of a given set of assertions Γ, the algorithm non-deterministically 'guesses'

(1) the 1-types that are realised in $\mathfrak{W}_1$ and the 2-types that are realised in $\mathfrak{W}_2$, where an i-type is a set of i-terms (constructed from Γ) satisfied by a domain element of $\mathfrak{W}_i$; and
(2) a binary relation e between the guessed sets of 1-types and 2-types.

Then it checks whether the guessed sets satisfy a set of integrity conditions. This check involves satisfiability tests of certain sets of $\mathcal{S}_i$-assertions ($i = 1, 2$) constructed from Γ—here we use the fact that the satisfiability problems for $\mathcal{S}_1$ and $\mathcal{S}_2$ are decidable. If the integrity conditions are satisfied it is possible to construct a model of Γ using models of the constructed sets of $\mathcal{S}_i$-assertions. If the integrity conditions are not satisfied Γ has no model.

This algorithm also provides an upper complexity bound for the satisfiability problem for $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$: the time complexity of our algorithm is one exponential higher than the time complexity of the original decision procedures for $\mathcal{S}_1, \ldots, \mathcal{S}_n$. Moreover, the combined decision procedure is non-deterministic. The provided complexity bound is not always optimal. For instance, the $\mathcal{E}$-connection of $\mathcal{ALC}$ with itself, $\mathcal{C}^{\mathcal{E}}(\mathcal{ALC}, \mathcal{ALC})$, can be simulated by a 'two-sorted' version of $\mathcal{ALC}$ and thus its satisfiability problem can be solved in EXPTIME, whereas the algorithm for the $\mathcal{E}$-connection provides an upper bound of 2NEXPTIME.

We can, however, show that there indeed exist cases where the complexity of the $\mathcal{E}$-connection is higher than the complexity of the combined formalisms, namely, growing from NP to EXPTIME.

Let $\mathcal{B} = (\mathcal{L}_{\mathcal{B}}, \mathcal{M}_{\mathcal{B}})$ be the ADS, where

- $\mathcal{L}_{\mathcal{B}}$ is the abstract description language *without* any function and relation symbols (but, by definition, with the Booleans, infinitely many set variables and infinitely many object variables);
- $\mathcal{M}_{\mathcal{B}}$ consists of all ADMs for $\mathcal{L}_{\mathcal{B}}$.

$\mathcal{B}$ can be regarded as the basic ADS from which all others are obtained by adding function and relation symbols and/or constraints on the ADMs. Obviously, the satisfiability problems for $\mathcal{B}$ and classical propositional logic are mutually reducible to each other. So we have:

LEMMA 5.2. *The satisfiability problem for $\mathcal{B}$ is* NP-*complete.*

On the other hand, the $\mathcal{E}$-connection of $\mathcal{B}$ with itself is quite powerful:

THEOREM 5.3. *The satisfiability problem for $\mathcal{C}^{\mathcal{E}}(\mathcal{B}, \mathcal{B})$ is* EXPTIME-*hard for any infinite set $\mathcal{E}$ of links.*

PROOF. We reduce the EXPTIME-complete satisfiability problem for $\mathcal{ALC}$-concepts relative to (general) TBoxes [Schild, 1991] to the satisfiability problem for $\mathcal{C}^{\mathcal{E}}(\mathcal{B}, \mathcal{B})$. Intuitively, $\mathcal{B}$ is used for the Boolean part of $\mathcal{ALC}$, while the link relations and link operators simulate roles and value- and exists-restrictions, respectively.

Select, for any role name $R \in \mathcal{R}$ of $\mathcal{ALC}$, two links E_1^R and E_2^R, set

$$\mathcal{E} = \{E_1^R, E_2^R \mid R \in \mathcal{R}\},$$

and associate with any concept name A_i of $\mathcal{ALC}$ a set variable X^{A_i} of the first component of $\mathcal{C}^{\mathcal{E}}(\mathcal{B},\mathcal{B})$. Now define a translation $.^{\dagger}$ by taking

$$A_i^{\dagger} = X^{A_i}; \qquad (C_1 \wedge C_2)^{\dagger} = C_1^{\dagger} \wedge C_2^{\dagger};$$
$$(\neg C)^{\dagger} = \neg C^{\dagger}; \qquad (\exists R.C)^{\dagger} = \left\langle E_1^R \right\rangle^1 (\left\langle E_2^R \right\rangle^2 (C^{\dagger}));$$
$$(C_1 \sqsubseteq C_2)^{\dagger} = C_1^{\dagger} \sqsubseteq C_2^{\dagger}; \qquad (a : C)^{\dagger} = a : C^{\dagger}.$$

We claim that for every set Γ of $\mathcal{ALC}$-assertions and the corresponding set of $\mathcal{C}^{\mathcal{E}}(\mathcal{B},\mathcal{B})$ assertions $\Gamma^{\dagger} = \{\varphi^{\dagger} \mid \varphi \in \Gamma\}$ we have

(♠) $\qquad \Gamma$ is $\mathcal{ALC}$-satisfiable $\iff$ $\Gamma^{\dagger}$ is $\mathcal{C}^{\mathcal{E}}(\mathcal{B},\mathcal{B})$-satisfiable.

For, assume that Γ is satisfied in an $\mathcal{ALC}$-model

$$\mathfrak{I} = \left\langle \Delta, A_1^{\mathfrak{I}}, \dots, R_1^{\mathfrak{I}}, \dots, a_1^{\mathfrak{I}}, \dots \right\rangle.$$

Define a model

$$\mathfrak{M} = \left\langle \mathfrak{M}_1, \mathfrak{M}_2, \{(E_1^R)^{\mathfrak{M}}, (E_2^R)^{\mathfrak{M}}\}_{R \in \mathcal{R}} \right\rangle,$$

where $\mathfrak{M}_1 = \left\langle \Delta, (X^{A_1})^{\mathfrak{M}_1}, \dots, a_1^{\mathfrak{M}_1}, \dots \right\rangle$ with $(X^{A_i})^{\mathfrak{M}_1} = A_i^{\mathfrak{I}}$ and $a_i^{\mathfrak{M}_1} = a_i^{\mathfrak{I}}$, $\mathfrak{M}_2$ is some arbitrary ADM for $\mathcal{B}$ with domain Δ, and

$$(E_1^R)^{\mathfrak{M}} = \{(x,x) \mid x \in \Delta\}, \quad (E_2^R)^{\mathfrak{M}} = \{(x,y) \mid (y,x) \in R^{\mathfrak{I}}\}.$$

Clearly, it suffices to show that, for any concept C of $\mathcal{ALC}$,

$$C^{\mathfrak{I}} = (C^{\dagger})^{\mathfrak{M}}.$$

The proof is by induction on the construction of C. We consider the case $C = \exists R.D$, leaving the remaining ones to the reader:

$$\begin{aligned}
(\left\langle E_1^R \right\rangle^1 (\left\langle E_2^R \right\rangle^2 (D^{\dagger})))^{\mathfrak{M}} &= \{x \mid \exists y.\, y \in (\left\langle E_2^R \right\rangle^2 (D^{\dagger}))^{\mathfrak{M}} \wedge (x,y) \in (E_1^R)^{\mathfrak{M}}\} \\
&= \{x \mid x \in (\left\langle E_2^R \right\rangle^2 (D^{\dagger}))^{\mathfrak{M}}\} \\
&= \{x \mid \exists y \in (D^{\dagger})^{\mathfrak{M}}\ (x,y) \in R^{\mathfrak{I}}\} \\
&= (\exists R.D)^{\mathfrak{I}}.
\end{aligned}$$

Conversely, suppose $\Gamma^{\dagger}$ is satisfied in a model $\mathfrak{M}$ of $\mathcal{C}^{\mathcal{E}}(\mathcal{B},\mathcal{B})$. Define a model $\mathfrak{I}$ of $\mathcal{ALC}$ by taking $A_i^{\mathfrak{I}} = (X^{A_i})^{\mathfrak{M}_1}$, $a_i^{\mathfrak{I}} = a_i^{\mathfrak{M}_1}$, and

$$R^{\mathfrak{I}} = \{(x,y) \mid \exists z \in \Delta_2.\, (x,z) \in (E_1^R)^{\mathfrak{M}} \wedge (y,z) \in (E_2^R)^{\mathfrak{M}}\}.$$

Again, it suffices to show that, for any concept C of $\mathcal{ALC}$, $C^{\mathfrak{I}} = (C^{\dagger})^{\mathfrak{M}}$. We consider only the case $C = \exists R.D$ of the inductive proof:

$$\begin{aligned}(\exists R.D)^{\mathfrak{I}} &= \{x \in \Delta_1 \mid \exists y \in D^{\mathfrak{I}}\ (x,y) \in R^{\mathfrak{I}}\}\\ &= \{x \in \Delta_1 \mid \exists y \in (D^{\dagger})^{\mathfrak{M}} \exists z \in \Delta_2.\ (x,z) \in (E_1^R)^{\mathfrak{M}} \wedge (y,z) \in (E_2^R)^{\mathfrak{M}}\}\\ &= \{x \in \Delta_1 \mid \exists z \in \Delta_2.\ z \in \left(\langle E_2^R \rangle^2 (D^{\dagger})\right)^{\mathfrak{M}} \wedge (x,z) \in (E_1^R)^{\mathfrak{M}}\}\\ &= \left(\langle E_1^R \rangle^1 (\langle E_2^R \rangle^2 (D^{\dagger}))\right)^{\mathfrak{M}}.\end{aligned}$$

This completes the proof. □

In the case of $\mathcal{E}$-connections that provide for Boolean operators on links the given complexity bound is in fact optimal, namely growing from NP to NEXPTIME, as proved in Theorem 5.16.

In contrast to full satisfiability, the decidability of A-satisfiability is not preserved under the formation of $\mathcal{E}$-connections. Consider the description logic $\mathcal{ALCF}$ which is the extension of $\mathcal{ALC}$ with functional roles and the feature agreement and disagreement constructors. More precisely, the set of role names of $\mathcal{ALCF}$ is partitioned into two sets R and F, where the elements of F (called **features**) are interpreted as partial functions. For any two sequences of features $p = f_1 \cdots f_k$ and $q = f'_1 \cdots f'_\ell$, $\mathcal{ALCF}$ provides the additional concept constructors $p \downarrow q$ (**feature agreement**) and $p \uparrow q$ (**feature disagreement**) with the following semantics:

$$(p \downarrow q)^{\mathfrak{I}} = \{w \in \Delta \mid \exists v (v = f_k(\cdots(f_1(w))) = f'_\ell(\cdots(f'_1(w))))\};$$
$$(p \uparrow q)^{\mathfrak{I}} = \{w \in \Delta \mid \exists v, v' (v = f_k(\cdots(f_1(w))) \wedge v' = f'_\ell(\cdots(f'_1(w))) \wedge v \neq v')\}.$$

It is straightforward to define a corresponding ADS $\mathcal{ALCF}^{\sharp}$ (see Baader et al. [2002] for details). The satisfiability of ABoxes with respect to (general) TBoxes is undecidable for $\mathcal{ALCF}$, while satisfiability of ABoxes (without TBoxes) is decidable [Hollunder and Nutt, 1990, Baader et al., 1993, Lutz, 1999]. Hence, the satisfiability problem for $\mathcal{ALCF}^{\sharp}$ is undecidable, while the A-satisfiability problem is decidable. Interestingly, in the $\mathcal{E}$-connection of $\mathcal{ALCF}^{\sharp}$ and $\mathcal{ALCO}^{\sharp}$, we can simulate general TBoxes, even in the case of A-satisfiability. Thus, we obtain the following theorem:

THEOREM 5.4. *Let $\mathcal{E}$ be an arbitrary non-empty set of link relations. Then the A-satisfiability problem for $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCF}^{\sharp}, \mathcal{ALCO}^{\sharp})$ is undecidable.*

PROOF. As noted above, the satisfiability problem for ABoxes relative to TBox axioms in $\mathcal{ALCF}$ is undecidable. For simplicity, however, we will consider the concept satisfiability problem relative to TBox axioms which is formulated as follows: given an $\mathcal{ALCF}$-concept C and a set Γ of $\mathcal{ALCF}$ TBox assertions of the form $D \sqsubseteq D'$, does there exist a model $\mathfrak{I}$ for Γ such that $C^{\mathfrak{I}} \neq \emptyset$? As shown in Baader et al. [1993], this problem is undecidable for $\mathcal{ALCF}$. To prove (1), we reduce this problem to the A-satisfiability problem for the connection $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCF}^{\sharp}, \mathcal{ALCO}^{\sharp})$.

Let C be an $\mathcal{ALCF}$-concept and Γ a set of $\mathcal{ALCF}$ TBox assertions. We use $\mathcal{R}$ to denote the set of roles occurring in C or Γ, and $[E]^i D$ as an abbreviation for $\neg \langle E \rangle^i \neg D$. Let a be an object variable of $\mathcal{ALCF}^\sharp$ and b an object variable of $\mathcal{ALCO}^\sharp$. Define the following set of $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCF}^\sharp, \mathcal{ALCO}^\sharp)$-object assertions:

$$\begin{aligned}\Gamma^* = {} & \{a : C^\sharp \wedge \langle E \rangle^1 \{b\}\} \\ & \cup \{b : [E]^2 (D^\sharp \to D'^\sharp) \mid D \sqsubseteq D' \in \Gamma\} \\ & \cup \{b : [E]^2 f_{\forall R}(\langle E \rangle^1 \{b\}) \mid R \in \mathcal{R}\},\end{aligned}$$

where E is some link from $\mathcal{E}$. We show that

$$\begin{aligned} C \text{ is satisfiable relative to } \Gamma \text{ in } \mathcal{ALCF} \quad & \iff \\ \iff \quad & \Gamma^* \text{ is A-satisfiable in } \mathcal{C}^{\mathcal{E}}(\mathcal{ALCF}^\sharp, \mathcal{ALCO}^\sharp).\end{aligned}$$

($\Longrightarrow$) Suppose that $\{a : C\} \cup \Gamma$ is satisfiable relative to Γ. Due to the correspondence between $\mathcal{ALCF}$ and the ADS $\mathcal{ALCF}^\sharp$, there is an $\mathcal{ALCF}^\sharp$-model $\mathfrak{W}_1$ of $\{a : C\} \cup \Gamma$ with domain Δ_1. Define a model $\mathfrak{M}$ for $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCF}^\sharp, \mathcal{ALCO}^\sharp)$ by taking an arbitrary $\mathcal{ALCO}^\sharp$-model $\mathfrak{W}_2$ with domain Δ_2 and putting $E^{\mathfrak{M}} = \Delta_1 \times \Delta_2$. It is easily checked that $\mathfrak{M} \models \Gamma^*$.

($\Longleftarrow$) Suppose $\mathfrak{M} \models \Gamma^*$ for a $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCF}^\sharp, \mathcal{ALCO}^\sharp)$-model $\mathfrak{M} = (\mathfrak{W}_1, \mathfrak{W}_2, E^{\mathfrak{M}})$. Let Δ be the domain of $\mathfrak{W}_1$. Denote by Δ' the minimal subset of Δ containing $a^{\mathfrak{M}}$ and satisfying the following closure condition for all $d, d' \in \Delta$:

$$\text{if } (d, d') \in S^{\mathfrak{M}} \text{ for some } d \in \Delta' \text{ and } S \in \mathcal{R}, \text{ then } d' \in \Delta'.$$

Let $\mathfrak{W}_1'$ be the substructure of $\mathfrak{W}_1$ induced by Δ'. Since it is straightforward to prove that $\mathcal{ALCF}^\sharp$ is invariant under taking generated substructures, we have $a^{\mathfrak{W}_1'} \in C^{\mathfrak{W}_1'}$. To show that $\mathfrak{W}_1'$ satisfies Γ, it obviously suffices to prove that, for every assertion of the form $D \sqsubseteq D' \in \Gamma$, we have $(D^\sharp)^{\mathfrak{W}} \cap \Delta' \subseteq (D'^\sharp)^{\mathfrak{W}} \cap \Delta'$. To this end, note that $d \in (D^\sharp \to D'^\sharp)^{\mathfrak{M}}$ whenever $(d, b^{\mathfrak{M}}) \in E^{\mathfrak{M}}$ due to the third component of Γ^*. Hence, it is sufficient to prove that, for all $d \in \Delta'$, we have $(d, b^{\mathfrak{M}}) \in E^{\mathfrak{M}}$. This, however, is an easy consequence of the facts that $(a^{\mathfrak{M}}, b^{\mathfrak{M}}) \in E^{\mathfrak{M}}$ and $\mathfrak{M}$ satisfies the third component of Γ^*. □

5.2. Link Operators on Object Variables

In some of the examples from Section 4.5, the connected ADSs have nominals. According to Definition 4.6, this means that, for each object name a, they provide terms $\{a\}$ such that, for every model $\mathfrak{W}$, we have $\{a\}^{\mathfrak{W}} = \{a^{\mathfrak{W}}\}$. This is the case, e.g., for $\mathcal{MSO}_D^\sharp$, $\mathcal{ALCO}^\sharp$, and $\mathbf{PTL}^\sharp$ (see Section 4.2). In $\mathcal{E}$-connections where the components do have nominals, it is often convenient to form terms such as $\langle E \rangle^i(\{a\})$ to state that the current element is connected to a *particular element* of the other component, namely, the one denoted by a. However, not all $\mathcal{E}$-connections considered in Section 4.5 are of this type, e.g., $\mathcal{C}^{\mathcal{E}}(\mathcal{SHIQ}^{\sharp}, \mathcal{ALCO}^{\sharp})$ from Section 4.5.3. In this connection, we are *not* allowed to build, say, the term comprising all of the countries where some person, Bob, has citizenship: since $\mathcal{SHIQ}^{\sharp}$ has no nominals, we cannot use

$$\mathsf{country} \sqcap \langle S \rangle^2 (\{Bob\}),$$

where *Bob* is an object variable of $\mathcal{SHIQ}^{\sharp}$. An addition of the nominal constructor to $\mathcal{SHIQ}$ does not seem to be a promising solution because, despite considerable efforts of the description logic community, no 'implementable' algorithms are known for $\mathcal{SHIQ}$ extended with nominals. A better idea is to allow applications of link operators directly to objects, even if nominals are not available in the component ADS. Indeed, we can show that this kind of $\mathcal{E}$-connection is still computationally robust.

DEFINITION 5.5 (LINK OPERATORS ON OBJECT VARIABLES).
Suppose that $\mathcal{S}_i = (\mathcal{L}_i, \mathcal{M}_i)$, $1 \leq i \leq n$, *are abstract description systems and* $\mathcal{E} = \{E_j \mid j \in J\}$ *is a set of n-ary relation symbols. Denote by*

$$\mathcal{C}^{\mathcal{E}}_{O}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$$

the $\mathcal{E}$*-connection in which the definition of i-terms is extended with the following clause, for* $1 \leq i \leq n$:

- *if* $(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ *is a sequence of object variables* a_j *from* $\mathcal{L}_j$, $j \neq i$, *then* $\langle E_k \rangle^i (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ *is an* ***i-term****, for* $k \in J$.

As for the semantics, given an ADM

$$\mathfrak{M} = \left\langle (\mathfrak{W}_i)_{i \leq n}, \mathcal{E}^{\mathfrak{M}} \right\rangle$$

and a tuple $\overline{a}_i = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$, *we set*

$$(\langle E_j \rangle^i (\overline{a}_i))^{\mathfrak{M}} = \{x \in W_i \mid (a_1^{\mathfrak{M}}, \ldots, a_{i-1}^{\mathfrak{M}}, x, a_{i+1}^{\mathfrak{M}}, \ldots, a_n^{\mathfrak{M}}) \in E_j^{\mathfrak{M}}\}.$$

The next result shows that applications of link operators to object variables do not influence the decidability of $\mathcal{E}$-connections. But before going into the details of the proof, let us introduce some notation that will be used not only in the proof of the next theorem, but also in the proofs of the transfer theorems that we will prove in the following sections. In particular, to make the presentation as clear as possible, and since the general case is proved in essentially the same way, we confine ourselves to the cases of $\mathcal{E}$-connections of only two ADSs $\mathcal{S}_1 = (\mathcal{L}_1, \mathcal{M}_1)$ and $\mathcal{S}_2 = (\mathcal{L}_2, \mathcal{M}_2)$. For this case, we will use the following notations, conventions, and terminology:

NOTATION 5.6 (TERMINOLOGY FOR DECIDABILITY TRANSFER PROOFS).
Let Γ *be a finite set of assertions of some* $\mathcal{E}$*-connection of* $\mathcal{S}_1 = (\mathcal{L}_1, \mathcal{M}_1)$ *and* $\mathcal{S}_2 = (\mathcal{L}_2, \mathcal{M}_2)$

(possibly allowing link operators on object variables and/or Boolean combinations of link relations). We use the following notations and conventions:

- *We use* $\overline{1}$ *to denote* 2*, and* $\overline{2}$ *to denote* 1.
- *We write* $ob_i(\Gamma)$ *to denote the set of object variables from* $\mathcal{L}_i$ *which occur in* Γ*, for* $i = 1, 2$.
- *We write* $\mathcal{X}_i(\Gamma)$ *to denote the set of object variables*

  $$\mathcal{X}_i \setminus (ob_i(\Gamma) \cup (\mathcal{X}_i)_{\mathcal{G}_i}),$$

  where $\mathcal{G}_i$ *is the set of function symbols of* $\mathcal{L}_i$ *which occur in* Γ *and* $(\mathcal{X}_i)_{\mathcal{G}_i}$ *is the set of object variables supplied by the closure condition of Definition 4.3 (ii), for* $i = 1, 2$.
- *In each of the decidability proofs, we will use* $\mathsf{cl}_i(\Gamma)$*,* $i = 1, 2$*, to refer to some finite closure of the set of i-terms occurring in* Γ*. Since different closures are required in different proofs, we do not fix the exact details here.*
- *We assume that, for every i-term t of the form* $\langle F \rangle^i (s)$ *occurring in* $\mathsf{cl}_i(\Gamma)$ *(where s is an* $\bar{i}$*-term or an object name of* $\mathcal{L}_{\bar{i}}$*,* $i = 1, 2$*, and F is a link symbol or a Boolean combination of such symbols), there exists a set variable* x_t *of* $\mathcal{L}_i$ *not occurring in* Γ*. Given an i-term t, denote by* $sur_i(t)$*—the* ***surrogate of*** *t—the term which results from t by replacing all subterms* t' *of the form* $\langle F \rangle^i (s)$ *that are not within the scope of another term* $\langle G \rangle^i (s)$ *with* $x_{t'}$*. Clearly,* $sur_i(t)$ *belongs to the language* $\mathcal{L}_i$.
- *The i-**consistency set*** $\mathfrak{C}_i(\Gamma)$ *is defined as the set* $\{t_c \mid c \subseteq \mathsf{cl}_i(\Gamma)\}$*, where*

  $$t_c = \bigwedge\{\chi \mid \chi \in c\} \wedge \bigwedge\{\neg\chi \mid \chi \in \mathsf{cl}_i(\Gamma) \setminus c\}.$$

  Sometimes we will identify $t \in \mathfrak{C}_i(\Gamma)$ *with the set of its conjuncts. Then* $s \in t$ *means that s is a conjunct of t.*
- *Recall that by* $\top_i$ *we denote* $x_i \vee \neg x_i$*, where* x_i *is a set variable from* $\mathcal{L}_i$.

We are now prepared for the proof of the first transfer result:

THEOREM 5.7 (TRANSFER OF DECIDABILITY FOR $\mathcal{C}^{\mathcal{E}}_{O}$).
Let $\mathcal{S}_1, \ldots, \mathcal{S}_n$ *be* ADSs *with decidable satisfiability problems. Then the satisfiability problem for any* $\mathcal{E}$*-connection* $\mathcal{C}^{\mathcal{E}}_{O}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ *is decidable as well.*

PROOF. As was said above, we confine ourselves to $\mathcal{E}$-connections of only *two* ADSs $\mathcal{S}_1$ and $\mathcal{S}_2$. Moreover, for simplicity we assume that the link set $\mathcal{E}$ contains only a *single* link symbol E. Thus, our aim is to prove the following variant of Theorem 5.7.

THEOREM 5.8. *Suppose the satisfiability problems for the* ADSs $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are decidable. Then the satisfiability problem for the* $\{E\}$*-connection* $\mathcal{C}^{\{E\}}_{O}(\mathcal{S}_1, \mathcal{S}_2)$ *is decidable as well.*

The reader should be able to extend the proofs to n-ary $\mathcal{E}$-connections with multiple link relations without any difficulty.

Observe that, since we restrict ourselves to the connection of only two ADSs, the additional function symbols $\langle E\rangle^1$ and $\langle E\rangle^2$ of the connection are unary. Since the connections treated in this section allow the application of link operators to object variables, we do not explicitly treat link assertions of the form $(a_1, a_2) : E$. Clearly, such a link assertion can be replaced with the equivalent object assertion $a_1 : \langle E\rangle^1 (a_2)$.

PROOF OF THEOREM 5.8. Fix two abstract description systems

$$\mathcal{S}_1 = (\mathcal{L}_1, \mathcal{M}_1) \text{ and } \mathcal{S}_2 = (\mathcal{L}_2, \mathcal{M}_2)$$

with decidable satisfiability problems, and let Γ be a finite set of assertions of the $\{E\}$-connection $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$. To define the **closure** $\mathsf{cl}_i(\Gamma)$ of i-terms occurring in Γ, we first introduce the abbreviation

$$o_i(\Gamma) = \{\langle E\rangle^i \neg \langle E\rangle^{\bar{i}} (a) \mid a \in ob_i(\Gamma)\},$$

for $i = 1, 2$. The set $o_i(\Gamma)$ contains i-terms that must be present in the closure $\mathsf{cl}_i(\Gamma)$ in order to ensure a proper treatment of link operators applied to object variables. Note that, given a model $\mathfrak{M}$ of the $\{E\}$-connection $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$,

$$(\langle E\rangle^i \neg \langle E\rangle^{\bar{i}} (a))^{\mathfrak{M}} = \{x \in W_i \mid \exists y \in W_{\bar{i}} \left((a, y) \notin E^{\mathfrak{M}} \wedge (x, y) \in E^{\mathfrak{M}}\right)\},$$

and so $a^{\mathfrak{M}} \notin (\langle E\rangle^i \neg \langle E\rangle^{\bar{i}} (a))^{\mathfrak{M}}$.

We now define $\mathsf{cl}_i(\Gamma)$, $i = 1, 2$, to be the closure under negation of the set of i-terms which occur in $\Gamma \cup o_i(\Gamma)$. Without loss of generality we can identify $\neg\neg t$ with t. Thus, $\mathsf{cl}_i(\Gamma)$ is finite.

The following lemma is the core component in the proof of Theorem 5.8: it provides us with a criterion of satisfiability of sets of $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$-assertions Γ which almost immediately implies decidability of the satisfiability problem for $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$.

LEMMA 5.9 (SATISFIABILITY CRITERION FOR $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$).
Let Γ be a $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$-knowledge base. Then Γ is satisfiable if and only if there exist

(i) *subsets $\Delta_1 \subseteq \mathfrak{C}_1(\Gamma)$ and $\Delta_2 \subseteq \mathfrak{C}_2(\Gamma)$,*

(ii) *a relation $e \subseteq \Delta_1 \times \Delta_2$, and*

(iii) *functions $\sigma_1 : ob_1(\Gamma) \longrightarrow \Delta_1$ and $\sigma_2 : ob_2(\Gamma) \longrightarrow \Delta_2$,*

such that, for $i = 1, 2$, the following conditions are satisfied:

(1) *for any $a \in ob_i(\Gamma)$, we have $\langle E\rangle^i \neg \langle E\rangle^{\bar{i}} (a) \notin \sigma_i(a)$;*

(2) *the union Γ_i of*

- $\{sur_i(\bigvee \Delta_i) = \top_i\}$,
- $\{a_t : sur_i(t) \mid t \in \Delta_i\}$,
- $\{a : sur_i(\sigma_i(a)) \mid a \in ob_i(\Gamma)\}$,
- $\{sur_i(t_1) \sqsubseteq sur_i(t_2) \mid t_1 \sqsubseteq t_2 \in \Gamma$ *is an i-term assertion*$\}$,
- $\{R_j(a_1, \ldots, a_{m_j}) \mid R_j(a_1, \ldots, a_{m_j}) \in \Gamma$ *is an i-object assertion*$\}$, *and*

– $\{(a : sur_i(t)) \mid (a : t) \in \Gamma$ *is an i-object assertion*$\}$

is $\mathcal{S}_i$*-satisfiable, where* $a_t \in \mathcal{X}_i(\Gamma)$ *is a fresh object variable for each* $t \in \Delta_i$*;*

(3) *for all* $t \in \Delta_1$ *and* $\langle E \rangle^1 (s) \in \mathsf{cl}_1(\Gamma)$ *with* s *a* 2*-term, we have*

$$\langle E \rangle^1 (s) \in t \iff \textit{there exists } t' \in \Delta_2 \textit{ with } (t, t') \in e \textit{ and } s \in t';$$

(4) *for all* $t \in \Delta_2$ *and* $\langle E \rangle^2 (s) \in \mathsf{cl}_2(\Gamma)$ *with* s *a* 1*-term, we have*

$$\langle E \rangle^2 (s) \in t \iff \textit{there exists } t' \in \Delta_1 \textit{ with } (t', t) \in e \textit{ and } s \in t';$$

(5) *for all* $t \in \Delta_1$ *and* $\langle E \rangle^1 (a) \in \mathsf{cl}_1(\Gamma)$ *with* $a \in ob_2(\Gamma)$*, we have*

$$\langle E \rangle^1 (a) \in t \iff (t, \sigma_2(a)) \in e;$$

(6) *for all* $t \in \Delta_2$ *and* $\langle E \rangle^2 (a) \in \mathsf{cl}_2(\Gamma)$ *with* $a \in ob_1(\Gamma)$*, we have*

$$\langle E \rangle^2 (a) \in t \iff (\sigma_1(a), t) \in e.$$

PROOF. We first prove the 'only if' part of the lemma.

$(\implies)$ Suppose Γ is $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$-satisfiable and $\mathfrak{M} = ((\mathfrak{W}_1, \mathfrak{W}_2), E^{\mathfrak{M}})$ is a model of Γ, with W_1 being the domain of $\mathfrak{W}_1$ and W_2 being the domain of $\mathfrak{W}_2$. For $i = 1, 2$ and each $d \in W_i$, define

$$t(d) = \bigwedge \{s \in \mathsf{cl}_i(\Gamma) \mid d \in s^{\mathfrak{M}}\}.$$

Then set $\Delta_i = \{t(d) \mid d \in W_i\}$ for $i = 1, 2$ and define $e \subseteq \Delta_1 \times \Delta_2$ by putting $(t, t') \in e$ if and only if there exist $d_1 \in W_1$ and $d_2 \in W_2$ such that $t = t(d_1)$, $t' = t(d_2)$, and $(d_1, d_2) \in E^{\mathfrak{M}}$. Finally, for $i = 1, 2$ and each $a \in ob_i(\Gamma)$, define

$$\sigma_i(a) = \bigwedge \{s \in \mathsf{cl}_i(\Gamma) \mid a^{\mathfrak{M}} \in s^{\mathfrak{M}}\} = t(a^{\mathfrak{M}}) \in \Delta_i.$$

It remains to check that Δ_1, Δ_2, e, σ_1, and σ_2 satisfy conditions (1)–(6).

(1) Suppose that there is an $a \in ob_i(\Gamma)$ such that $\langle E \rangle^i \neg \langle E \rangle^{\bar{i}} (a) \in \sigma_i(a)$. Then, by the definition of σ_i, $a^{\mathfrak{M}} \in (\langle E \rangle^i \neg \langle E \rangle^{\bar{i}} (a))^{\mathfrak{M}}$, which is impossible.

(2) We have to show that the Γ_i are $\mathcal{S}_i$-satisfiable. The models

$$\mathfrak{W}_i = \left\langle W_i, \mathcal{V}_i^{\mathfrak{W}_i}, \mathcal{X}_i^{\mathfrak{W}_i}, \mathcal{R}_i^{\mathfrak{W}_i}, \mathcal{F}_i^{\mathfrak{W}_i} \right\rangle$$

are almost as required: we just have to give appropriate values to the fresh set variables x_t (which result from taking surrogates) and the fresh object names a_t from $\mathcal{X}_i(\Gamma)$. To this end, put

$$x_s^{\mathfrak{W}'_i} = s^{\mathfrak{M}}$$

for every term $s \in \mathsf{cl}_i(\Gamma)$ of the form $\langle E \rangle^i (s')$ and $x^{\mathfrak{W}'_i} = x^{\mathfrak{W}_i}$ for the remaining variables. For every $t \in \Delta_i$, choose $a_t^{\mathfrak{W}'_i}$ such that

$$a_t^{\mathfrak{W}'_i} \in t^{\mathfrak{M}}$$

and set $a^{\mathfrak{W}'_i} = a^{\mathfrak{W}_i}$ for the remaining object names. Note that

$$\mathfrak{W}'_i = \left\langle W_i, \mathcal{V}_i^{\mathfrak{W}'_i}, \mathcal{X}_i^{\mathfrak{W}'_i}, \mathcal{R}_i^{\mathfrak{W}_i}, \mathcal{F}_i^{\mathfrak{W}'_i} \right\rangle \in \mathcal{M}_i$$

for some interpretation $\mathcal{F}_i^{\mathfrak{W}'_i}$ of the function symbols in $\mathcal{F}_i$ such that $f^{\mathfrak{W}_i} = f^{\mathfrak{W}'_i}$ for all function symbols f of Γ (due to the closure condition for the class $\mathcal{M}_i$ formulated in Definition 4.3). Using induction on the term structure of s, it is straightforward to show that

$$d \in (sur_i(s))^{\mathfrak{W}'_i} \quad \Longleftrightarrow \quad d \in s^{\mathfrak{M}}$$

for all $d \in W_i$ and $s \in \mathsf{cl}_i(\Gamma)$. By considering the construction of Γ_i, it is readily checked that this implies $\mathfrak{W}'_i \models \Gamma_i$. Hence Γ_i is $(\mathcal{L}_i, \mathcal{M}_i)$-satisfiable.

(3) Let $t \in \Delta_1$ and $\langle E \rangle^1 (s) \in \mathsf{cl}_1(\Gamma)$ with s a 2-term. Since $t \in \Delta_1$, there is a $d \in W_1$ such that $t(d) = t$. First assume that $\langle E \rangle^1 (s) \in t$. By definition, this means that there exists a $d' \in W_2$ with $(d, d') \in E^{\mathfrak{M}}$ and $d' \in s^{\mathfrak{M}}$. This, in turn, clearly yields $s \in t(d')$ and $(t, t(d')) \in e$, as required. Now assume that $(t, t') \in e$ and $s \in t'$. Then there exist $d \in W_1$ and $d' \in W_2$ such that $t = t(d)$, $t' = t(d')$, and $(d, d') \in E^{\mathfrak{M}}$. We have $d' \in s^{\mathfrak{W}}$, and so $d \in (\langle E \rangle^1 (s))^{\mathfrak{M}}$, from which $\langle E \rangle^1 (s) \in t$, as required.

(4) is proved similarly to (3).

(5) Let $t \in \Delta_1$ and $\langle E \rangle^1 (a) \in \mathsf{cl}_1(\Gamma)$ with $a \in ob_2(\Gamma)$. Since $t \in \Delta_1$, there is a $d \in W_1$ such that $t(d) = t$. First assume $\langle E \rangle^1 (a) \in t$. By definition, we then have $(d, a^{\mathfrak{M}}) \in E^{\mathfrak{M}}$. Hence $(t, t(a^{\mathfrak{M}})) \in e$, i.e., $(t, \sigma_2(a)) \in e$, as required. Conversely, suppose $(t, \sigma_2(a)) \in e$. condition (1) yields $\langle E \rangle^2 \neg \langle E \rangle^1 (a) \notin \sigma_2(a)$. By condition (4), we have $\neg \langle E \rangle^1 (a) \notin t$. Hence $\langle E \rangle^1 (a) \in t$, as required.

(6) is proved similarly to (5).

We next prove the 'if' part of the lemma:

($\Longleftarrow$) Conversely, suppose that Δ_1, Δ_2, e, σ_1, and σ_2 satisfy the conditions of the theorem. By (2), there exist a model $\mathfrak{W}_1 \in \mathcal{M}_1$ of Γ_1 and a model $\mathfrak{W}_2 \in \mathcal{M}_2$ of Γ_2. For $i = 1, 2$, let $\mathfrak{W}_i$ be based on the domain W_i. For each $d \in W_i$, we set

$$t(d) = \bigwedge \{ t \in \mathsf{cl}_i(\Gamma) \mid d \in (sur_i(t))^{\mathfrak{W}_i} \} \in \mathfrak{C}_i(\Gamma).$$

Now define the extension $E^{\mathfrak{M}} \subseteq W_1 \times W_2$ of the link symbol E by taking:

$$E^{\mathfrak{M}} = \{ (d, d') \mid (t(d), t(d')) \in e \}.$$

In the following, we prove that $\mathfrak{M} = (\mathfrak{W}_1, \mathfrak{W}_2, E^{\mathfrak{M}})$ is a model of Γ. Using the construction of the Γ_i, it is readily checked that it suffices to show that

$$(*) \qquad d \in (sur_i(s))^{\mathfrak{W}_i} \quad \Longleftrightarrow \quad d \in s^{\mathfrak{M}}$$

for $i = 1, 2$, all $d \in W_i$, and all $s \in \mathsf{cl}_i(\Gamma)$.

The proof of this claim is by induction on the term structure of s, simultaneously for $i = 1, 2$. For set variables, the claim is an immediate consequence of the definition of $\mathfrak{M}$. The cases of the Boolean operators and the function symbols of $\mathcal{L}_i$, $i = 1, 2$, are trivial. Thus, it remains to consider the cases of

(a) $s = \langle E \rangle^i (s')$, with s' an $\bar{i}$-term, and
(b) $s = \langle E \rangle^i (a)$, with $a \in ob_{\bar{i}}(\Gamma)$.

We assume $i = 1$, since the case $i = 2$ is dual.

(a) $s = \langle E \rangle^1 (s')$ with s' a 2-term. Let $d \in (sur_1(\langle E \rangle^1 s'))^{\mathfrak{W}_1}$. Then we have $\langle E \rangle^1 (s') \in t(d)$. Since $\mathfrak{W}_1$ is a model of Γ_1,

$$\mathfrak{W}_1 \models sur_1(\bigvee \Delta_1) = \top_1.$$

Thus $t(d) \in \Delta_1$. By condition (3), we find a $t' \in \Delta_2$ with $(t(d), t') \in e$ and $s' \in t'$. By the definition of Γ_2, we have

$$\mathfrak{W}_2 \models a_{t'} : sur_2(t'),$$

and so there is a $d' \in W_2$ such that $t' = t(d')$. Hence we have $(d, d') \in E^{\mathfrak{M}}$ by the definition of $E^{\mathfrak{M}}$. From $s' \in t'$, we obtain $d' \in (sur_2(s'))^{\mathfrak{M}_2}$, and therefore the induction hypothesis yields $d' \in s'^{\mathfrak{M}}$. Thus, $d \in (\langle E \rangle^1 (s'))^{\mathfrak{M}}$ by definition.

Conversely, suppose $d \in (\langle E \rangle^1 (s'))^{\mathfrak{M}}$. We find $d' \in W_2$ with $(d, d') \in E^{\mathfrak{M}}$ and $d' \in s^{\mathfrak{M}}$. By the induction hypothesis, $d' \in (sur_2(s'))^{\mathfrak{W}_2}$ and so $s' \in t(d')$. The definition of $E^{\mathfrak{M}}$ together with $(d, d') \in E^{\mathfrak{M}}$ yields $(t(d), t(d')) \in e$. Finally, by (3), we obtain $\langle E \rangle^1 (s') \in t(d)$ which implies $d \in (sur_1(\langle E \rangle^1 s'))^{\mathfrak{W}_1}$.

(b) $s = \langle E \rangle^1 (a)$ with $a \in ob_2(\Gamma)$. Let $d \in (sur_1(\langle E \rangle^1 (a)))^{\mathfrak{W}_1}$. This implies that $\langle E \rangle^1 (a) \in t(d)$. As in the previous case, we have $t(d) \in \Delta_1$. By condition (5), we thus obtain $(t(d), \sigma_2(a)) \in e$. Also, as in the previous case, we know that

$$\mathfrak{W}_2 \models a : sur_2(\sigma_2(a)).$$

Hence $(t(d), \sigma_2(a)) \in e$ and the definition of $E^{\mathfrak{M}}$ yields $(d, a^{\mathfrak{W}_2}) \in E^{\mathfrak{M}}$, which implies $d \in (\langle E \rangle^1 (a))^{\mathfrak{M}}$.

Conversely, suppose $d \in (\langle E \rangle^1 (a))^{\mathfrak{M}}$. Then $(d, a^{\mathfrak{M}}) \in E^{\mathfrak{M}}$ by definition, and so $(t(d), t(a^{\mathfrak{M}})) \in e$ by the definition of $E^{\mathfrak{M}}$. We have $t(a^{\mathfrak{M}}) = \sigma_2(a)$, and therefore $(t(d), \sigma_2(a)) \in e$. Together with condition (5), this yields $\langle E \rangle^1 (a) \in t(d)$ which clearly implies $d \in (sur_1(\langle E \rangle^1 (a)))^{\mathfrak{W}_1}$.

This completes the proof of Lemma 5.9 □

Theorem 5.8 follows from Lemma 5.9. Indeed, since the sets $\mathfrak{C}_i(\Gamma)$ are finite, Lemma 5.9 provides us with a decision procedure for the connection $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$ if decision procedures for $\mathcal{S}_1$ and $\mathcal{S}_2$ are known. To decide whether a set Γ of $\mathcal{C}_O^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$-assertions is satisfiable, we 'guess' sets $\Delta_1 \subseteq \mathfrak{C}_1(\Gamma)$ and $\Delta_2 \subseteq \mathfrak{C}_2(\Gamma)$, a relation

$e \subseteq \Delta_1 \times \Delta_2$, and functions $\sigma_i : ob_i(\Gamma) \to \Delta_i$, $i = 1, 2$, and then check whether they satisfy the conditions listed in the formulation of the theorem. $\square$

To estimate the complexity of the obtained decision procedure, note that the cardinality of the sets $\mathfrak{C}_i(\Gamma)$ is exponential in the size of Γ. Thus, the same holds true for the sets Δ_1 and Δ_2 and for the constructed sets of assertions Γ_1 and Γ_2 which are passed to decision procedures for $\mathcal{S}_i$-satisfiability. This means that the time complexity of the obtained decision procedure for $\mathcal{C}^{\{E\}}_O(\mathcal{S}_1, \mathcal{S}_2)$-satisfiability is one exponential higher than the time complexity of the original decision procedures for $\mathcal{S}_1$ and $\mathcal{S}_2$-satisfiability. Moreover, the combined decision procedure is non-deterministic: if, for example, $\mathcal{S}_1$ and $\mathcal{S}_2$-satisfiability are in EXPTIME, then our algorithm yields a 2NEXPTIME decision procedure for $\mathcal{C}^{\{E\}}_O(\mathcal{S}_1, \mathcal{S}_2)$-satisfiability.

This result is somewhat surprising, since the addition of nominals to an arbitrary ADS with a decidable satisfiability problem sometimes results in an undecidable one; for an example see Theorem 4.17 in Section 5.4.

In Theorem 5.4, we connected the ADSs $\mathcal{ALCF}^\sharp$ and $\mathcal{ALCO}^\sharp$ to obtain a counter-example for the transfer of decidability of A-satisfiability. The choice of $\mathcal{ALCO}^\sharp$ was motivated by the fact that this ADS has nominals. Now that we are allowed to apply the link operators to object variables, we can strengthen this result: *any* connection (of the type considered in this section) involving $\mathcal{ALCF}^\sharp$ as one of its components has an undecidable A-satisfiability problem.

THEOREM 5.10. *Let $\mathcal{E}$ be an arbitrary non-empty set of link relations and $\mathcal{S}$ an* ADS. *Then the A-satisfiability problem for $\mathcal{C}^{\mathcal{E}}_O(\mathcal{ALCF}^\sharp, \mathcal{S})$ is undecidable.*

PROOF. The proof of Theorem 5.4 depends essentially on the possibility of applying link operators to the nominals of the connection's second component. We can therefore simply repeat this proof. Given an $\mathcal{ALCF}$-concept C and a set Γ of $\mathcal{ALCF}$ TBox assertions of the form $D \sqsubseteq D'$, we now define Γ^* by

$$\begin{aligned}\Gamma^* = \{&a : C^\sharp \wedge \langle E\rangle^1 (b)\}\\ &\cup \{b : [E]^2(D^\sharp \to D'^\sharp) \mid D \sqsubseteq D' \in \Gamma\}\\ &\cup \{b : [E]^2 f_{\forall R}(\langle E\rangle^1 (b)) \mid R \in \mathcal{R}\},\end{aligned}$$

where E is some link from $\mathcal{E}$, and prove, as before, that

$$C \text{ is satisfiable relative to } \Gamma \text{ in } \mathcal{ALCF} \iff$$

$$\iff \Gamma^* \text{ is A-satisfiable in } \mathcal{C}^{\mathcal{E}}_O(\mathcal{ALCF}^\sharp, \mathcal{S}).$$

$\square$

5.3. Boolean Operations on Links

The two variants of $\mathcal{E}$-connections introduced so far do not allow any interaction between links, which is a rather severe restriction. To illustrate this, we again consider the connection $\mathcal{C}^{\mathcal{E}}(\mathcal{SHIQ}^\sharp, \mathcal{ALCO}^\sharp)$ from Section 4.5.3. Recall that $\mathcal{E} = \{C, R, W\}$, where the link C represents citizenship (of people in EU countries) and R represents the place of residence. In the $\mathcal{E}$-connections $\mathcal{C}^{\mathcal{E}}(\mathcal{SHIQ}^\sharp, \mathcal{ALCO}^\sharp)$ and $\mathcal{C}^{\mathcal{E}}_{O}(\mathcal{SHIQ}^\sharp, \mathcal{ALCO}^\sharp)$, we cannot describe a concept such as

(iv) 'People taking residence in the country of their citizenship'.

To do this, we need the intersection of the links C and R:

$$\mathsf{Human\_being} \sqcap \langle C \cap R \rangle^1 (\mathsf{Country}).$$

Similarly, suppose that we are in the estate agent's framework of Section 4.5.1 and want to describe the set of points in space (say, Liverpool) which are served by *all* mobile phone providers. This can be naturally done using the complement operator on a link S (representing 'serves'):

$$\neg \langle \neg S \rangle^2 (\mathsf{Mobile\_phone\_provider}).$$

Note that $\langle \neg S \rangle^2 (\mathsf{Mobile\_phone\_provider})$ is the set of points that are not served by some mobile phone provider.

These simple examples motivate the following definition:

DEFINITION 5.11 (BOOLEAN LINKS).
Suppose that $\mathcal{S}_i = (\mathcal{L}_i, \mathcal{M}_i)$, $1 \le i \le n$, are ADSs *and that $\mathcal{E} = \{E_j \mid j \in J\}$ is a set of n-ary relation symbols. Denote by*

$$\mathcal{C}^{\mathcal{E}}_{B}(\mathcal{S}_1, \dots, \mathcal{S}_n)$$

the $\mathcal{E}$-connection with the smallest set $\overline{\mathcal{E}}$ of links such that

- $\mathcal{E} \subseteq \overline{\mathcal{E}}$;
- *if $F \in \overline{\mathcal{E}}$, then $\neg F \in \overline{\mathcal{E}}$;*
- *if $F, G \in \overline{\mathcal{E}}$, then $F \wedge G \in \overline{\mathcal{E}}$.*

Given an ADM

$$\mathfrak{M} = \left\langle (\mathfrak{W}_i)_{i \le n}, \mathcal{E}^{\mathfrak{M}} \right\rangle,$$

we interpret the links $F \in \overline{\mathcal{E}}$ as relations $F^{\mathfrak{M}} \subseteq W_1 \times \cdots \times W_n$ (with W_i being the domain of $\mathfrak{W}_i$) in the obvious way:

$$(F \wedge G)^{\mathfrak{M}} = F^{\mathfrak{M}} \cap G^{\mathfrak{M}}, \quad (\neg F)^{\mathfrak{M}} = (W_1 \times \cdots \times W_n) \setminus F^{\mathfrak{M}}.$$

The Boolean operations on links allow us to express **link inclusion assertions** of the form $F \sqsubseteq G$, where F and G are links, and $\mathfrak{M} \models F \sqsubseteq G$ if and only if $F^{\mathfrak{M}} \subseteq G^{\mathfrak{M}}$.

Such assertions are called **role hierarchies** in the area of description logics. Indeed, $F \sqsubseteq G$ can be equivalently rewritten as

$$\top_1 \sqsubseteq \neg \langle F \wedge \neg G \rangle^1 \top_2,$$

where $\top_i = x_i \vee \neg x_i$, for some set variable x_i of $\mathcal{L}_i$.

We denote by $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ the $\mathcal{E}$-connection which allows Boolean operations on links as well as applications of link operators to object variables.

We can now prove the analogue of Theorem 5.7. The intuition behind the proof is similar to the basic case: we again reduce the satisfiability problem for $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ to the satisfiability problem for its components. This time, however, the reduction is not so straightforward because the interaction between (complex) links has to be taken into account. For this reason, it is not enough to simply guess the 1-types and 2-types realised in a potential model together with a binary relation between them, but we have to guess a so-called *pre-model* which involves a relational structure *between elements* (rather than between types) and can be understood as the 'irregular core' of an otherwise 'regular' model. Fortunately, the size of this irregular core is at most exponential in the size of the input.

THEOREM 5.12 (TRANSFER OF DECIDABILITY FOR $\mathcal{C}^{\mathcal{E}}_{OB}$).
Let $\mathcal{S}_1, \ldots, \mathcal{S}_n$ be ADSs *with decidable satisfiability problems. Then the satisfiability problem for any $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ is decidable as well.*

PROOF. As before, we confine ourselves to considering $\mathcal{E}$-connections of only two ADSs. In contrast to the proof of Theorem 5.7, however, we admit an arbitrary number of link relations, since otherwise the Boolean operators on link relations cannot deploy their full power. Under these restrictions, Theorem 5.12 reads as follows:

THEOREM 5.13. *Suppose the satisfiability problems for* ADSs $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are decidable. Then the satisfiability problem for any $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$ is decidable as well.*

Let us fix two ADSs $\mathcal{S}_1 = (\mathcal{L}_1, \mathcal{M}_1)$ and $\mathcal{S}_2 = (\mathcal{L}_2, \mathcal{M}_2)$ with decidable satisfiability problems and a set of link symbols $\mathcal{E}$. Let Γ be a finite set of assertions of the $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$. We start by defining some notions:

- In contrast to the proof of Theorem 5.7, the **closures** $\mathsf{cl}_i(\Gamma)$ (for $i = 1, 2$) are now defined as the closure under negation of the set of i-terms occurring in Γ. As before, we identify $\neg\neg t$ with t, and so $\mathsf{cl}_i(\Gamma)$ is finite.
- By $rel(\Gamma)$ we denote the set of link symbols used in Γ. A **link type** for Γ is a set $T \subseteq rel(\Gamma)$. We use $\mathfrak{T}(\Gamma)$ to denote the set of all link types for Γ. If we interpret the symbols of $rel(\Gamma)$ as propositional variables, then a link type T for Γ can clearly be viewed as a propositional logic interpretation. Thus, given a link type T and a link F, we use the notation $T \models F$ if T is a model of F.

- For $t \in \mathfrak{C}_i(\Gamma)$, $t' \in \mathfrak{C}_{\bar{i}}(\Gamma)$, and T a link type for Γ, we write $t \rightsquigarrow^T t'$ if the following conditions are satisfied:
  (1) for all $\neg \langle F \rangle^i (s) \in t$ with s $\bar{i}$-term and $T \models F$, we have $s \notin t'$;
  (2) for all $\neg \langle F \rangle^{\bar{i}} (s) \in t'$ with s i-term and $T \models F$, we have $s \notin t$.
- Let S_1, S_2, and S_3 be sets. We call a total function

$$f : (S_1 \times S_2) \cup (S_2 \times S_1) \to S_3$$

a **symmetric function** from S_1, S_2 to S_3 if for all $(x_1, x_2) \in S_1 \times S_2$ we have

$$f(x_1, x_2) = f(x_2, x_1).$$

We assume without loss of generality that $\mathcal{S}_1$ and $\mathcal{S}_2$ support assertions of the form $a = a'$ and $a \neq a'$, where a and a' are object names. An assertion $a = a'$ ($a \neq a'$) is satisfied by a model $\mathfrak{W}$ if and only if $a^{\mathfrak{W}} = a'^{\mathfrak{W}}$ ($a^{\mathfrak{W}} \neq a'^{\mathfrak{W}}$). It should be clear that reasoning with such assertions can be reduced to reasoning without them: first perform appropriate substitutions of object names to eliminate all assertions of the form $a = a'$. Then introduce a fresh set variable x from the respective language for every assertion of the form $a \neq a'$ and replace $a \neq a'$ with $\{a : x, a' : \neg x\}$. As in the proof of Theorem 5.7, we assume that link assertions $(a_1, a_2) : E$ are replaced by the equivalent object assertion $a_1 : \langle E \rangle^1 (a_2)$.

Our aim is to formulate a criterion of satisfiability of sets of $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$-assertions Γ similar to Lemma 5.9, from which we will derive decidability of the satisfiability problem for $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$.

However, in the presence of the Boolean operators on link relations, things are somewhat more complicated. To see why this is the case, consider the ($\Longleftarrow$) direction of the proof of Lemma 5.9 in which we 'connect' the models for the sets Γ_1 and Γ_2 to a model for Γ. Whenever an element $d \in W_i$ should satisfy a term $\langle E \rangle^i(s)$, properties (3) to (6) ensure that there is a $t \in \Delta_{\bar{i}}$ such that (i) $s \in t$, and (ii) $s' \notin t$ for all $\langle E \rangle^i(s')$ that d should not satisfy. Moreover, $\Gamma_{\bar{i}}$ ensures that t is 'realised' at least once in $\mathfrak{W}_{\bar{i}}$, and thus we can connect d to an appropriate witness via the relation E. This simple strategy does not work with Boolean operators on link relations: since the element $d \in W_i$ may need a witness for the term s for many complex link relations $E_1, \ldots, E_k$ that are mutually exclusive (the simplest case is a an atomic link relation and its negation), it does not suffice to ensure that there is only one appropriate $t \in \Delta_{\bar{i}}$ that is realised only once in $\mathfrak{W}_{\bar{i}}$. The requirement of having enough witnesses for each term is in conflict with the fact that the involved ADSs may not allow certain terms to be realised an arbitrary number of times.

Our solution is to view models of $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$ as having a core of complex structure which is 'surrounded' by a shell of more regular structure. Intuitively, the core provides a 'sufficient' number of witnesses required for the model construction: witness requirements inside the core are satisfied inside the core, and witness requirements of elements outside the core (whose existence may be enforced by the class of models of the involved ADMs) are also satisfied inside the core.

In what follows, *pre-models* are used to describe the core part of models.

DEFINITION 5.14 (PRE-MODELS).
*Let $\Delta_1 \subseteq \mathfrak{C}_1(\Gamma)$ and $\Delta_2 \subseteq \mathfrak{C}_2(\Gamma)$. A **pre-model** for Δ_1, Δ_2 is a structure*

$$\langle P_1, P_2, t_1, t_2, e, \sigma_1, \sigma_2 \rangle ,$$

where

- *P_1 and P_2 are disjoint sets,*
- *t_i is a surjective function mapping each $p \in P_i$ to an element of Δ_i,*
- *e is a symmetric function from P_1, P_2 to $\mathfrak{T}(\Gamma)$, and*
- *σ_i is a function mapping each $a \in ob_i(\Gamma)$ to an element of P_i,*

such that, for $i \in \{1,2\}$, the following conditions are satisfied:

(1) *for all $p \in P_i$: if $\langle F \rangle^i (s) \in t_i(p)$, then there is a $p' \in P_{\bar{i}}$ such that $e(p,p') \models F$ and $s \in t_{\bar{i}}(p')$;*
(2) *for all $p \in P_i$: if $\langle F \rangle^i (a) \in t_i(p)$, then $e(p, \sigma_{\bar{i}}(a)) \models F$;*
(3) *for all $p \in P_i$ and $p' \in P_{\bar{i}}$: $t_i(p) \leadsto^{e(p,p')} t_{\bar{i}}(p')$;*
(4) *for all $p \in P_i$: if $\neg \langle F \rangle^i (a) \in t_i(p)$, then $e(p, \sigma_{\bar{i}}(a)) \not\models F$.*

We are now in a position to formulate a satisfiability criterion for sets of $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$-assertions.

LEMMA 5.15 (SATISFIABILITY CRITERION FOR $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$).
Let Γ be a $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$-knowledge base. Then Γ is satisfiable if and only if there exist subsets

$$\Delta_1 \subseteq \mathfrak{C}_1(\Gamma) \text{ and } \Delta_2 \subseteq \mathfrak{C}_2(\Gamma),$$

and a pre-model

$$\mathfrak{P} = \langle P_1, P_2, t_1, t_2, e, \sigma_1, \sigma_2 \rangle$$

for Δ_1, Δ_2, such that, for $i \in \{1,2\}$, the following conditions are satisfied:

(i) $|P_i| \leq (2^\delta + 1) \cdot 4\delta^4$, *where*

$$\delta = \max(|ob_1(\Gamma)|, |ob_2(\Gamma)|, |\mathsf{cl}_1(\Gamma)|, |\mathsf{cl}_2(\Gamma)|);$$

(ii) *the union Γ_i of the sets*
   - $\{sur_i(\bigvee \Delta_i) = \top_i\}$,
   - $\{a_p : sur_i(t_i(p)) \mid p \in P_i\}$,
   - $\{a_p = a \mid \sigma_i(a) = p\}$,

– $\{a_p \neq a_{p'} \mid p, p' \in P_i$ *and* $p \neq p'\}$,
– $\{sur_i(t_1) \sqsubseteq sur_i(t_2) \mid t_1 \sqsubseteq t_2 \in \Gamma$ *an i-term assertion*$\}$,
– $\{R_j(a_1, \ldots, a_{m_j}) \mid R_j(a_1, \ldots, a_{m_j}) \in \Gamma$ *an i-object assertion*$\}$, *and*
– $\{(a : sur_i(t)) \mid (a : t) \in \Gamma$ *an i-object assertion*$\}$,

is $\mathcal{S}_i$*-satisfiable for* $i \in \{1, 2\}$, *where* a_p *is a fresh object name from* $\mathcal{X}_i(\Gamma)$ *for each* $p \in P_i$.

PROOF. ($\Longrightarrow$) Let $\mathfrak{M} = \left\langle \mathfrak{W}_1, \mathfrak{W}_2, (E_i^{\mathfrak{M}})_{i \leq k} \right\rangle$ be a model for Γ, where $\mathfrak{W}_1$ has domain W_1 and $\mathfrak{W}_2$ has domain W_2. We use $\mathfrak{M}$ to choose sets Δ_1 and Δ_2 and define a pre-model $\mathfrak{P}$ satisfying the conditions given in the theorem: for $i \in \{1, 2\}$ and $d \in W_i$, put

$$t(d) = \bigwedge \{s \in \mathsf{cl}_i(\Gamma) \mid d \in s^{\mathfrak{M}}\}.$$

Further, for $d \in W_1$ and $d' \in W_2$, define their link type $ct(d, d')$ as

$$ct(d, d') = \{E \in rel(\Gamma) \mid (d, d') \in E^{\mathfrak{M}}\} \in \mathfrak{T}(\Gamma).$$

Then set

$$\Delta_i = \{t(d) \mid d \in W_i\}.$$

The construction of $\mathfrak{P} = \langle P_1, P_2, t_1, t_2, e, \sigma_1, \sigma_2 \rangle$ requires a bit more effort. We proceed in several steps:

1. Choose a set $L_1 \subseteq W_1$ such that the following conditions are satisfied:
   (a) for $t \in \Delta_1$ and $\Sigma_t = \{d \in W_1 \mid t(d) = t$ and $a^{\mathfrak{M}} \neq d$ for all $a \in ob_1(\Gamma)\}$ we let
   $$\{d \in W_1 \mid t(d) = t\} \subseteq L_1,$$
   if $|\Sigma_t| = |\mathsf{cl}_2(\Gamma)|$, and, otherwise, choose a set
   $$\Sigma' \subseteq \Sigma_t \text{ with } |\Sigma'| = |\mathsf{cl}_2(\Gamma)| \text{ and let } \Sigma' \subseteq L_1;$$
   (b) for all $a \in ob_1(\Gamma)$, we have $a^{\mathfrak{M}} \in L_1$;
   (c) $|L_1| \leq |\Delta_1| \cdot |\mathsf{cl}_2(\Gamma)| + |ob_1(\Gamma)|$.

It is easy to see that such a set exists.

2. Choose a set $R_1 \subseteq W_2$ satisfying the following conditions:
   (a) for each $t \in \Delta_2$, there is a $d \in R_1$ such that $t(d) = t$;
   (b) for all $a \in ob_2(\Gamma)$, we have $a^{\mathfrak{M}} \in R_1$;
   (c) for each $d \in L_1$ and $\langle F \rangle^1 (s) \in t(d)$, there exists a $d' \in R_1$ such that $(d, d') \in F^{\mathfrak{M}}$ and $s \in t(d')$;
   (d) $|R_1| \leq |L_1| \cdot |\mathsf{cl}_1(\Gamma)| + |\Delta_2| + |ob_2(\Gamma)|$.

Such a set exists since property (2.c) can clearly be satisfied by choosing at most $|L_1| \cdot |\mathsf{cl}_1(\Gamma)|$ elements of W_2 for R_1.

3. Choose a set $L_2 \subseteq W_1$ such that the following conditions are satisfied:

(a) $L_1 \cap L_2 = \emptyset$;

(b) for each $d \in R_1$ and $\langle F \rangle^2 (s) \in t(d)$, there exists a $d' \in L_1 \cup L_2$ such that $(d, d') \in F^{\mathfrak{M}}$ and $s \in t(d')$;

(c) $|L_2| \leq |R_1| \cdot |\mathsf{cl}_2(\Gamma)|$.

4. Choose a set $R_2 \subseteq W_2$ such that the following conditions are satisfied:

(a) $R_1 \cap R_2 = \emptyset$;

(b) for each $d \in L_2$ and $\langle F \rangle^1 (s) \in t(d)$, there exists a $d' \in R_1 \cup R_2$ such that $(d, d') \in F^{\mathfrak{M}}$ and $s \in t(d')$;

(c) $|R_2| \leq |L_2| \cdot |\mathsf{cl}_1(\Gamma)|$.

5. Choose a function K from $L_1 \times R_2$ to $\mathfrak{T}(\Gamma)$ such that the following conditions are satisfied:

(a) for each $d \in R_2$ and each $\langle F \rangle^2 (s) \in t(d)$, there exists a $d' \in L_1$ such that $K(d', d) \models F$ and $s \in t(d')$;

(b) for each $d \in R_2$ and $\langle F \rangle^2 (a) \in t(d)$, we have $K(a^{\mathfrak{M}}, d) \models F$;

(c) for all $(d, d') \in L_1 \times R_2$, we have $d \rightsquigarrow^{K(d,d')} d'$;

(d) for each $d \in R_2$ and $\neg \langle F \rangle^2 (a) \in t(d)$, we have $K(a^{\mathfrak{M}}, d) \not\models F$.

Let us show that such a function does exist. First, for each $d \in R_2$ we fix a subset $\tau(d) \subseteq W_1$ of cardinality $\leq |\mathsf{cl}_2(\Gamma)|$ such that, for each $\langle F \rangle^2 (s) \in t(d)$, there exists a $d' \in \tau(d)$ such that $(d', d) \in F^{\mathfrak{M}}$ and $s \in t(d')$. Due to properties (1.a) and (1.b) of L_1, we can find a map

$$\pi : \bigcup_{d \in R_2} \tau(d) \rightarrow L_1$$

whose restriction to $\tau(d)$ is injective for each $d \in R_2$ and such that, for all d' in the domain of π, we have

(i) $t(d') = t(\pi(d'))$,

(ii) $d' = a^{\mathfrak{M}}$ for some $a \in ob_1(\Gamma)$ implies $d' = \pi(d')$, and

(iii) $d' \neq a^{\mathfrak{M}}$ for all $a \in ob_1(\Gamma)$ implies $\pi(d') \neq a^{\mathfrak{M}}$ for all $a \in ob_1(\Gamma)$.

We now define K in three steps:

(1) for each $a \in ob_1(\Gamma)$ and $d \in R_2$, set $K(a^{\mathfrak{M}}, d) = ct(a^{\mathfrak{M}}, d)$;

(2) for each $d \in R_2$ and $d' \in \tau(d)$, set $K(\pi(d'), d) = ct(d', d)$;

(3) for each $d \in L_1$ and each $d' \in R_2$ such that $K(d, d')$ is undefined, we set $K(d, d') = ct(d, d')$.

Due to properties (ii) and (iii) of π, K is well-defined. It is straightforward to verify that K satisfies properties (5.a) to (5.d).

6. We now define the pre-model $\mathfrak{P}$ as follows:

(1) Set $P_1 = L_1 \cup L_2$ and $P_2 = R_1 \cup R_2$.
(2) For $i = 1, 2$, set $t_i(d) = t(d)$ for all $d \in P_i$. In view of property (1.a) of L_1 and property (3.a) of L_2, it is clear that the t_i are surjective.
(3) Let $d \in P_1$ and $d' \in P_2$. If $d \notin L_1$ or $d' \notin R_2$ set $e(d, d') = e(d', d) = ct(d, d')$. If $d \in L_1$ and $d' \in R_2$ set $e(d, d') = e(d', d) = K(d, d')$.
(4) For $i = 1, 2$ and $a \in ob_i(\Gamma)$, set $\sigma_i(a) = a^{\mathfrak{M}}$ (we do not 'leave' P_1 and P_2 due to property (1.b) of L_1 and property (3.b) of L_2).

A lengthy but easy computation yields the upper bound $|P_i| \leq (2^\delta + 1) \cdot 4\delta^4$ for the size of the sets P_i. Next, we show that $\mathfrak{P}$ is indeed a pre-model, i.e., that it satisfies properties (1)–(4) from Definition 5.14:

(1) Let $d \in L_1$ and $\langle F \rangle^1 (s) \in t_1(d)$. Since $t_1(d) = t(d)$ by the definition of $\mathfrak{P}$, property (2.c) of R_1 yields a $d' \in R_1$ such that $(d, d') \in F^{\mathfrak{M}}$ and $s \in t(d')$. By the definition of $\mathfrak{P}$, we have $e(d, d') = ct(d, d')$ and $t_2(d') = t(d')$. Thus, $e(d, d') \models F$ and $s \in t_2(d')$, as required.

In the case $d \in R_1$ and $\langle F \rangle^2 (s) \in t_2(d)$, we may use an analogous argument employing property (3.b) of L_2 instead of property (2.c) of R_1. Similarly, in the case $d \in L_2$ we may use property (4.b) of R_2.

Now let $d \in R_2$ and $\langle F \rangle^2 (s) \in t_2(d)$. By property (5.a) of K, there exists a $d' \in L_1$ such that $K(d', d) \models F$ and $s \in t(d')$. By the definition of $\mathfrak{P}$, we have $e(d', d) = K(d', d)$ and $t_1(d') = t(d')$. Thus, $e(d, d') \models F$ and $s \in t_1(d')$.

(2) Let $d \in L_1 \cup L_2$ and $\langle F \rangle^1 (a) \in t_1(d)$. By property (2.b) of R_1, we have $a^{\mathfrak{M}} \in R_1$. Moreover, by the definition of $\mathfrak{P}$, we have $t_1(d) = t(d)$. Thus, $\langle F \rangle^1 (a) \in t(d)$ which implies $ct(d, a^{\mathfrak{M}}) \models F$. Since $e(d, a^{\mathfrak{M}}) = ct(d, a^{\mathfrak{M}})$ and $\sigma_2(a) = a^{\mathfrak{M}}$ by the definition of $\mathfrak{P}$, we obtain $e(d, \sigma_2(a)) \models F$, as required.

In the case $d \in R_1$ and $\langle F \rangle^2 (a) \in t_2(d)$, we may use an analogous argument employing property (1.b) of L_1 instead of property (2.b) of R_1.

Now let $d \in R_2$ and $\langle F \rangle^2 (a) \in t_2(d)$. By property (1.b) of L_1, we have $a^{\mathfrak{M}} \in L_1$. By property (5.b) of K, we have $K(a^{\mathfrak{M}}, d) \models F$. Since $e(a^{\mathfrak{M}}, d) = K(a^{\mathfrak{M}}, d)$ and $\sigma_1(a) = a^{\mathfrak{M}}$ by the definition of $\mathfrak{P}$, we obtain $e(\sigma_1(a), d) = e(d, \sigma_1(a)) \models F$, as required.

(3) As the definition of $\rightsquigarrow$ is symmetric, it suffices to show $t_1(d_1) \rightsquigarrow^{e(d_1,d_2)} t_2(d_2)$ for all $d_1 \in P_1$ and $d_2 \in P_2$. First, let $d_1 \in P_1$ and $d_2 \in R_1$. The definition of $\mathfrak{P}$ implies $e(d_1, d_2) = ct(d_1, d_2)$. By the definition of $\rightsquigarrow$, we need to show two properties:

- Let $\neg \langle F \rangle^1 (s) \in t_1(d_1)$ and $e(d_1, d_2) \models F$. Since $t_1(d_1) = t(d_1)$, we have $\neg \langle F \rangle^1 (s) \in t(d_1)$. Since $e(d_1, d_2) = ct(d_1, d_2)$ and $e(d_1, d_2) \models F$, we obtain $s \notin t(d_2)$. Now $t(d_2) = t_2(d_2)$ implies $s \notin t_2(d_2)$, as required.
- The case of $\neg \langle F \rangle^2 (s) \in t_2(d_2)$ and $e(d_1, d_2) \models F$ is considered analogously.

Now let $d_1 \in P_1$ and $d_2 \in R_2$. The definition of $\mathfrak{P}$ implies $e(d_1, d_2) = K(d_1, d_2)$. Again we need to show two properties:

- Let $\neg \langle F \rangle^1 (s) \in t_1(d_1)$ and $e(d_1, d_2) \models F$. Since $t_1(d_1) = t(d_1)$, we have $\neg \langle F \rangle^1 (s) \in t(d_1)$. Since $e(d_1, d_2) = K(d_1, d_2)$, we obtain $s \notin t(d_2)$ by property (5.c) of K. Now $t(d_2) = t_2(d_2)$ implies $s \notin t_2(d_2)$ as required.
- The case of $\neg \langle F \rangle^2 s \in t_2(d_2)$ and $e(d_1, d_2) \models F$ is considered analogously.

(4) Let $d \in L_1 \cup L_2$ and $\neg \langle F \rangle^1 (a) \in t_1(d)$. By property (2.b) of R_1, $a^{\mathfrak{M}} \in R_1$. Moreover, by the definition of $\mathfrak{P}$ we have $t_1(d) = t(d)$. Thus $\neg \langle F \rangle^1 (a) \in t(d)$, which implies $ct(d, a^{\mathfrak{M}}) \not\models F$. Since $e(d, a^{\mathfrak{M}}) = ct(d, a^{\mathfrak{M}})$ and $\sigma_2(a) = a^{\mathfrak{M}}$ by the definition of $\mathfrak{P}$, we obtain $e(d, \sigma_2(a)) \not\models F$, as required.

In the case $d \in R_1$ and $\langle F \rangle^2 (a) \in t_2(d)$, we may use an analogous argument employing property (1.b) of L_1 instead of property (2.b) of R_1.

Now let $d \in R_2$ and $\neg \langle F \rangle^2 (a) \in t_2(d)$. By property (1.b) of L_1, we have $a^{\mathfrak{M}} \in L_1$. By property (5.d) of K, $K(d, a^{\mathfrak{M}}) \not\models F$. Since $e(d, a^{\mathfrak{M}}) = K(d, a^{\mathfrak{M}})$ and $\sigma_1(a) = a^{\mathfrak{M}}$ by the definition of $\mathfrak{P}$, we obtain $e(d, \sigma_1(a)) \not\models F$ as required.

To complete the proof of the 'only if' direction, it remains to show that the sets Γ_i are $\mathcal{S}_i$ satisfiable, which is done as in Lemma 5.9 by additionally setting $(a_p)^{\mathfrak{M}'_i} = p$ for all $p \in P_i$.

($\Longleftarrow$) Suppose that Δ_1, Δ_2, and $\mathfrak{P} = \langle P_1, P_2, t_1, t_2, e, \sigma_1, \sigma_2 \rangle$ satisfying the conditions of the theorem are given. We construct a model satisfying Γ. To this end, take models $\mathfrak{W}_i \in \mathcal{M}_i$ with domain W_i satisfying Γ_i, for $i = 1, 2$. Let, for $d \in W_i$,

$$t(d) = \bigwedge \{ s \in \mathsf{cl}_i(\Gamma) \mid d \in (sur_i(s))^{\mathfrak{W}_i} \}.$$

By the definition of Γ_i, we clearly have $t(d) \in \Delta_i$ for each $d \in W_i$. Now fix an element $\rho(d) \in P_i$ for each $d \in W_i$ such that $t(d) = t_i(\rho(d))$ and $d = a_p^{\mathfrak{W}_i}$ implies $\rho(d) = p$, for all $p \in P_i$. This is possible, since the functions t_i of $\mathfrak{P}$ are surjective and $t(a_p^{\mathfrak{W}_i}) = t_i(p)$ by the definition of Γ_i. Let $rel(\Gamma) = \{E_1, \ldots, E_k\}$. For $1 \leq j \leq k$, we define the extension $E_j^{\mathfrak{M}}$ of a link relation E_j by setting

$$d \, E_j^{\mathfrak{M}} \, d' \quad \iff \quad E_i \in e(\rho(d), \rho(d')).$$

The proof of the following claim is straightforward and left to the reader:

(♣) For all links F, $d_1 \in W_1$, and $d_2 \in W_2$,

$$(d_1, d_2) \in F^{\mathfrak{M}} \quad \iff \quad e(\rho(d_1), \rho(d_2)) \models F.$$

We now show that $\mathfrak{M} = \left\langle \mathfrak{W}_1, \mathfrak{W}_2, (E_j^{\mathfrak{M}})_{j \leq k} \right\rangle$ is a model for Γ. It clearly suffices to prove that

$$d \in sur_i(s)^{\mathfrak{W}_i} \quad \iff \quad d \in s^{\mathfrak{M}}$$

for all $d \in W_i$, $s \in \mathsf{cl}_i(\Gamma)$, and $i \in \{1,2\}$, which can be done by simultaneous structural induction. We only consider the interesting cases, i.e., Case (i): $t = \langle F \rangle^i (s')$, and further, Case (ii): $t = \langle F \rangle^i (a)$, for $i = 1$ (the case $i = 2$ is symmetric).

(i) Assume $t = \langle F \rangle^1 (s')$. Let $d \in sur_1(\langle F \rangle^1 s')^{\mathfrak{W}_1}$. This implies $\langle F \rangle^1 (s') \in t(d)$, and so $\langle F \rangle^1 (s') \in t_1(\rho(d))$. By property (1) of pre-models, there exists a $p \in P_2$ such that $e(\rho(d), p) \models F$ and $s' \in t_2(p)$. By the choice of ρ, we have $\rho(a_p^{\mathfrak{W}_2}) = p$. Since $e(\rho(d), p) \models F$, we thus obtain $(d, a_p^{\mathfrak{W}_2}) \in F^{\mathfrak{M}}$ from (♣). Moreover, $s' \in t_2(p)$ and $\rho(a_p^{\mathfrak{W}_2}) = p$ yield $s' \in t(a_p^{\mathfrak{W}_2})$, and hence $a_p^{\mathfrak{W}_2} \in sur_2(s')^{\mathfrak{W}_2}$, from which we obtain $a_p^{\mathfrak{W}_2} \in s'^{\mathfrak{M}}$ by the induction hypotheses. To sum up, $d \in (\langle F \rangle^1 (s'))^{\mathfrak{M}}$.

For the 'if' direction, we show the contrapositive. Let $d \notin sur_1(\langle F \rangle^1 (s'))^{\mathfrak{W}_1}$. We need to prove that $d \notin (\langle F \rangle^1 (s'))^{\mathfrak{M}}$. Fix a $d' \in W_2$ with $(d, d') \in F^{\mathfrak{M}}$. By (♣), we have $e(\rho(d), \rho(d')) \models F$, and $d \notin sur_1(\langle F \rangle^1 (s'))^{\mathfrak{W}_i}$ yields $\neg \langle F \rangle^1 (s') \in t(d)$ and furthermore $\neg \langle F \rangle^1 (s') \in t_1(\rho(d))$. Thus, we have $s' \notin t_2(\rho(d'))$ by property (3) of pre-models and the definition of $\rightsquigarrow$. This clearly yields $s' \notin t(d')$ and thus $d' \notin sur_2(s')^{\mathfrak{W}_2}$, which implies $d' \notin s'^{\mathfrak{M}}$ by the induction hypotheses. Since this holds independently of the choice of d', we obtain $d \notin (\langle F \rangle^1 (s'))^{\mathfrak{M}}$, as required.

(ii) Let $t = \langle F \rangle^1 (a)$ and $d \in sur_1(\langle F \rangle^1 (a))^{\mathfrak{W}_1}$. This implies $\langle F \rangle^1 (a) \in t(d)$ and so $\langle F \rangle^1 (a) \in t_1(\rho(d))$. By property (2) of pre-models, $e(\rho(d), \sigma_2(a)) \models F$. By the construction of Γ_2, there is a $p \in P_2$ such that $p = \sigma_2(a)$ and $a_p^{\mathfrak{W}_2} = a^{\mathfrak{W}_2}$. By the choice of ρ, we then have $\rho(a_p^{\mathfrak{W}_2}) = \sigma_2(a)$. Since $e(\rho(d), \sigma_2(a)) \models F$, we thus obtain $(d, a_p^{\mathfrak{W}_2}) \in F^{\mathfrak{M}}$ from (♣). Hence, $d \in (\langle F \rangle^1 (a))^{\mathfrak{M}}$.

For the 'if' direction, we show the contrapositive. Let $d \notin sur_1(\langle F \rangle^1 (a))^{\mathfrak{W}_1}$. We need to prove that $d \notin (\langle F \rangle^1 (a))^{\mathfrak{M}}$. Fix a $d' \in W_2$ such that $(d, d') \in F^{\mathfrak{M}}$. By the claim, we have $e(\rho(d), \rho(d')) \models F$. Moreover, $d \notin sur_1(\langle F \rangle^1 (a))^{\mathfrak{W}_1}$ yields $\neg \langle F \rangle^1 (a) \in t(d)$ and $\neg \langle F \rangle^1 (a) \in t_1(\rho(d))$. Thus, $e(\rho(d), \sigma_2(a)) \not\models F$, i.e., $\rho(d') \neq \sigma_2(a)$, by property (4) of pre-models, and so $d' \neq a^{\mathfrak{W}_2}$ by the definition of Γ_2 and the choice of ρ. Thus $d' \neq a^{\mathfrak{M}}$. Since this holds independently of the choice of d', we obtain $d \notin (\langle F \rangle^1 (a))^{\mathfrak{M}}$, as required.

This completes the proof of Lemma 5.15. □

Similarly to the proof of Theorem 5.7, Lemma 5.15 almost immediately provides us with a decision procedure for the connection $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$ if decision procedures for $\mathcal{S}_1$ and $\mathcal{S}_2$ are known: since the sets $\mathfrak{C}_i(\Gamma)$ are finite and $|P_i| \leq (2^\delta + 1) \cdot 4\delta^4$, to decide whether a set Γ of $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \mathcal{S}_2)$-assertions is satisfiable, we may 'guess' sets $\Delta_1 \subseteq \mathfrak{C}_1(\Gamma)$ and $\Delta_2 \subseteq \mathfrak{C}_2(\Gamma)$ and a pre-model $\mathfrak{P}$, and then check whether they satisfy the conditions listed in the formulation of the theorem. □

As before, a non-deterministic upper time bound for the satisfiability problem for the $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}_{OB}(\mathcal{S}_1, \dots, \mathcal{S}_n)$ is obtained by adding one exponential to the maximal time complexity of the components. The following result shows that this upper bound cannot be improved, in general, since the satisfiability problem for the basic ADS $\mathcal{B}$ introduced in Section 4.4 is NP-complete (cf. Lemma 5.2).

The proof is by reduction of the NEXPTIME-complete satisfiability problem for the modal logic **S5** $\times$ **S5** [Marx, 1999] (i.e., the full binary product of modal **S5** with itself) to satisfiability in $\mathcal{C}^{\mathcal{E}}_{B}(\mathcal{B}, \mathcal{B})$. Since the ADS $\mathcal{B}$ is rather trivial, while **S5** $\times$ **S5** is known to be a variant of the two-variable fragment of first-order logic[1], this result demonstrates the considerable expressive power which the Boolean operators on links add to $\mathcal{E}$-connections.

THEOREM 5.16. *The satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{B}(\mathcal{B}, \mathcal{B})$ *is* NEXPTIME*-complete, for any infinite* $\mathcal{E}$.

PROOF. Recall that **S5** $\times$ **S5**-formulae are composed from propositional variables $p_1, p_2, \dots$ by means of the Boolean operators and the modal operators $\Box_1$ and $\Box_2$. The models of **S5** $\times$ **S5** are structures $\mathfrak{N} = \langle W_1 \times W_2, \mathfrak{V} \rangle$ that consist of the Cartesian product of two non-empty sets W_1 and W_2 and a valuation $\mathfrak{V}$ which maps any propositional variable to a subset of $W_1 \times W_2$. The extension $\varphi^{\mathfrak{N}}$ of an **S5** $\times$ **S5**-formula φ in $\mathfrak{N}$ is computed inductively as follows:

$$p_i^{\mathfrak{N}} = \mathfrak{V}(p_i), \ \ (\psi_1 \wedge \psi_2)^{\mathfrak{N}} = \psi_1^{\mathfrak{N}} \cap \psi_2^{\mathfrak{N}}, \ \ (\neg\psi)^{\mathfrak{N}} = (W_1 \times W_2) \setminus \psi^{\mathfrak{N}},$$

$$(\Box_1\psi)^{\mathfrak{N}} = \{(w_1, w_2) \mid \forall v \in W_1 \ (v, w_2) \in \psi^{\mathfrak{N}}\},$$

$$(\Box_2\psi)^{\mathfrak{N}} = \{(w_1, w_2) \mid \forall v \in W_2 \ (w_1, v) \in \psi^{\mathfrak{N}}\}.$$

A formula φ is **S5** $\times$ **S5**-satisfiable if there exists an **S5** $\times$ **S5**-model in which φ has a non-empty extension.

Suppose now that φ is an **S5** $\times$ **S5**-formula. Denote by $sub(\varphi)$ the set of all subformulae of φ. For any $\psi \in sub(\varphi)$, take a link $E_\psi \in \mathcal{E}$ and let the $\mathcal{C}^{\mathcal{E}}_{B}(\mathcal{B}, \mathcal{B})$-knowledge base Γ consist of:

(1) $E_{\psi_1 \wedge \psi_2} = E_{\psi_1} \wedge E_{\psi_2};$ $\qquad (\psi_1 \wedge \psi_2 \in sub(\varphi))$

(2) $E_{\neg\psi} = \neg E_\psi;$ $\qquad (\neg\psi \in sub(\varphi))$

(3) $\langle \neg E_\psi \rangle^2 (\top_1) = [E_{\Box_1\psi}]^2(\bot_1);$
$[E_{\Box_1\psi}]^2(\bot_1) = \langle \neg E_{\Box_1\psi} \rangle^2 (\top_1);$ $\qquad (\Box_1\psi \in sub(\varphi))$

(4) $\langle \neg E_\psi \rangle^1 (\top_2) = [E_{\Box_2\psi}]^1(\bot_2);$
$[E_{\Box_2\psi}]^1(\bot_2) = \langle \neg E_{\Box_2\psi} \rangle^1 (\top_2).$ $\qquad (\Box_2\psi \in sub(\varphi))$

[1] To be more precise, the two-variable substitution free fragment, Gabbay et al. [2003].

It was shown on Page 150 that such equations can be added to the vocabulary when working with connections allowing the Boolean closure of links. More precisely, an equation of the form $F = G$ is shorthand for the conjunction of the two link inclusions $F \sqsubseteq G$ and $G \sqsubseteq F$. Moreover, we use expressions of the form $[E]^i\varphi$ as abbreviations for the formula $\neg \langle E \rangle^i \neg\varphi$.

We now claim that

$$(\clubsuit) \quad \varphi \text{ is } \mathbf{S5} \times \mathbf{S5}\text{-satisfiable} \iff \Gamma \cup \{a : \langle E_\varphi \rangle^1 (\top_2)\} \text{ is satisfiable in } \mathcal{C}^{\mathcal{E}}_B(\mathcal{B}, \mathcal{B}),$$

where a is an object name of the first component of $\mathcal{C}^{\mathcal{E}}_B(\mathcal{B}, \mathcal{B})$.

To prove ($\clubsuit$), assume first that φ is satisfied in $\mathfrak{N} = \langle W_1 \times W_2, \mathfrak{V} \rangle$. We construct a model $\mathfrak{M} = \left\langle \mathfrak{M}_1, \mathfrak{M}_2, \{E^{\mathfrak{M}}_\psi\}_{\psi \in sub(\varphi)} \right\rangle$ that satisfies $\Gamma \cup \{a : \langle E_\varphi \rangle^1 (\top_2)\}$. Let $\mathfrak{M}_2$ be any model for $\mathcal{B}$ with domain W_2. By assumption, $\varphi^{\mathfrak{N}} \neq \varnothing$, so we can pick some $(u, v) \in \varphi^{\mathfrak{N}}$ and choose $\mathfrak{M}_1$ to be any model for $\mathcal{B}$ with domain W_1, where $a^{\mathfrak{M}_1} = u$. Finally, we can define $E^{\mathfrak{M}}_\psi = \psi^{\mathfrak{N}} \subseteq W_1 \times W_2$, for every $\psi \in sub(\varphi)$. By construction, $\mathfrak{M} \models a : \langle E_\varphi \rangle^1 (\top_2)$, so it suffices to show that equations (1)–(4) hold in $\mathfrak{M}$, which can be done by structural induction.

If $\psi_1 \wedge \psi_2 \in sub(\varphi)$, then

$$E^{\mathfrak{M}}_{\psi_1 \wedge \psi_2} = (\psi_1 \wedge \psi_2)^{\mathfrak{N}} = \psi_1^{\mathfrak{N}} \cap \psi_2^{\mathfrak{N}} = E^{\mathfrak{M}}_{\psi_1} \cap E^{\mathfrak{M}}_{\psi_2}.$$

Equation (2) is shown in the same way. To prove (3), notice that the following equivalences hold:

$$\begin{aligned} v \in (\langle \neg E_\psi \rangle^2 (\top_1))^{\mathfrak{M}} \iff \exists u\,(u, v) \notin E^{\mathfrak{M}}_\psi \iff \exists u\,(u, v) \notin \psi^{\mathfrak{N}} \iff \\ \iff \forall u\,(u, v) \notin (\Box_1 \psi)^{\mathfrak{N}} \iff \forall u\,(u, v) \notin E^{\mathfrak{M}}_{\Box_1 \psi} \iff v \in ([E_{\Box_1 \psi}]^2 (\bot_1))^{\mathfrak{M}}, \end{aligned}$$

and

$$\begin{aligned} v \in ([E_{\Box_1 \psi}]^2 (\bot_1))^{\mathfrak{M}} \iff \forall u\,(u, v) \notin (\Box_1 \psi)^{\mathfrak{N}} \iff \exists u\,(u, v) \notin (\Box_1 \psi)^{\mathfrak{N}} \\ \iff \exists u\,(u, v) \notin (E_{\Box_1 \psi})^{\mathfrak{M}} \iff v \in (\langle \neg E_{\Box_1 \psi} \rangle^2 (\top_1))^{\mathfrak{M}}. \end{aligned}$$

The equations in (4) are proved in exactly the same way.

Conversely, assume that $\Gamma \cup \{a : \langle E_\varphi \rangle^1 (\top_2)\}$ is satisfied in a model $\mathfrak{M}$, where $\mathfrak{M} = \left\langle \mathfrak{M}_1, \mathfrak{M}_2, \{E^{\mathfrak{M}}_\psi\}_{\psi \in sub(\varphi)} \right\rangle$ is based on the domains W_1 and W_2. We define a model $\mathfrak{N}$ for $\mathbf{S5} \times \mathbf{S5}$ based on the domain $W_1 \times W_2$ by letting $p_i^{\mathfrak{N}} = E^{\mathfrak{M}}_{p_i}$ for $p_i \in sub(\varphi)$ and arbitrary otherwise. It can now be shown by induction that, for all $\psi \in sub(\varphi)$,

$$(\heartsuit) \qquad E^{\mathfrak{M}}_\psi = \psi^{\mathfrak{N}}.$$

The base case, $\psi = p_i$, follows from the definition of $\mathfrak{N}$. If $\psi = \psi_1 \wedge \psi_2$ we obtain $(\psi_1 \wedge \psi_2)^{\mathfrak{N}} = \psi_1^{\mathfrak{N}} \cap \psi_2^{\mathfrak{N}} = E^{\mathfrak{M}}_{\psi_1} \cap E^{\mathfrak{M}}_{\psi_2} = E^{\mathfrak{M}}_{\psi_1 \wedge \psi_2}$ by (1). The case of $\psi = \neg\chi$ is shown in the same way using (2). The case of $\psi = \Box_1\chi$ is shown using (3) as follows:

$$\begin{aligned}
(u,v) \notin (\Box_1\psi)^{\mathfrak{N}} &\iff \exists \hat{u} \in W_1\ (\hat{u},v) \notin \psi^{\mathfrak{N}} \\
&\iff \exists \hat{u} \in W_1\ (\hat{u},v) \notin E_{\psi}^{\mathfrak{M}} \text{ (by induction)} \\
&\iff v \in (\langle \neg E_{\psi} \rangle^2 (\top_1))^{\mathfrak{M}} \\
&\iff v \in ([E_{\Box_1\psi}]^2(\bot_1))^{\mathfrak{M}} \text{ (by (3.1))} \\
&\iff \forall \hat{u} \in W_1\ (\hat{u},v) \notin (E_{\Box_1\psi})^{\mathfrak{M}} \\
&\implies (u,v) \notin (E_{\Box_1\psi})^{\mathfrak{M}},
\end{aligned}$$

and

$$\begin{aligned}
(u,v) \notin (E_{\Box_1\psi})^{\mathfrak{M}} &\implies \exists \hat{u} \in W_1\ (\hat{u},v) \notin (E_{\Box_1\psi})^{\mathfrak{M}} \\
&\iff v \in (\langle \neg E_{\Box_1\psi} \rangle^2 (\top_1))^{\mathfrak{M}} \\
&\iff v \in ([E_{\Box_1\psi}]^2(\bot_1))^{\mathfrak{M}} \text{ (by (3.2))} \\
&\iff (u,v) \notin (\Box_1\psi)^{\mathfrak{N}} \text{ (from above).}
\end{aligned}$$

The case of $\psi = \Box_2\chi$ is treated in the same way. This shows (♡).

As $\mathfrak{M} \models a : \langle E_{\varphi} \rangle^1 (\top_2)$, there is a $v \in W_2$ such that $(a^{\mathfrak{M}}, v) \in E_{\varphi}^{\mathfrak{M}} = \varphi^{\mathfrak{N}} \neq \varnothing$. It follows that φ is satisfied in $\mathfrak{N}$, which proves (♣). □

5.4. Number Restrictions on Links

An obviously desirable expressive capability when dealing with $\mathcal{E}$-connections is the ability to constrain the *number* of objects that are connected via the link relations. For example, in the real estate agent's application, we may want to say that—according to the chosen granularity of the spatial domain—the spatial extension of any house consists of precisely one point in space. Thus, the corresponding link relation should be a partial function. The concept constructors employed in description logic to represent this kind of constraints are known as (**qualified**) **number restrictions**[2]; in modal logic they are called **graded modalities**[3]. So let us investigate what happens if we introduce similar constructors for links in $\mathcal{E}$-connections.

DEFINITION 5.17 (NUMBER RESTRICTIONS ON LINKS).
Suppose that $\mathcal{S}_i = (\mathcal{L}_i, \mathcal{M}_i)$ *are* ADSs, $1 \le i \le n$, *and that* $\mathcal{E} = \{E_j \mid j \in J\}$ *is a set of n-ary relation symbols. Denote by*

$$\mathcal{C}^{\mathcal{E}}_{\mathcal{Q}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$$

the $\mathcal{E}$*-connection in which the definition of i-terms,* $1 \le i \le n$, *is extended with the following clause, for every natural number r:*

[2] Cf. Hollunder and Baader [1991], De Giacomo and Lenzerini [1996], and Horrocks et al. [1999].

[3] Cf. Fine [1972], de de Rijke [2000], and Tobies [2001b].

- *if* $\bar{t}_i = (t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$ *is a sequence of j-terms* t_j, *for* $j \neq i$, *and* $k \in J$, *then* $\langle \leq rE_k \rangle^i (\bar{t}_i)$ *and* $\langle \geq rE_k \rangle^i (\bar{t}_i)$ *are* i-terms.

The semantics of the new constructors, called ***number restrictions on links****, is defined as follows. Let*

$$\mathfrak{M} = \left\langle (\mathfrak{W}_i)_{i \leq n}, \mathcal{E}^{\mathfrak{M}} \right\rangle$$

be a model for $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$. *Then*

$$x \in (\langle \leq rE_j \rangle^i (\bar{t}_i))^{\mathfrak{M}} \iff \left| \{ \bar{x}_i \mid (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) \in E_j^{\mathfrak{M}} \wedge x_k \in t_k^{\mathfrak{M}} \} \right| \leq r$$

and

$$x \in (\langle \geq rE_j \rangle^i (\bar{t}_i))^{\mathfrak{M}} \iff \left| \{ \bar{x}_i \mid (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n) \in E_j^{\mathfrak{M}} \wedge x_k \in t_k^{\mathfrak{M}} \} \right| \geq r,$$

where $\bar{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$.

Combinations of $\mathcal{C}^{\mathcal{E}}_{Q}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ with previous extensions are denoted by the obvious names, e.g., $\mathcal{C}^{\mathcal{E}}_{QB}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ stands for the extension of basic $\mathcal{E}$-connections with both number restrictions and the Boolean operators on links.

Unfortunately, it turns out that, in general, decidability does not transfer from ADSs $\mathcal{S}_1, \ldots, \mathcal{S}_n$ to their $\mathcal{E}$-connection with number restrictions $\mathcal{C}^{\mathcal{E}}_{Q}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$.

5.4.1. Undecidable $\mathcal{E}$-Connections with Number Restrictions. In Section 4.3, we introduced the reasoning problem of singleton satisfiability of terms, and we showed in Theorem 4.17 that there are number tolerant ADSs with a decidable satisfiability problem for which singleton satisfiability is undecidable.

Apart from ADSs for which singleton satisfiability is undecidable, there is one more ADS that will play an important role in this section:

DEFINITION 5.18. *The* ADS $\mathcal{B}_1 = (\mathcal{L}_{\mathcal{B}}, \mathcal{M}_{\mathcal{B}_1})$ *is defined as follows:*

- $\mathcal{L}_{\mathcal{B}}$ *is the* ADL *(as defined on Page 138) without function symbols (apart from the Booleans) and relation symbols;*
- $\mathcal{M}_{\mathcal{B}_1}$ *consists of all* ADMs *of the signature of* $\mathcal{L}_{\mathcal{B}}$ *based on a singleton domain.*

It is obviously trivial to decide satisfiability in $\mathcal{B}_1$. Note also that $\mathcal{B}_1$ is *not* number tolerant. We are now in a position to prove the undecidability results concerning $\mathcal{E}$-connections that allow for qualified number restrictions.

THEOREM 5.19. *There exist* ADSs $\mathcal{S}_1$ *and* $\mathcal{S}_2$, *not both number tolerant, with decidable satisfiability problems and such that the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{Q}(\mathcal{S}_1, \mathcal{S}_2)$ *is undecidable, even if* $\mathcal{E}$ *is a singleton.*

PROOF. We first prove a technical lemma. This, together with Theorem 4.17, implies the result:

LEMMA 5.20. *Let* $\mathcal{S} = (\mathcal{L}, \mathcal{M})$ *be an* ADS *for which singleton satisfiability is undecidable and let* $\mathcal{E}$ *be a non-empty set of link symbols. Then the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{Q}(\mathcal{S}, \mathcal{B}_1)$ *is undecidable.*

PROOF. By Theorem 4.17, there exist ADS $\mathcal{S}$ for which singleton satisfiability is undecidable but which have a decidable satisfiability problem. Take any such $\mathcal{S}$. We prove the lemma by reducing singleton satisfiability in $\mathcal{S}$ to satisfiability in $\mathcal{C}^{\mathcal{E}}_{Q}(\mathcal{S}, \mathcal{B}_1)$: it is readily checked that an $\mathcal{L}$-term t is singleton satisfiable if and only if the set of $\mathcal{C}^{\mathcal{E}}_{Q}(\mathcal{S}, \mathcal{B}_1)$-assertions (consisting of a 1-assertion and a 2-assertion)

$$\{t \sqsubseteq \langle E\rangle^1 (\top_2)\,, \quad \top_2 \sqsubseteq \langle{=}\,1\,E\rangle^2 (t)\}$$

is satisfiable, where E is a link relation from $\mathcal{E}$ and $\langle{=}\,1\,E\rangle^i (t)$ is an abbreviation for $\langle\leq 1\,E\rangle^i (t) \wedge \langle\geq 1\,E\rangle^i (t)$. □

□

The intuitive reason for this 'negative' result is that number restrictions on links allow the transfer of 'counting capabilities' from one component to another. For example, in $\mathcal{C}^{\{E\}}_{Q}(\mathcal{SHIQ}^\sharp, \mathcal{ALCO}^\sharp)$, we can 'export' the nominals of $\mathcal{ALCO}^\sharp$ to $\mathcal{SHIQ}^\sharp$: the assertions

$$\top_2 = \langle\leq 1E\rangle^2 (\top_1),\ \ \top_2 = \langle\geq 1E\rangle^2 (\top_1),\ \ \top_1 = \langle\leq 1E\rangle^1 (\top_2),\ \ \top_1 = \langle\geq 1E\rangle^1 (\top_2)$$

state that E is a bijective function, and so we can use $\langle E\rangle^1 (\{a\})$, a an object variable of $\mathcal{ALCO}^\sharp$, as a nominal in $\mathcal{SHIQ}^\sharp$.

When introducing number restrictions on links, it is thus natural to confine ourselves to ADSs which, intuitively, 'cannot count'. Indeed, as the decidability transfer result from the next section shows, the fact that one of the ADSs used in the proof of Theorem 5.19 was not number tolerant, is essential.

5.4.2. Decidable $\mathcal{E}$-Connections with Number Restrictions. Fortunately, number tolerance is precisely what we need in order to preserve decidability in the presence of number restrictions on links.

The proof of the next result is similar to that of Theorem 5.1: we guess sets of 1-types and 2-types to be realised in a potential model. Additionally, for each i-type t we need to guess the number and type of witnesses for the link operators $\langle\geq rE\rangle^i(s)$ such that none of the link operators $\langle\leq rE\rangle^i(s)$ of t is violated. Similarly to the previous variants of $\mathcal{E}$-connections, we get a non-deterministic upper time bound for the satisfiability problem that is obtained by adding one exponential to the maximal time complexity of the component ADSs.

THEOREM 5.21 (TRANSFER OF DECIDABILITY FOR $\mathcal{C}^{\mathcal{E}}_{Q}$).
Let $\mathcal{S}_1, \dots, \mathcal{S}_n$ be number tolerant ADSs *with decidable satisfiability problems. Then the satisfiability problem is also decidable for any $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}_{Q}(\mathcal{S}_1, \dots, \mathcal{S}_n)$.*

PROOF. As in the proofs of Theorems 5.7 and 5.12, we restrict ourselves to two ADSs and a single link relation E. For simplicity, we will therefore write number restrictions as $\langle \geq r \rangle^i (s)$ and $\langle \leq r \rangle^i (s)$, thus omitting the link symbol E.

Here is the variant of Theorem 5.21 obtained by the two restrictions:

THEOREM 5.22. *Suppose that the satisfiability problems for the* ADSs *$\mathcal{S}_1$ and $\mathcal{S}_2$ are decidable and both, $\mathcal{S}_1$ and $\mathcal{S}_2$, are number tolerant. Then the satisfiability problem for the $\{E\}$-connection $\mathcal{C}^{\{E\}}_{Q}(\mathcal{S}_1, \mathcal{S}_2)$ is decidable as well.*

Fix two ADSs $\mathcal{S}_1 = (\mathcal{L}_1, \mathcal{M}_1)$ and $\mathcal{S}_2 = (\mathcal{L}_2, \mathcal{M}_2)$ with decidable satisfiability problems. Note that for any model $\mathfrak{M}$ of $\mathcal{C}^{\{E\}}_{Q}(\mathcal{S}_1, \mathcal{S}_2)$ and any i-term s of $\mathcal{S}_i$ ($i = 1, 2$) we have

$$(\langle \leq r \rangle^{\bar{i}} (s))^{\mathfrak{M}} = (\neg \langle \geq r+1 \rangle^{\bar{i}} (s))^{\mathfrak{M}} \text{ for all } r \in \mathbb{N}, \text{ and}$$
$$(\langle E \rangle^{\bar{i}} (s))^{\mathfrak{M}} = (\langle \geq 1 \rangle^{\bar{i}} (s))^{\mathfrak{M}}.$$

Therefore, without loss of generality we may assume that we do not have terms of the form $\langle \leq r \rangle^{\bar{i}} (s)$ and $\langle E \rangle^{\bar{i}} (s)$. Let us fix some notational conventions:

- We use $\mathsf{cl}_i(\Gamma)$, $i = 1, 2$, to denote the **closure** under negation of the set of i-terms occurring in Γ. Without loss of generality we can identify $\neg\neg t$ with t, and thus $\mathsf{cl}_i(\Gamma)$ is finite.
- For an i-term t, we define a **surrogate** $sur_i(t)$ as described on Page 142, but now replacing subterms s of the form $\langle \geq r \rangle^i (s')$ with surrogate variables x_s.
- For $i \in \{1, 2\}$, we use $deg_i(\Gamma)$ to denote the maximum number r such that $\langle \geq r \rangle^i (s) \in \mathsf{cl}_i(\Gamma)$, for some term s.
- Given domain elements $d \in W_i$ and $d' \in W_{\bar{i}}$ (or object variables a of $\mathcal{S}_i$ and b of $\mathcal{S}_{\bar{i}}$) we use the expression $[d, d']$ (or $[a, b]$) to denote the pair (d, d') (respectively, (a, b)), if $i = 1$, and the pair (d', d) (or (b, a)), if $i = 2$.

As observed on Page 151, without loss of generality we may assume that the ADSs $\mathcal{S}_1$ and $\mathcal{S}_2$ support assertions of the form $a = a'$ and $a \neq a'$, where a and a' are object names. Note that, since we do not allow the application of link operators on object variables, we cannot replace link assertions with object assertions as in the previous decidability transfer proofs. Hence, we will treat link assertions $(a, b) : E$ explicitly in this proof.

We can now reduce satisfiability for the connection $\mathcal{C}^{\{E\}}_{Q}(\mathcal{S}_1, \mathcal{S}_2)$ to satisfiability for the components $\mathcal{S}_1$ and $\mathcal{S}_2$.

LEMMA 5.23 (SATISFIABILITY CRITERION FOR $\mathcal{C}^{\{E\}}_{Q}(\mathcal{S}_1, \mathcal{S}_2)$).
Let Γ be a $\mathcal{C}^{\{E\}}_{Q}(\mathcal{S}_1, \mathcal{S}_2)$-knowledge base, where the $\mathcal{S}_i$ are number tolerant. Then Γ is satisfiable if and only if there are subsets

$$\Delta_1 \subseteq \mathfrak{C}_1(\Gamma) \quad \textit{and} \quad \Delta_2 \subseteq \mathfrak{C}_2(\Gamma)$$

and equivalence relations

$$\sim_1 \subseteq ob_1(\Gamma) \times ob_1(\Gamma) \quad \textit{and} \quad \sim_2 \subseteq ob_2(\Gamma) \times ob_2(\Gamma)$$

such that, for $i \in \{1, 2\}$, the following conditions are satisfied:

(1) *For each $t \in \Delta_i$, there exists a set $\mathcal{W}_t = \{(Z_1, \gamma_1), \ldots, (Z_{k(t)}, \gamma_{k(t)})\}$, where $Z_j \subseteq \Delta_{\bar{i}}$ and the γ_j are functions $\gamma_j : Z_j \longrightarrow \{1, \ldots, deg_i(\Gamma)\}$ such that, for $(Z_j, \gamma_j) \in \mathcal{W}_t$, we have the following:*
   (a) *For each term $\langle \geq r \rangle^i (s) \in \mathsf{cl}_i(\Gamma)$, we have*

$$\langle \geq r \rangle^i (s) \in t \quad \Longleftrightarrow \quad \sum_{\{t' \in Z_j | s \in t'\}} \gamma_j(t') \geq r;$$

   (b) *for each $t' \in Z_j$, there exists $(Z, \gamma) \in \mathcal{W}_{t'}$ such that $t \in Z$.*

(2) *For each equivalence class C of $\sim_i$, there exist a type $t_C \in \Delta_i$, a set of types $Z_C \subseteq \Delta_{\bar{i}}$, and a function $\gamma_C : Z_C \longrightarrow \{1, \ldots, deg_i(\Gamma)\}$ such that*
   (a) *for each term $\langle \geq r \rangle^i (s) \in \mathsf{cl}_i(\Gamma)$, we have*

$$\langle \geq r \rangle^i (s) \in t_C \iff \sum_{\{t' \in Z_C | s \in t'\}} \gamma_C(t') + \left| \{C' \in conn_\Gamma(C) \mid s \in t_{C'}\} \right| \geq r,$$

   where the set $conn_\Gamma(C)$ contains precisely those equivalence classes C' of $\sim_{\bar{i}}$ for which $[a, b] : E \in \Gamma$, for some $a \in C$ and $b \in C'$;
   (b) *for each $t' \in Z_C$, there is $(Z, \gamma) \in \mathcal{W}_{t'}$ such that $t_C \in Z$.*

(3) *The union Γ_i of the sets*
   – $\{sur_i(\bigvee \Delta_i) = \top_i\}$,
   – $\{a_t : sur_i(t) \mid t \in \Delta_i\}$,
   – $\{a = a' \mid a \sim_i a'\}$,
   – $\{a \neq a' \mid a \not\sim_i a'\}$,
   – $\{a : sur_i(t_{[a]_i}) \mid a \in ob_i(\Gamma)\}$,
   – $\{sur_i(s_1) \sqsubseteq sur_i(s_2) \mid (s_1 \sqsubseteq s_2) \in \Gamma$ *an i-term assertion*$\}$,
   – $\{R_j(a_1, \ldots, a_{m_j}) \mid R_j(a_1, \ldots, a_{m_j}) \in \Gamma$ *an i-object assertion*$\}$, *and*
   – $\{a : sur_i(s) \mid (a : s) \in \Gamma$ *an i-object assertion*$\}$,

   is $\mathcal{S}_i$-satisfiable, where $[a]_i$ denotes the equivalence class of a with respect to $\sim_i$ and a_t is a fresh object name from $\mathcal{X}_i(\Gamma)$ for each $t \in \Delta_i$.

PROOF. ($\Longrightarrow$) Let $\mathfrak{M} = \langle \mathfrak{W}_1, \mathfrak{W}_2, E^{\mathfrak{M}} \rangle$ be a model for Γ, where $\mathfrak{W}_1$ is based on the domain W_1 and $\mathfrak{W}_2$ is based on the domain W_2. We use $\mathfrak{M}$ to choose sets Δ_1 and Δ_2 and equivalence relations $\sim_1$ and $\sim_2$ satisfying the conditions given in the formulation of the theorem.

We start with some preliminaries. A domain element $d \in W_i$ is called **anonymous** if $d \neq a^{\mathfrak{M}}$ for all $a \in ob_i(\Gamma)$. For $i \in \{1,2\}$, $d \in W_i$, and $t' \in \mathfrak{C}_{\bar{i}}(\Gamma)$, define the abbreviations

$$\begin{aligned} t(d) &= \bigwedge\{s \in \mathsf{cl}_i(\Gamma) \mid d \in s^{\mathfrak{M}}\}; \\ R(d) &= \{d' \in W_{\bar{i}} \mid [d,d'] \in E^{\mathfrak{M}}\}; \\ P(d) &= \{t(d') \mid d' \in R(d)\}; \\ P_A(d) &= \{t(d') \mid d' \in R(d) \text{ is anonymous}\}; \\ c(d,t') &= \min\{deg_i(\Gamma), \left|\{d' \in R(d) \mid t(d') = t'\}\right|\}; \\ c_A(d,t') &= \min\{deg_i(\Gamma), \left|\{d' \in R(d) \mid t(d') = t' \text{ and } d' \text{ is anonymous}\}\right|\}. \end{aligned}$$

Then we set

- $\Delta_i = \{t(d) \mid d \in W_i\}$;
- $\sim_i = \{(a,b) \in ob_i(\Gamma) \times ob_i(\Gamma) \mid a^{\mathfrak{M}} = b^{\mathfrak{M}}\}$;
- $\mathcal{W}_t = \{(P(d), \gamma_d) \mid d \in W_i \text{ and } t(d) = t\}$ for each $t \in \Delta_i$, where

$$\gamma_d = \{t' \mapsto c(d,t') \mid t' \in P(d)\};$$

- $t_C = t(a^{\mathfrak{M}})$, with $a \in C$, for each equivalence class C of $\sim_i$;
- $Z_C = P_A(a^{\mathfrak{M}})$, with $a \in C$, for each equivalence class C of $\sim_i$;
- $\gamma_C = \{t' \mapsto c_A(a^{\mathfrak{M}}, t') \mid t' \in P_A(a^{\mathfrak{M}})\}$, with $a \in C$, for each equivalence class C of $\sim_i$.

Note that t_C, Z_C, and γ_C are well-defined by the definition of the relations $\sim_i$. It remains to show that these definitions satisfy conditions (1)–(3) from the formulation of the theorem. We only do this for $i = 1$, since the case $i = 2$ is symmetric.

1. Fix terms $\langle \geq r \rangle^1 (s) \in \mathsf{cl}_1(\Gamma)$, $t \in \Delta_1$, and a pair $(Z, \gamma) \in \mathcal{W}_t$. Then there is a $d \in W_1$ such that $t(d) = t$, $Z = P(d)$, and $\gamma = \gamma_d$. Let

$$\Sigma_d^s = \{d' \in W_2 \mid (d,d') \in E^{\mathfrak{M}} \text{ and } s \in t(d')\}.$$

By definition we have

$$\langle \geq r \rangle^1 (s) \in t \quad \Longleftrightarrow \quad d \in (\langle \geq r \rangle^1 (s))^{\mathfrak{M}} \quad \Longleftrightarrow \quad \left|\Sigma_d^s\right| \geq r.$$

By the definition of $P(d)$ and γ_d,

$$\sum_{\{t' \in Z \mid s \in t'\}} \gamma(t') = \left|\{d' \in W_2 \mid (d,d') \in E^{\mathfrak{M}} \text{ and } s \in t(d')\}\right|$$

if for all $t' \in Z$ with $s \in t'$ we have $\left|\{d' \in R(d) \mid t(d') = t'\}\right| < deg_i(\Gamma)$, and

$$\sum_{\{t' \in Z \mid s \in t'\}} \gamma(t') \geq deg_2(\Gamma) \geq r$$

otherwise. The latter case implies $\left|\Sigma_d^s\right| \geq deg_2(\Gamma) \geq r$. We thus obtain

$$\left|\Sigma_d^s\right| \geq r \quad \Longleftrightarrow \quad \sum_{\{t' \in Z \mid s \in t'\}} \gamma(t') \geq r,$$

which gives (1.a).

To prove (1.b), let $t' \in Z$. Then there exists a $d' \in W_2$ such that $(d, d') \in E^{\mathfrak{M}}$ and $t(d') = t'$. It is readily checked that $(P(d'), \gamma_{d'}) \in \mathcal{W}_{t'}$ is as required, i.e., $t \in P(d')$.

2. Fix an equivalence class C of $\sim_1$, an $a \in C$ and a term $\langle \geq r \rangle^1 (s) \in \mathrm{cl}_1(\Gamma)$. Let

$$\Sigma_a^s = \{d' \in W_2 \mid (a^{\mathfrak{M}}, d') \in E^{\mathfrak{M}} \text{ and } s \in t(d')\}.$$

As above, we have by definition that $\langle \geq r \rangle^1 (s) \in t_C$ if and only if $\left|\Sigma_a^s\right| \geq r$ and, moreover,

$$\begin{aligned} \left|\Sigma_a^s\right| \quad = \quad & \left|\{d' \in W_2 \mid (a^{\mathfrak{M}}, d') \in E^{\mathfrak{M}}, s \in t(d'), \text{ and } d' \text{ anonymous}\}\right| \; + \\ & \left|\{d' \in W_2 \mid (a^{\mathfrak{M}}, d') \in E^{\mathfrak{M}}, s \in t(d'), \text{ and } d' \text{ not anonymous}\}\right|. \end{aligned}$$

By the definition of P_A, c_A, $\sim_1$, Z_C, and γ_C, the sum

$$\sum_{\{t' \in Z_C \mid s \in t'\}} \gamma_C(t')$$

is equal to the former component of $\left|\Sigma_a^s\right|$ or is at least $deg_2(\Gamma)$. Further, by the definition of $\sim_1$ and t_C, the second component is equal to

$$\left|\{C' \in conn_\Gamma(C) \mid s \in t_{C'}\}\right|.$$

Thus, as in the proof of (1.a), we obtain

$$\left|\Sigma_a^s\right| \geq r \quad \Longleftrightarrow \quad \sum_{\{t' \in Z_C \mid s \in t'\}} \gamma_C(t') + \left|\{C' \in conn_\Gamma(C) \mid s \in t_{C'}\}\right| \geq r$$

which gives (2.a).

To prove (2.b), let $t' \in Z_C$. Then there is a $d' \in W_2$ such that $(a^{\mathfrak{M}}, d') \in E^{\mathfrak{M}}$ and $t(d') = t'$. It is readily checked that $(P(d'), \gamma_{d'}) \in \mathcal{W}_{t'}$ is as required, i.e., $t_C \in P(d')$.

3. Take the model $\mathfrak{W}_1$ and extend it as follows:

- for each surrogate variable x_s occurring in Γ_1 with s of the form $\langle \geq r \rangle^i (s')$, set $x_s^{\mathfrak{W}_1} = s^{\mathfrak{W}}$;
- for each newly introduced object name a_t (with $t \in \Delta_1$), set $a_t^{\mathfrak{W}_1}$ to some element of $t^{\mathfrak{W}}$.

Note that the resulting model $\mathfrak{W}_1'$ can be found in the set of models $\mathcal{M}_1$ by the closure conditions that are required to hold for $\mathcal{M}_1$. It is easy to prove by induction that, for all $d \in W_1$ and $s \in \mathsf{cl}_1(\Gamma)$, we have $d \in sur_1(s)^{\mathfrak{W}_1'}$ if and only if $d \in s^{\mathfrak{M}}$; details are left to the reader. Using this fact, in turn, it is straightforward to verify that $\mathfrak{W}_1'$ is a model of Γ_1.

($\Longleftarrow$) Suppose that there exist Δ_1, Δ_2, $\sim_1$, and $\sim_2$ satisfying the conditions of the theorem. Hence, there also exist sets $\mathcal{W}_t$, for $t \in \Delta_i$, and types t_C, sets of types Z_C, and functions γ_C, for equivalence classes C of $\sim_i$, satisfying conditions (1.a), (1.b) and (2.a), (2.b). Our aim is to construct a model satisfying Γ. For each ADS $\mathcal{S}_i$, $i = 1, 2$, let κ_i denote the cardinal number for $\mathcal{S}_i$ from the definition of 'number tolerance.' Take an infinite cardinal κ such that $\kappa \geq \kappa_i$, for $i = 1, 2$, and models $\mathfrak{W}_i \in \mathcal{M}_i$ with domains W_i satisfying Γ_i, for $i = 1, 2$. Let, for $d \in W_i$,

$$t(d) = \bigwedge \{s \in \mathsf{cl}_i(\Gamma) \mid d \in (sur_i(s))^{\mathfrak{W}_i}\}.$$

By the definition of the Γ_i, we clearly have $t(d) \in \Delta_i$ for each $d \in W_i$. Since $\mathcal{S}_1$ and $\mathcal{S}_2$ are number tolerant and $t \in \Delta_i$ implies the existence of some $d \in W_i$ such that $t(d) = t$ by the definition of the Γ_i, by the choice of κ we may assume that

$$(*) \qquad \big|\{d \in W_i \mid t(d) = t\}\big| = \kappa \text{ for each } t \in \Delta_i.$$

Again, a domain element $d \in W_i$ is called *anonymous* if $d \neq a^{\mathfrak{M}}$ for all $a \in ob_i(\Gamma)$. We now show that there exists a relation $E^{\mathfrak{M}} \subseteq W_1 \times W_2$ satisfying the following conditions:

(I) For all $a \in ob_1(\Gamma)$ and $b \in ob_2(\Gamma)$, we have

$$(a^{\mathfrak{W}_1}, b^{\mathfrak{W}_2}) \in E^{\mathfrak{M}} \quad \Longleftrightarrow \quad \text{there are } a' \in [a]_1, b' \in [b]_2 \text{ such that } (a', b') : E \in \Gamma.$$

(II) For all $i \in \{1, 2\}$ and $a \in ob_i(\Gamma)$, we have
- $[a^{\mathfrak{W}_i}, d'] \in E^{\mathfrak{M}}$ implies $t(d') \in Z_{[a]_i}$;
- for each $t \in Z_{[a]_i}$,

$$\gamma_{[a]_i}(t) = \big|\{d' \in W_{\bar{i}} \mid [a^{\mathfrak{W}_i}, d'] \in E^{\mathfrak{M}}, d' \text{ anonymous and } t(d') = t\}\big|.$$

(III) For all $i \in \{1, 2\}$ and $d \in W_i$, there exists a $(Z, \gamma) \in \mathcal{W}_{t(d)}$ such that
- $[d, d'] \in E^{\mathfrak{M}}$ implies $t(d') \in Z$;
- for each $t \in Z$,

$$\gamma(t) = \big|\{d' \in W_{\bar{i}} \mid [d, d'] \in E^{\mathfrak{M}} \text{ and } t(d') = t\}\big|.$$

Since there are only finitely many types $t \in \Delta_i$ and each t is of the form $t(d)$ for some $d \in W_i$, we have $|W_i| = |\Delta_i| \cdot \kappa = \kappa$. Hence, we can assume that the sets W_i are ordered by $<_i$ such that $(\kappa, \in)$ is order-isomorphic to $(W_i, <_i)$ (i.e., $<_i$ is a well-ordering on W_i

such that no $<_i$-initial subset of W_i is of cardinality κ). We construct the relation $E^{\mathfrak{M}}$ by transfinite induction as

$$E^{\mathfrak{M}} = \bigcup_{\alpha<\kappa} E^{\mathfrak{M}}_{\alpha},$$

and simultaneously define (partial) functions π_i^{α}, $\alpha < \kappa$, $i = 1, 2$, that take anonymous domain elements $d \in W_i$ to elements of $\mathcal{W}_{t(d)}$. We start with $\alpha = 0, 1$:

- Set $E^{\mathfrak{M}}_0 = \{(a^{\mathfrak{W}_1}, b^{\mathfrak{W}_2}) \mid (a, b) : E \in \Gamma\}$ and $\pi_1^0 = \pi_2^0 = \emptyset$.
- For all $i \in \{1, 2\}$, $a \in ob_i(\Gamma)$, $t \in Z_{[a]_i}$, and j, $1 \leq j \leq \gamma_{[a]_i}(t)$, choose an anonymous element $d_{a,t,j} \in W_{\bar{i}}$ with $t(d_{a,t,j}) = t$ such that $(a, t, j) \neq (a', t', j')$ implies $d_{a,t,j} \neq d_{a',t',j'}$—this is possible since $Z_{[a]_i} \subseteq \Delta_{\bar{i}}$ and in view of $(*)$. Then set, for each a, t, j as above, $\pi_i^1(d_{a,t,j})$ to some $(Z, \gamma) \in \mathcal{W}_t$ such that $t_{[a]} \in Z$, which exists by property (2.b). Further, set

$$E^{\mathfrak{M}}_1 = E^{\mathfrak{M}}_0 \cup \bigcup_{i\in\{1,2\}} \; \bigcup_{a\in ob_i(\Gamma)} \; \bigcup_{t\in Z_{[a]_i}} \; \bigcup_{1\leq j\leq\gamma_{[a]_i}(t)} \{[a^{\mathfrak{W}_i}, d_{a,t,j}]\}.$$

- Suppose that $\alpha < \kappa$ is the minimal ordinal for which $E^{\mathfrak{M}}_{\alpha}$ is not yet defined. If α is a limit ordinal set

$$E^{\mathfrak{M}}_{\alpha} = \bigcup_{\beta<\alpha} E^{\mathfrak{M}}_{\beta} \quad \text{and} \quad \pi_i^{\alpha} = \bigcup_{\beta<\alpha} \pi_i^{\beta} \text{ for } i = 1, 2.$$

  Now suppose that $\alpha = \alpha' + 1$. Let β be the largest limit ordinal which is smaller than α, or 0 if no such limit ordinal exists. If $\alpha = \beta + 2n$ for some natural number n, set $i = 1$. Otherwise set $i = 2$. Choose the $<_i$-minimal domain element $d \in W_i$ such that

  (i) $\pi_i^{\alpha'}(d)$ is undefined, or
  (ii) $\pi_i^{\alpha'}(d) = (Z, \gamma)$ and there is a $t' \in Z$ such that

$$\left|\{d' \in W_{\bar{i}} \mid [d, d'] \in E^{\mathfrak{M}}_{\alpha'} \text{ and } t(d') = t'\}\right| < \gamma(t').$$

  In case (i), set

$$E^{\mathfrak{M}}_{\alpha} = E^{\mathfrak{M}}_{\alpha'}, \quad \pi_i^{\alpha} = \pi_i^{\alpha'} \cup \{(d, (Z, \gamma))\}, \quad \pi_{\bar{i}}^{\alpha} = \pi_{\bar{i}}^{\alpha'},$$

  where (Z, γ) is an element of $\mathcal{W}_{t(d)}$. In case (ii), we do the following: choose an anonymous element $d' \in W_{\bar{i}}$ with $t(d') = t'$ and $[d, d'] \notin E^{\mathfrak{M}}_{\alpha'}$ such that $\pi_{\bar{i}}^{\alpha'}(d')$ is undefined—this is possible since $Z \subseteq \Delta_{\bar{i}}$ and by $(*)$. Then set

$$E^{\mathfrak{M}}_{\alpha} = E^{\mathfrak{M}}_{\alpha'} \cup \{[d, d']\}, \quad \pi_i^{\alpha} = \pi_i^{\alpha'}, \quad \pi_{\bar{i}}^{\alpha} = \pi_{\bar{i}}^{\alpha'} \cup \{(d', (Z, \gamma))\},$$

  for some $(Z, \gamma) \in \mathcal{W}_{t'}$ such that $t(d) \in Z$, which is possible by property (1.b).

It is not hard to verify that the relation $E^{\mathfrak{M}} = \bigcup_{\alpha<\kappa} E^{\mathfrak{M}}_{\alpha}$ constructed in this way indeed satisfies properties (I)–(III).

We now show that $\mathfrak{M} = \langle \mathfrak{W}_1, \mathfrak{W}_2, E^{\mathfrak{M}} \rangle$ is a model for Γ. Since $(a,b) : E \in \Gamma$ implies $(a^{\mathfrak{M}}, b^{\mathfrak{M}}) \in E^{\mathfrak{M}}$ by property (I) of $E^{\mathfrak{M}}$, it clearly suffices to show that

$$d \in sur_i(s)^{\mathfrak{W}_i} \quad \Longleftrightarrow \quad d \in s^{\mathfrak{M}}$$

for all $d \in W_i$, $s \in \mathrm{cl}_i(\Gamma)$, and $i \in \{1,2\}$, which can be done by simultaneous structural induction. The case of set variables and the Boolean cases are trivial, so we only consider the case $s = \langle \geq r \rangle^i (s')$ and $i = 1$.

Let $s = \langle \geq r \rangle^1 (s')$ for s' a 2-term. First assume that $d \in sur_1(s)^{\mathfrak{W}_1}$, i.e., $s \in t(d)$, and consider the case where d is not anonymous, i.e., there exists an $a \in ob_1(\Gamma)$ such that $a^{\mathfrak{M}} = d$. By condition (2.a), we then have

$$r \leq \sum_{\{t \in Z_{[a]_1} \mid s' \in t\}} \gamma_{[a]_1}(t) + \left| \{C' \in conn_\Gamma([a]_1) \mid s' \in t_{C'}\} \right|.$$

By the definitions of Γ_2 and of $\mathfrak{M}$, we have $b^{\mathfrak{M}} = b'^{\mathfrak{M}}$ if and only if $b \sim_2 b'$ for all $b, b' \in ob_2(\Gamma)$. Thus, property (I) of $E^{\mathfrak{M}}$ and the definition of Γ_2 yield

$$\begin{aligned}
&\left| \{d' \in W_2 \mid (d,d') \in E^{\mathfrak{M}},\ s' \in t(d'),\ \text{and } d' \text{ not anonymous}\} \right| = \\
&\left| \{[b]_2 \mid s' \in t_{[b]_2} \text{ and } (a',b') : E \in \Gamma \text{ for some } a' \in [a]_1, b' \in [b]_2\} \right| = \\
&\left| \{C' \in conn_\Gamma([a]_1) \mid s' \in t_{C'}\} \right|.
\end{aligned}$$

By property (II) of $E^{\mathfrak{M}}$, we have for each $t \in Z_{[a]_1}$:

$$\gamma_{[a]_1}(t) = \left| \{d' \in W_2 \mid (a^{\mathfrak{W}_1}, d') \in E^{\mathfrak{M}}, d' \text{ anonymous and } t(d') = t\} \right|.$$

Moreover, $(a^{\mathfrak{W}_1}, d') \in E^{\mathfrak{M}}$ implies $t(d') \in Z_{[a]_1}$. This yields

$$\left| \{d' \in W_2 \mid (d,d') \in E^{\mathfrak{M}} \text{ and } s' \in t(d')\} \right| \geq r.$$

Since, by the induction hypotheses, $s' \in t(d')$ if and only if $d' \in s'^{\mathfrak{M}}$, this yields $d \in s^{\mathfrak{M}}$, as required.

Now assume that $d \in sur_1(s)^{\mathfrak{W}_1}$ and d is anonymous. Then $s \in t(d)$, property (1.a), and property (III) of $E^{\mathfrak{M}}$ yield

$$\left| \{d' \in W_2 \mid (d,d') \in E^{\mathfrak{M}} \text{ and } s' \in t(d')\} \right| \geq r,$$

which is equivalent to $d \in s^{\mathfrak{M}}$, and we are done.

Conversely, assume that $d \in s^{\mathfrak{M}}$. By definition and the induction hypotheses, we have that

$$\left| \Sigma_d^{s'} \right| \geq r, \quad \text{where} \quad \Sigma_d^{s'} = \{d' \in W_2 \mid (d,d') \in E^{\mathfrak{M}} \text{ and } s' \in t(d')\}.$$

Assume first that $d = a^{\mathfrak{M}}$ for some $a \in ob_1(\Gamma)$. Clearly, for each $d' \in \Sigma_d^{s'}$ that is not anonymous, i.e., $b^{\mathfrak{M}} = d'$ for some $b \in ob_2(\Gamma)$, there are $a' \in [a]_1$ and $b' \in [b]_2$ such that

$(a', b') : E \in \Gamma$ and $s' \in t_{[b']_2}$, by condition (I) and the definition of Γ_2. By condition (II) of $E^{\mathfrak{M}}$ we further have that for all $d' \in \Sigma_d^{s'}$, $s' \in t(d') \in Z_{[a]_1}$ and for any $t \in Z_{[a]_1}$,

$$\gamma_{[a]_1}(t) = \left|\{d' \in W_2 \mid (a^{\mathfrak{W}_1}, d') \in E^{\mathfrak{M}}, d' \text{ anonymous and } t(d') = t\}\right|.$$

Hence

$$r \leq \left|\Sigma_d^{s'}\right| = \sum_{\{t \in Z_{[a]_1} | s' \in t\}} \gamma_{[a]_1}(t) + \left|\{C' \in conn_\Gamma([a]_1) \mid s' \in t_{C'}\}\right|.$$

By condition (2.a), $s \in t_{[a]_1}$. Since, by the definition of Γ_1, we have $t(d) = t_{[a]_1}$, this yields $d \in sur_1(s)^{\mathfrak{W}_1}$, as required.

Assume now that d is anonymous. By condition (III) of $E^{\mathfrak{M}}$ there exists some $(Z, \gamma) \in \mathcal{W}_{t(d)}$ such that $s' \in t(d') \in Z$ for all $d' \in \Sigma_d^{s'}$. As above we obtain

$$r \leq \left|\Sigma_d^{s'}\right| = \sum_{\{t \in Z | s' \in t\}} \gamma(t),$$

and so $s \in t(d)$ by condition (1.a), which completes the proof of Lemma 5.23. □

Assuming that there exist decision procedures for $\mathcal{S}_1$ and $\mathcal{S}_2$, it is now easy to use Lemma 5.23 to derive a decision procedure for the connection $\mathcal{C}_Q^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$. Since the sets $\mathfrak{C}_i(\Gamma)$ are finite, to decide whether a set Γ of $\mathcal{C}_Q^{\{E\}}(\mathcal{S}_1, \mathcal{S}_2)$-assertions is satisfiable, we may 'guess' sets $\Delta_1 \subseteq \mathfrak{C}_1(\Gamma)$ and $\Delta_2 \subseteq \mathfrak{C}_2(\Gamma)$, equivalence relations $\sim_1$ and $\sim_2$, sets $\mathcal{W}_t$ for each $t \in \Delta_1 \cup \Delta_2$, and types t_C, sets Z_C, and functions γ_C for each equivalence class C of $\sim_1$ and $\sim_2$, and then check whether they satisfy the conditions listed in the formulation of the theorem.

□

The time complexity of the obtained decision procedure is the same as in the previous two transfer theorems: it is one exponential higher than the complexity of the original decision procedures for $\mathcal{S}_1$- and $\mathcal{S}_2$-satisfiability. Moreover, the decision procedure for the connection is non-deterministic.

Now, for example, the $\mathcal{E}$-connection $\mathcal{C}_Q^{\mathcal{E}}(\mathcal{SHIQ}^\sharp, \mathsf{S4}_u^\sharp)$ is decidable, since both components are number tolerant. But then it is a natural question to ask whether an $\mathcal{E}$-connection allowing number restrictions on links can be extended by also allowing link operators on objects variables and/or Boolean operators on links, without losing the transfer of decidability.

Unfortunately, this is not the case.

5.4.3. $\mathcal{E}$-Connections of Type $\mathcal{C}_{QB}^{\mathcal{E}}$ and $\mathcal{C}_{QO}^{\mathcal{E}}$. Adding Boolean operators on links to a decidable $\mathcal{E}$-connection of type $\mathcal{C}_Q^{\mathcal{E}}$, or additionally allowing the application of link operators on object variables, leads, in general, to undecidability. The proof of the following theorem is similar to the proof of Theorem 5.19:

THEOREM 5.24 (FAILURE OF TRANSFER FOR $\mathcal{C}^{\mathcal{E}}_{QB}$ AND $\mathcal{C}^{\mathcal{E}}_{QO}$).

(i) *There exist number tolerant* ADSs $\mathcal{S}_1, \mathcal{S}_2$ *with decidable satisfiability problems such that the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{QB}(\mathcal{S}_1, \mathcal{S}_2)$ *is undecidable even if* $\mathcal{E}$ *is a singleton.*

(ii) *There exist number tolerant* $\mathcal{S}_1, \mathcal{S}_2$ *with decidable satisfiability problems such that the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{QO}(\mathcal{S}_1, \mathcal{S}_2)$ *is undecidable even if* $\mathcal{E}$ *is a singleton.*

PROOF. As in the proof of Theorem 5.19, we give a technical lemma similar to Lemma 5.20 from which, together with Theorem 4.17, the result follows:

LEMMA 5.25.

(i) *Let* $\mathcal{S}_1 = (\mathcal{L}, \mathcal{M})$ *be an* ADS *for which singleton satisfiability is undecidable and* $\mathcal{E}$ *a non-empty set of link symbols. Then the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{QB}(\mathcal{S}_1, \mathcal{S}_2)$ *is undecidable for any* ADS $\mathcal{S}_2$.

(ii) *Let* $\mathcal{S}_1 = (\mathcal{L}, \mathcal{M})$ *be an* ADS *for which singleton satisfiability is undecidable and* $\mathcal{E}$ *a non-empty set of link symbols. Then the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{QO}(\mathcal{S}_1, \mathcal{S}_2)$ *is undecidable for any* ADS $\mathcal{S}_2$.

PROOF. As the proofs for (i) and (ii) are similar to the proof of Lemma 5.20, we give only the set of reduction assertions which varies with the type of $\mathcal{E}$-connection under consideration.

For a proof of (i) we use the following set, which consists only of a single 2-assertion:

$$\{b : (\langle E\rangle^2 (t) \wedge \langle{=}1\, E\rangle^2 (\top_1) \wedge \neg \langle\neg E\rangle^2 (t))\}.$$

To prove (ii), we can use the following set of assertions (one 1-assertion and one 2-assertion):

$$\{t \sqsubseteq \langle E\rangle^1 (b)\ , \quad b : \langle{=}1\, E\rangle^2 (t)\}.$$

Here, b is an object variable from $\mathcal{L}_2$. □

□

Thus, by Theorem 5.24, if number restrictions on links are important for a certain application, we have to disallow Boolean operators on link relations and the use of link operators on object variables in order to avoid the danger of possibly using an undecidable $\mathcal{E}$-connection. This, of course, does not mean that any *specific* $\mathcal{E}$-connection of type $\mathcal{C}^{\mathcal{E}}_{QBO}(\mathcal{S}_1, \mathcal{S}_2)$ is undecidable, but that a proof of its decidability requires some extra effort.

CHAPTER 6

Expressivity, Link Constraints, and DDL

This chapter studies a number of questions related to the expressivity of basic $\mathcal{E}$-connections. In Section 6.1, we show that while basic $\mathcal{E}$-connections are expressive enough to simulate distributed description logics (DDLs) without so-called *complete individual correspondences*, to be able to treat DDL in general within the framework of $\mathcal{E}$-connections we have to add a new kind of assertion to basic $\mathcal{E}$-connections. We provide a number of transfer results for DDLs and show where transfer of decidability fails in general.

In Section 6.2, we study the expressivity of $\mathcal{E}$-connections more systematically by defining a notion of *bisimulation* for $\mathcal{E}$-connections which enables us to give a number of examples of properties of $\mathcal{E}$-connections that are not definable in the basic language. We take this as an opportunity to study so-called *link constraints* in Section 6.3, which allow us to express some of those undefinable properties without losing the general transfer of decidability.

Finally, in Section 6.4, we will compare the methodology of $\mathcal{E}$-connections with related combination techniques for logics, namey with multi-dimensional formalism, independent fusions, fibrings, and description logics with concrete domains.

6.1. $\mathcal{E}$-Connections and Distributed Description Logics

Let us recall the knowledge base regulating relations between people in the EU from Section 4.5.3. We proposed to employ the $\mathcal{E}$-connection $\mathcal{C}(\mathcal{SHIQ}^\sharp, \mathcal{ALCO}^\sharp)$: the $\mathcal{SHIQ}^\sharp$ component was used to talk about people and their relations, and the $\mathcal{ALCO}^\sharp$ component to talk about the EU countries. Apart from computational considerations, there is another important motivation for such a separation of various aspects of a large application: we may think of the components as independently maintained databases which are constantly updated, systematically linked, and import information from each other. This leads us to a discussion of distributed DLs (DDLs) introduced by Borgida and Serafini [2002] and further studied in Borgida and Serafini [2003], who observed that in some cases functional correspondences between different information systems are not enough to capture important information and who provided a number of examples illustrating this point. They also stressed that, unlike other approaches relating databases, a suitable logic-based approach enlarges the possible inferences we may draw from a combined knowledge base.

In this section, we show that the distributed description logics of Borgida and Serafini can be regarded as a special case of $\mathcal{E}$-connections linking a finite number of DLs. In what follows, all DLs are considered as their ADS representations.

6.1.1. The DDL Formalism. We start with a brief, but self-contained, description of the DDL formalism. Suppose that n description logics $DL_1, \ldots, DL_n$ are given. A sequence $\mathfrak{D} = (DL_i)_{i \le n}$ is then called a **distributed description logic (DDL)**. We use subscripts to indicate that some concept C_i belongs to the language of the description logic DL_i. Two types of assertions—bridge rules and individual correspondence—are used to establish interconnections between the components of a DDL.

DEFINITION 6.1 (BRIDGE RULES). *Let C_i and C_j be concepts from DL_i and DL_j, respectively. A **bridge rule** is an expression of the form*

(into rule) $$C_i \xrightarrow{\sqsubseteq} C_j$$

or of the form

(onto rule) $$C_i \xrightarrow{\sqsupseteq} C_j.$$

*Let a_i be an object name of DL_i and b_j, $b_j^1, \ldots, b_j^n$ object names of DL_j. A **partial individual correspondence** is an expression of the form*

(PIC) $$a_i \mapsto b_j.$$

*A **complete individual correspondence** is an expression of the form*

(CIC) $$a_i \overset{=}{\mapsto} \{b_j^1, \ldots, b_j^n\}.$$

*A **distributed TBox** $\mathfrak{T}$ consists of TBoxes T_i of DL_i together with a set of bridge rules. A **distributed ABox** $\mathfrak{A}$ consists of ABoxes A_i of DL_i together with a set of partial and complete individual correspondences. A **distributed knowledge base** is a pair $(\mathfrak{T}, \mathfrak{A})$.*

The semantics of distributed knowledge bases is defined as follows.

DEFINITION 6.2 (SEMANTICS FOR DDL). *A **distributed interpretation** $\mathfrak{I}$ of a distributed knowledge base $(\mathfrak{T}, \mathfrak{A})$ as above is a pair $(\{\mathfrak{I}_i\}_{i \le n}, \mathcal{R})$, where each $\mathfrak{I}_i$ is a model for the corresponding DL_i and $\mathcal{R}$ is a function associating with every pair (i, j), $i \neq j$, a binary relation $r_{ij} \subseteq W_i \times W_j$ between the domains W_i and W_j of $\mathfrak{I}_i$ and $\mathfrak{I}_j$, respectively. Given a point $u \in W_i$ and a subset $U \subseteq W_i$, we set*

$$r_{ij}(u) = \{v \in W_j \mid (u, v) \in r_{ij}\}, \qquad r_{ij}(U) = \bigcup_{u \in U} r_{ij}(u).$$

The ***truth-relation*** *is standard for formulae of the component DLs. For bridge rules and individual correspondences it is defined as follows:*

- $\mathfrak{I} \models C_i \xrightarrow{\sqsubseteq} C_j \iff r_{ij}(C_i^{\mathfrak{I}}) \subseteq C_j^{\mathfrak{I}}$;
- $\mathfrak{I} \models C_i \xrightarrow{\sqsupseteq} C_j \iff r_{ij}(C_i^{\mathfrak{I}}) \supseteq C_j^{\mathfrak{I}}$;
- $\mathfrak{I} \models a_i \mapsto b_j \iff b_j^{\mathfrak{I}} \in r_{ij}(a_i^{\mathfrak{I}})$;
- $\mathfrak{I} \models a_i \overset{=}{\mapsto} \{b_j^1, \ldots, b_j^n\} \iff r_{ij}(a_i^{\mathfrak{I}}) = \{(b_j^1)^{\mathfrak{I}}, \ldots, (b_j^n)^{\mathfrak{I}}\}$.

As usual, $\mathfrak{T} \models C \sqsubseteq D$ means that for every distributed interpretation $\mathfrak{I}$, if $\mathfrak{I} \models \varphi$ for all $\varphi \in \mathfrak{T}$, then $\mathfrak{I} \models C \sqsubseteq D$. The same definition applies to ABoxes $\mathfrak{A}$ and individual assertions.

It is of interest to note that, unlike $\mathcal{E}$-connections, DDLs do not provide new concept-formation operators to link the components of the DDL: both bridge rules and individual correspondences are assertions, and so atoms of knowledge bases, but not part of the concept language.

The satisfiability problem for distributed knowledge bases without complete individual correspondences (CIC) is easily reduced to the satisfiability problem for basic $\mathcal{E}$-connections. Indeed, fix a DDL $\mathfrak{D} = (DL_i)_{i \leq n}$ and associate with it the $\mathcal{E}$-connection $\mathfrak{D}^\sharp = \mathcal{C}^{\mathcal{E}}(DL_1^\sharp, \ldots, DL_n^\sharp)$, where $\mathcal{E} = \{E_{ij} \mid i,j \leq n, i \neq j\}$ consists of $n \times (n-1)$ many n-ary relations. To define a translation $\cdot^\sharp$ of $\mathfrak{D}$-assertions into $\mathfrak{D}^\sharp$-assertions, we mainly have to take care of the fact that DDL relations are binary, while $\mathcal{E}$-connection links are n-ary.

DEFINITION 6.3 (TRANSLATION). *Suppose that $\mathfrak{K} = (\mathfrak{T}, \mathfrak{A})$ is a distributed knowledge base for $\mathfrak{D} = (DL_i)_{i \leq n}$ without complete individual correspondences. We define a translation $.^\sharp$ from $\mathfrak{D}$-assertions to $\mathfrak{D}^\sharp$-assertions as follows:*

- *if φ is neither a bridge rule nor an individual correspondence, then $\varphi^\sharp$ is defined by translating the concepts in φ using the $.^\sharp$ translation from Section 4.2.2;*
- $(C_i \xrightarrow{\sqsubseteq} C_j)^\sharp = \langle E_{ij} \rangle^j (\top_1, \ldots, C_i^\sharp, \ldots, \top_n) \sqsubseteq C_j^\sharp$;
- $(C_i \xrightarrow{\sqsupseteq} C_j)^\sharp = \langle E_{ij} \rangle^j (\top_1, \ldots, C_i^\sharp, \ldots, \top_n) \sqsupseteq C_j^\sharp$;
- $(a_i \mapsto a_j)^\sharp = (a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n) : E_{ij}$,
  where a_k, for $k \neq i, j$, are fresh object variables of DL_k.

Finally, we put $\mathfrak{T}^\sharp = \{\varphi^\sharp \mid \varphi \in \mathfrak{T}\}$, $\mathfrak{A}^\sharp = \{\varphi^\sharp \mid \varphi \in \mathfrak{A}\}$ and $\mathfrak{K}^\sharp = \mathfrak{T}^\sharp \cup \mathfrak{A}^\sharp$.

Note that we only need simple link assertions to translate partial individual correspondences: no application of link operators to object variables is required. The theorem below follows now easily from the definition of the translation $\cdot^\sharp$:

THEOREM 6.4 (REDUCTION). *A distributed knowledge base $\mathfrak{K}$ for a DDL $\mathfrak{D}$ without complete individual correspondences is satisfiable if and only if $\mathfrak{K}^\sharp$ is satisfiable in a model of the basic $\mathcal{E}$-connection $\mathfrak{D}^\sharp$.*

PROOF. Suppose first that $\mathfrak{K}$ is satisfied in the interpretation $\mathfrak{I} = (\{\mathcal{I}_i\}_{i<n}, \mathcal{R})$ of DDL $\mathfrak{D} = (\mathrm{DL_i})_{\mathrm{i<n}}$. Define a model $\mathfrak{M}$ of the associated $\mathcal{E}$-connection $\mathfrak{D}^\sharp$ by defining, for $\mathcal{R}((i,j)) = r_{ij} \subseteq W_i \times W_j$ and all $u_1 \in W_1, \ldots, u_n \in W_n$:

$$(u_1, \ldots, u_i, \ldots, u_j, \ldots, u_n) \in E_{ij} :\Longleftrightarrow (u_i, u_j) \in r_{ij}.$$

We claim that for any $\varphi \in \mathfrak{K}$, $\mathfrak{M} \models \varphi^\sharp$. Indeed, if φ is in some DL_k, the claim is trivial since the local interpretations are the same. Due to the definition of E_{ij}, we have that

$$(u_i, u_j) \in r_{ij} \iff \forall_{k \neq i,j} u_k \in W_k : (u_1, \ldots, u_i, \ldots, u_j, \ldots, u_n) \in E_{ij} \tag{12}$$

$$\iff \exists_{k \neq i,j} u_k \in W_k : (u_1, \ldots, u_i, \ldots, u_j, \ldots, u_n) \in E_{ij}. \tag{13}$$

Hence, for any concept C_i of DL_i,

$$r_{ij}(C_i^{\mathfrak{I}}) = \{u_j \mid \exists u_i \in C_i^{\mathfrak{I}} \text{ such that } (u_i, u_j) \in r_{ij}\} = (\langle E_{ij} \rangle^j (\top_1, \ldots, C_i, \ldots, \top_n))^{\mathfrak{M}}$$

by equivalence (13). This shows the case of bridge rules. Finally, suppose $\varphi = a_i \mapsto a_j$ is a partial individual correspondence in $\mathfrak{K}$. Then $\mathfrak{I} \models \varphi$ implies $\mathfrak{M} \models \varphi^\sharp$ by (12).

Conversely, suppose that $\mathfrak{K}^\sharp$ is satisfied in a model $\mathfrak{M}$. We define the relations r_{ij} of the associated DDL by letting:

$$(u_i, u_j) \in r_{ij} :\Longleftrightarrow \exists_{k \neq i,j} u_k \in W_k : (u_1, \ldots, u_i, \ldots, u_j, \ldots, u_n) \in E_{ij}.$$

This defines an interpretation $\mathfrak{I}$ for the DDL in the obvious way. We claim that for any $\varphi \in \mathfrak{K}, \mathfrak{I} \models \varphi$. The proof is as above with the exception that we need (13) to prove the implication for (PIC). □

COROLLARY 6.5. *The satisfiability problem for DDLs $(DL_i)_{i \leq n}$ without complete individual correspondences is decidable whenever the satisfiability problem for ABoxes relative to TBoxes is decidable for each of the DL_i.*

Unfortunately, complete individual correspondences cannot be translated into basic $\mathcal{E}$-connections, and Corollary 6.5 does not hold for arbitrary distributed description logics with knowledge bases including complete individual correspondences. To be able to deal with these as well, we introduce another extension of $\mathcal{E}$-connections.

6.1.2. Complete Individual Correspondence in $\mathcal{E}$-Connections. In this section, we extend the basic $\mathcal{E}$-connections of n ADSs with an analogue of complete individual correspondences.

DEFINITION 6.6 ($\mathcal{E}$-CONNECTIONS WITH CIC).
Suppose that $\mathcal{S}_i = (\mathcal{L}_i, \mathcal{M}_i)$, $1 \leq i \leq n$, *are* ADSs *and that* $\mathcal{E} = \{E_j \mid j \in J\}$ *is a set of n-ary relation symbols. We denote by*

$$\mathcal{C}^{\mathcal{E}}_{I}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$$

the $\mathcal{E}$*-connection in which the set of i-object assertions is extended with **(CIC) i-assertions** of the form*

$$\langle E_k \rangle^i (a_j) = B_i,$$

where $1 \leq i \leq n$, $k \in J$, B_i *is a finite set of object variables of* $\mathcal{L}_i$*, and* a_j *is an object variable of* $\mathcal{L}_j$*, for some* $j \neq i$.

The truth-relation for the new assertions is defined as follows. Given an ADM

$$\mathfrak{M} = \left\langle (\mathfrak{W}_i)_{i \leq n}, \mathcal{E}^{\mathfrak{M}} \right\rangle,$$

we put

$$\mathfrak{M} \models \langle E_k \rangle^i (a_j) = B_i \iff$$
$$\{x_i \in W_i \mid \exists_{l \neq i,j} x_l \in W_l (x_1, \ldots, a_j^{\mathfrak{M}}, \ldots, x_n) \in E_k^{\mathfrak{M}}\} = \{b_i^{\mathfrak{M}} \mid b_i \in B_i\}.$$

Obviously, (CIC) i-assertions of the form $\langle E_k \rangle^i (a_j) = B_i$ can be expressed in the basic $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ if all its components have nominals: if $B_i = \{b_i^1, \ldots, b_i^r\}$ then $\langle E_k \rangle^i (a_j) = B_i$ is equivalent to

$$\langle E_k \rangle^i (\top_1, \ldots, \{a_j\}, \ldots, \top_n) = \{b_i^1\} \sqcup \cdots \sqcup \{b_i^r\}.$$

Therefore, as a consequence of Theorem 5.12 we obtain:

THEOREM 6.7 (CIC AND NOMINALS). *Suppose that* $\mathcal{S}_1, \ldots, \mathcal{S}_n$ *are* ADSs *with decidable satisfiability problems and that each of them has nominals. Then the satisfiability problem for any* $\mathcal{E}$*-connection* $\mathcal{C}^{\mathcal{E}}_{OBI}(\mathcal{S}_1, \ldots, \mathcal{S}_n)$ *is decidable as well.*

Moreover, there exists a connection to number restrictions on links: if we consider the connection of two ADSs $\mathcal{S}_1$ and $\mathcal{S}_2$, then $\langle E_k \rangle^1 (a) = B$, where a is an object variable of $\mathcal{S}_2$ and $B = \{b_1^1, \ldots, b_1^r\}$ is a set of object variables of $\mathcal{S}_1$, is equivalent to the set of object assertions

$$\{(b_1^1, a) : E_k, \ldots, (b_1^r, a) : E_k,\ a : (\leq r E_k)^2 \top_1\}$$

if we adopt the **unique name assumption** (**UNA**), i.e., assume that $(b_k^i)^{\mathfrak{M}} \neq (b_k^j)^{\mathfrak{M}}$ for any distinct b_k^i and b_k^j and any model $\mathfrak{M}$. It should be clear that this assumption can be made without loss of generality: reasoning without UNA can be reduced to reasoning with UNA by first 'guessing' an equivalence relation on the set of object names of each $\mathcal{S}_i$, then choosing a representative of each equivalence class, and finally replacing each object name with the representative of its class. We thus obtain from Theorem 5.21:

THEOREM 6.8 (CIC AND NUMBER RESTRICTIONS). *Let* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *be number tolerant* ADSs *with decidable satisfiability problems. Then the satisfiability problem for any* $\mathcal{E}$*-connection* $\mathcal{C}^{\mathcal{E}}_{QI}(\mathcal{S}_1, \mathcal{S}_2)$ *is decidable as well.*

Let us now transfer these results to distributed description logics. Obviously, the translation $\cdot^\sharp$ can be extended to a map from distributed knowledge bases which possibly contain CICs into the set of assertions of the corresponding $\mathcal{E}$-connection by taking

- $(a_i \overset{=}{\mapsto} \{b_j^1, \ldots, b_j^n\})^\sharp = \langle E_{ij} \rangle^j (a_i) = \{b_j^1, \ldots, b_j^n\}$.

We then obtain the following transfer results for DDLs:

COROLLARY 6.9 (TRANSFER RESULTS FOR DDL).
(i) *The satisfiability problem for DDLs* $\mathfrak{D} = (DL_i)_{i \leq n}$ *is decidable whenever the satisfiability problem for ABoxes relative to TBoxes is decidable for each of the* DL_i*, and all of them have nominals.*

(ii) *The satisfiability problem for distributed description logics* $\mathfrak{D} = (DL_1, DL_2)$ *is decidable whenever the satisfiability problem for ABoxes relative to TBoxes is decidable for each of the* DL_i*, and both of them are number tolerant.*

Although we were able to identify some natural cases in which decidability transfers from $\mathcal{S}_1, \mathcal{S}_2$ to $\mathcal{C}^{\mathcal{E}}_{I}(\mathcal{S}_1, \mathcal{S}_2)$, the transfer of decidability fails in general. The proof of the following theorem is similar to the proofs of Theorems 5.19 and 5.24:

THEOREM 6.10 (CIC AND UNDECIDABILITY).
(i) *There exist* ADS $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *with decidable satisfiability problems such that the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{I}(\mathcal{S}_1, \mathcal{S}_2)$ *is undecidable even if* $\mathcal{E}$ *is a singleton.*

(ii) *There exist number tolerant* $\mathcal{S}_1, \mathcal{S}_2$ *with decidable satisfiability problems such that the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{IB}(\mathcal{S}_1, \mathcal{S}_2)$ *is undecidable even if* $\mathcal{E}$ *is a singleton.*

(iii) *There exist number tolerant* $\mathcal{S}_1, \mathcal{S}_2$ *with decidable satisfiability problems such that the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{IO}(\mathcal{S}_1, \mathcal{S}_2)$ *is undecidable even if* $\mathcal{E}$ *is a singleton.*

PROOF. Again, as in Theorem 5.19, this follows from Lemma 4.17 that states that there exist (number tolerant) ADSs with an undecidable singleton satisfiability problem, together with the following technical lemma. The definition of the ADS $\mathcal{B}_1$ can be found on Page 161.

LEMMA 6.11.
(i) *Let* $\mathcal{S} = (\mathcal{L}, \mathcal{M})$ *be an* ADS *for which singleton satisfiability is undecidable and* $\mathcal{E}$ *a non-empty set of link symbols. Then the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{I}(\mathcal{S}, \mathcal{B}_1)$ *is undecidable.*

(ii) *Let* $\mathcal{S}_1 = (\mathcal{L}, \mathcal{M})$ *be an* ADS *for which singleton satisfiability is undecidable and* $\mathcal{E}$ *a non-empty set of link symbols. Then the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{IB}(\mathcal{S}_1, \mathcal{S}_2)$ *is undecidable for any* ADS $\mathcal{S}_2$.
(iii) *Let* $\mathcal{S}_1 = (\mathcal{L}, \mathcal{M})$ *be an ADS for which singleton satisfiability is undecidable and* $\mathcal{E}$ *a non-empty set of link symbols. Then the satisfiability problem for* $\mathcal{C}^{\mathcal{E}}_{IO}(\mathcal{S}_1, \mathcal{S}_2)$ *is undecidable for any* ADS $\mathcal{S}_2$.

PROOF. The proofs of (i), (ii) and (iii) are similar to the proof of Lemma 5.20, whence we only give the respective reduction sets:

(i) We can use the following set of assertions (three 1-assertions):

$$\{a : t\,, \quad t \sqsubseteq \langle E \rangle^1 (\top_2)\,, \quad \langle E \rangle^1 b = \{a\}\}.$$

Here, the last assertion is a CIC assertion, a is an object variable from $\mathcal{L}_1$, and b is an object variable from $\mathcal{L}_{\mathcal{B}}$.

(ii) Use the following set of assertions (two 1-assertions and one 2-assertion):

$$\{a : t\,, \quad \langle E \rangle^1 (b) = \{a\}\,, \quad b : \neg \langle \neg E \rangle^2 (t)\}.$$

Here, a is an object variable from $\mathcal{L}_1$ and b is an object variable from $\mathcal{L}_2$.

(iii) Use the following set of assertions (three 1-assertions):

$$\{a : t\,, \quad t \sqsubseteq \langle E \rangle^1 (b)\,, \quad \langle E \rangle^1 (b) = \{a\}\}.$$

Again, a is an object variable from $\mathcal{L}_1$ and b is such a variable from $\mathcal{L}_2$. □

□

6.2. Expressivity of $\mathcal{E}$-Connections

Given that the basic $\mathcal{E}$-connection of any finite number of decidable ADSs is decidable as well (Theorem 5.1), it is clear that the interaction between the components has to be rather limited. Yet, it is not obvious what exactly can and what cannot be expressed in the combined language. In Section 4.5, we have gone into great depth to provide examples of potentially useful $\mathcal{E}$-connections. This section is devoted to shedding some light on the question of expressivity.

Recall the example from Section 4.5.3, where we constructed a knowledge base containing information about relationships between people, companies etc. and countries in the EU, based on an $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}(\mathcal{SHIQ}^\sharp, \mathcal{ALCO}^\sharp)$. We used $\mathcal{SHIQ}$ to represent knowledge about people etc., $\mathcal{ALCO}$ to talk about countries, and used link relations being interpreted as, e.g., 'has citizenship in'.

Suppose we want to extend this knowledge base with the following information:[1]

(1) 'Children have the citizenship of their parents';

(2) 'If a company cooperates with another company then the countries from which they operate have diplomatic relations'.

Assume we have link relations C for 'having citizenship in' and O for 'operating from', as well as roles $\mathsf{has\_child}$ and $\mathsf{cooperate}$ of $\mathcal{SHIQ}$, and a role $\mathsf{diplomatic}$ of $\mathcal{ALCO}$. Then these constraints can easily be expressed in the language of first-order logic, compare Figure 6.1, as:

(1)′ $\forall x \forall y \forall z \,\big((x \; \mathsf{has\_child} \; y \wedge x\, C\, z) \rightarrow y\, C\, z)\big);$

(2)′ $\forall x \forall y \forall x' \forall y' \big((x \; \mathsf{cooperate} \; y \wedge x\, O\, x' \wedge y\, O\, y') \rightarrow x' \; \mathsf{diplomatic} \; y'\big).$

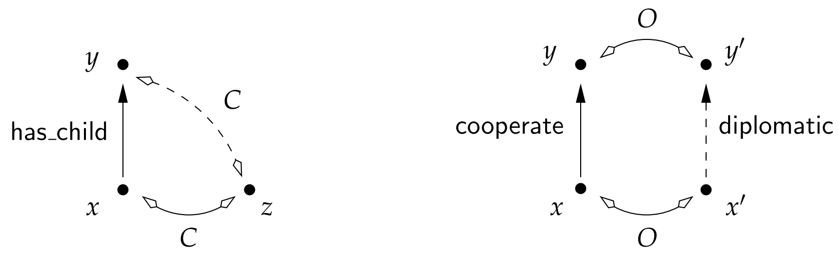


Figure 6.1: Undefinable Properties

Unfortunately, as we will show below, the basic $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}(\mathcal{SHIQ}^\sharp, \mathcal{ALCO}^\sharp)$ is not expressive enough to enforce these conditions. Here, we will work with the following definition of **definability** in $\mathcal{E}$-connections, where by **property** we shall mean any condition specified in the $\mathcal{E}$-connections analogue of a first-order correspondence language, compare Page 83.

DEFINITION 6.12 (DEFINABILITY IN $\mathcal{E}$-CONNECTIONS). *Let $\mathcal{C}$ be an $\mathcal{E}$-connection. A property $\mathcal{P}$ of models of $\mathcal{C}$ is called **definable in** $\mathcal{C}$ if there exists a finite set Γ of assertions of $\mathcal{C}$ such that, for all models $\mathfrak{M}$ of $\mathcal{C}$, the following holds:*

$$\mathfrak{M} \text{ has } \mathcal{P} \iff \mathfrak{M} \models \Gamma.$$

As is well known, undefinability results in modal logic—such as the undefinability of the irreflexivity of a Kripke frame—are usually gained by the concept of **bisimulation**, compare for instance Proposition 2.6 on Page 43. In what is to follow, we first lift

[1]These constraints are not true empirically: for instance, (1) is false according to US law, but true according to German law. Nevertheless, they can serve as prescriptions or regulations, e.g., of the new EU constitution.

this concept to cover ADSs, and then generalise bisimulations to $\mathcal{E}$-connections. Finally, we are able to derive that properties like (1) and (2) from above are not definable properties of $\mathcal{E}$-connections.

6.2.1. Bisimulations for ADSs. As ADSs abstract from the concrete definition of a given logic, it is difficult to come up with a notion of bisimulation that is non-trivial in the sense that it reflects certain properties of the logics under investigation—as is the case with bisimulations for modal logics, where the semantic definition of modal operators is reflected in the definition of bisimulations.

So let us use a rather straightforward definition of bisimulation that simply pins down exactly what is needed to ensure that two models are indistinguishable.[2]

DEFINITION 6.13 (BISIMULATIONS FOR ADSs). *Let* $(\mathcal{L}, \mathcal{M})$ *be an* ADS, *and let*

$$\mathfrak{W}_k = \left\langle W_k, \mathcal{V}^{\mathfrak{W}_k}, \mathcal{X}^{\mathfrak{W}_k}, \mathcal{F}^{\mathfrak{W}_k}, \mathcal{R}^{\mathfrak{W}_k} \right\rangle,$$

$k = 1, 2$, *be two abstract description models from the class* $\mathcal{M}$.

We say that $\mathfrak{W}_1$ *and* $\mathfrak{W}_2$ *are* ***locally bisimilar***, *in symbols* $\mathfrak{W}_1 \rightleftharpoons_L \mathfrak{W}_2$, *if there exists a non-empty binary relation* $\rightleftharpoons \subseteq W_1 \times W_2$, *such that the following holds:*

(a) *For all object variables* $a, a_1, \ldots, a_n \in \mathcal{X}$ *and* n*-ary relation symbols* $R \in \mathcal{R}$:
- $a^{\mathfrak{W}_1} \rightleftharpoons a^{\mathfrak{W}_2}$, *and*
- $R(a_1^{\mathfrak{W}_1}, \ldots, a_n^{\mathfrak{W}_1}) \iff R(a_1^{\mathfrak{W}_2}, \ldots, a_n^{\mathfrak{W}_2})$;

(b) *For all set variables* $x \in \mathcal{V}$, $u \in W_1$ *and* $v \in W_2$:
- *If* $u \rightleftharpoons v$ *then* $u \in x^{\mathfrak{W}_1} \iff v \in x^{\mathfrak{W}_2}$;

(c) *For* m_f*-ary function symbols* $f \in \mathcal{F}$, *terms* t_i, $i \leq m_f$, *of* $\mathcal{L}$, *and* $u \in W_1, v \in W_2$:
- *If* $u \rightleftharpoons v$ *then* $u \in f_i^{\mathfrak{W}_1}(t_1^{\mathfrak{W}_1}, \ldots, t_m^{\mathfrak{W}_1}) \iff v \in f_i^{\mathfrak{W}_2}(t_1^{\mathfrak{W}_2}, \ldots, t_m^{\mathfrak{W}_2})$.

Further, we say that $\mathfrak{W}_1$ *and* $\mathfrak{W}_2$ *are* ***globally bisimilar***, *in symbols* $\mathfrak{W}_1 \rightleftharpoons_G \mathfrak{W}_2$, *if they are locally bisimilar and the relation* $\rightleftharpoons \subseteq W_1 \times W_2$ *is* ***global*** *in the sense that for all* $u \in W_1$ *there is some* $v \in W_2$ *such that* $u \rightleftharpoons v$, *and, conversely, for all* $v \in W_2$ *there is some* $u \in W_1$ *such that* $u \rightleftharpoons v$.

The next proposition states the basic properties we expect from a notion of bisimulation between ADS.

PROPOSITION 6.14 (BISIMILAR MODELS IN ADSs).
Let $\mathcal{S} = (\mathcal{L}, \mathcal{M})$ *be an* ADS *and* $\mathfrak{W}_k$, $k = 1, 2$, *any two* ADMs *from the class* $\mathcal{M}$ *that are locally bisimilar,* $\mathfrak{W}_1 \rightleftharpoons_L \mathfrak{W}_2$. *Then:*

(i) *For all terms* t *of* $\mathcal{L}$ *and all points* $u \in W_1$, $v \in W_2$, *we have:*

$$\text{if } u \rightleftharpoons v \text{ then } u \in t^{\mathfrak{W}_1} \iff v \in t^{\mathfrak{W}_2}.$$

[2] If one is interested in particular classes of logics, this definition of bisimulation can be accordingly strengthened.

(ii) *For all object assertions φ of $\mathcal{L}$:*

$$\mathfrak{W}_1 \models \varphi \iff \mathfrak{W}_2 \models \varphi.$$

(iii) *For all term assertions $t_1 \sqsubseteq t_2$:*

$$\text{if } \mathfrak{W}_1 \rightleftharpoons_G \mathfrak{W}_2 \text{ then } \mathfrak{W}_1 \models t_1 \sqsubseteq t_2 \iff \mathfrak{W}_2 \models t_1 \sqsubseteq t_2.$$

PROOF. Claim (i) follows directly from items (b) and (c) of Definition 6.13 (with the Boolean connectives being a trivial inductive step) and (ii) follows from (i) and item (a).

For the proof of the third claim, suppose that $\rightleftharpoons$ is global and that $\mathfrak{W}_1 \models t_1 \sqsubseteq t_2$. If $t_1^{\mathfrak{W}_2} = \varnothing$ then $\mathfrak{W}_2 \models t_1 \sqsubseteq t_2$ follows. So assume otherwise. Take any $v \in t_1^{\mathfrak{W}_2}$. By globality, there is a $u \in W_1$ such that $u \rightleftharpoons v$. Hence $u \in t_1^{\mathfrak{W}_1}$ by (i). By assumption, $u \in t_2^{\mathfrak{W}_1}$ as well, whence $v \in t_2^{\mathfrak{W}_2}$ by (i). ☐

6.2.2. Bisimulations for $\mathcal{E}$-Connections. We will now extend the notion of bisimulation for ADSs to $\mathcal{E}$-connections.

DEFINITION 6.15 (BISIMULATIONS FOR $\mathcal{E}$-CONNECTIONS). *Let $\mathcal{S}_i = (\mathcal{L}_i, \mathcal{M}_i)$, for $i = 1, \dots, n$, be n* ADSs *and let $\mathcal{C} = \mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \dots, \mathcal{S}_n)$ be an $\mathcal{E}$-connection. Two models*

$$\mathfrak{M}^1 = \left\langle (\mathfrak{W}_i^1)_{i\le n}, (E_j^{\mathfrak{M}^1})_{j\le m} \right\rangle \quad \text{and} \quad \mathfrak{M}^2 = \left\langle (\mathfrak{W}_i^2)_{i\le n}, (E_j^{\mathfrak{M}^2})_{j\le m} \right\rangle$$

*for $\mathcal{C}$ are called $\mathcal{E}$-**bisimilar**, symbolically $\mathfrak{M}^1 \rightleftharpoons_{\mathcal{E}} \mathfrak{M}^2$, if there are global relations $\rightleftharpoons_i$, for $i = 1, \dots, n$, satisfying conditions (a)–(c) from Definition 6.13 such that the following holds for any j:*

(d) *If $\langle u_1, \dots, u_n \rangle \in E_j^{\mathfrak{M}^1}$ and $u_i \rightleftharpoons_i v_i$ for some i, then there exist points $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n$ such that $\langle v_1, \dots, v_n \rangle \in E_j^{\mathfrak{M}^2}$ and $u_k \rightleftharpoons_k v_k$ for all k.*

(e) *If $\langle v_1, \dots, v_n \rangle \in E_j^{\mathfrak{M}^2}$ and $u_i \rightleftharpoons_i v_i$ for some i, then there exist points $u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n$ such that $\langle u_1, \dots, u_n \rangle \in E_j^{\mathfrak{M}^1}$ and $u_k \rightleftharpoons_k v_k$ for all k.*

Again, the following proposition tells us that we have defined the notion of bisimulation for $\mathcal{E}$-connections appropriately.

PROPOSITION 6.16 (BISIMILAR MODELS IN $\mathcal{E}$-CONNECTIONS). *Let $\mathcal{C} = \mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \dots, \mathcal{S}_n)$ be an $\mathcal{E}$-connection and suppose that $\mathfrak{M}^1 \rightleftharpoons_{\mathcal{E}} \mathfrak{M}^2$. Then for all assertions φ of $\mathcal{C}$ it holds that:*

$$\mathfrak{M}^1 \models \varphi \iff \mathfrak{M}^2 \models \varphi,$$

i.e., $\mathcal{E}$-bisimilar models are indistinguishable by means of assertions.

PROOF. For simplicity, let us assume that the $\mathcal{E}$-connection is 2-dimensional, i.e., $\mathcal{C} = \mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \mathcal{S}_2)$, and that $\mathcal{E}$ contains only a single link relation E. Suppose $\varphi = \langle E \rangle^1 (t)$, with t a 2-term. We prove that for all $u_1^1 \in W_1^1$ and $u_1^2 \in W_1^2$ with $u_1^1 \rightleftharpoons_1 u_1^2$

$$u_1^1 \in \left(\langle E \rangle^1 (t)\right)^{\mathfrak{M}^1} \iff u_1^2 \in \left(\langle E \rangle^1 (t)\right)^{\mathfrak{M}^2}$$

The case $\varphi = \langle E \rangle^2 (s)$, s a 1-term, can be treated similarly. From this and the fact that the $\rightleftharpoons_i$ are global bisimulations, the claim follows.

So suppose that $u_1^1 \in \left(\langle E \rangle^1 (t)\right)^{\mathfrak{M}^1}$, i.e., that there is a $u_2^1 \in t^{\mathfrak{M}^1}$ such that we have $\langle u_1^1, u_2^1 \rangle \in E^{\mathfrak{M}^1}$. Since $u_1^1 \rightleftharpoons_1 u_1^2$, it follows that there is a u_2^2 such that $\langle u_1^2, u_2^2 \rangle \in E^{\mathfrak{M}^2}$ and $u_2^1 \rightleftharpoons_2 u_2^2$. Hence, $u_2^2 \in t^{\mathfrak{M}^2}$ by induction hypothesis, whence $u_1^2 \in \left(\langle E \rangle^1 (t)\right)^{\mathfrak{M}^2}$. $\square$

6.2.3. Undefinability in $\mathcal{E}$-Connections. We will now give examples of undefinable properties of $\mathcal{E}$-connections. For brevity, we will restrict the examples to the case of 2-dimensional $\mathcal{E}$-connections with a single link relation E and show that the properties (1) and (2) given earlier are not definable in basic $\mathcal{E}$-connections.

THEOREM 6.17 (UNDEFINABILITY IN $\mathcal{E}$-CONNECTIONS).
(i) Let $\mathcal{C}$ be any of the $\mathcal{E}$-connections $\mathcal{C}^{\{E\}}(\mathcal{ALC}^\sharp, \mathcal{MSO}_D^\sharp)$, $\mathcal{C}^{\{E\}}(\mathcal{ALCO}^\sharp, \mathbf{S4}_\mathbf{u}^\sharp)$ or $\mathcal{C}^{\{E\}}(\mathcal{SHIQ}^\sharp, \mathcal{ALCO}^\sharp)$ and R be a role name of, respectively, $\mathcal{ALC}$, $\mathcal{ALCO}$, or $\mathcal{SHIQ}$. Then the property

$$(\dagger) \quad \forall x \forall y \forall z (xRy \rightarrow (xEz \rightarrow yEz))$$

is not definable in $\mathcal{C}$.
(ii) Let $\mathcal{C}$ be the $\mathcal{E}$-connection $\mathcal{C}^{\{E\}}(\mathcal{SHIQ}^\sharp, \mathcal{ALCO}^\sharp)$, R a role name of $\mathcal{SHIQ}$, and S a role name of $\mathcal{ALCO}$. Then the property

$$(\ddagger) \quad \forall x \forall y \forall x' \forall y' (xRy \wedge xEx' \wedge yEy' \rightarrow x'Sy')$$

is not definable in $\mathcal{C}$.

PROOF. Our strategy will be to give appropriate pairs of models for the respective $\mathcal{E}$-connections, one model satisfying the given property, the other not, and to provide an $\mathcal{E}$-bisimulation between them. This shows that the properties ($\dagger$) and ($\ddagger$) are not definable in those $\mathcal{E}$-connections.

Let us prove (i) and consider first the case of the $\mathcal{E}$-connection $\mathcal{C}^{\{E\}}(\mathcal{SHIQ}, \mathcal{ALCO})$. We treat this case rather detailed and will be briefer in the remaining cases. To visualise the models we define below, compare Figure 6.2.

Let $\mathfrak{M}^1 = \langle \mathfrak{W}_1^1, \mathfrak{W}_2^1, E^{\mathfrak{M}_1} \rangle$ be a model for $\mathcal{C}^{\{E\}}(\mathcal{SHIQ}, \mathcal{ALCO})$ with

$$\mathfrak{W}_i^1 = \left\langle W_i^1, \mathcal{V}^{\mathfrak{W}_i^1}, \mathcal{X}^{\mathfrak{W}_i^1}, \mathcal{F}^{\mathfrak{W}_i^1}, \mathcal{R}^{\mathfrak{W}_i^1} \right\rangle,$$

where

$$W_1^1 := \{a_1, a_2, b_1, b_2, o_1\}, \quad x^{\mathfrak{W}_1^1} := \varnothing, \text{ for all } x \in \mathcal{V}, \quad a^{\mathfrak{W}_1^1} = o_1, \text{ for all } a \in \mathcal{X}.$$

There is one (non-trivial) role name R with

$$R^{\mathfrak{W}_1^1} := \{\langle a_1, b_1\rangle, \langle a_2, b_2\rangle\}.$$

For other roles S we assume $S^{\mathfrak{W}_1^1} \doteq \varnothing$. The function symbols $f_{\exists R}$, $f_{\exists R^{-1}}$, $f_{\exists_{\leq n} R}$ ($n \in \mathbb{N}$), etc. have the usual $\mathcal{SHIQ}$ interpretation, compare Page 115.

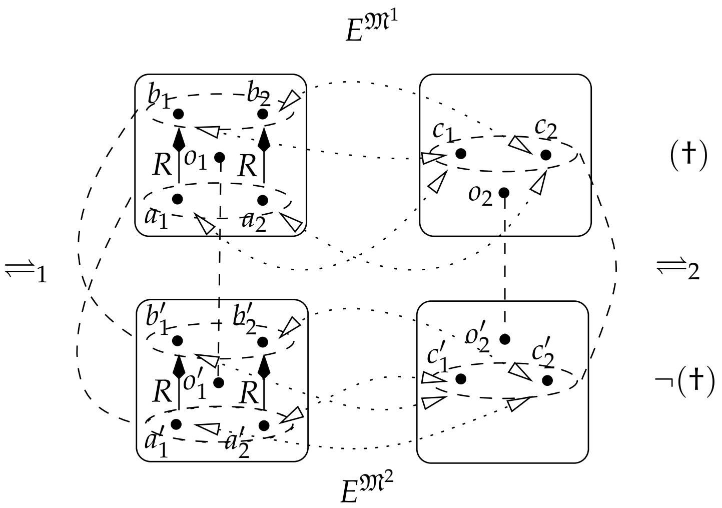


Figure 6.2: $\mathcal{E}$-bisimilar models of $\mathcal{C}(\mathcal{SHIQ}, \mathcal{ALCO})$ for (†)

Further, let

$$W_2^1 := \{c_1, c_2, o_2\}, \quad x^{\mathfrak{W}_2^1} := \varnothing, \text{ for all } x \in \mathcal{V}, \quad a^{\mathfrak{W}_2^1} := o_2, \text{ for all } a \in \mathcal{X}.$$

We assume that in $\mathfrak{W}_2^1$, all roles are interpreted by the empty set. For every nominal o of $\mathcal{ALCO}$ we have a 0-ary function symbol f_o which we interpret as $f_o^{\mathfrak{W}_2^1} = \{o_2\}$. Finally, we define

$$E^{\mathfrak{M}_1} := \{\langle a_1, c_1\rangle, \langle a_2, c_2\rangle, \langle b_1, c_1\rangle, \langle b_2, c_2\rangle\}.$$

Next, let the model $\mathfrak{M}_2 = \left\langle \mathfrak{W}_1^2, \mathfrak{W}_2^2, E^{\mathfrak{M}_2}\right\rangle$ with

$$W_1^2 := \{a'_1, a'_2, b'_1, b'_2, o'_1\} \quad \text{and} \quad W_2^2 := \{c'_1, c'_2, o'_2\}$$

be defined just like in $\mathfrak{M}_1$, except for $E^{\mathfrak{M}_2}$, which is given by

$$E^{\mathfrak{M}_2} := \{\langle a'_1, c'_2\rangle, \langle a'_2, c'_1\rangle, \langle b'_1, c'_1\rangle, \langle b'_2, c'_2\rangle\}.$$

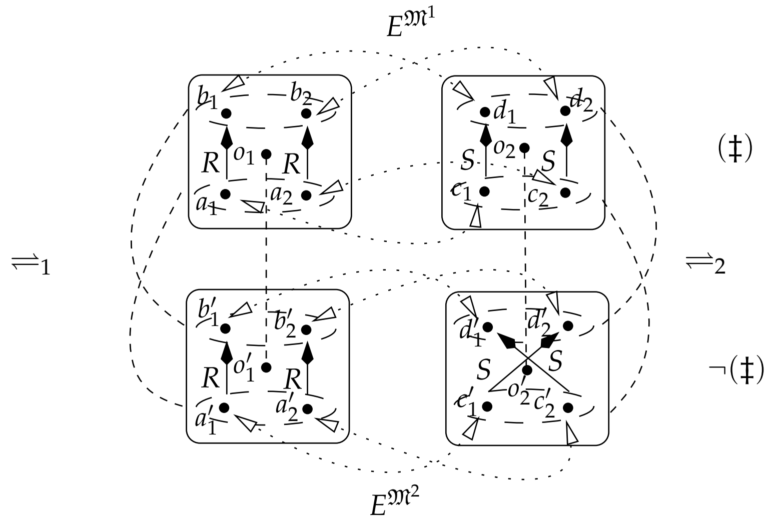


Figure 6.3: $\mathcal{E}$-bisimilar models of $\mathcal{C}(\mathcal{SHIQ}, \mathcal{ALCO})$ for (‡)

It should be obvious that $\mathfrak{M}_1$ satisfies (†), while $\mathfrak{M}_2$ does not. We claim that the relation $\rightleftharpoons_1$ and $\rightleftharpoons_2$ defined by

$$\begin{aligned}
\rightleftharpoons_1 \;:=\; & \{\langle a_1, a_1' \rangle, \langle a_1, a_2' \rangle, \langle a_2, a_1' \rangle, \langle a_2, a_2' \rangle, \\
& \langle b_1, b_1' \rangle, \langle b_1, b_2' \rangle, \langle b_2, b_1' \rangle, \langle b_2, b_2' \rangle, \\
& \langle o_1, o_1' \rangle\}, \qquad \text{and} \\
\rightleftharpoons_2 \;:=\; & \{\langle c_1, c_1' \rangle, \langle c_1, c_2' \rangle, \langle c_2, c_1' \rangle, \langle c_2, c_2' \rangle, \\
& \langle o_2, o_2' \rangle\}
\end{aligned}$$

are global relations satisfying conditions (a)–(e) of the definition of $\{E\}$-bisimulation for $\mathfrak{M}_1$ and $\mathfrak{M}_2$. First, it is obvious that the relations $\rightleftharpoons_i$, $i = 1, 2$, are global. Further, it should also be rather obvious that conditions (a), (b) and (c) from Definition 6.13 are satisfied for $\rightleftharpoons_i$, $i = 1, 2$.

It remains to establish that conditions (d) and (e) of the definition of $\mathcal{E}$-bisimulation hold. We show only (d). There are a total of 16 cases to be considered. We go through some of them and leave the rest to the reader.

We have $\langle a_1, c_1 \rangle \in E^{\mathfrak{M}^1}$ and four possibilities to instantiate the antecedent of (d). In the cases $a_1 \rightleftharpoons_1 a_1'$ and $c_1 \rightleftharpoons_2 c_2'$, we have $\langle a_1', c_2' \rangle \in E^{\mathfrak{M}^2}$ and $c_1 \rightleftharpoons_1 c_2'$ and $a_1 \rightleftharpoons_1 a_1'$. In the cases $a_1 \rightleftharpoons_1 a_2'$ and $c_1 \rightleftharpoons_2 c_1'$, we have $\langle a_2', c_1' \rangle \in E^{\mathfrak{M}^2}$ and $c_1 \rightleftharpoons_1 c_1'$ and $a_1 \rightleftharpoons_1 a_2'$. In all cases, (d) is satisfied.

It should be clear how to check the remaining cases. Thus, we have shown that the models $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are $\{E\}$-bisimilar.

Let us now briefly discuss which modifications are needed in the cases of the $\mathcal{E}$-connections $\mathcal{C}(\mathcal{ALC}, \mathcal{MS})$ and $\mathcal{C}(\mathcal{ALCO}, \mathsf{S4}_u)$. Roughly, we can use the same models as before. But this time, to interpret the distance operators of $\mathcal{MSO}_D$ and the interior/closure operators as well as the universal modality of $\mathsf{S4}_u$, we need to specify metrics d/d' (in the case of $\mathcal{MS}$) and topologies $\mathfrak{T}/\mathfrak{T}'$ (in the case of $\mathsf{S4}_u$) for $\mathfrak{M}^1/\mathfrak{M}^2$, respectively. A straightforward solution to this problem is to choose the discrete metric in the case of $\mathcal{MSO}_D$—i.e., the metric d such that $d(x, y) = 0$ if and only if $x = y$ and $d(x, y) = 1$, otherwise—and to choose the topology induced by the discrete metric in the case of $\mathsf{S4}_u$. The above proof can then be mimicked without further major modifications.

For the proof of the second claim, we ask the reader to follow the lines of the proof of (i) and restore the details from Figure 6.3. □

6.3. Link Constraints

We have seen in the last section that rather natural requirements having the logical form of constraint (†) from the last section, such as

- 'The spatial extension of the capital of every country is included in the spatial extension of that country', and
- 'The EU will never contract',

and which can be formalised in the language of first-order logic as

(i) $\forall x \forall y \forall z \; \big(x \; \textit{capital-of} \; y \rightarrow (xEz \rightarrow yEz)\big)$,

(ii) $\forall x \forall y \forall z \; \big(y < z \rightarrow (E(EU, x, y) \rightarrow E(EU, x, z))\big)$,

are not expressible in the language of basic $\mathcal{E}$-connections, for instance in the $\mathcal{E}$-connections $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCO}^{\sharp}, \mathsf{S4}_u^{\sharp})$ and $\mathcal{C}^{\mathcal{E}}(\mathcal{ALCO}^{\sharp}, \mathsf{S4}_u, \mathsf{PTL}^{\sharp})$. However, we can of course add these kinds of constraints as new *primitive* assertions to $\mathcal{E}$-connections, obtaining various ways of increasing the expressive power of $\mathcal{E}$-connections. Thus, it is an interesting question to find out what kinds of first-order constraints are 'harmless' from the computational point of view.

A general investigation of this question seems to be rather complex. Here, we only consider constraints of the form (i) and (ii) above which have the same structure in the sense that they enforce a new E-link between the models under certain conditions. In the following, we will show that, under some weak assumptions, constraints of this form do not harm the transfer of decidability. We begin by introducing link constraints formally.

DEFINITION 6.18 (LINK CONSTRAINTS). *Suppose* $\mathcal{S}_i = (\mathcal{L}_i, \mathcal{M}_i)$, $n \geq 2$, *are* ADSs, R *is a binary relation symbol of* $\mathcal{L}_1$, $\overline{a} = a_3, \dots, a_n$ *are object variables in* $\mathcal{L}_3, \dots, \mathcal{L}_n$, *respectively, and* $E \in \mathcal{E}$. *Then the formula*

$$\forall x \forall y \forall z \left(xRy \rightarrow (E(x,z,\overline{a}) \rightarrow E(y,z,\overline{a}))\right)$$

is called a ***link constraint*** *for* $\mathcal{C}^{\mathcal{E}}(\mathcal{S}_1, \dots, \mathcal{S}_n)$.

We say that the binary relation R *of* $\mathcal{L}_1$ *is* ***describable*** *in* $\mathcal{S}_1$ *if there exists a term* t_R *in* $\mathcal{L}_1$ *such that, for every model* $\mathfrak{M} \in \mathcal{M}_1$ *with domain* W, *every* $x \in W$ *and every* $X \subseteq W$, *we have*

$$x \in t_R^{\mathfrak{M}}(X) \quad \Longleftrightarrow \quad \forall y \in W \; (xR^{\mathfrak{M}}y \rightarrow y \in X).$$

A link constraint with describable R *is called a* ***describable link constraint***.

Clearly, the relations R and $<$ in link constraints (1) and (2) above are describable by the $\mathcal{ALC}^\sharp$- and $\mathsf{PTL}^\sharp$-terms corresponding to the 'box operators' $\forall R.C$ and $\Box_F p$, respectively. In what follows, we only consider those link constraints that are describable.

DEFINITION 6.19 ($\mathcal{E}$-CONNECTIONS WITH LINK CONSTRAINTS). *Suppose that we are given* n *abstract description systems* $\mathcal{S}_i = (\mathcal{L}_i, \mathcal{M}_i)$, $1 \leq i \leq n$, *and that* $\mathcal{E} = \{E_j \mid j \in J\}$ *is a set of* n*-ary relation symbols. We denote by*

$$\mathcal{C}^{\mathcal{E}}_{LO}(\mathcal{S}_1, \dots, \mathcal{S}_n)$$

the $\mathcal{E}$*-connection in which the set of link assertions is extended with describable link constraints and the link operators can be applied to object variables. The truth-relation for the new* $\mathcal{E}$*-connection is defined in the obvious way; in particular, satisfiability of link constraints is defined via the standard first-order reading of these constraints.*

The following transfer theorem is proved by appropriately extending the proof of Theorem 5.7.

THEOREM 6.20 (TRANSFER OF DECIDABILITY FOR $\mathcal{C}^{\mathcal{E}}_{LO}$).
Let $\mathcal{S}_1, \dots, \mathcal{S}_n$ *be* ADSs *with decidable satisfiability problems. Then the satisfiability problem for any* $\mathcal{E}$*-connection* $\mathcal{C}^{\mathcal{E}}_{LO}(\mathcal{S}_1, \dots, \mathcal{S}_n)$ *is decidable as well.*

PROOF. As before, we prove a simplified version of the theorem involving only two ADSs, $\mathcal{S}_1$ and $\mathcal{S}_2$, and a single link relation E. Here is the simplified variant of this theorem:

THEOREM 6.21. *Suppose the satisfiability problems for the* ADSs $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are decidable. Then the satisfiability problem for the* $\{E\}$*-connection* $\mathcal{C}^{\{E\}}_{LO}(\mathcal{S}_1, \mathcal{S}_2)$ *is decidable as well.*

We prove this theorem by modifying the proof of Theorem 5.8: we extend Lemma 5.9 and its proof to take into account constraints, thus obtaining a proof of Theorem 6.21.

Let Γ be a $\mathcal{C}^{\{E\}}_{LO}(\mathcal{S}_1, \mathcal{S}_2)$-knowledge base containing a finite set Φ of link constraints talking only about the link relation E such that the relations $R_1, \dots, R_k$ occurring in Φ are describable in $\mathcal{S}_1$. Observe that no vectors of object variables $\overline{a}$ appear in the constraints, as we are concerned with the connection of only two ADSs. We need the following modifications of the notions used in Lemma 5.9 on Page 144.

- We redefine the closure $\mathsf{cl}_1(\Gamma)$ as follows (but keep the definition of $\mathsf{cl}_2(\Gamma)$):

  let Θ_0 denote the closure under negation of the set of 1-terms occurring in Γ and $o_1(\Gamma)$. Then set

  $$\begin{aligned}\Theta_1 &= \Theta_0 \cup \{\langle E\rangle^1 (s) \mid s = \neg \langle E\rangle^2 (s') \in \mathsf{cl}_2(\Gamma) \text{ or } s = \neg \langle E\rangle^2 (a) \in \mathsf{cl}_2(\Gamma)\},\\ \Theta_2 &= \Theta_1 \cup \bigcup_{1\le j\le k} \{t_{R_j}(s) \mid s = \langle E\rangle^1 (s') \in \Theta_1 \text{ or } s = \langle E\rangle^1 (a) \in \Theta_1\},\end{aligned}$$

  where, for $1 \le j \le k$, t_{R_j} is the $\mathcal{L}_1$-term describing the relation R_j (cf. the definition of 'describable' in Definition 6.18). Finally, define $\mathsf{cl}_1(\Gamma)$ to be the closure of Θ_2 under subformulae and negation; again we identify $\neg\neg t$ with t, so that the closure is finite.

The formulation of the satisfiability criterion is largely identical to that of Lemma 5.9. We only add some extra 1-term assertions to the definition of Γ_1 in Condition (2), but leave Γ_2 unchanged:

LEMMA 6.22 (SATISFIABILITY CRITERION FOR $\mathcal{C}^{\{E\}}_{LO}(\mathcal{S}_1, \mathcal{S}_2)$).
Let Γ be a $\mathcal{C}^{\{E\}}_{LO}(\mathcal{S}_1, \mathcal{S}_2)$-knowledge base containing a finite set Φ of link constraints for E such that the relations $R_1, \dots, R_k$ occurring in Φ are describable in $\mathcal{S}_1$. Then Γ is satisfiable if and only if there exist

(i) *subsets $\Delta_1 \subseteq \mathfrak{C}_1(\Gamma)$ and $\Delta_2 \subseteq \mathfrak{C}_2(\Gamma)$,*

(ii) *a relation $e \subseteq \Delta_1 \times \Delta_2$, and*

(iii) *functions $\sigma_1 : ob_1(\Gamma) \longrightarrow \Delta_1$ and $\sigma_2 : ob_2(\Gamma) \longrightarrow \Delta_2$,*

such that, for $i = 1, 2$, the following conditions are satisfied:

(1) *for any $a \in ob_i(\Gamma)$, we have $\langle E\rangle^i \neg \langle E\rangle^{\bar{i}} (a) \notin \sigma_i(a)$;*

(2) *the union Γ_i of*

- $\{sur_i(\bigvee \Delta_i) = \top_i\}$,
- $\{a_t : sur_i(t) \mid t \in \Delta_i\}$,
- $\{a : sur_i(\sigma_i(a)) \mid a \in ob_i(\Gamma)\}$,
- $\{sur_i(t_1) \sqsubseteq sur_i(t_2) \mid t_1 \sqsubseteq t_2 \in \Gamma$ *is an i-term assertion*$\}$,
- $\{R_j(a_1, \dots, a_{m_j}) \mid R_j(a_1, \dots, a_{m_j}) \in \Gamma$ *is an i-object assertion*$\}$, *and*

– $\{(a : sur_i(t)) \mid (a : t) \in \Gamma$ *is an i-object assertion*$\}$,

and where Γ_1 *contains additionally*

– $\{sur_1(\langle E\rangle^1(s)) \sqsubseteq t_R(sur_1(\langle E\rangle^1(s))) \mid \langle E\rangle^1(s) \in \mathsf{cl}_1(\Gamma)\}$,

– $\{sur_1(\langle E\rangle^1(a)) \sqsubseteq t_R(sur_1(\langle E\rangle^1(a))) \mid \langle E\rangle^1(a) \in \mathsf{cl}_1(\Gamma)\}$,

is $\mathcal{S}_i$*-satisfiable, where* $a_t \in \mathcal{X}_i(\Gamma)$ *is a fresh object variable for each* $t \in \Delta_i$*;*

(3) *for all* $t \in \Delta_1$ *and* $\langle E\rangle^1(s) \in \mathsf{cl}_1(\Gamma)$ *with* s *a* 2*-term, we have*

$$\langle E\rangle^1(s) \in t \iff \text{there exists } t' \in \Delta_2 \text{ with } (t,t') \in e \text{ and } s \in t';$$

(4) *for all* $t \in \Delta_2$ *and* $\langle E\rangle^2(s) \in \mathsf{cl}_2(\Gamma)$ *with* s *a* 1*-term, we have*

$$\langle E\rangle^2(s) \in t \iff \text{there exists } t' \in \Delta_1 \text{ with } (t',t) \in e \text{ and } s \in t';$$

(5) *for all* $t \in \Delta_1$ *and* $\langle E\rangle^1(a) \in \mathsf{cl}_1(\Gamma)$ *with* $a \in ob_2(\Gamma)$*, we have*

$$\langle E\rangle^1(a) \in t \iff (t, \sigma_2(a)) \in e;$$

(6) *for all* $t \in \Delta_2$ *and* $\langle E\rangle^2(a) \in \mathsf{cl}_2(\Gamma)$ *with* $a \in ob_1(\Gamma)$*, we have*

$$\langle E\rangle^2(a) \in t \iff (\sigma_1(a), t) \in e.$$

PROOF OF LEMMA. The proof of the lemma remains largely unchanged. The ($\Longrightarrow$) direction can be proved as before. Only in the ($\Longleftarrow$) direction, the definition of the link relation $E^{\mathfrak{M}}$ has to be modified.

Thus, suppose that Δ_1, Δ_2, e, σ_1, and σ_2 satisfy the conditions of the theorem. By (2), there exist a model $\mathfrak{W}_1 \in \mathcal{M}_1$ of Γ_1 and a model $\mathfrak{W}_2 \in \mathcal{M}_2$ of Γ_2. For $i = 1,2$, let $\mathfrak{W}_i$ be based on the domain W_i. For each $d \in W_i$, we set

$$t(d) = \bigwedge\{t \in \mathsf{cl}_i(\Gamma) \mid d \in (sur_i(t))^{\mathfrak{W}_i}\} \in \mathfrak{C}_i(\Gamma).$$

Now, define the extension $E^{\mathfrak{M}} \subseteq W_1 \times W_2$ of the link symbol E by taking:

$$\begin{aligned} E_0^{\mathfrak{M}} &= \{(d,d') \mid (t(d),t(d')) \in e\};\\ E_{n+1}^{\mathfrak{M}} &= E_n^{\mathfrak{M}} \cup \{(d,d') \mid \exists d''\ (d'',d) \in R_j^{\mathfrak{W}_1} \text{ with } 1 \le j \le k \text{ and } (d'',d') \in E_n^{\mathfrak{M}}\};\\ E^{\mathfrak{M}} &= \bigcup_{n\ge 0} E_n^{\mathfrak{M}}. \end{aligned}$$

It is easy to see that $E^{\mathfrak{M}}$ satisfies all of the constraints in Φ. We have to show that $\mathfrak{M} = (\mathfrak{W}_1, \mathfrak{W}_2, E^{\mathfrak{M}})$ is a model of Γ. As in the proof of Lemma 5.9, it suffices to show that

$$(*) \qquad d \in (sur_i(s))^{\mathfrak{W}_i} \iff d \in s^{\mathfrak{M}}$$

for $i = 1,2$, all $d \in W_i$, and all $s \in \mathsf{cl}_i(\Gamma)$.

The proof of this claim is by induction on the term structure of s, simultaneously for $i = 1,2$. For set variables, the claim is an immediate consequence of the definition of $\mathfrak{M}$. The cases of the Boolean operators and the function symbols of $\mathcal{L}_i$, $i = 1, 2$, are trivial. Thus, it remains to consider the cases of

(a) $s = \langle E \rangle^i (s')$, with s' an $\bar{i}$-term, and

(b) $s = \langle E \rangle^i (a)$, with $a \in ob_{\bar{i}}(\Gamma)$.

The proofs of

$$(\dagger) \qquad d \in (sur_i(s))^{\mathfrak{W}_i} \implies d \in s^{\mathfrak{M}}$$

for the Cases (a) and (b) remain unchanged. However, the proofs of

$$(\ddagger) \qquad d \in s^{\mathfrak{M}} \implies d \in (sur_i(s))^{\mathfrak{W}_i}$$

for (a) and (b) have to be modified. Let us prove the following auxiliary lemma:

LEMMA 6.23. *Let s be a 1-term, s' a 2-term, a an object variable of $\mathcal{L}_1$, and a' an object variable of $\mathcal{L}_2$ with $\{\langle E \rangle^1 (s'), \langle E \rangle^1 (a')\} \subseteq \mathsf{cl}_1(\Gamma)$ and $\{\langle E \rangle^2 (s), \langle E \rangle^2 (a)\} \subseteq \mathsf{cl}_2(\Gamma)$.*

If $(d, d') \in E^{\mathfrak{M}}$, then the following holds:

(i) *$s' \in t(d')$ implies $\langle E \rangle^1 (s') \in t(d)$;*

(ii) *$s \in t(d)$ implies $\langle E \rangle^2 (s) \in t(d')$;*

(iii) *${a'}^{\mathfrak{M}} = d'$ implies $\langle E \rangle^1 (a') \in t(d)$;*

(iv) *$a^{\mathfrak{M}} = d$ implies $\langle E \rangle^2 (a) \in t(d')$.*

PROOF. The proof is by induction on n. Let $n = 0$. Then $(d, d') \in E_0^{\mathfrak{M}}$ implies $(t(d), t(d')) \in e$. Thus, (i) is an immediate consequence of condition (3), (ii) is an immediate consequence of (4), (iii) of (5), and (iv) of (6).

Let $n > 0$. Then $(d, d') \in E_n^{\mathfrak{M}}$ implies that either $(d, d') \in E_{n-1}^{\mathfrak{M}}$ or there exists a d'' such that $(d'', d) \in R_j^{\mathfrak{W}_1}$ for some j with $1 \leq j \leq k$ and $(d'', d') \in E_{n-1}^{\mathfrak{M}}$. In the former case, (i)–(iv) follow by the induction hypotheses. Let us consider the latter one.

(i) Let $s' \in t(d')$. By the induction hypotheses and since $(d'', d') \in E_{n-1}^{\mathfrak{M}}$, we have $\langle E \rangle^1 (s') \in t(d'')$ and so $d'' \in sur_1(\langle E \rangle^1 (s'))^{\mathfrak{W}_1}$. Due to the new components of Γ_1 and the fact that $(d'', d) \in R_j^{\mathfrak{W}_1}$, we then have $d \in sur_1(\langle E \rangle^1 (s'))^{\mathfrak{W}_1}$, which yields $\langle E \rangle^1 (s') \in t(d)$, as required.

(ii) Assume by contraposition that $\neg \langle E \rangle^2 (s) \in t(d')$. By induction hypotheses (and since we extended the closure $\mathsf{cl}_1(\Gamma)$), we obtain $\langle E \rangle^1 \neg \langle E \rangle^2 (s) \in t(d'')$ using (i), and thus $d'' \in sur_1(\langle E \rangle^1 \neg \langle E \rangle^2 (s))^{\mathfrak{W}_1}$. Due to the new components of Γ_1, this yields $d'' \in t_R(sur_1(\langle E \rangle^1 \neg \langle E \rangle^2 (s)))^{\mathfrak{W}_1}$. Because $(d'', d) \in R_j^{\mathfrak{W}_1}$, we obtain that we have $d \in sur_1(\langle E \rangle^1 \neg \langle E \rangle^2 (s))^{\mathfrak{W}_1}$, and hence $\langle E \rangle^1 \neg \langle E \rangle^2 (s) \in t(d)$. By conditions (3) and (4), we then have $s \notin t(d)$, which had to be shown.

Finally, (iii) is proved analogously to (i), and (iv) is proved analogously to (ii). □

We can now adapt the proof of (‡) for the cases of (a) and (b). As before, we restrict ourselves to the case $i = 1$.

(a) $s = \langle E \rangle^1 (s')$ with s' a 2-term. Suppose $d \in (\langle E \rangle^1 (s'))^{\mathfrak{M}}$. We find $d' \in W_2$ with $(d, d') \in E^{\mathfrak{M}}$ and $d' \in s^{\mathfrak{M}}$. By the induction hypothesis, $d' \in (sur_2(s'))^{\mathfrak{W}_2}$ and so $s' \in t(d')$. As $(d, d') \in E^{\mathfrak{M}}$, part (i) of Lemma 6.23 yields $\langle E \rangle^1 (s') \in t(d)$, which implies $d \in (sur_1(\langle E \rangle^1 (s')))^{\mathfrak{W}_1}$.

(b) $s = \langle E \rangle^1 (a)$ with $a \in ob_2(\Gamma)$. Let $d \in (\langle E \rangle^1 (a))^{\mathfrak{M}}$. Then $(d, a^{\mathfrak{M}}) \in E^{\mathfrak{M}}$ by definition and thus $\langle E \rangle^1 (a) \in t(d)$ by part (iii) of Lemma 6.23. This obviously implies $d \in (sur_1(\langle E \rangle^1 (a)))^{\mathfrak{W}_1}$, as required.

The case $i = 2$ is similar and uses parts (ii) and (iv) of Lemma 6.23 instead of parts (i) and (iii).

This completes the proof of Lemma 6.22. As before, Theorem 6.21—and thus Theorem 6.20—is an immediate consequence. □

As already noted, a general investigation of first-order constraints seems to be rather complex. As to the link constraints of the form above, we conjecture that by dropping the describability condition one destroys the (general) transfer of decidability. The combination of link constraints with other variants of $\mathcal{E}$-connections and the computational properties of different kinds of first-order constraints are left for future work.

6.4. Comparison with Other Combination Methodologies

We now briefly compare $\mathcal{E}$-connections with three other (families of) combination methodologies which are important in knowledge representation and reasoning.

6.4.1. Multi-dimensional Systems. The formation of multi-dimensional systems out of one-dimensional ones is probably the most frequently employed methodology of combining knowledge representation and reasoning formalisms. Given n languages $L_1, \dots, L_n$ interpreted in domains $D_1, \dots, D_n$, we take the *union* L of the L_i and interpret it in the Cartesian product $D_1 \times \cdots \times D_n$ consisting of all n-tuples $(d_1, \dots, d_n)$, $d_i \in D_i$. (The combined language L contains no new constructors as compared with the original languages L_i.) Typical examples of such multi-dimensional formalisms are:

- **temporal epistemic logics** for reasoning about multi-agent systems—these are based on the Cartesian product of a flow of time and a set of possible states of a system (see Fagin et al. [1995], Halpern and Vardi [1989] and references therein);
- **first-order modal and temporal logics** based on the Cartesian product of a set of possible worlds or moments of time and a domain of first-order individuals [Fitting and Mendelsohn, 1998, Gabbay et al., 2003];

- **spatio-temporal logics** based on the Cartesian product of a flow of time and a model of space (see, e.g., Wolter and Zakharyaschev [2000, 2002]);
- **modal and temporal description logics** based on the Cartesian product of a set of possible worlds and a description logic domain [Laux, 1994, Baader and Laux, 1995, Baader and Ohlbach, 1995, Bettini, 1997, Artale and Franconi, 1998, Wolter and Zakharyaschev, 1998, 1999].

The main difference between multi-dimensional systems and $\mathcal{E}$-connections is the range of the quantifiers: while the former quantify (at least implicitly) over the set of n-tuples, in $\mathcal{E}$-connections we can quantify only over one-dimensional objects which form a component of a link. This seems to be the main reason why $\mathcal{E}$-connections exhibit a much more robust computational behaviour than multi-dimensional combinations (see, e.g., Gabbay et al. [2003] and references therein). In the multi-dimensional setting, even the two-dimensional combination of simple, say, NP-complete logics, can be highly undecidable [Spaan, 1993]. In contrast to $\mathcal{E}$-connections, no general transfer results are available for multi-dimensional combinations: their algorithmic behaviour is governed by rather subtle features of the component logics, so that the concept of abstract description systems is 'too abstract' to be useful in this context. On the contrary, it has been proved that three-dimensional products of standard unimodal logics (and even the two-dimensional products of CTL^* with standard unimodal logics) are usually undecidable [Hirsch et al., 2002, Hodkinson et al., 2002]. In this respect, $\mathcal{E}$-connections do not 'feel' the number of combined formalisms.

6.4.2. Independent Fusions and Gabbay's Fibring Methodology. Another way of combining formalisms without adding new constructors to the union of the languages is known as the formation of **independent fusions** or **joins** [Kracht and Wolter, 1991, Spaan, 1993, Fine and Schurz, 1996, Wolter, 1998, Baader et al., 2002]. In this case, it is assumed that the component languages L_i actually speak about the *same domain* D. In other words, the expressive capabilities of the L_i are combined by the independent fusion in order to reason about the same objects, yet viewed from different perspectives. As in the case of multi-dimensional systems, no new constructors are added.

A typical example of an independent fusion is the standard multi-modal **epistemic logic** modelling knowledge of $n > 1$ agents [Halpern and Moses, 1992], where we simply join n epistemic logics for a *single* agent. Sometimes, **temporal epistemic logics** degenerate to fusions of temporal and epistemic logics [Halpern and Vardi, 1989].

Independent fusions have also been suggested in the context of **description logics** [Baader et al., 2002], where constructors of different DLs may be required to represent knowledge about certain domains. Note that putting the constructors of different DLs together to form a new DL often results in an undecidable logic, even if the components are decidable.[3] It has been shown in Baader et al. [2002] that independent fusions form a more robust (but, of course, less expressive) way of combining the constructors of different DLs than multi-dimensional combinations.

In contrast to $\mathcal{E}$-connections, independent fusions behave 'badly' if the class of models is not closed under the formation of disjoint unions (the corresponding ADS is not local), for instance, when nominals or negations of roles are present [Baader et al., 2002] or when we combine logics of time and space—while linear orders are natural models of time, their disjoint unions are certainly not.

The **fibring methodology** of Gabbay [1999] is a generalisation of independent fusions: when constructing the fibring of two formalisms L_1 and L_2, their models are not matched, but combined by a so-called **fibring function** F which associates with any element of the domain D_i of a model M_i for L_i a model $M_{\bar{i}}$ of the other formalism. The truth-values of formulae at point w are computed inductively: the Boolean operators are treated as usual, and the inductive step for a given constructor of L_i depends on whether w is a member of a model M_i for L_i—in which case it is computed as in M_i—or a member of a model $M_{\bar{i}}$ for the other logic $L_{\bar{i}}$, in which case the truth-value is computed in the model $F(w)$ for L_i.

In contrast to $\mathcal{E}$-connections and similarly to multi-dimensional systems and independent fusions, the fibring formalisms do not add any new constructors to the combined languages, but are based on their unions. Also, in contrast to $\mathcal{E}$-connections, the atoms of the component languages are supposed to be identical. Finally, because of the guarded quantification in $\mathcal{E}$-connections in 'any direction' of a link relation, the interaction between the fibred components is much weaker than the interaction between the $\mathcal{E}$-connected ones.

6.4.3. Description Logics with Concrete Domains. As demonstrated in Section 4.5.1, $\mathcal{E}$-connections can be used to connect a description logic, such as $\mathcal{ALC}$, with another logic, such as $\mathcal{MSO}_D$, which is evaluated in a *single* model, say, a map of Liverpool. This idea—to fix a single model in one of the combined formalisms—also underlies the extension of description logics with so-called **concrete domains**: since 'classical' description logics represent knowledge at a rather abstract logical level, concrete domains have been proposed to cope with applications that require predefined predicates or temporal and spatial dimensions [Baader and Hanschke, 1991, Lutz, 2003].

[3] As an example, consider the DLs $\mathcal{ALCF}$ (introduced on Page 140) and $\mathcal{ALC}^{+,\circ,\sqcup}$ (extending $\mathcal{ALC}$ with transitive closure, composition, and union of roles). For both DLs, the subsumption of concept descriptions is known to be decidable [Hollunder and Nutt, 1990, Schild, 1991, Baader, 1991]. However, the subsumption problem for their **union** $\mathcal{ALCF}^{+,\circ,\sqcup}$ is undecidable [Baader et al., 1993].

Examples of concrete domains include the natural numbers equipped with predicates like $=_{17}$, $<$, and $+$ [Baader and Hanschke, 1992, Lutz, 2002], Allen's interval algebra [Allen, 1983, Lutz, 2001a], and the RCC-8 calculus discussed in Section 4.2.4 [Haarslev et al., 1998].

However, the expressive power provided by concrete domains is largely orthogonal to the expressive power of $\mathcal{E}$-connections. First, in DLs with concrete domains, the coupling of the two formalisms is 'one way,' i.e., we can only talk about the concrete domain in the description logic, but not vice versa. Clearly, $\mathcal{E}$-connections are 'two way' in this sense. Second, in DLs with concrete domains, the description logic is equipped with operators which allow us to make statements about relations (of arbitrary arity) between 'concrete elements'. In contrast, $\mathcal{E}$-connections allow us only to express that formulae (i.e., *unary* predicates) are satisfied by domain elements of other components. It should also be noted that the addition of a concrete domain to a DL is a rather sensitive operation as far as the preservation of computational properties is concerned: even 'weak' DLs combined with rather 'weak' concrete domains can become undecidable, see, e.g., Baader and Hanschke [1992], Haarslev et al. [1998], Lutz [2001b]. In fact, except for a result in Baader et al. [2002] which treats extremely inexpressive concrete domains, no general decidability transfer results for the extension of description logics with concrete domains are known. Indeed, investigating the computational properties of DLs with concrete domains is a cumbersome task which involves the development of new and specialised techniques, consult, e.g., the survey Lutz [2003].

DISCUSSION

We summarise our main results, list open problems and possible extensions to the formalisms introduced, and point to interesting application areas.

Logics of Distance

In Part 1 of the thesis, we systematically investigated first-order, modal, and Boolean modal languages intended for reasoning about distances. The structures in which these languages were interpreted are the class $\mathcal{MS}$ of *metric spaces*, or the more general classes of arbitrary, symmetric, and triangular *distance spaces*. The main motivation for considering these languages was a lack of knowledge representation formalisms capable of representing and reasoning with numerical, quantitative concepts of distance, where we understood distance in a rather general, not necessarily spatial way.

In Chapter 1, we compared the *expressive power* of these languages over different classes of distance spaces and showed that the modal language $\mathcal{LO}_F[M]$ is expressively complete over symmetric distance spaces for the two-variable fragment $\mathcal{LF}_2[M]$. Furthermore, we proved that $\mathcal{LO}_F[M]$ is expressively complete for the Boolean distance language $\mathcal{LOB}[M]$ over arbitrary distance spaces, and noted that the languages $\mathcal{L}_D[M]$ and $\mathcal{L}_F[M]$ can *simulate nominals* and define the *universal modality*.

In Chapter 2, we investigated the computational behaviour of these languages and showed that while both $\mathcal{LF}_2[\mathbb{Q}^+]$-satisfiability and $\mathcal{LO}[\mathbb{Q}^+]$-satisfiability are decidable for the class of all (symmetric) distance spaces, even weaker languages turn out to have an undecidable satisfiability problem for the class of metric spaces and the class $\mathcal{D}^t$ of triangular distance spaces. We also singled out a natural and expressive fragment $\mathcal{LO}_D[M]$ of $\mathcal{LO}[M]$ which has the finite model property and is decidable, both for metric and triangular spaces.

In Chapter 3, we studied logical properties of the modal distance logics introduced, gave Hilbert style axiomatisations for the logics $\mathcal{MS}_D^i$, axiomatised the modal counterpart $\mathcal{LO}_F$ of the two-variable first-order distance logic interpreted in metric spaces using a weak form of the *covering rule*, and discussed compactness and the interpolation property.

The decidability results from Chapter 2 concerning the language $\mathcal{L}_D[\mathbb{Q}^+]$ obviously imply corresponding upper complexity bounds for the satisfiability problems. For instance, the satisfiability problem for $\mathcal{L}_D[\mathbb{Q}^+]$-formulae in metric spaces is, according to the proof of Theorem 2.14, solvable in NEXPTIME if the parameters from $\mathbb{Q}^+$ are coded unary, and it is solvable in 2NEXPTIME if the parameters are coded binary. However, we have not discussed these results in detail since it is known that, in fact, the satisfiability problem of $\mathcal{L}_D[\mathbb{Q}^+]$ in metric spaces is EXPTIME-complete for unary encoding [Wolter, 2004]. A proof of EXPTIME-completeness for the language $\mathcal{LO}_{L_1}[\mathbb{Q}^+]$ interpreted in metric spaces and using binary encoding can be found in Wolter and Zakharyaschev [2003]. It is, however, an open problem whether $\mathcal{L}_D[\mathbb{Q}^+]$ satisfiability is still EXPTIME-complete if we encode parameters from $\mathbb{Q}^+$ in binary.

As was noted in Section 1.1, logics of distance spaces were conceived and investigated primarily in view of their possible applications in knowledge representation and reasoning. In this respect, the following extensions and open problems seem to be of special interest.

- The presented decision procedure for the language $\mathcal{L}_D[\mathbb{Q}^+]$ based on the finite model property does not appear to be 'practical'. While a tableaux based decision procedure has been developed for the language $\mathcal{LO}_{L_1}[\mathbb{Q}^+]$ in Wolter and Zakharyaschev [2003], no tableaux or resolution based algorithms for the language $\mathcal{LO}_D[\mathbb{Q}^+]$ are known.
- As concerns the logical properties of the distance logics discussed, the following problems seem interesting. Can the class of frame-companions of metric spaces in language $\mathcal{L}_F[M]$ be characterised in an elegant way for parameter sets such as $M = \mathbb{N}, \mathbb{Q}^+, \mathbb{R}^+$? What modal languages are appropriate for characterising such classes axiomatically? Is the modal logic of the class of Euclidean spaces axiomatisable in the language $\mathcal{LO}_F[\mathbb{Q}^+]$? Is the rule ($\mathrm{COV}_0$) conservative in $\mathsf{MSO}_F[M]$? Is the Beth property more widespread in distance logics than interpolation?
- We have briefly mentioned some extensions of the languages of distance logics, namely the extension with variables over distances [Wolter and Zakharyaschev, 2003] and the extension with topological operators [Wolter and Zakharyaschev, 2004]. Other interesting extensions that have not been studied are the following. What is the computational and logical behaviour of spatial analogues of the temporal binary modalities Since and Until? For instance, we could define

$$\langle \mathfrak{B}, u \rangle \models \varphi \mathcal{U} \psi \iff \langle \mathfrak{B}, v \rangle \models \varphi \text{ for all } v \text{ with } d(u,v) < \inf(u,\psi),$$

where $\inf(u,\psi) = \inf(\{a = d(u,w) \mid \langle \mathfrak{B}, w\rangle \models \psi\})$; compare also Aiello and van Benthem [2002] for topological variants of this operator. Furthermore, some applications might require that we have some algebraic structure on the parameter sets different to the reals, such as finite groups, which we could also incorporate into the formal language.

- Logics of distance spaces reflect only one aspect of possible application domains. We envisage these logics as components of more complex, many-dimensional representation formalisms involving, for instance, also logics of time and conceptual knowledge. This can be realised, e.g., by constructing appropriate $\mathcal{E}$-connections, where a specific application might call for a suitable fine-tuning of the expressive means of the $\mathcal{E}$-connection.
- Similarly to logics of time, spatial knowledge representation often requires to capture information at different layers of metric granularity [Montanari, 1996]. Thus, can we construct (computationally well-behaved) formalism reflecting these different levels of granularity by constructing appropriate $\mathcal{E}$-connections? Note that naively adding, e.g., number restrictions to an $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}(\mathcal{MS}^{\sharp}_{D}, \mathcal{MS}^{\sharp}_{D})$ quickly results in an undecidable system (and decidability transfer does not apply since $\mathcal{MS}^{\sharp}_{D}$ is not number tolerant). So how can we devise more subtle 'granularity' operators?

$\mathcal{E}$-Connections

Part 2 of the thesis was concerned with a new combination technique for knowledge representation and reasoning formalisms, called $\mathcal{E}$-connections. The key idea of the methodology was to keep the domains of the combined formalisms disjoint and to introduce 'link relations' representing correspondences between objects in different domains. Typical interpretations of link relations were:

- 'x is in the spatial extension of y';
- 'x belongs to the lifespan of y';
- 'Object x in information system IS_1 corresponds to object y in information system IS_2';
- 'Object x is at location y at time z'.

The new methodology was introduced in Chapter 4 within the framework of *abstract description systems* in order to provide coverage of a wide range of KR&R formalisms such as description logics, temporal logics, modal logics of space, epistemic logics, etc.

The main theoretical results were shown in Chapter 5, where we proved a number of theorems showing that the formation of various kinds of $\mathcal{E}$-connections is computationally robust, even if we allow expressive link operators such as qualified number restrictions and Boolean combinations of link relations. We continued in Chapter 6 to show that the methodology of distributed description logics (DDLs) can be understood as a special case of $\mathcal{E}$-connections, analysed the expressive power of the language of basic $\mathcal{E}$-connections, and provided examples of undefinable first-order constraints on models of an $\mathcal{E}$-connection. Finally, we showed that adding new, primitive assertions expressing such undefinable properties to basic $\mathcal{E}$-connections, again, preserves the transfer of decidability.

As we argued in Section 4.1, the investigation of combination methods for KR&R-formalisms consists, to a large extent, in an analysis of the trade-off between possible interactions of the components in the combined system and its computational properties. In this respect, our complexity and undecidability results show that we have achieved a rather promising compromise between predictable computational behaviour of $\mathcal{E}$-connections and expressive interaction between its components: the complexity results show that the various kinds of $\mathcal{E}$-connections do add a considerable amount of interaction between the components, while the undecidability results illustrate the limits of automatic transfer of decidability.

However, even if we have 'practical' algorithms for the components of a certain kind of $\mathcal{E}$-connection, the rather abstract model-theoretic proofs of transfer of decidability do not directly provide a 'practical' algorithm for the $\mathcal{E}$-connection itself. Moreover, it is unlikely that such 'practical' versions of the general transfer results exist at all. Rather, it is to be expected that the design of efficient reasoning systems for $\mathcal{E}$-connections cannot be specified at this level of generality, but depends on specific features of the combined formalisms. Nevertheless, the proofs of the decidability transfer results indicate that in many cases existing practical decision procedures for the components can be combined so as to obtain practical decision procedures for the $\mathcal{E}$-connection—first steps in this direction have been undertaken in Serafini and Tamilin [2004] for the case of DDLs, and in Grau and Parsia [2004] for weak fragments of basic $\mathcal{E}$-connections.

Although we have considered in-depth various extensions of the basic $\mathcal{E}$-connections, many interesting problems remain open. Here are some of them:

- Starting from the decidability transfer results obtained, develop 'practical' decision procedures for particularly interesting $\mathcal{E}$-connections like, for instance, $\mathcal{C}^{\mathcal{E}}(\mathcal{SHIQ}^\sharp, \mathcal{ALCO}^\sharp)$, $\mathcal{C}^{\mathcal{E}}(\mathcal{SHIQ}^\sharp, \mathcal{MSO}^\sharp_{L_1})$, or $\mathcal{C}^{\mathcal{E}}(\mathcal{SHIQ}^\sharp, \mathbf{S4}^\sharp_\mathbf{u}, \mathbf{PTL}^\sharp)$. In all these cases, efficient decision procedures for the components are known. Is it possible to devise decision procedures for the $\mathcal{E}$-connections which are modular and integrate the existing decision procedures for the components without substantial modifications? Compare the performance of implemented algorithms for the $\mathcal{E}$-connections with the performance of decision procedures for their components.

- Similarly to DDLs, $\mathcal{E}$-connections can be employed as a tool for reasoning with distributed ontologies in the 'semantic web'. But in such a set-up, we obviously do not want that a local inconsistency, say, in a knowledge base of description logic DL_1, spreads via the $\mathcal{E}$-connection to the knowledge bases of description logics $DL_2 \dots DL_n$. While there is some work on inconsistency-tolerant description logics, e.g. Odintsov and Wansing [2003], and some ideas on how to semantically treat inconsistency in the framework of DDLs, e.g. Borgida and Serafini [2003], there is no developed theory of paraconsistent reasoning with $\mathcal{E}$-connections, and in particular no analysis of its computational behaviour.
- Consider more general first-order constraints for the link relations and classify them according to their algorithmic behaviour. This can also lead to a deeper analysis of the structural properties of ADSs because more subtle conditions than describability are required for decidability transfer results covering larger classes of first-order constraints.
- Introduce elements of 'fuzziness' to link relations between different domains in order to reflect the fact that spatial extensions or 'corresponding' objects in distributed databases can often be specified only approximately. It would, therefore, be of interest to allow link operators stating, for example, that 'the probability that y belongs to the spatial extension of x is not more than 75%'.
- The embedding of the product logic $\mathbf{S5} \times \mathbf{S5}$ into the $\mathcal{E}$-connection $\mathcal{C}^{\mathcal{E}}_{B}(\mathcal{B}, \mathcal{B})$ allowing Boolean operations on links provides the first evidence that there might be an interesting and useful hierarchy of formalisms between the 'weak' basic $\mathcal{E}$-connections and multi-dimensional formalisms. For example, we can take the closure of the set of link relations $\mathcal{E}$ not only under the Booleans, but also under the operations $\langle R \rangle E$ and $[R]E$ defined by taking

$$(\langle R \rangle E)^{\mathfrak{M}} = \{\langle x, y\rangle \mid \exists z\, (\langle x, z\rangle \in R^{\mathfrak{M}} \wedge \langle z, y\rangle \in E^{\mathfrak{M}})\},$$
$$([R]E)^{\mathfrak{M}} = \{\langle x, y\rangle \mid \forall z\, (\langle x, z\rangle \in R^{\mathfrak{M}} \rightarrow \langle z, y\rangle \in E^{\mathfrak{M}})\},$$

for every binary relation symbol R of the first component of a binary $\mathcal{E}$-connection $\mathcal{C}$ (and similarly for the binary relations of the second component).

Using these new constructors, we can easily ‘simulate’ most of multi-dimensional formalisms. Useful and interesting intermediate formalisms could be obtained by restricting applications of the Boolean operators to links.

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