Logic in Computer Science (CS2142) *

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Based on a course by Prof. Andrei Voronkov

* Times: Thursdays at 10.00 and Fridays at 15.00 Room: 1.3

Course Structure

- Lectures: Thursdays at 10.00 and Fridays at 15.00; Room 1.3
- Assignments: Strict Deadlines!
- Exams

Course Material:

- slides
- handouts

Why logic?

- Formal specification – no ambiguity
- Formal reasoning – prove properties of systems
- Tools for automation of reasoning

Why logic?

- Formal specification – no ambiguity
- Formal reasoning – prove properties of systems
- Tools for automation of reasoning

Computer Science is about developing programs and hardware.

Logic in Computer Science is used in:

- Design of safe and reliable software and hardware
- Verification of existing programs and hardware designs
- Providing suitable formalism for automation

Logic in computer science

- circuit design
- constraint satisfaction
- planning
- software and hardware verification:
  - model checking
  - Hoare's logics
  - higher-order logics
- databases
- theorem proving in mathematics

This course

- propositional logic
- satisfiability checking in propositional logic
- semantic tableaux
- binary decision diagrams (BDDs)
- quantified boolean formulas
- propositional logic of finite domains
- state-changing systems and transition systems
- temporal logic
- model checking

What is logic?

- Syntax: formal language
- Semantics: meaning for the language
- Reasoning:
  - Proof theory
  - Model theory

Why Propositional Logic?

- Propositional logic is one of the simplest logics
- Propositional logic has direct applications e.g. circuit design
- There are efficient algorithms for reasoning in propositional logic
- Propositional logic is a foundation for most of the more expressive logics

Our next goal is to study properties of propositional formulas and devise algorithms for reasoning in propositional logic.

Propositional (Boolean) Logic

Example: ”If I study hard and I complete all assignments then I will get a good grade.”

Atomic propositions (can be true or false):

- I study hard
- I complete all assignments
- I will get a good grade

From atomic propositions we can construct more complex propositions (formulas) using boolean connectives (and, or, not,...).

Next: Syntax and Semantics

Syntax: Propositional Formulas

Propositional (boolean) variables usually denoted as $p, q, s, \ldots$.

Connectives: $\wedge$ "and", $\vee$ "or", $\neg$ "not", $\longrightarrow$ "implies", $\leftrightarrow$ "equivalent"

Propositional formula:

- Every propositional variable is a formula, also called atomic formula, or simply atom.
- $\top$ (called truth) and $\bot$ (false) are formulas.
- If $A_1, \ldots, A_n$ are formulas, where $n \geq 2$, then $(A_1 \wedge \ldots \wedge A_n)$ and $(A_1 \vee \ldots \vee A_n)$ are formulas.
- If A is a formula, then $\neg A$ is a formula.
- If A and B are formulas, then $(A \longrightarrow B)$ and $(A \leftrightarrow B)$ are formulas.

Subformulas

Example: $((p \wedge q) \rightarrow (q \vee \neg p \vee s))$

Immediate Subformulas:

$(p \wedge q)$ and $(q \vee \neg p \vee s)$

Subformulas:

$((p \wedge q) \rightarrow (q \vee \neg p \vee s))$;
$(p \wedge q)$ and $(q \vee \neg p \vee s)$;
p; q; $\neg p$; s

Notation: $A[B]$ means B occurs in A as a subformula.

Connectives

Example: $((p \wedge q) \rightarrow (q \vee \neg p \vee s))$ (too many brackets...)

Connective	Name	Priority
$\neg$	negation	4
$\wedge$	conjunction	3
$\vee$	disjunction	3
$\rightarrow$	implication	2
$\leftrightarrow$	equivalence	1

Now we can replace

$((p \wedge q) \rightarrow (q \vee \neg p \vee s))$ with $p \wedge q \rightarrow q \vee \neg p$.

Semantics: Interpretation

An interpretation I assigns truth values to propositional variables

$$I : P \to \{1, 0\}$$

$0, 1$ are called truth values or also boolean values.

- If $I(p) = 1$, then p is called true in I.
- If $I(p) = 0$, then p is called false in I.

Interpretations are also called truth assignments. Example:
$I(p) = 0;\ I(q) = 1;\ I(s) = 0$

Truth value

Extend I to all formulas:

1. $I(\top) = 1$ and $I(\bot) = 0$.
2. $I(A_1 \wedge \ldots \wedge A_n) = 1$ if and only if $I(A_i) = 1$ for all i.
3. $I(A_1 \vee \ldots \vee A_n) = 1$ if and only if $I(A_i) = 1$ for some i.
4. $I(\neg A) = 1$ if and only if $I(A) = 0$.
5. $I(A \rightarrow B) = 1$ if and only if $I(A) = 0$ or $I(B) = 1$.
6. $I(A \leftrightarrow B) = 1$ if and only if $I(A) = I(B)$.

Notation: $I \models A$ if $I(A) = 1$ (A is true in I)

$I \not\models A$ if $I(A) = 0$ (A is false in I)

Truth Tables

A	B
0	0
1	0
0	1
1	1

Truth Tables

A	B	$A \wedge B$
0	0	0
1	0	0
0	1	0
1	1	1

Truth Tables

A	B	$A \wedge B$	$A \vee B$
0	0	0	0
1	0	0	1
0	1	0	1
1	1	1	1

Truth Tables

A	B	$A \wedge B$	$A \vee B$	$\neg A$
0	0	0	0	1
1	0	0	1	0
0	1	0	1	1
1	1	1	1	0

Truth Tables

A	B	$A \wedge B$	$A \vee B$	$\neg A$	$A \rightarrow B$
0	0	0	0	1	1
1	0	0	1	0	0
0	1	0	1	1	1
1	1	1	1	0	1

Truth Tables

A	B	$A \wedge B$	$A \vee B$	$\neg A$	$A \rightarrow B$	$A \leftrightarrow B$
0	0	0	0	1	1	1
1	0	0	1	0	0	0
0	1	0	1	1	1	0
1	1	1	1	0	1	1

Operation tables

$\wedge$	1	0
1	1	0
0	0	0

$\vee$	1	0
1	1	1
0	1	0

$\neg$	
1	0
0	1

$\rightarrow$	1	0
1	1	0
0	1	1

$\leftrightarrow$	1	0
1	1	0
0	0	1

How to evaluate a formula?

Let's evaluate the formula

$$(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$$

in the interpretation

$$\{p \mapsto 1, q \mapsto 0, r \mapsto 1\}.$$

Evaluating a formula. 1

	formula	value
1	$(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$	

Evaluating a formula. 2

	formula	value
1	$(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$	
2	$p \to r$	
3	$(p \to q) \wedge (p \wedge q \to r)$	

Evaluating a formula. 3

	formula	value
1	$(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$	
2	$p \to r$	
3	$(p \to q) \wedge (p \wedge q \to r)$	
4	$p \wedge q \to r$	
5	$p \to q$	

Evaluating a formula. 4

	formula	value
1	$(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$	
2	$p \to r$	
3	$(p \to q) \wedge (p \wedge q \to r)$	
4	$p \wedge q \to r$	
5	$p \to q$	
6	$p \wedge q$	
7		
9	r	

Evaluating a formula. 5

	formula	value
1	$(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$	
2	$p \to r$	
3	$(p \to q) \wedge (p \wedge q \to r)$	
4	$p \wedge q \to r$	
5	$p \to q$	
6	$p \wedge q$	
7	p	
9	r r	

Evaluating a formula. 6

	formula	value
1	$(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$	
2	$p \to r$	
3	$(p \to q) \wedge (p \wedge q \to r)$	
4	$p \wedge q \to r$	
5	$p \to q$	
6	$p \wedge q$	
7	p p	
8	q	
9	r r	

Evaluating a formula. 7

	formula	value
1	$(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$	
2	$p \to r$	
3	$(p \to q) \wedge (p \wedge q \to r)$	
4	$p \wedge q \to r$	
5	$p \to q$	
6	$p \wedge q$	
7	p p p	
8	q q	
9	r r	

Evaluating a formula. 8

	formula	value
1	$(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$	
2	$p \to r$	
3	$(p \to q) \wedge (p \wedge q \to r)$	
4	$p \wedge q \to r$	
5	$p \to q$	
6	$p \wedge q$	
7	p p p	1
8	q q	0
9	r r	1

Evaluating a formula. 9

	formula	value
1	$(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$	
2	$p \to r$	
3	$(p \to q) \wedge (p \wedge q \to r)$	
4	$p \wedge q \to r$	
5	$p \to q$	
6	$p \wedge q$	0
7	p p p	1
8	q q	0
9	r r	1

Evaluating a formula. 10

	formula	value
1	$(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$	
2	$p \to r$	
3	$(p \to q) \wedge (p \wedge q \to r)$	
4	$p \wedge q \to r$	
5	$p \to q$	0
6	$p \wedge q$	0
7	p p p	1
8	q q	0
9	r r	1

Evaluating a formula. 11

	formula	value
1	$(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$	
2	$p \to r$	
3	$(p \to q) \wedge (p \wedge q \to r)$	
4	$p \wedge q \to r$	1
5	$p \to q$	0
6	$p \wedge q$	0
7	p p p	1
8	q q	0
9	r r	1

Evaluating a formula. 12

	formula	value
1	$(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$	
2	$p \to r$	
3	$(p \to q) \wedge (p \wedge q \to r)$	0
4	$p \wedge q \to r$	1
5	$p \to q$	0
6	$p \wedge q$	0
7	p p p	1
8	q q	0
9	r r	1

Evaluating a formula. 13

	formula	value
1	$(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$	
2	$p \to r$	1
3	$(p \to q) \wedge (p \wedge q \to r)$	0
4	$p \wedge q \to r$	1
5	$p \to q$	0
6	$p \wedge q$	0
7	p p p	1
8	q q	0
9	r r	1

Evaluating a formula. 14

	formula	value
1	$(p \to q) \wedge (p \wedge q \to r) \to (p \to r)$	1
2	$p \to r$	1
3	$(p \to q) \wedge (p \wedge q \to r)$	0
4	$p \wedge q \to r$	1
5	$p \to q$	0
6	$p \wedge q$	0
7	p p p	1
8	q q	0
9	r r	1

Summary

We started studying propositional logic:

- Syntax – propositional formulas
- Semantics – Interpretations assigning truth values

Next: satisfiability, validity, equivalence