Forgetting Concept and Role Symbols in $\mathcal{ALCH}$-Ontologies*

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Abstract. We develop a resolution-based method for forgetting concept and role symbols in $\mathcal{ALCH}$ ontologies, or for computing uniform interpolants in $\mathcal{ALCH}$. Uniform interpolants use only a restricted set of symbols, while preserving logical consequences of the original ontology involving these symbols. While recent work towards practical methods for uniform interpolation in expressive description logics limits attention to forgetting concept symbols, we believe most applications would benefit from the possibility to forget both role and concept symbols. We focus on the description logic $\mathcal{ALCH}$, which allows for the formalisation of role hierarchies. Our approach is based on a recently developed resolution-based calculus for forgetting concept symbols in $\mathcal{ALC}$ ontologies, which we extend by redundancy elimination techniques to make it practical for larger ontologies. Experiments on $\mathcal{ALCH}$ fragments of real life ontologies suggest that our method is applicable in a lot of real-life applications.

1 Introduction

Ontologies model a domain of interest using description logics by describing the vocabulary of this domain in terms of roles and concepts. Reflecting the different applications and contexts in which ontologies are used, ontologies are modelled using different description logics that vary in expressivity and complexities of common reasoning tasks. In the development of complex ontologies, it is often desirable to restrict the vocabulary of an ontology to a smaller set of symbols. Uniform interpolation, also known as forgetting, establishes this by constructing a new ontology that only uses a predefined set of symbols, such that all logical consequences of the original ontology using these symbols are preserved. Examples where this is useful are: (i) *Ontology Reuse.* When constructing larger ontologies, it can be useful to reuse parts from existing ontologies. Using uniform interpolation, one can restrict the vocabulary of the reused ontology to the symbols that are known and interesting for the new application. (ii) *Predicate Hiding.* When publishing or sharing an ontology, it is often desirable to hide confidential parts from the ontology, without affecting the intended meaning of the remaining vocabulary [6]. (iii) *Exhibiting Hidden Relations.* Relations between symbols are often stated indirectly in an ontology and only become visible through the use

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of reasoners. With increased complexity of the ontology, this makes it hard to get a deeper understanding of the ontology and to maintain ontology changes. The uniform interpolant over a set of symbols makes the relations between these symbols explicit. (iv) *Logical difference.* In the development of evolving ontologies, it is important for ontology engineers to ensure that modifications do not interfere with the meaning of existing terms. This can be achieved by computing the uniform interpolants of two versions of an ontology over the common set of used symbols, or over a set of symbols under consideration, and checking whether the resulting ontologies are equivalent [13].

Uniform interpolation has been extensively investigated for simpler description logics such as $\mathcal{EL}$ or DL-Lite [9, 24, 16, 14]. Recently, practical algorithms for forgetting concept symbols in the more expressive description logic $\mathcal{ALC}$ have been developed [13, 12]. In this paper, we investigate forgetting of role symbols as well, and supplement earlier presented work with optimisation techniques to make it practical on larger ontologies. Since roles play a larger role in this context, we focus on the description logic $\mathcal{ALCH}$, which extends $\mathcal{ALC}$ with role hierarchies. It is known that already in the description logic $\mathcal{ALC}$ uniform interpolants cannot be finitely expressed in the language of the logic [15]. This also applies to $\mathcal{ALCH}$. For this reason our method computes uniform interpolants for the target language $\mathcal{ALCH}\mu$, which extends $\mathcal{ALCH}$ with fixpoint operators, thus enabling us to always compute finite representations. If fixpoints are used in the uniform interpolant, it is possible to represent it in $\mathcal{ALCH}$ by extending the signature of the interpolant.

Our work is based on a recently developed method for forgetting concept symbols in $\mathcal{ALC}$-ontologies [12]. The method is based on a resolution-based decision procedure for $\mathcal{ALCH}$. In order to analyse the practicality of our approach, we undertake an experimental evaluation on $\mathcal{ALCH}$-fragments of a set of real-life ontologies. The results suggest that uniform interpolation can be used for the presented applications in a lot of real-life situations.

Proofs of all theorems can be found in the accompanying technical report [10].

2 Preliminaries

Let N_c, N_r be two disjoint sets of *concept symbols* and *role symbols*. *Concepts* in $\mathcal{ALCH}$ are of the following form:

$$\bot \mid \top \mid A \mid \neg C \mid C \sqcup D \mid C \sqcap D \mid \exists r.C \mid \forall r.C,$$

where $A \in N_c$, $r \in N_r$ and C and D are arbitrary concepts. $\top$, $C \sqcap D$ and $\forall r.C$ are defined as abbreviations: $\top$ stands for $\neg\bot$, $C \sqcap D$ for $\neg(\neg C \sqcup \neg D)$ and $\forall r.C$ for $\neg\exists r.\neg C$.

A TBox is a set of *concept axioms* of the forms $C \sqsubseteq D$ (*concept inclusion*) and $C \equiv D$ (*concept equivalence*), where C and D are concepts. An RBox is a set of *role axioms* of the form $r \sqsubseteq s$ (*role inclusion*) and $r \equiv s$ (*role equivalence*), where r and s are role symbols. $C \equiv D$ is a short-hand for the two concept axioms $C \sqsubseteq D$ and $D \sqsubseteq C$, and $r \equiv s$ is a short-hand for the two role axioms $r \sqsubseteq s$ and $s \sqsubseteq r$. We assume an ontology consists of a TBox and an RBox. Given an ontology $\mathcal{O}$, we define $\sqsubseteq_{\mathcal{O}}$ to be the reflexive transitive closure of the role inclusions in $\mathcal{O}$.

The semantics of $\mathcal{ALCH}$ is defined as follows. An *interpretation* is a pair $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, where the *domain* $\Delta^{\mathcal{I}}$ is a nonempty set and the *interpretation function* $\cdot^{\mathcal{I}}$ assigns to each concept symbol $A \in N_c$ a subset of $\Delta^{\mathcal{I}}$ and to each role symbol $r \in N_r$ a subset of $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$. The interpretation function is extended to concepts as follows.

$$\begin{array}{lll} \bot^{\mathcal{I}} := \emptyset & (\neg C)^{\mathcal{I}} := \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} & (C \sqcup D)^{\mathcal{I}} := C^{\mathcal{I}} \cup D^{\mathcal{I}} \\ \multicolumn{3}{l}{(\exists r.C)^{\mathcal{I}} := \{x \in \Delta^{\mathcal{I}} \mid \exists y : (x,y) \in r^{\mathcal{I}} \land y \in C^{\mathcal{I}}\}} \end{array}$$

A concept inclusion $C \sqsubseteq D$ is *true* in an interpretation $\mathcal{I}$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. $\mathcal{I}$ is model of a TBox $\mathcal{T}$ if all concept inclusions in $\mathcal{T}$ are true in $\mathcal{I}$. A TBox $\mathcal{T}$ is *satisfiable* if there exists a model for $\mathcal{T}$, otherwise it is *unsatisfiable*. $\mathcal{T} \models C \sqsubseteq D$ holds iff in every model of $\mathcal{T}$ we have $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. Two TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$ are *equi-satisfiable* if every model of $\mathcal{T}_1$ can be extended to a model of $\mathcal{T}_2$, and vice versa. The definitions of truth, model, satisfiability and equi-satisfiability extend to roles, RBoxes and ontologies in a similar way. Observe that $\mathcal{O} \models r \sqsubseteq s$ iff $r \sqsubseteq_{\mathcal{O}} s$.

In order to define $\mathcal{ALCH}\mu$, we extend the language with a set N_v of *concept variables*. $\mathcal{ALCH}\mu$ extends $\mathcal{ALCH}$ with concepts of the form $\mu X.C$ and $\nu X.C$, where $X \in N_v$, and C is a concept in which X occurs as a concept symbol only positively (under an even number of negations). $\mu X.C$ denotes the *least fixpoint* of C on X and $\nu X.C$ the *greatest fixpoint*.

A concept variable X is *bound* if it occurs in the scope C of a fixpoint expression $\mu X.C$ or $\nu X.C$. Otherwise it is *free*. A concept is *closed* if it does not contain any free variables. Axioms in $\mathcal{ALCH}\mu$ are of the form $C \sqsubseteq D$ and $C \equiv D$, where C and D are closed concepts.

Following [3], we define the semantics of fixpoint expressions. Let $\mathcal{V}$ be an *assignment function* that maps concept variables to subsets of $\Delta^{\mathcal{I}}$. $\mathcal{V}[X \mapsto W]$ denotes $\mathcal{V}$ modified by setting $\mathcal{V}(X) = W$. $C^{\mathcal{I},\mathcal{V}}$ is the interpretation of C taking into account this assignment, and when $\mathcal{V}$ is defined for all variables in C, $C^{\mathcal{I},\mathcal{V}} = C^{\mathcal{I}}$. The semantics of fixpoint concepts is defined as follows:

$$\begin{aligned} (\mu X.C)^{\mathcal{I},\mathcal{V}} &:= \bigcap \{W \subseteq \Delta^{\mathcal{I}} \mid C^{\mathcal{I},\mathcal{V}[X \mapsto W]} \subseteq W\} \\ (\nu X.C)^{\mathcal{I},\mathcal{V}} &:= \bigcup \{W \subseteq \Delta^{\mathcal{I}} \mid W \subseteq C^{\mathcal{I},\mathcal{V}[X \mapsto W]}\}. \end{aligned}$$

The *size* of an ($\mathcal{ALCH}$- or $\mathcal{ALCH}\mu$-)axiom is defined recursively as follows: $\mathit{size}(A) = 1$, where A is a concept symbol, $\mathit{size}(\neg C) = \mathit{size}(C) + 1$, $\mathit{size}(\exists r.C) = \mathit{size}(\forall r.C) = \mathit{size}(C) + 2$, $\mathit{size}(C \sqcup D) = \mathit{size}(C \sqcap D) = \mathit{size}(C) + \mathit{size}(D) + 1$, $\mathit{size}(\mu X.C) = \mathit{size}(\nu X.C) = \mathit{size}(C) + 2$ and $\mathit{size}(C \sqsubseteq D) = \mathit{size}(C \equiv D) = \mathit{size}(C) + \mathit{size}(D) + 1$.

A *signature* Σ is a subset of $N_c \cup N_r$. $\mathit{sig}(E)$ denotes the concept and role symbols occurring in E, where E ranges over concept descriptions, axioms, TBoxes, RBoxes and ontologies. Given two ontologies $\mathcal{O}_1$ and $\mathcal{O}_2$ and a signature Σ, we say $\mathcal{O}_1$ and $\mathcal{O}_2$ are *Σ-inseparable*, in symbols $\mathcal{O}_1 \equiv_\Sigma \mathcal{O}_2$, iff for every concept or role inclusion α with $sig(\alpha) \subseteq \Sigma$, $\mathcal{O}_1 \models \alpha$ implies $\mathcal{O}_2 \models \alpha$, and vice versa. Given an ontology $\mathcal{O}$ and a signature Σ, $\mathcal{O}'$ is a *uniform interpolant of* $\mathcal{O}$ if $sig(\mathcal{O}') \subseteq \Sigma$ and $\mathcal{O} \equiv_\Sigma \mathcal{O}'$. From this definition, it follows that uniform interpolants for a given ontology and signature are unique modulo logical equivalence. For a given ontology $\mathcal{O}$ and signature Σ, we will therefore speak of *the* uniform interpolant and denote it by $\mathcal{O}^\Sigma$. Given an ontology $\mathcal{O}$ and a concept or role symbol σ, the result of *forgetting σ in $\mathcal{O}$*, denoted by $\mathcal{O}^{-\sigma}$, is the uniform interpolant $\mathcal{O}^\Sigma$, where $\Sigma = sig(\mathcal{O}) \setminus \{\sigma\}$.

3 Overview of the Method

We reduce the problem of computing uniform interpolants to the problem of forgetting single symbols. In order to compute the uniform interpolant for any signature Σ, we forget each symbol in $sig(\mathcal{O}) \setminus \Sigma$ one by one. The method for computing $\mathcal{O}^{-\sigma}$, where σ is either a role or a concept symbol, consists of three phases:

Phase 1: Eliminate the symbol using a resolution-based calculus, obtaining $\mathcal{O}' = \mathcal{F}^\sigma_{\mathcal{ALCH}}(\mathcal{O})$.
Phase 2: Eliminate the newly introduced symbols, obtaining $\mathcal{O}^{-\sigma} = \mathcal{F}_D(\mathcal{O}')$.
Phase 3: Apply simplifications and represent clauses as proper concept inclusions.

Central to the method is a new resolution-based calculus which works on a structural transformation based normal form. The calculus is described in Section 4. Depending on whether the symbol to be forgotten is a role or a concept symbol, in Phase 1 a different method based on this resolution calculus is used to derive consequences on the selected symbol. This is described in Section 5. The result is a finitely bounded set N of axioms such that $\sigma \notin sig(N)$ and $N \equiv_\Sigma \mathcal{O}$ for $\Sigma = sig(\mathcal{O}) \setminus \{\sigma\}$, but N may use new symbols due to structural transformation. These symbols, called definers, all occur in a form that allows for elimination in a simple and uniform way, following a known principle first presented in [17]. This is performed in Phase 2 and described in Section 6. Depending on whether the aim is to compute a representation in $\mathcal{ALCH}\mu$ or in $\mathcal{ALCH}$, the result may involve fixpoint operators or extend the signature of the original ontology.

After Phase 2, the uniform interpolant is already computed, but we add a third phase that makes the resulting ontology more accessible by applying several equivalence-preserving transformations. The following main theorem of this paper states the correctness of the method.

Theorem 1. *For any $\mathcal{ALCH}$-ontology $\mathcal{O}$ and any symbol σ, our method terminates and returns the uniform interpolant of $\mathcal{O}$ over $sig(\mathcal{O}) \setminus \{\sigma\}$ in $\mathcal{ALCH}\mu$. If the result does not make use of a fixpoint operator, it is the uniform interpolant of $\mathcal{O}$ over $sig(\mathcal{O}) \setminus \{\sigma\}$ in $\mathcal{ALCH}$.*

4 The Underlying Calculus

Our method for forgetting concept and role symbols is based on a resolution calculus $\mathcal{R}_{\mathcal{ALCH}}$ which provides a decision procedure for $\mathcal{ALCH}$-ontology satisfiability. The calculus extends a calculus introduced in [12] by incorporating the role hierarchy. In order to make our method practical for larger ontologies, we extend $\mathcal{R}_{\mathcal{ALCH}}$ with redundancy elimination techniques, resulting in the calculus $\mathcal{R}^s_{\mathcal{ALCH}}$.

Both calculi operate on sets of clauses, which are defined as follows. Let $N_D \subseteq N_c$ be a set of *definer symbols* or *definers*, which do not occur in any input ontology.

Definition 1. *An $\mathcal{ALCH}$-*literal *is a concept description of the form A, $\neg A$, $\forall r.D$ or $\exists r.D$, where A is a concept symbol, r a role symbol and D is a definer.*

*A TBox is in $\mathcal{ALCH}$-*conjunctive normal form *if every axiom is of the form $\top \sqsubseteq L_1 \sqcup ... \sqcup L_n$, where each L_i is an $\mathcal{ALCH}$-literal. The right part of such a concept inclusion is called $\mathcal{ALCH}$-*clause*. In the following we assume $\mathcal{ALCH}$-clauses are represented as sets of literals (this means no clause contains the same literal more than once and the order of the literals does not matter). The empty clause is denoted by $\perp$ and represents a contradiction.*

For our method it is crucial that any $\mathcal{ALCH}$-TBox is transformed into an equisatisfiable TBox in $\mathcal{ALCH}$-conjunctive normal form using structural transformation as follows. First the input TBox is transformed into negation normal form. Then every concept C that occurs immediately below a role restriction is replaced by a definer D, and we add the axiom $D \sqsubseteq C$ for each such subconcept. The resulting TBox does not contain any nested role restrictions and can be brought into $\mathcal{ALCH}$-conjunctive normal form by applying standard CNF-transformation techniques. For an ontology $\mathcal{O}$, let *clauses*$(\mathcal{O})$ refer to the set of clauses generated in this way from the TBox of $\mathcal{O}$.

The calculus $\mathcal{R}_{\mathcal{ALCH}}$ uses the rules shown in Figure 1. Since the normal form has to be preserved, the role propagation rule may require the introduction of a new definer symbol D_3 representing the conjunction of the definers D_1 and D_2 occurring in the premises. This is done by adding new clauses $\neg D_3 \sqcup D_1$ and $\neg D_3 \sqcup D_2$ to the clause set. Observe that the resolution rule also applies to definer literals. This way for each pair of clauses $\neg D_1 \sqcup C_1$ and $\neg D_2 \sqcup C_2$ we derive the clauses $\neg D_3 \sqcup C_1$ and $\neg D_3 \sqcup C_2$, for which the side conditions of the rules are satisfied.

In order to ensure termination, it is necessary to reuse definers whenever possible. For this we define an identification function for introduced definers, that identifies definers with the context from which they have been created, and whose range is finitely bounded. The function $id(D)$ is defined as follows. (i) If D is introduced by the initial normal form transformation, then $id(D) = \{D\}$. (ii) If D is required by the role propagation rule and the respective role restrictions are $\forall s.D_1$ and $\mathsf{Q}r.D_2$, then $id(D) = id(D_1) \cup id(D_2)$.

If the role propagation rule requires a new definer D, we first check whether a definer D' with $id(D) = id(D')$ is already present, and reuse it in this case.

Resolution:

$$\frac{C_1 \sqcup A \qquad C_2 \sqcup \neg A}{C_1 \sqcup C_2}$$

provided $C_1 \sqcup C_2$ does not contain more than one negative definer literal.

Role Propagation:

$$\frac{C_1 \sqcup \forall s.D_1 \qquad C_2 \sqcup \mathsf{Q}r.D_2}{C_1 \sqcup C_2 \sqcup \mathsf{Q}r.D_3} \quad r \sqsubseteq_{\mathcal{O}} s$$

where $\mathsf{Q} \in \{\exists, \forall\}$ and D_3 is a (possibly new) definer representing $D_1 \sqcap D_2$, provided $C_1 \sqcup C_2$ does not contain more than one negative definer literal.

Existential Role Restriction Elimination:

$$\frac{C \sqcup \exists r.D \qquad \neg D}{C}$$

Fig. 1. Rules of the calculus $\mathcal{R}_{\mathcal{ALCH}}$.

Otherwise we introduce a new definer in the way described above. Observe that the domain of *id* is bounded by 2^n, where n is the number of definers introduced by the initial normal form transformation. Therefore the number of clauses that can possibly be derived is limited by a double-exponential bound. We can prove:

Theorem 2. *$\mathcal{R}_{\mathcal{ALCH}}$ is sound and refutationally complete, and provides a decision procedure for $\mathcal{ALCH}$-ontology satisfiability.*

As in traditional resolution-based decision procedures, it is possible to extend the method with redundancy elimination and further simplification techniques. For this purpose, it is possible to exploit the structure imposed by the introduced definers. Note that new definers are introduced by adding clauses of the form $\neg D_1 \sqcup D_2$. $\neg D_1 \sqcup D_2$ is equivalent to the concept inclusion $D_1 \sqsubseteq D_2$. This concept inclusion can be transferred to subsumption between existential and universal role restrictions, and to subsumption between clauses.

Definition 2. *A literal l_1 is* subsumed by a literal l_2 *($l_1 \sqsubseteq_l l_2$) if either $l_1 = l_2$ or if $l_1 = \mathsf{Q}r.D_1$ and $l_2 = \mathsf{Q}r.D_2$ for $\mathsf{Q} \in \{\exists, \forall\}$ and there is a clause $\neg D_1 \sqcup D_2$ in the current clause set. A clause C_1 is* subsumed by a clause C_2 *($C_1 \sqsubseteq_C C_2$) if every literal $l_1 \in C_1$ is subsumed by a literal $l_2 \in C_2$. A clause C is* redundant *with respect to a clause set N, if N contains a clause C' with $C' \sqsubseteq_C C$. The* reduction *of a clause C, red(C), is obtained from C by removing every literal that is subsumed by another literal in C.*

Example 1 (Subsumption and reduction). Assume D_3 represents $D_1 \sqcap D_2$, which means we have the clauses $\neg D_3 \sqcup D_1$ and $\neg D_3 \sqcup D_2$. Then $\neg A \sqcup B$ is subsumed by

Tautology deletion:	$\dfrac{N \cup \{C \sqcup A \sqcup \neg A\}}{N}$	
Subsumption deletion:	$\dfrac{N \cup \{C,\ D\}}{N \cup \{C\}}$	provided $C \sqsubseteq_C D$
Reduction:	$\dfrac{N \cup \{C\}}{N \cup \{red(C)\}}$	

Fig. 2. Simplification rules.

$\neg A \sqcup B \sqcup C$, $\exists r.D_3$ is subsumed by $\exists r.D_1$, $\forall r.D_3 \sqcup B$ is subsumed by $\forall r.D_1 \sqcup A \sqcup B$ and $red(A \sqcup \exists r.D_3 \sqcup \exists r.D_2) = A \sqcup \exists r.D_2$.

In addition to subsumption and reduction, we also detect tautological clauses which contain pairs of contradictory literals. This leads to a set of simplification rules shown in Figure 2. We denote the calculus $\mathcal{R}_{\mathcal{ALCH}}$ extended with these rules by $\mathcal{R}^s_{\mathcal{ALCH}}$. It can be shown that these rules preserve soundness and refutational completeness, as stated by the following theorem.

Theorem 3. *$\mathcal{R}^s_{\mathcal{ALCH}}$ is sound and refutationally complete and provides a decision procedure for $\mathcal{ALCH}$-ontology satisfiability.*

5 Forgetting Concept and Role Symbols

In this section, we describe the methods $\mathcal{F}^A_{\mathcal{ALCH}}$ and $\mathcal{F}^r_{\mathcal{ALCH}}$ for forgetting respectively concept symbols and role symbols. Both methods are based on $\mathcal{R}^s_{\mathcal{ALCH}}$.

For any definer D, we say D is *connected to* A, if D either co-occurs with A in a clause or if D co-occurs in a clause with another definer D' that is connected to A. If the aim is to forget a concept symbol, we restrict the rules of $\mathcal{R}^s_{\mathcal{ALCH}}$ by adding the following conditions:

Resolution: A is the symbol we want to forget or a definer.
Role Propagation: D_1 and D_2 are connected to the symbol we want to forget.

For a concept symbol A, $\mathcal{F}^A_{\mathcal{ALCH}}$ denotes the calculus $\mathcal{R}^s_{\mathcal{ALCH}}$ with these modifications for A. For any ontology $\mathcal{O}$, $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O})$ denotes the ontology consisting of the RBox of $\mathcal{O}$ and the TBox represented by *clauses*$(\mathcal{O})$ saturated using the rules of $\mathcal{F}^A_{\mathcal{ALCH}}$, after removing all clauses containing A or positive definer literals that are not role restrictions.

Theorem 4. *Given an ontology $\mathcal{O}$, $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O})$ is a clausal representation of $\mathcal{O}^{-A}$, that is, $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O}) \equiv_\Sigma \mathcal{O}$, where $\Sigma = sig(\mathcal{T}) \setminus \{A\}$, and every symbol in $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O})$ is either a definer or in Σ.*

Role hierarchy:

$$\frac{s \sqsubseteq r \qquad r \sqsubseteq t}{s \sqsubseteq t}$$

Universal role restriction monotonicity:

$$\frac{C \sqcup \forall r.D}{C \sqcup \forall s.D} \qquad s \sqsubseteq r \in \mathcal{O}$$

Existential role restriction monotonicity:

$$\frac{C \sqcup \exists r.D}{C \sqcup \exists s.D} \qquad r \sqsubseteq s \in \mathcal{O}$$

Role restriction resolution:

$$\frac{C_0 \sqcup \forall r.D_0 \quad ... \quad C_n \sqcup \forall r.D_n \qquad C \sqcup \exists r.D}{C_0 \sqcup ... \sqcup C_n \sqcup C} \qquad \mathcal{O} \models D_0 \sqcap ... \sqcap D_n \sqcap D \sqsubseteq \bot$$

provided (i) there is no role s with $r \sqsubseteq s \in \mathcal{O}$ and (ii) $C_0 \sqcup ... \sqcup C_n \sqcup C$ does not contain more than one negative definer literal.

Fig. 3. Rules for forgetting role symbol r.

The method $\mathcal{F}^A_{\mathcal{ALCH}}$ provides a focused way to forget the concept symbol A. In order to forget role symbols, a few modifications have to be made. Since role symbols also occur in the RBox of an ontology, the RBox has to be processed as well. Additionally, we need rules that compute all derivations on a selected role symbol in a focused way.

The rules in Figure 3, together with the rules of $\mathcal{R}_{\mathcal{ALCH}}$, where the resolution rule is restricted to only resolve on definer literals, constitute the method $\mathcal{F}^r_{\mathcal{ALCH}}$, where r is the role symbol to be forgotten. The role hierarchy rule is the only rule applied on the RBox of the input ontology, and makes implicit role inclusions around the role symbol to be forgotten explicit. The universal and existential role monotonicity rules compute inferences on the basis of clauses and RBox axioms. If there is no role inclusion $s \sqsubseteq r$, the universal role monotonicity rule cannot be applied and we have to apply role propagation on that role exhaustively in order to preserve all consequences when forgetting r.

If there is no role inclusion $r \sqsubseteq s$, we can neither apply the existential role restriction monotonicity rule nor role propagation. Instead we use the role restriction resolution rule in this case, which is similarly motivated as the resolution rule, but works on larger sets of clauses. This rule is formulated to allow the use of an external reasoner to check satisfiability of concepts (even though in theory $\mathcal{R}_{\mathcal{ALCH}}$ can be used for this as well).

Non-cyclic definer elimination:

$$\frac{\mathcal{T} \cup \{D \sqsubseteq C\}}{\mathcal{T}^{[D \mapsto C]}} \qquad \text{provided } D \notin \mathit{sig}(C)$$

Definer purification:

$$\frac{\mathcal{T}}{\mathcal{T}^{[D \mapsto \top]}} \qquad \text{provided } D \text{ occurs only positively in } \mathcal{T}$$

Cyclic definer elimination:

$$\frac{\mathcal{T} \cup \{D \sqsubseteq C[D]\}}{\mathcal{T}^{[D \mapsto \nu X.C[X]]}} \qquad \text{provided } D \in \mathit{sig}(C[D])$$

Fig. 4. Rules for eliminating definer concept symbols

For any ontology $\mathcal{O}$, we define $\mathcal{F}^r_{\mathcal{ALCH}}(\mathcal{O})$ as the ontology consisting of the RBox of $\mathcal{O}$ and the TBox represented by *clauses*$(\mathcal{O})$ saturated using the rules of $\mathcal{F}^r_{\mathcal{ALCH}}$, after removing all the axioms and clauses that use the symbol r or contain a positive definer literal that is not a role restriction.

Theorem 5. *For any ontology $\mathcal{O}$, $\mathcal{F}^r_{\mathcal{ALCH}}(\mathcal{O})$ is a clausal representation of $\mathcal{O}^{-r}$.*

6 Definer Elimination

In Phase 2, the symbols introduced by the normal form transformation or the role propagation rule are eliminated. Note that we only derive clauses that contain at most one negative definer literal in Phase 1. This means we can for each definer D group the clauses of the form $\neg D \sqcup C_i$, $0 \leq i \leq n$, into a single axiom of the form $D \sqsubseteq \bigsqcap_{0 \leq i \leq n} C_i$ that can be seen as a *definition* of the definer. This definition can be used to undo the structural transformation and eliminate the remaining definers. If a definition is cyclic, we use a fixpoint operator in the result. Figure 4 shows the rules for definer elimination. The rules are justified by Ackermann's Lemma and its generalisation to the fixpoint case [1, 17].

If the output of the algorithm contains fixpoints, we can represent it in $\mathcal{ALCH}$ by extending the desired signature Σ by the cyclic definers. This is done by omitting the cyclic definer elimination rule.

7 Examples

To illustrate the presented method this section includes two examples of respectively forgetting concept and role symbols.

Example 2 (Forgetting Concept Symbols). Let $\mathcal{O}_1$ be the following ontology.

$$A \sqsubseteq B \sqcup C \qquad B \sqsubseteq \exists r.B \qquad C \sqsubseteq \forall s.\neg B \qquad r \sqsubseteq s$$

We want to compute $\mathcal{O}_1^{-B}$. We obtain the following clause set $\mathit{clauses}(\mathcal{O}_1)$.

1. $\neg A \sqcup B \sqcup C$ 2. $\neg B \sqcup \exists r.D_1$ 3. $\neg D_1 \sqcup B$
4. $\neg C \sqcup \forall s.D_2$ 5. $\neg D_2 \sqcup \neg B$

We first apply the resolution rule.

6. $\neg A \sqcup C \sqcup \exists r.D_1$ (*resolution on 2 and 1*)
7. $\neg D_1 \sqcup \exists r.D_1$ (*resolution on 2 and 3*)
8. $\neg D_2 \sqcup \neg A \sqcup C$ (*resolution on 5 and 1*)

We cannot resolve on clauses 3 and 5, since the conclusion would contain more than one negative definer literal. We can however apply role propagation on clauses 2 and 4, which makes further applications of the resolution rule possible.

~~9.~~ $\neg B \sqcup \neg C \sqcup \exists r.D_3$ (*role propagation on 2 and 4*, $\mathit{id}(D_3) = \{D_1, D_2\}$)
~~10.~~ $\neg D_3 \sqcup D_1$
~~11.~~ $\neg D_3 \sqcup D_2$
~~12.~~ $\neg D_3 \sqcup B$ (*resolution on 10 and 3*)
~~13.~~ $\neg D_3 \sqcup \neg B$ (*resolution on 11 and 5*)
14. $\neg D_3$ (*resolution on 12 and 13*)

Clause 14 makes clauses 10–13 become redundant, and existential role restriction elimination on Clause 9 possible.

15. $\neg B \sqcup \neg C$ (*exist. role restr. elimination on 9 and 14*)

Clause 15 makes Clause 9 become redundant. We saturate the remaining clauses.

~~16.~~ $\neg A \sqcup C \sqcup \neg C$ (*resolution on 15 and 1, tautology*)
17. $\neg D_1 \sqcup \neg C$ (*resolution on 15 and 3*)

Only clauses that do not contain B or a positive definer are included in $\mathcal{F}^B_{\mathcal{ALCH}}(\mathcal{O}_1)$. These are the clauses 4, 6, 7, 8, 14 and 17. Eliminating the definers and expressing clauses as concept inclusions (Phases 2 and 3) results in the following ontology $\mathcal{O}_1^{-B}$:

$$A \sqsubseteq C \sqcup \exists r.\nu X.(\neg C \sqcap \exists r.X) \qquad C \sqsubseteq \forall s.(\neg A \sqcup C)$$

Example 3 (Forgetting Role Symbols). Let $\mathcal{O}_2$ contain the following axioms. We want to compute $\mathcal{O}_2^{-r}$.

$$A \sqsubseteq \exists r.(A \sqcup B) \qquad B \sqsubseteq \forall r.\neg A$$
$$C \sqsubseteq \forall r.\neg B \qquad s \sqsubseteq r$$

We obtain the following clausal representation $\mathit{clauses}(\mathcal{O}_2)$:

1. $\neg A \sqcup \exists r.D_1$ 2. $\neg D_1 \sqcup A \sqcup B$
3. $\neg B \sqcup \forall r.D_2$ 4. $\neg D_2 \sqcup \neg A$
5. $\neg C \sqcup \forall r.D_3$ 6. $\neg D_3 \sqcup \neg B$

We observe that there is no role r' with $r \sqsubseteq r'$ and that $D_1 \sqcap D_2 \sqcap D_3$ is unsatisfiable, which means we can apply role restriction resolution on 1, 3 and 5:

7. $\neg A \sqcup \neg B \sqcup \neg C$ (*role restriction resolution on 1, 3 and 5*)

Furthermore, we do have a role r' with $r' \sqsubseteq r$, namely s which means we can apply universal role restriction monotonicity:

8. $\neg B \sqcup \forall s.D_2$ (*universal role restriction monotonicity on 3*)
9. $\neg C \sqcup \forall s.D_3$ (*universal role restriction monotonicity on 5*)

Omitting all clauses containing r and applying Phases 2 and 3 leads to the uniform interpolant $\mathcal{O}_2^{-r}$ consisting of the following axioms:

$$A \sqcap B \sqcap C \sqsubseteq \bot \qquad B \sqsubseteq \forall s.\neg A \qquad C \sqsubseteq \forall s.\neg B$$

8 Experimental Evaluation

In order to investigate the practicality of our approach, we implemented our method in Scala[1] using the OWL-API[2] and evaluated it on $\mathcal{ALCH}$-fragments of ontologies from the NCBO Bioportal ontology repository.[3] The ontologies of this corpus are known to have diverse complexity, size and structure [8]. For the role restriction resolution rule, we made use of the HermiT reasoner Version 1.3.6 [19] for checking satisfiability of conjunctions of definer concepts.

It turns out that several additional optimisations are necessary to make the method perform well on larger ontologies. Especially the role propagation rule creates a lot of unnecessary derivations when applied in its unrestricted form. This can be reduced by analysing the structure of the clause set before applying the rule to see in which cases it actually leads to new derivations on the symbol we want to forget. We further used module extraction in order to reduce the size of the input ontologies. Given an ontology $\mathcal{O}$, the $\top\bot*$-module of $\mathcal{O}$ over Σ contains a subset of the axioms of $\mathcal{O}$ that preserves all consequences of $\mathcal{O}$ in Σ, given $\mathcal{O}$ is consistent [18]. In order to compute $\mathcal{O}^{\Sigma}$, it is therefore sufficient to apply our method on the $\top\bot*$-module of $\mathcal{O}$ over Σ. In order to keep the clauses small, we further apply structural transformation to replace every subconcept C in the TBox that does not contain the symbol we want to forget by a new symbol X, which reduces the number of clauses a lot [13]. These symbols are replaced by the original subconcepts in the final result. For a complete overview of optimisations used we refer to the paper [11] on practical aspects of computing uniform interpolants in $\mathcal{ALC}$.

The corpus for our experiments was created as follows. From the NCBO Bioportal repository, we selected those ontologies that contain role hierarchies, and for which parsing and module extraction using the OWL-API was possible. We then restricted the selected ontologies to $\mathcal{ALCH}$ by removing all axioms that

[1] `http://www.scala-lang.org`
[2] `http://owlapi.sourceforge.net`
[3] `http://bioportal.bioontology.org`

are not expressible in $\mathcal{ALCH}$ using simple reformulations. This led to a corpus of 115 ontologies, on which we ran our experiments.

The experiments were conducted on an Intel Core i5-2400 CPU with four cores running at 3.10 GHz and 8 GB of RAM. Since our implementation does not make use of multi-threading (except for computations of the HermiT reasoner), we ran several experiments in parallel, taking care that experiments do not affect each other due to use of resources.

We started with a series of experiments to evaluate the perfomance of forgetting small sets of symbols, which may for example be interesting for predicate hiding or for computing logical differences between ontology versions, as mentioned in the Introduction. First, we evaluated the performance of concept forgetting. For this, we randomly selected samples of 5, 50, 100 and 150 concept symbols for each ontology and computed the result of forgetting these, with a timeout set to 100 seconds. In 4.5% of the cases, our implementation was not able to compute the uniform interpolant in the given time limit, and in 16.7% of the remaining cases, fixpoints where used in the result. Even though it known that uniform interpolants can be of size triple exponential of the size of the input ontology [15], in our experiments uniform interpolants were much smaller. In fact, in 62.8% of the cases where a uniform interpolant could be computed, the uniform interpolant was smaller than the input ontology. In the worst case however, the uniform interpolant was 104 times bigger than the input ontology. The difference also becomes more apparent when looking at the axiom size.

On average, the average axiom size of the uniform interpolant was 1.8 times bigger than in the input ontology, and the largest axiom size 10.3 times bigger. This effect was to be expected since more information about the role structure of the ontology and indirect concept relations has to be presented in the definitions of fewer concepts. Considering that in the input ontologies the average axiom size was only 3.48, and the average maximal axiom size was 15.21, this still means most axioms were not overly complex. However, in the worst case, the computed uniform interpolant contained an axiom that was 1,406 times bigger than the largest one in the input.

Next we evaluated forgetting of role symbols. Since the role restriction resolution rule makes use of an external reasoner, and can have more than two clauses as premises, one could expect that forgetting role symbols is much more expensive than forgetting concept symbols. On the other hand, since most ontologies have much fewer role symbols than concept symbols, it seems reasonable to conduct the experiments with smaller sets of symbols to be forgotten. We therefore compared how forgetting 5 role symbols performed in comparison with forgetting 5 concept symbols, again with a timeout of 100 seconds. Forgetting role symbols could be performed in 86.6% of the cases in the given time frame, whereas forgetting concept symbols succeeded in 99.8% of the cases. The impact on the ontology size was on the other hand less apparent. In only 3.8% of the cases the uniform interpolant was actually bigger than the input ontology (10.5% for concept symbols), and on average the interpolant was 93% of the size of the input ontology (97% for concept symbols). The largest axiom per ontology was on

$\|\Sigma\|$	Timeouts	Fixpoints	Interpolant Size	Axiom Size	Max. Axiom Size	Average Duration
50	15.12%	6.99%	22.50%	799.37%	1,053.68%	24.2 sec.
100	18.38%	11.57%	45.21%	646.32%	847.36%	21.0 sec.
150	22.25%	13.58%	76.55%	837.66%	5,657.87%	23.7 sec.
All	18.38%	10.44%	45.74%	757.69%	2,309.08%	23.0 sec.

Table 1. Results for computing uniform interpolants.

average 1.58 times larger than in the input ontology (1.18 for concept symbols), and in the worst case 51.1 times larger (360.3 for concept symbols). One might suspect that this result is partly due to the exploitation of role hierarchies using the role restriction monotonicity rules. But it turned out that when ignoring the RBox, the results were nearly unchanged, and even slightly better.

To evaluate our complete method, we computed uniform interpolants for small signatures of size 50, 100 and 150. This corresponds to the applications exhibiting hidden relations and ontology reuse mentioned in the Introduction, as well as predicate hiding, if only a small part of the ontology is to be published.

For these uniform interpolants, usually a large number of symbols, including both role and concept symbols, had to be forgotten from the input ontology, even though module extraction already performs part of the job. For this reason we set a higher timeout of 1,000 seconds. The results are summarised in Table 1. It shows the percentage of experimental runs where a timeout occured, the percentage in the remaining set where fixpoints were used in the result, the ontology size, average axiom size and maximal axiom size of each uniform interpolant compared to the respective values of the input ontologies, and the average duration. In 18.38% of the cases the uniform interpolant could not be computed in 1,000 seconds, and in only 10.44% of the remaining cases, it made use of fixpoint operators. Despite the relatively high number of timeouts, the average duration was only 23 seconds, and the cumulative distribution of durations shows (Figure 5), that around 1,600 out of 2,911 runs (more than half of them) could be performed in less than a second. This suggests that computing uniform interpolants is in most cases a cheap operation.

It is known that uniform interpolants of $\mathcal{ALC}$-ontologies can be in the worst case be triple exponential in the size of the input ontology [15]. When fixpoints are used, the worst case complexity is better, but still double exponential. This bound was not at all reflected in the empirical results, where the average interpolant is less than half the size of the input ontology. In only 6.05% of the cases the uniform interpolant was bigger (see also Figure 5). The axioms in the uniform interpolant were usually around 8–10 times larger than in the input ontology, which is still a reasonable size for ontology analysis considering that in the input ontologies the average axiom size was less than 4.

It should be noted that randomly drawn samples of signatures not neccessarily reflect realistic use cases. One might assume that it is most often desirable

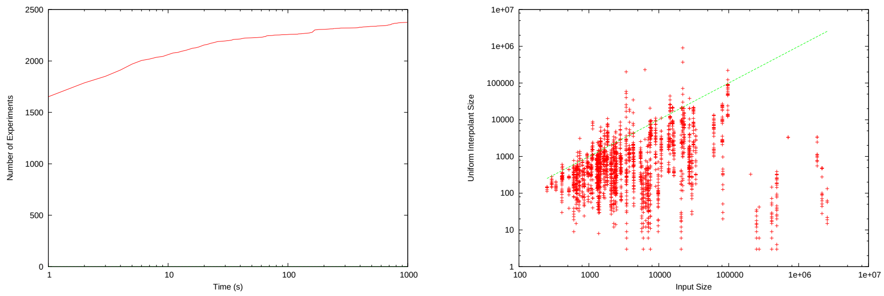

Fig. 5. Cumulative distribution of durations of experimental runs and sizes of the computed uniform interpolants.

to forget or preserve symbols that are closer related to each other, whereas randomly selected symbols are more likely to be randomly distributed along the whole ontology, which can contain thousands of symbols. We therefore believe that our method would perform even better in realistic use cases.

9 Related Work

Most previous work has focused on uniform interpolation in simpler description logics like $\mathcal{EL}$ and DL-Lite (see for example [9, 24, 16, 14]). In [23, 22], one of the first approaches for a more expressive description logic, namely $\mathcal{ALC}$, is presented. Their method uses a tableaux-reasoner to add inferences from the input ontology in an incremental way. Regular checking for TBox-equivalence is used to decide whether the uniform interpolant is computed and the process can stop. By using tableaux-reasoning as a basis, the authors hope to make their method easily extendable with known techniques from existing tableau-reasoners. Its less focused way of deriving inferences make it however unfeasible for large ontologies. In [15], it was discovered that the method is incomplete. The solution offered can be seen as an extension of the original method, even though tableau-reasoning is not stated explicitly. The resulting method can be used to compute all uniform interpolants that can be finitely represented in $\mathcal{ALC}$, but offers no solutions for ontologies where the interpolant cannot be represented without fixpoints.

A more practical approach for forgetting concept symbols in $\mathcal{ALC}$ is presented in [13]. A resolution-based method influenced by [7] is used to derive consequences on the selected concept symbol in a focused way. Experiments on modified ontologies from the NCBO Bioportal ontology show the practicality of this approach under certain restrictions. Since their approach does not use structural transformation, a calculus based on meta-rules is used to make resolutions on nested concepts expressions possible. A disadvantage is that the method does not terminate if infinite chains of nested role-restrictions are derivable. The solution offered is to approximate interpolants by a given (lower) bound instead.

A method using fixpoints for the description logic $\mathcal{EL}$ was presented in [16]. This method aims at forgetting concept symbols, and computes derivation graphs for least common subsumers and most general subsumees of the concept to be eliminated. This graph is analysed to decide whether fixpoint operators are necessary in the result or not. In [14], an automata based representation is used to make finite representations of uniform interpolants possible. The computed automata can be used to decide whether a finite representation in pure $\mathcal{EL}$ is possible and can be translated into corresponding $\mathcal{EL}$-TBoxes in this case.

The method presented in this paper is an extension of a recently introduced method for forgetting concept symbols in $\mathcal{ALC}$-ontologies [12], which is evaluated in [11]. Both methods take ideas from second-order quantifier elimination techniques presented in [5], especially from the resolution-based method SCAN [4] and a method based on a generalised version of Ackermann's Lemma [17]. The latter technique has first been applied for description logics in [21]. Like the methods presented in [13] and [16], the method presented in [12] focuses on forgetting concept symbols. Our current method adds redundancy elimination techniques and is the first practical algorithm for forgetting role symbols from ontologies in expressive description logics.

10 Conclusion and Future Work

We presented a method for forgetting concept and role symbols from $\mathcal{ALCH}$-ontologies, or for computing uniform interpolants of $\mathcal{ALCH}$-ontologies. Since uniform interpolants cannot always be represented in a finite way, the resulting ontology may use fixpoint operators, which can be simulated in $\mathcal{ALCH}$ by extending the signature of the interpolant. Our experimental results suggest that the method is already applicable in a lot of real life situations.

An open point regards the use of fixpoints. One can construct easy examples where our method computes an interpolant with fixpoints, even though the uniform interpolant can be represented in $\mathcal{ALCH}$. Reasons for this are interactions between different fixpoint expressions in the ontology and indirect knowledge encoded in the remaining part of the ontology. For example, it is possible that the cyclic relation expressed by a fixpoint expression is already covered by a set of axioms that was not touched by the method, or that the fixpoint can be represented in a finite way due to entailments from the remaining ontology. Of course this leaves also the question on whether optimal use of fixpoint is actually practical on large ontologies, since an approach focused solely on the symbols we want to forget would not be sufficient here.

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Appendix

A Proofs of Theorems

A.1 The Underlying Calculus

Lemma 1. *The calculus $\mathcal{R}_{\mathcal{ALCH}}$ is sound, that is, for any ontology $\mathcal{O}$, the saturation of clauses($\mathcal{O}$) using the rules of $\mathcal{R}_{\mathcal{ALCH}}$ is equi-satisfiable with the TBox of $\mathcal{O}$, and if the empty clause can be derived, $\mathcal{O}$ is unsatisfiable.*

Proof. The soundness of the resolution rule and the role propagation rule follow from the logical entailment $\models ((A \sqcup B) \sqcap (C \sqcup D)) \sqsubseteq (A \sqcup C \sqcup (B \sqcap D))$. More specifically, for the resolution rule, we have $\top \sqsubseteq ((C_1 \sqcup A) \sqcap (C_2 \sqcup \neg A)) \models \top \sqsubseteq (C_1 \sqcup C_2 \sqcup (A \sqcap \neg A)) \models \top \sqsubseteq (C_1 \sqcup C_2)$. For the role propagation rule, we have $\top \sqsubseteq ((C_1 \sqcup \forall s.D_1) \sqcap (C_2 \sqcup \mathsf{Q}r.D_2)) \models \top \sqsubseteq (C_1 \sqcup C_2 \sqcup (\forall s.D_1 \sqcap \mathsf{Q}r.D_2) \models \top \sqsubseteq (C_1 \sqcup C_2 \sqcup \mathsf{Q}r.(D_1 \sqcap D_2)$.

The soundness of the use of the identification function for the reuse of introduced definers can be shown by induction on the definers in $id(D)$ and the clauses introduced for each new definer. □

Lemma 2. *Any derivation using the calculus $\mathcal{R}_{\mathcal{ALCH}}$ terminates.*

Proof. The number of introduced definers is bounded by the range of the identification function id. Let N_D* be the definers that have been introduced by the structural transformation of the input ontology. Since every element in $id(D)$ is either an element of N_D* or a universal role restriction over a subset of N_D*, the range of id is bounded by the number of roles occurring in $\mathcal{O}$ and the number of definers in N_D*. Since clauses are represented as sets, the number of clauses that can possibly be derived is bounded by the number of possible literals, which is bounded by the number of concept names, roles and introduced definers. This gives a finite bound on the number of definers introduced.

In order to prove completeness of $\mathcal{R}_{\mathcal{ALCH}}$, we use a candidate model construction approach similar to the one used to prove refutational completeness for ordered resolution [2] and refutational completeness for consequence-driven reasoners for description logics [20]. We show that for each set of clauses saturated using the rules of our calculus and not containing the empty clause, we can construct a candidate model which is actually a model for the set. This proof is very similar to the one offered in [12], since we only added role inclusions to the rules, but we repeat it here in full since it makes it easier to prove the correctness of the simplification rules, and thus to prove the soundness and refutational completeness of $\mathcal{R}^s_{\mathcal{ALCH}}$.

The model construction is done in the following way. For each satisfiable definer concept D, we create a set I^D of literals that have to be satisfied by a domain element in order to satisfy the definer. A special definer ϵ is used to represent concepts that do not occur under a role restriction. We then create a domain element x_D for each definer D and construct an interpretation in such a way that every atomic concept and every existential restriction in each I^D is satisfied. We show that the resulting interpretation is indeed a model for the saturated set of clauses.

Let N_s denote a set of clauses saturated using the rules of $\mathcal{R}_{\mathcal{ALCH}}$. The set $\mathbf{D}$ consists of all definers used in N_s and the special symbol ϵ. N_s is partitioned into a set of *definition sets*: the function $d : \mathbf{D} \longrightarrow 2^{N_s}$ maps each definer $D \in \mathbf{D}$ to the subset of clauses in N_s which have $\neg D$ as only negative definer literal, and ϵ to all clauses that do not contain negative definer literals. $d(D)$ contains all clauses that make up the *definition* of D, in the sense that they can be represented by an axiom of the form $D \sqsubseteq ...$, hence we use the terminology *definition set* for $d(D)$. $d(\epsilon)$ contains all the remaining clauses, which are not related to the definition of any definer. Let $d^e(D) = d(D) \cup d(\epsilon)$ be the definition set extended with these clauses. If a domain element satisfies D, it also has to satisfy all clauses in $d^e(D)$, and it suffices to check the clauses in $d^e(D)$ to check whether an instance satisfies D or not.

We define a partial ordering $\sqsubseteq_{\mathbf{D}}$ on definers in the following way: $D_1 \sqsubseteq_{\mathbf{D}} D_2$ iff $\neg D_1 \sqcup D_2 \in N_s$.

We define an ordering $\prec_L$ on literals that satisfies the following constraints:

- $D \prec_L \neg D \prec_L A \prec_L \neg A \prec_L \exists r.D' \prec_L \forall r.D''$ for all atomic concepts A that are not definers, for all roles r and for all definers D, D', D''.
- If $D_1 \sqsubseteq_{\mathbf{D}} D_2$ and $r \sqsubseteq_{\mathcal{O}} s$, then $D_1 \prec_L D_2$, $\exists r.D_1 \prec_L \exists s.D_2$ and $\forall r.D_1 \prec_L \forall s.D_2$. (From this follows that if D represents $D_1 \sqcap D_2$, then $\exists r.D \prec_L \exists r.D_1$ and $\forall r.D \prec_L \forall r.D_1$.)

It can be shown that an ordering with these constraints always exists. $\prec_L$ is extended to an ordering $\prec_C$ between clauses using the multi-set extension $(\prec_L)_{mul}$ of $\prec_L$. This ordering provides an enumeration of the clauses in each $d^e(D)$: C_i^D denotes the ith clause in $d^e(D)$ according to $\prec_C$, starting from the smallest clause.

Following this enumeration, we define a set I^D of positive literals for each element $D \in \mathbf{D}$ (including ϵ) such that, if a domain element x satisfies every literal in I^D, it also satisfies $d^e(D)$.

Definition 3. *For a set I of positive literals, I* satisfies a literal l, taking into account the subsumption hierarchy on $\mathbf{D}$*, written $I \models_{\mathbf{D}} l$, iff*

1. *l is a positive literal of the form A and $A \in I$,*
2. *l is a negative literal of the form $\neg A$ and $A \notin I$,*
3. *l is of the form $\exists r.D$ and there is a $\exists s.D' \in I$ with $D' \sqsubseteq_{\mathbf{D}} D$ and $r \sqsubseteq_{\mathcal{O}} s$ or*
4. *l is of the form $\forall r.D$ and for every literal of the form $\exists s.D' \in I$ with $s \sqsubseteq_{\mathcal{O}} r$ we have $D' \sqsubseteq_{\mathbf{D}} D$.*

We say I satisfies a clause C*, written $I \models_{\mathbf{D}} C$, if there is a literal $l \in C$ such that $I \models_{\mathbf{D}} l$.*

We define I^D formally in five steps:

1. If $\neg D \in d(D)$, set $I^D = \emptyset$. Otherwise, let
2. $I_0^D = \{D\}$ if $D \neq \epsilon$ and $I_0^D = \emptyset$ if $D = \epsilon$.
3. $I_i^D = I_{i-1}^D \cup \{L\}$, if $I_{i-1}^D \not\models_{\mathbf{D}} C_{i-1}^D$ and the maximal literal L of C_{i-1}^D is a positive literal of the form A or $\exists r.D'$, and
4. $I_i^D = I_{i-1}^D$ otherwise.
5. $I^D = I_n^D$, where n is the number of clauses in $d^e(D)$.

Lemma 3. *If I^D is nonempty, then $I^D \models_{\boldsymbol{D}} C_i^D$ for all clauses C_i^D in $d^e(D)$.*

Proof. We validate that for each C_i^D we have $I^D \models_{\mathbf{D}} C_i^D$. Observe that because of how $d^e(D)$ is defined, every clause in $d^e(D)$ contains either no negative definer literal or $\neg D$ is the only negative definer literal (no clauses with more than one negative definer literal can be derived). This means, for any two clauses $C_i^D, C_j^D \in d^e(D)$, the side conditions of the rules are satisfied (the union never has more than one negative definer literal). We do the proof by contradiction. Assume i is the smallest i with $I^D \not\models_{\mathbf{D}} C_i^D$.

1. If the maximal literal in C_i^D is of the form A or $\exists r.D'$, then the clause is satisfied due to Step 3 in the construction of I^D, which contradicts our assumption.
2. If the maximal literal in C_i^D is of the form $\neg A$, we have $I^D \not\models \neg A$ and therefore $I^D \models A$. This means there must be a clause C_j^D where A is maximal in C_j^D and $I_j^D \not\models C_j^D \setminus \{A\}$, otherwise A is not added to I^D. But then, due to the resolution rule, we also have a clause $C = (C_i^D \cup C_j^D) \setminus \{A, \neg A\}$, which is also in $d^e(D)$. Since $\prec_C$ is the multi-set extension of the ordering between literals, C is smaller than C_i^D, since $\neg A \in C_i^D$ and $\neg A$ is larger than all elements in C ($\neg A$ is maximal in C_i^D).
   Since both $I^D \not\models_{\mathbf{D}} C_j^D \setminus \{A\}$ and $I^D \not\models_{\mathbf{D}} C_i^D \setminus \{\neg A\}$, we have $I^D \not\models_{\mathbf{D}} C$. Because C belongs to $d^e(D)$ and is smaller than C_i^D, there is a $k < i$ with $C = C_k^D$, which contradicts our initial assumption that i is the smallest i with $I^D \not\models_{\mathbf{D}} C_i^D$.
3. If the maximal literal in C_i^D is of the form $\forall r.D'$, we have $I^D \not\models_{\mathbf{D}} C_i^D \setminus \{\forall r.D'\}$ and $I^D \not\models_{\mathbf{D}} \forall r.D'$. The only way the latter can be true is due to a literal $\exists s.D_2 \in I^D$ with $s \sqsubseteq_{\mathcal{O}} r$ and $D_2 \not\sqsubseteq_{\mathbf{D}} D'$. If D_2 is not subsumed by D', $\exists s.D_2$ is a counter-example for $\forall r.D'$.
   If $\exists s.D_2 \in I^D$, there must be a clause C_j^D such that the maximal literal in C_j^D is $\exists s.D_2$ and $I_j^D \not\models_{\mathbf{D}} C_j^D$. Because of the role propagation rule, we then also have a clause $C_k^D = (C_i^D \cup C_j^D \cup \{\exists s.D_3\}) \setminus \{\forall r.D', \exists s.D_2\}$, where D_3 represents $D' \sqcap D_2$. In our ordering, $\exists s.D_3$ is smaller than both $\forall r.D'$ and $\exists s.D_2$, and therefore $C_k^D \prec_C C_j^D$, $C_k^D \prec_C C_i^D$ and $k < j < i$. We obtain that $I^D \not\models_{\mathbf{D}} C_k^D$ because (i) $I^D \not\models_{\mathbf{D}} C_i^D \setminus \{\forall r.D'\}$, (ii) $I^D \not\models_{\mathbf{D}} C_j^D \setminus \{\exists s.D_2\}$, and (iii) $I^D \not\models_{\mathbf{D}} \exists s.D_3$ (for else $I_j^D \models_{\mathbf{D}} \exists s.D_2$ as $D_3 \sqsubseteq_{\mathbf{D}} D_2$ and $C_k^D \prec_C C_j^D$, and $\exists s.D_3$ cannot be maximal in any clause larger than C_j^D). However, $I^D \not\models_{\mathbf{D}} C_k^D$ contradicts our initial assumption. $\square$

Based on I^D, we construct the candidate model $\mathcal{I}_c = \langle \Delta^{\mathcal{I}_c}, \cdot^{\mathcal{I}_c} \rangle$ with:

1. $\Delta^{\mathcal{I}_c} = \{x_D \mid D \in \mathbf{D} \text{ and } I^D \text{ is not empty}\}$,
2. for every atomic concept A, $A^{\mathcal{I}_c} = \{x_D \mid A \in I^D\}$, and
3. for every atomic role r, $r^{\mathcal{I}_c} = \{(x_{D_1}, x_{D_2}) \mid \exists s.D_2 \in I^{D_1}, s \sqsubseteq_{\mathcal{O}} r, I^{D_2} \neq \emptyset\}$.

Lemma 4. *If $\bot \notin N_s$, then $\mathcal{I}_c$ is a model of N_s.*

Proof. Assume C_i^D is the smallest clause in N_s that is not satisfied in $\mathcal{I}_s$. Observe that from Lemma 3 follows that each maximal literal in C_i^D is satisfied by I^D. We check the cases for the maximal literal in C_i^D.

1. The maximal literal in C_i^D is of the form A, then $\mathcal{I}_s \models C_i^D$ follows immediately from Lemma 3, which contradicts our initial assumption.
2. The maximal literal in C_i^D is of the form $\exists r.D_2$. Since $I^D \models \exists r.D_2$, we have $\exists s.D_3 \in I^D$, where $s \sqsubseteq_{\mathcal{O}} r$ and $D_3 \sqsubseteq_{\mathbf{D}} D_2$ (see Definition 3). (i) Assume $I_3^D \neq \emptyset$. Then we have $(x_D, x_{D_3}) \in s^{\mathcal{I}_s}$ and also $(x_D, x_{D_3}) \in r^{\mathcal{I}_s}$ due to Step 3 of the construction of $\mathcal{I}_s$. Note that if $D_3 \sqsubseteq_{\mathbf{D}} D$, either $D_3 = D$ or $\neg D_3 \sqcup D \in N_s$. In both cases, we will have $D \in I^{D_3}$ and $x_{D_3} \in D^{\mathcal{I}_s}$, and therefore $x_D \in (\exists r.D_3)^{\mathcal{I}_s}$, which contradicts our initial assumption. (ii) Assume $I^{D_3} = \emptyset$. Then there is a clause $\neg D_3$. Observe that since $\exists s.D_3 \in I^D$, there must be a clause C_j^D where $\exists s.D_3$ is the maximal literal, such that $I_j^D \not\models C_j^D$ (Step 3 in the construction of I_D). Applying the existential role restriction elimination rule on $\neg D_3$ and C_j^D produces the clause $C_k^D = C_j^D \setminus \{\exists s.D_3\}$, which is smaller than C_j. We obtain that $I^D \not\models_{\mathbf{D}} C_k^D$ because $I^D \not\models_{\mathbf{D}} C_j^D \setminus \{\exists s.D_3\}$ (for else $I_j^D \models_{\mathbf{D}} C_j$ as $C_k^D \prec_C C_j^D$, and $\exists s.D_3$ cannot be maximal in any clause larger than C_k^D). However, $I^D \not\models_{\mathbf{D}} C_k^D$ contradicts our initial assumption.
3. The maximal literal in C_i^D is of the form $\forall r.D_2$. $x_D \notin (\forall r.D_2)^{\mathcal{I}_c}$ implies there is a relation $(x_D, x_{D_3}) \in r^{\mathcal{I}_c}$ such that $x_{D_3} \notin D_2^{\mathcal{I}_c}$. Due to Step 3 of the construction of $\mathcal{I}_c$, this implies $\exists s.D_3 \in I^D$, $s \sqsubseteq_{\mathcal{O}} r$. But this contradicts Lemma 3, since then $I^D \not\models C_i^D$. □

We can now prove refutational soundness and completeness of $\mathcal{R}_{\mathcal{ALCH}}$.

Theorem 6 (Theorem 2). *$\mathcal{R}_{\mathcal{ALCH}}$ is sound and refutationally complete, and provides a decision procedure for $\mathcal{ALCH}$-TBox satisfiability.*

Proof. Soundness and termination were already established in Lemmas 1 and 2. Therefore if $N \vdash \bot$, N is unsatisfiable. Suppose $N \not\vdash \bot$. Then we can construct a model for N_s (N saturated by $\mathcal{R}_{\mathcal{ALCH}}$) using the method described above (Lemma 4), and since N_s is equi-satisfiable with N (Lemma 1), N is satisfiable. □

Theorem 7 (Theorem 7). *$\mathcal{R}^s_{\mathcal{ALCI}}$ is refutationally sound and complete. Given a TBox $\mathcal{T}$, clauses($\mathcal{T}$) $\vdash_{\mathcal{R}^s_{\mathcal{ALCH}}} \bot$ iff $\mathcal{T}$ is unsatisfiable.*

Proof. Since the tautology deletion and the subsumption deletion rule only remove clauses, they do not affect the soundness of the calculus. Reduction is sound, since (i) $D_1 \sqsubseteq D_2$ implies $\mathsf{Q}r.D_1 \sqsubseteq \mathsf{Q}r.D_2$, $\mathsf{Q} \in \{\exists, \forall\}$, (ii) $\mathsf{Q}r.D_1 \sqsubseteq \mathsf{Q}r.D_2$ implies $\mathsf{Q}r.D_1 \sqcup \mathsf{Q}r.D_2 \sqsubseteq \mathsf{Q}r.D_2 \sqcup \mathsf{Q}r.D_2$, (iii) $\top \models \mathsf{Q}r.D_2 \sqcup \mathsf{Q}r.D_2 \sqsubseteq \mathsf{Q}r.D_2$ and (iv) $C \sqsubseteq D$ implies $\top \sqsubseteq C \models \top \sqsubseteq D$.

We show that the refutational completeness is preserved by each rule by referring to the candidate model construction in Lemma 3.

Tautology deletion: Assume $C = C' \sqcup A \sqcup \neg A$. Since both $A \in C$ and $\neg A \in C$, for any literal set I^D we have $I^D \models_D C$, therefore the presence of C does not affect the candidate model construction, and it can as well be removed from the clause set.

Subsumption deletion: Observe that, according to the orderings used to prove Lemma 3, $l_1 \sqsubseteq_l l_2$ implies either $l_1 = l_2$ or $l_1 \prec_l l_2$. Hence $C_1 \sqsubseteq_C C_2$ implies either $C_1 = C_2$ or $C_1 \prec_C C_2$. Observe also that due to Definition 3 for any $D \in \mathbf{D}$, if $C_1 \sqsubseteq_C C_2$, $I^D \models C_1$ implies $I^D \models C_2$. Therefore C_2 does not affect the candidate model construction and can as well be removed from the clause set.

Reduction: Since $red(C) \sqsubseteq_C C$ and subsumption deletion preserves refutational completeness, reduction preserves refutational completeness as well. □

A.2 Forgetting Concept Symbols

The proof for forgetting concepts from $\mathcal{ALCH}$-ontologies using $\mathcal{F}^A_{\mathcal{ALCH}}$ follows the same line as our proof for concept forgetting in $\mathcal{ALC}$ presented in [12]. In order to prove the correctness of our uniform interpolation method, we have to show that every consequence $C \sqsubseteq D$ in the desired signature is preserved by the uniform interpolant. For Phase 2, we use our decision procedure to show that these consequences are preserved by $F^A_{ALCH}(\mathcal{O})$. This is done by generating a set of clauses M for each consequence $C \sqsubseteq D$, such that $\mathcal{O} \models C \sqsubseteq D$ iff $\mathcal{O} \cup M$ is unsatisfiable. We first show that for any clause set M over the desired signature, $N \cup M$ is satisfiable iff $\mathcal{F}_{ALCH}(\mathcal{O}) \cup M$ is satisfiable.

Lemma 5. *For any concept symbol A, any ontology $\mathcal{O}$ and any set M of clauses, such that $A \notin sig(M)$, let N_s be the result of saturating clauses$(O) \cup M$ and N'_s be the result of saturating $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O}) \cup M$ using $\mathcal{R}^s_{\mathcal{ALCH}}$. It is possible to create a candidate model for N'_s iff it is possible to create a candidate model for N_s.*

Proof. We define the orderings $\prec_L$ and $\prec_C$ as in the last section with the additional constraint that $\neg B \prec_L A$ for any concept symbol $B \neq A$, A being the concept symbol eliminated in N'_s.

We first point out the following properties of clause sets N saturated using $\mathcal{R}^s_{\mathcal{ALCH}}$ regarding definer symbols D: (i) If $|id(D)| > 1$ (D is an introduced definer), we have $\neg D \sqcup D_i \in N$ for every $D_i \in id(D)$ (due to role propagation and possibly subsequent resolution steps). Due to further resolution applications, this implies (ii) for every $D_i \sqsubseteq_{\mathbf{D}} D$, we have $d^e(D) \subseteq d^e(D_i)^{D_i \mapsto D}$, where $d^e(D_i)^{D_i \mapsto D}$ denotes the result of replacing every D_i in $d^e(D_i)$ with D. (iii) There is maximally one definer in $id(D)$ that occurs under an existential role restriction, and every pair of definers in $id(D)$ occurs under contexts that allow for their combination via the role propagation rule (role propagation is only applied if at least one literal is a universal role restriction and if the side conditions are not violated). (iv) Every nonempty subset of $id(D)$ is represented by a definer (this is a consequence of (iii)).

Now, observe that in the proof for Lemma 3 only rule applications on maximal literals and definer literals are needed. Resolution on definer literals is assumed indirectly in the proof by how we define satisfaction for literal sets I^D taking into account $\sqsubseteq_{\mathbf{D}}$. This means it is sufficient to perform inferences on maximal literals or definer literals.

A difference between N_s and N'_s is that N'_s does not contain any clauses using A. If we can show that nevertheless conclusions of resolving on A literals occurring in N_s also occur in N'_s, we are done, since in $\mathcal{R}^s_{\mathcal{ALCH}}$ rules are applied unrestricted and in $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O})$ only clauses containing A are removed. If A is not crucial in deriving the empty clause, it is safe to remove clauses containing it. If A is crucial, the only derivations we lose when removing clauses containing A are conclusions of inference steps involving A.

For pairs of clauses that do not contain any definers, or that only contain definers that are also in $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O})$, their resolvents on A are in N'_s since they are in $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O})$. Assume we have a clause $C = \neg D \sqcup C_1 \sqcup C_2 \in N_s$ that is the resolvent of two clauses $\neg D \sqcup C_1 \sqcup A, \neg D \sqcup C_2 \sqcup \neg A \in N_s$, such that $C \notin N'_s$, and assume further that C is the largest clause according to $\prec_C$ with this property. As already mentioned, D cannot be in $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O})$, since otherwise C is also in $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O})$. Hence, D can only co-occur with A due to resolution on clauses of the form $\neg D \sqcup D_i$, where D_i co-occurs with A. This means there are two clauses $\neg D \sqcup D_1, \neg D \sqcup D_2 \in N_s$, where at least one of D_1 and D_2 co-occurs with A in a clause (observe that $D \sqsubseteq_{\mathbf{D}} D_1$ and $D \sqsubseteq_{\mathbf{D}} D_2$). We have two cases:

1. $\neg D_1 \sqcup C_1 \sqcup A, \neg D_1 \sqcup C_2 \sqcup \neg A \in N_s$. Then $\neg D_1 \sqcup C_1 \sqcup C_2 \in N'_s$ (due to our assumption that C is the largest resolvent not in N'_s), and due to resolution on D_1 we have $C \in N'_s$, which contradicts our assumption that $C \notin N'_s$.
2. $\neg D_1 \sqcup C_1 \sqcup A, \neg D_2 \sqcup C_2 \sqcup \neg A \in N_s$. Since at least one definer is not in $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O})$ (otherwise D would be in $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O})$ as well), there must be clauses $\neg D'_1 \sqcup C_1 \sqcup A$ and $\neg D_1 \sqcup D'_1$. But then, due to (iv), we also have a definer D' representing $D'_1 \sqcap D_2$, and $D \sqsubseteq_{\mathbf{D}} D'$. Due to our assumption, $\neg D' \sqcup C_1 \sqcup C_2 \in N'_s$. Due to (ii), we also have $\neg D \sqcup C_1 \sqcup C_2 = C \in N'_s$, thus contradicting our assumption that $C \notin N'_s$. $\square$

Now we can prove Theorem 4, which states that for any concept symbol A and ontology $\mathcal{O}$, $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O})$ can be computed in finitely bounded time and preserves all consequences over $sig(N) \setminus \{A\}$.

Theorem 8 (Theorem 4). *Given an ontology $\mathcal{O}$, $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O})$ computes the clausal representation of $\mathcal{O}^{-A}$, that is, $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O}) \equiv_\Sigma \mathcal{O}$, where $\Sigma = sig(\mathcal{T}) \setminus \{A\}$, and every symbol in $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O})$ is either a definer or in Σ.*

Proof. The fact that $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O})$ can always be computed in finitely bounded time follows from Lemma 2. Since in $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O})$ all clauses containing A are removed, all symbols in $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O})$ are either definers or in $sig(N)\setminus\{A\}$. Hence we only have to check the second condition of the definition of uniform interpolants: $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O}) \models C \sqsubseteq D$ iff $N \models C \sqsubseteq D$, for any $\mathcal{ALCH}$ concept subsumption not containing A.

$N \models C \sqsubseteq D$ can be proven by showing that $C \sqcap \neg D$ is unsatisfiable in N, or by showing that $N' = N \cup \{\top \sqsubseteq \exists r^*.(C \sqcap \neg D)\} \models \bot$, where r^* is a new role not occurring in N. Set $M = clauses(\{\top \sqsubseteq \exists r^*.(C \sqcap \neg D)\}$. Since $A \notin sig(M)$ and due to Lemma 5, we have $N \cup M \models \bot$ iff $\mathcal{F}^A_{\mathcal{ALCH}}(\mathcal{O}) \cup M \models \bot$. □

A.3 Forgetting Role Symbols

Theorem 9 (Theorem 5). *For any ontology $\mathcal{O}$, $\mathcal{F}^r_{\mathcal{ALCH}}(\mathcal{O})$ is a clausal representation of $\mathcal{O}^{-r}$.*

Proof. We have to show that for every subsumption $C \sqsubseteq D$ not using r, we have $\mathcal{O} \models C \sqsubseteq D$ iff $\mathcal{F}^r_{\mathcal{ALCH}}(\mathcal{O}) \models C \sqsubseteq D$.

If $\mathcal{O} \models C \sqsubseteq D$, no model of $\mathcal{O}$ contains an individual that satisfies $C \sqcap \neg D$. This is equivalent to saying $\mathcal{O} \cup \alpha$ is unsatisfiable, where $\alpha = \{\top \sqsubseteq \exists r^*.(C \sqcap \neg D)\}$, r^* being a new role not occurring in $\mathcal{O}$. (The new axiom can only be true if an individual satisfying $C \sqcap \neg D$ is possible.) Since $\mathcal{R}^s_{\mathcal{ALCH}}$ is refutationally sound and complete, this can be reduced to showing $\bot \in \mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{O} \cup \alpha)$ iff $\bot \in \mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{F}^r_{\mathcal{ALCH}}(\mathcal{O}) \cup \alpha)$, where $\mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{O})$ represents the clause set obtained by saturating $clauses(\mathcal{O})$ using $\mathcal{R}^s_{\mathcal{ALCH}}$. If $\mathcal{O}$ or α are inconsistent, this is trivially the case, so we assume $\mathcal{O}$ and α to be both consistent.

(i) Assume r is not related to any other role via a role inclusion in $\mathcal{O}$. The only interesting case is when the role r is involved in deriving the empty clause from $\mathcal{O} \cup \{\alpha\}$. This means, the definers that occur under a role restriction for r in $\mathcal{O}$ must play a crucial role, otherwise $\bot \in \mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{O}' \cup \alpha)$ holds trivially. Denote the definers occurring in $\mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{O} \cup \alpha)$ under a role restriction for r by D^r_i. Observe that no rule application between clauses containing $\neg D^r_i$ and clauses from $\mathcal{R}^s_{\mathcal{ALCH}}(\alpha)$ is possible. This is because the only clause in $\mathcal{R}^s_{\mathcal{ALCH}}(\alpha)$ without a negative definer literal is of the form $\forall r^*.D$ and r^* does not occur in $\mathcal{T}$. Furthermore, no definer which is a combination of D^r_i and a definer in $\mathcal{R}^s_{\mathcal{ALCH}}(\alpha)$ is in $\mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{T} \cup \alpha)$, since $\mathcal{R}^s_{\mathcal{ALCH}}(\alpha)$ does not contain r, and therefore no role propagation between any role restriction for r and any role restriction in $\mathcal{R}^s_{\mathcal{ALCH}}(\alpha)$ is possible. Therefore, if the empty clause can be derived from $\mathcal{T} \cup \alpha$, this is due to rule applications between clauses in $\mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{O})$ and $\mathcal{R}^s_{\mathcal{ALCH}}(\alpha)$ which do not contain r. (Any other rule application will produce a clause which only leads to the derivation of clauses that are either redundant or contain r as well and are therefore not empty). But every clause in $\mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{O})$ which does not contain r is already included in $\mathcal{F}_{\mathcal{ALCH}}(\mathcal{O})$, and therefore, the empty clause is only in $\mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{O} \cup \alpha)$ iff it is also in $\mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{F}_{\mathcal{ALCH}}(\mathcal{O}) \cup \alpha)$.

(ii) Assume there are role inclusions of the form $r \sqsubseteq s$ or $s \sqsubseteq r$ in $\mathcal{O}$. The role hierarchy rule ensures that the induced role hierarchy in $\mathcal{F}^r_{\mathcal{ALCH}}(\mathcal{O})$ contains all role inclusions not using r that are derivable from $\mathcal{O}$. Assume (a) α contains a role s with $r \sqsubseteq s$. Then, due to the universal role restriction rule, for every clause $C \sqcup \forall s.D \in \mathcal{R}^s_{\mathcal{ALCH}}(\alpha)$ we will have $C \sqcup \forall r.D \in \mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{T} \cup \alpha)$. On the other hand, due to the existential role restriction rule, for every clause $C \sqcup \exists r.D \in \mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{T})$ we will have $C \sqcup \exists s.D$ both in $\mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{T})$ and in $\mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{T}')$. This means, for every pair of definers $\forall r.D_1, \exists r.D_2 \in \mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{T} \cup \alpha)$ between which the role propagation rule is applicable, we also have a pair of definers $\forall s.D_1, \exists s.D_2 \in \mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{T} \cup \alpha)$. Everything derivable on the basis of $\forall r.D_1, \exists r.D_2$ is also derivable on the basis of $\forall s.D_1, \exists s.D_2$, which means role propagation between literals $\forall r.D_1$ and $\forall r.D_2$ is redundant.

Assume (b) α contains a role s with $s \sqsubseteq r$. Here, we have a similar situation: this time for every $C \sqcup \exists s.D_1 \in \alpha$ we have a corresponding $C \sqcup \exists r.D_1 \in \mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{T} \cup \alpha)$, but for every corresponding $C \sqcup \forall r.D_2 \in \mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{T})$ we also have $C \sqcup \forall s.D_2$ both in $\mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{T})$ and in $\mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{T}')$, which again renders all role propagations between literals of the form $\forall r.D_1$ and $\exists r.D_2$ redundant. The other cases (no $r \sqsubseteq s$ resp. $s \sqsubseteq r$) reduce to (i), which means, we can perform all derivations in $\mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{F}^r_{\mathcal{ALCH}}(\mathcal{O}) \cup \alpha)$ as well, and therefore $\bot \in \mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{O} \cup \alpha)$ iff $\bot \in \mathcal{R}^s_{\mathcal{ALCH}}(\mathcal{F}^r_{\mathcal{ALCH}}(\mathcal{O}) \cup \alpha)$. □

Theorem 10 (Theorem 1). *For any $\mathcal{ALCH}$-ontology $\mathcal{O}$ and any symbol σ, our method terminates and returns a finite representation of $\mathcal{O}^{-\sigma}$ in the description logic $\mathcal{ALCH}\mu$. If the result does not make use of the greatest fixpoint operator, the result is the uniform interpolant of $\mathcal{O}$ over $sig(\mathcal{O}) \setminus \{\sigma\}$ in $\mathcal{ALCH}$.*

Proof. Because of the Theorems 4 and 5, Phase 2 of the method computes the clausal representation of the uniform interpolant in finite time. The remaining phases are the same than in the original method for forgetting concept symbols in $\mathcal{ALC}$, for which we refer to [12]. □