Satisfiability, validity

- If a formula A is true in I we say that I satisfies A and that I is a model of A, denoted by $I \models A$.
- A is satisfiable if it is true in some interpretation.
- A is valid if it is true in every interpretation.
- Two formulas A and B are called equivalent, denoted $A \equiv B$ if they have the same models.

We will study: algorithms for evaluating formulas on interpretations, for checking satisfiability, validity and equivalence of formulas.

Truth Tables

Consider $A = p \wedge \neg q \rightarrow q \vee \neg p$.

We know how to calculate the truth value of A in a given interpretation I.

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p	q	$p \wedge \neg q \rightarrow q \vee \neg p$
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Is this formula satisfiable?

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1	1	1

Is this formula satisfiable? Yes

Is this formula valid? No

Truth Tables

p	q	$p \wedge \neg q \longrightarrow q \vee \neg p$
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0	1	1
1	0	0
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p	q	$p \wedge \neg q \rightarrow q \vee \neg p$	$\neg p \vee q$
0	0	1	
0	1	1	
1	0	0	
1	1	1	

Truth Tables

p	q	$p \land \neg q \to q \lor \neg p$	$\neg p \lor q$
0	0	1	1
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Summary: Using truth tables we can check satisfiability, validity and equivalence.

Limitations: For modest number of variables truth tables are unacceptably huge!

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Later: more practical algorithms

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4. Formulas A and B are equivalent if and only if the formula $A \leftrightarrow B$ is valid.

Some useful equivalences

For all formulas A and B, the following equivalences hold.

$$\neg\neg A \equiv A \quad (1)$$

$$A \wedge B \equiv B \wedge A \quad (2)$$

$$A \vee B \equiv B \vee A \quad (3)$$

$$(A \vee B) \wedge C \equiv (A \wedge C) \vee (B \wedge C) \quad (4)$$

$$(A \wedge B) \vee C \equiv (A \vee C) \wedge (B \vee C) \quad (5)$$

$$A \rightarrow B \equiv \neg A \vee B \quad (6)$$

$$\neg(A \wedge B) \equiv \neg A \vee \neg B \quad (7)$$

$$\neg(A \vee B) \equiv \neg A \wedge \neg B \quad (8)$$

$$A \leftrightarrow B \equiv (A \rightarrow B) \wedge (B \rightarrow A) \quad (9)$$

A few more equivalences

$$A \lor \bot \equiv \tag{10}$$

$$A \lor \top \equiv \tag{11}$$

$$A \land \bot \equiv \tag{12}$$

$$A \land \top \equiv \tag{13}$$

$$A \rightarrow \bot \equiv \tag{14}$$

$$\bot \rightarrow A \equiv \tag{15}$$

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\begin{aligned}
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A \land \bot &\equiv \bot && (12)\\
A \land \top &\equiv A && (13)\\
A \rightarrow \bot &\equiv \neg A && (14)\\
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Example: $(D \land C \rightarrow \neg D) \rightarrow C$ Apply: $A \rightarrow B \equiv \neg A \lor B$

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$\neg(D \land C) \lor \neg D \rightarrow C$ Apply: $\neg(A \land B) \equiv \neg A \lor \neg B$

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Example:	$(D \land C \to \neg D) \to C$	Apply:	$A \to B \equiv \neg A \lor B$
	$(D \land C \to \neg D) \to C$		
	$\neg(D \land C) \lor \neg D \to C$	Apply:	$\neg(A \land B) \equiv \neg A \lor \neg B$
	$\neg D \lor \neg C \lor \neg D \to C$	Apply:	$A \to B \equiv \neg A \lor B$
	$\neg(\neg D \lor \neg C \lor \neg D) \lor C$	Apply...	Apply...

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	$\neg(D \wedge C) \vee \neg D \rightarrow C$	Apply:	$\neg(A \wedge B) \equiv \neg A \vee \neg B$
	$\neg D \vee \neg C \vee \neg D \rightarrow C$	Apply:	$A \rightarrow B \equiv \neg A \vee B$
	$\neg(\neg D \vee \neg C \vee \neg D) \vee C$	Apply...	Apply...
	$(D \wedge C) \vee C$		

A purely syntactic algorithm for evaluation

We know a semantic algorithm for evaluation of a formula on an interpretation I.

Now we will study syntactic algorithm for evaluation.

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Assume I and we evaluate formula $A[p]$ on I.

Observe:

If $I \models p$, then $I \models p \leftrightarrow \top$;

If $I \not\models p$, then $I \models p \leftrightarrow \bot$;

Since we can replace equivalent formulas by equivalent, we can replace every atom p by either $\top$ or $\bot$, depending on the value of p in I.

Rewrite rules for evaluating a formula

$$\top \wedge \ldots \wedge \top \;\Rightarrow\; \top$$
$$\bot \wedge A_1 \wedge \ldots \wedge A_n \;\Rightarrow\; \bot$$

$$\top \vee A_1 \vee \ldots \vee A_n \;\Rightarrow\; \top$$
$$\bot \vee \ldots \vee \bot \;\Rightarrow\; \bot$$

$$\neg\top \;\Rightarrow\; \bot$$
$$\neg\bot \;\Rightarrow\; \top$$

$$A \rightarrow \top \;\Rightarrow\; \top$$
$$\bot \rightarrow A \;\Rightarrow\; \top$$
$$\top \rightarrow \bot \;\Rightarrow\; \bot$$

$$\top \leftrightarrow \top \;\Rightarrow\; \top$$
$$\top \leftrightarrow \bot \;\Rightarrow\; \bot$$
$$\bot \leftrightarrow \top \;\Rightarrow\; \bot$$
$$\bot \leftrightarrow \bot \;\Rightarrow\; \top$$

Algorithm for evaluating a formula

<u>procedure</u> $evaluate(G, I)$

<u>input</u>: formula G, interpretation I

<u>output</u>: a boolean value

<u>begin</u>

  <u>forall</u> atoms p occurring in G

    <u>if</u> $I(p) = 1$

      <u>then</u> replace all occurrences of p in G by $\top$;

      <u>else</u> replace all occurrences of p in G by $\bot$;

  rewrite G into a normal form using the rewrite rules

  <u>if</u> $G = \top$ <u>then</u> <u>return</u> 1 <u>else</u> <u>return</u> 0

<u>end</u>

Example

Let's evaluate the formula

$$(p \rightarrow q) \wedge (p \wedge q \rightarrow r) \rightarrow (p \rightarrow r)$$

in the interpretation

$$\{p \mapsto 1, q \mapsto 0, r \mapsto 1\}.$$

We obtain

$$(\top \rightarrow \bot) \wedge (\top \wedge \bot \rightarrow \top) \rightarrow (\top \rightarrow \top).$$

Apply rewrite rules

$$(\top \rightarrow \bot) \wedge (\top \wedge \bot \rightarrow \top) \rightarrow (\top \rightarrow \top) \Rightarrow$$

$$\color{red}{(\top \rightarrow \bot)} \wedge (\top \wedge \bot \rightarrow \top) \rightarrow (\top \rightarrow \top) \Rightarrow$$

$$\bot \wedge (\top \wedge \bot \rightarrow \top) \rightarrow (\top \rightarrow \top) \Rightarrow$$

$$\color{red}{\bot \wedge (\top \wedge \bot \rightarrow \top)} \rightarrow (\top \rightarrow \top) \Rightarrow$$

$$\bot \rightarrow (\top \rightarrow \top) \Rightarrow$$

$$\color{red}{\bot \rightarrow (\top \rightarrow \top)} \Rightarrow$$

$$\top$$

Russian spy puzzle

There are three persons: Stirlitz, Müller, and Eismann. It is known that exactly one of them is Russian, while the other two are Germans. Moreover, every Russian must be a spy.

When Stirlitz meets Müller in a corridor, he makes the following joke: “you know, Müller, you are as German as I am Russian”. It is known that Stirlitz always says the truth when he is joking.

We have to establish that Eismann is not a Russian spy.

Formalisation

Introduce propositions XY, where

$$X \in \{R, G, S\}$$ (denoting Russian, German, Spy)

$$Y \in \{S, M, E\}$$ (denoting Stirlitz, Müller, Eismann)

For example,

SE : Eismann is a Spy

RS : Stirlitz is Russian

Formalisation

(1) $(RS \wedge GM \wedge GE) \vee (GS \wedge RM \wedge GE) \vee (GS \wedge GM \wedge RE)$

(2) $(RS \rightarrow SS) \wedge (RM \rightarrow SM) \wedge (RE \rightarrow SE)$

(3) $RS \leftrightarrow GM$

(4) $RS \leftrightarrow \neg GS$

(5) $RM \leftrightarrow \neg GM$

(6) $RE \leftrightarrow \neg GE$

---

(7) $RE \wedge SE$

Summary

We have studied notions of:

- ▷ satisfiability, validity, equivalence
- ▷ Using a semantic method of truth tables we can solve the above problems for a small number of variables
  - ▷ for a large number of variables truth tables are impractical
- ▷ We introduced a syntactic method for evaluation of a formula

Next, more practical algorithms for satisfiability.