Chapter 4

The Two-Variable Fragment with Counting

4.1 Normal forms

In this chapter, we consider the 2-variable fragment of first-order logic with counting quantifiers. The main result of this chapter is that the satisfiability and finite satisfiability problems for this logic are both $\text{NExpTime}$-complete.

Formally, we take $C^2$ to be the set of function-free formulas of first-order logic with counting quantifiers, but employing only the variables $x$ and $y$. For convenience, we restrict attention to purely relational signatures with predicates of arities 1 or 2: it is routine to show that adding individual constants, nullary predicates or predicates of arity 3 or more does not affect the results presented here. (By contrast, adding even unary function-symbols to the signature would result in an undecidable logic.) In the presence of counting quantifiers, equality is definable by the formula

$$\forall x.e(x, x) \land \forall x \exists \leq 1 y.e(x, y).$$

Consequently, we shall assume that the equality predicate $\approx$ is available. The Löwenheim-Skolem-Tarski theorem guarantees that, if a first-order formula has a model, then it has a finite or countably infinite model. Since our concern in this chapter is the problems $\text{Sat-C}^2$ and $\text{Fin-Sat-C}^2$, we shall henceforth silently take all structures to be either finite or countably infinite.

The fragment $C^2$ presents us with much greater challenges than any of the fragments considered so far. For a start, $C^2$ lacks the finite model property.

Example 4.1. The $C^2$-formula $\varphi$ given by

$$\forall x \exists y.s(x, y) \land \forall x \exists \leq 1 y.s(y, x) \land \exists x \forall y.\neg s(y, x)$$

is satisfiable, but not finitely satisfiable. For let $A = \mathbb{N}$ and $s^A = \{(i, i + 1) \mid i \in \mathbb{N}\}$. Then $A \models \varphi$. On the other hand, suppose $\mathfrak{B} \models \varphi$. Since $\mathfrak{B} \models \forall x \exists y.s(x, y)$, let $f : B \rightarrow B$ be a function such that $f \subseteq s^B$. Since $\mathfrak{B} \models \forall x \exists \leq 1 y.s(y, x)$, $f$ is injective. Since $\mathfrak{B} \models \exists x \forall y.\neg s(y, x)$, $f$ is not onto. Therefore $B$ is infinite. $\blacksquare$
Even though $C^2$ lacks the finite model property, we may still ask whether, for a finitely satisfiable formula $\varphi$, we can bound the size of the smallest satisfying model, as a function of $\|\varphi\|$. The following example shows that any such bound must dominate a doubly exponential function.

**Example 4.2.** For $n \geq 0$, let $T_n$ be the complete, binary tree of depth $2^n - 1$, as depicted in Fig. 4.1. (Note: if $a$ is a node in any tree $T$, we take the depth of $a$, denoted $d(a)$, to be the number of edges on the unique path from $a$ to the root of $T$; and we take the depth of $T$ to be the depth of the deepest node.) Thus, $T_n$ contains $2^{2^n} - 1$ elements. Recalling the representation of a natural number $n \leq 2^{2^n} - 1$ as a string of binary digits $d_{n-1}, \ldots, d_0$ (where the zeroth digit $d_0$ is the least significant), we interpret unary predicates $X_0, \ldots, X_{n-1}$ and $X^*_{0}, \ldots, X^*_{n}$ and binary predicate $r$ over $T_n$ to form a model $\mathfrak{T}_n$, as follows. Let $X_i$ be satisfied by a node $a$ just in case the $i$th digit of $d(a)$ is 1. Likewise, for $i < n$, let $X^*_i$ be satisfied by $a$ just in case $d(a) < 2^n - 1$ and the least significant zero-digit in $d(a)$ is the $i$th digit; and let $X^*_n$ be satisfied by $a$ just in case $d(a) = 2^n - 1$. Finally, let $r$ be satisfied by the pair of nodes $\langle a, b \rangle$ just in case $b$ is a daughter of $a$.

Let $\varphi_{n,0}$ be the conjunction of all formulas

$$\forall x (X^*_i(x) \iff \neg X_i(x) \land \bigwedge_{0 \leq j < i} X_j(x)),$$

for $0 \leq i < n$, together with the formula

$$\forall x (X^*_n(x) \iff \bigwedge_{0 \leq j < n} X_j(x)).$$

Thus, $\varphi_{n,0}$ fixes the interpretations of the $X^*_i$ in terms of the $X_i$ in the required way. Let $\varphi_{n,1}$ be the conjunction of the following formulas

$$\forall x \forall y (r(x, y) \rightarrow (X^*_i(x) \rightarrow (\neg X_j(y) \land X_i(y))))$$

for $0 \leq j < i < n$, and

$$\forall x \forall y (r(x, y) \rightarrow (X^*_i(x) \rightarrow (X_j(x) \iff X_j(y))))$$
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for $0 \leq i < j < n$. Recalling the standard algorithm for incrementing binary numerals, we see that $\varphi_{n,1}$ asserts that the depth of a daughter of a node in $T$ is one greater than the depth of that node. Finally, let $\varphi_{n,2}$ be the conjunction of the formulas

$$\exists x \left( \bigwedge_{0 \leq i < n} \lnot X_i(x) \right) \quad \forall x \left( \lnot X_n^*(x) \rightarrow \exists \geq 2 y. r(x, y) \right) \quad \forall x \exists \leq 1 y. r(y, x).$$

Thus, $\varphi_{n,2}$ asserts that there is a node of depth zero, that each node of depth less than $2^n - 1$ has at least two daughters, and that no node has more than one mother. Let $\varphi_n = \varphi_{n,0} \land \varphi_{n,1} \land \varphi_{n,2}$. We have shown that $T_n \models \varphi_n$. Conversely, it is evident that every model of $\varphi_n$ contains an isomorphic copy of $T_n$.

We have shown that each $\varphi_n$ is finitely satisfiable, but not satisfiable over any domain containing fewer than $2^{2^n}$ elements. On the other hand, inspection of the above formulas shows that $\parallel \varphi_n \parallel$ grows as a polynomial function of $n$.

Example 4.1 shows that the problems Sat-$\mathcal{C}^2$ and Fin-Sat-$\mathcal{C}^2$ do not coincide. Obviously, then, we cannot hope to establish the decidability of Sat-$\mathcal{C}^2$ by describing a procedure for guessing models of bounded size. Example 4.2 shows that even finitely satisfiable $\mathcal{C}^2$-formulas in general require models of doubly-exponential size. Again, it follows that we cannot hope to establish that Fin-Sat-$\mathcal{C}^2$ is in NEXPTIME by describing a procedure for guessing models of exponential size. We remark that, in the course of the chapter, a matching upper bound is obtained to Example 4.2: any finitely satisfiable $\mathcal{C}^2$ formula $\varphi$ is shown to have a model whose size is bounded by a doubly exponential function of $\parallel \varphi \parallel$. (A priori, this is by no means obvious.)

One technique of previous chapters that does usefully apply to $\mathcal{C}^2$, however, is the existence of equisatisfiable Scott-form formulas.

**Lemma 4.1.** Let $\varphi$ be a $\mathcal{C}^2$-formula. We can generate, in time bounded by a polynomial function of $\parallel \varphi \parallel$, a quantifier-free $\mathcal{L}^2$-formula $\alpha$, a list of positive integers $C_1, \ldots, C_m$ and a list of binary predicates $f_1, \ldots, f_m$ such that the formulas $\varphi$ and

$$\psi = \forall x \forall y (\alpha \lor x \approx y) \land \bigwedge_{1 \leq h \leq m} \forall x \exists \leq C_h y. (f_h(x, y) \land x \neq y)$$

are satisfiable over the same domains containing at least $C + 1$ elements, where $C = \max_h (C_1, \ldots, C_m)$.

**Proof.** We construct the formula $\psi$ in stages.

**Stage 1:** Let $\varphi_0 = \forall x \forall y \exists y. \exists x. \varphi$. (Thus, $\varphi_0$ and $\varphi$ are satisfied in exactly the same structures.) In the sequel, we take $u, v$ to denote the variables $x, y$, in either order, and, as usual, we silently convert between $\exists x. \theta$ and $\exists \geq 1 x. \theta$ and also between $\forall x. \theta$ and $\forall = 0 x. \lnot \theta$ as necessary. Suppose $\varphi_0$ possesses a subformula
\( \theta(u) = \exists x D v. \chi \), with \( \chi \) quantifier-free. Let \( p \) be a new unary predicate, and \( r_1, r_2 \) new binary predicates. Define \( \varphi_1 := \varphi_0[p(u)/\theta(u)] \) and 
\[
\psi_1 := \forall u \exists \leq D v (p(u) \rightarrow \chi) \land \forall u \exists \geq D +1 v (\neg p(u) \rightarrow \neg \chi).
\]
Since \( \psi_1 \) is evidently logically equivalent to \( \forall u(p(u) \leftrightarrow \exists \leq D v \chi) \), it is clear that \( \varphi_0 \) and \( \varphi_1 \land \psi_1 \) are satisfiable over the same domains. (If \( \varphi_0 \) possesses no subformula \( \exists \leq D v. \chi \), with \( \chi \) quantifier-free, but instead, a subformula of the forms \( \exists \geq D u. \chi \) or \( \exists = D u. \chi \); then we proceed analogously, subject to the obvious adjustments.)

Now process \( \varphi_1 \) in the same way, and continue until some formula \( \varphi_k \) is reached having the form \( \forall x \forall y. \chi \), with \( \chi \) quantifier-free. Set 
\[
\psi' := \varphi_k \land (\psi_k \land \psi_{k-1} \land \ldots \land \psi_1).
\]
Thus, \( \psi' \) and \( \varphi_0 \) (and hence \( \varphi \)) are satisfiable over the same domains.

**Stage 2:** By performing trivial logical simplifications if necessary, we may take \( \psi' \) to be a conjunction of formulas of the four forms
\[
\forall x \forall y. \pi \land \forall x \exists \leq D y\pi(x,y) \land \forall x \exists \geq D y\pi(x,y),
\]
where \( \pi \) is a quantifier-free \( L^2_{\geq} \)-formula, and \( D > 0 \). Now replace any conjunct \( \forall x \exists \leq D y\pi(x,y) \) by the formula \( \xi \) consisting of the conjunction of the following four formulas:
\[
\forall x \exists = D-1 y (s(x,y) \land x \neq y) \land \forall x \forall y (\pi(x,x) \rightarrow (\pi(x,y) \rightarrow s(x,y)))
\]
\[
\forall x \exists = D y (s'(x,y) \land x \neq y) \land \forall x \forall y (\neg \pi(x,x) \rightarrow (\pi(x,y) \rightarrow s'(x,y))),
\]
where \( s \) and \( s' \) are new binary predicates. It is straightforward to check that \( \models \xi \rightarrow \forall x \exists \leq D y \pi(x,y) \). Conversely, suppose \( \mathfrak{A} \models \forall x \exists \leq D y \pi(x,y) \), and \( |A| > D \); we expand \( \mathfrak{A} \) to a structure \( \mathfrak{A}' \) by interpreting the new binary predicates \( s \) and \( s' \) as follows. If \( \mathfrak{A} \models \pi[a,a] \), let \( B \subseteq A \setminus \{a\} \) be such that \( |B| = D-1 \), and \( B \) includes all the elements \( b \in A \setminus \{a\} \) such that \( \mathfrak{A} \models \pi[a,b] \); and let \( B' \subseteq A \setminus \{a\} \) be such that \( |B'| = D \). On the other hand, if \( \mathfrak{A} \not\models \pi[a,a] \), let \( B \subseteq A \setminus \{a\} \) be such that \( |B| = D-1 \), and let \( B' \subseteq A \setminus \{a\} \) be such that \( |B'| = D \), and \( B \) includes all the elements \( b \in A \setminus \{a\} \) such that \( \mathfrak{A} \models \pi[a,b] \). In either case, set \( \mathfrak{A}' \models s[a,b] \) if and only if \( b \in B \cup \{a\} \), and \( \mathfrak{A}' \models s'[a,b] \) if and only if \( b \in B' \). It is obvious that \( \mathfrak{A}' \models \xi \).

Conjuncts of the form \( \forall x \exists \geq D y \pi(x,y) \) may be treated analogously. By continuing this process until all unwanted quantifiers have been eliminated, and carrying out some further re-arrangement, we may write \( \psi'' \) equisatisfiably as 
\[
\psi'' = \forall x \forall y \alpha'(x,y) \land \bigwedge_{1 \leq h \leq m} \forall x \exists = c_h y(f_h(x,y) \land x \neq y),
\]
where \( \alpha'(x,y) \) is a quantifier-free \( L^2_{\geq} \)-formula. Thus, \( \varphi \) and \( \psi'' \) are satisfiable over the same domains containing at least \( C+1 \) elements, where \( C = \max_h c_h \).

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Stage 3: It remains only to reform the occurrences of \( \approx \) in \( \alpha'(x,y) \). We proceed exactly as in Lemma 2.3, obtaining a quantifier-free \( \mathcal{L}_2 \)-formula \( \alpha \) such that, restricting attention to domains containing at least 2 elements,

\[
\forall x \forall y. \alpha'(x,y) \equiv \forall x \forall y (\alpha(x,y) \lor x \approx y)
\]

For such domains, then, \( \psi'' \) is logically equivalent to the formula \( \psi \) given in (4.1). Moreover, it is routine to check that the above computation can be effected in time bounded by a polynomial function of \( \|\varphi\| \).

Our approach to Sat-\( C^2 \) and FinSat-\( C^2 \) will employ essentially the same methods as our approach to Sat-\( C^2 \) = FinSat-\( C^2 \) in Chapter 3: we reduce the problems to the existence of solutions to systems of linear equations. Since \( C^2 \) lacks the finite model property, however, we must extend our system of arithmetic to accommodate countably infinite cardinalities.

**Definition 4.1.** Let \( \mathbb{N}^* \) denote the set \( \mathbb{N} \cup \{\aleph_0\} \). We refer to elements of \( \mathbb{N}^* \) as extended natural numbers, taking the operations + and \( \cdot \) to be defined in the usual way for cardinals, and extending the usual ordering on \( \mathbb{N} \) so that \( n < \aleph_0 \) for all \( n \in \mathbb{N} \).

**Remark 4.1.** We have: \( \aleph_0 + \aleph_0 = \aleph_0 \cdot \aleph_0 = \aleph_0 \) and \( 0 \cdot \aleph_0 = \aleph_0 \cdot 0 = 0 \); \( n + \aleph_0 = \aleph_0 + n = \aleph_0 \) for all \( n \in \mathbb{N} \); and \( n \cdot \aleph_0 = \aleph_0 \cdot n = \aleph_0 \) for all \( n \in \mathbb{N} \) such that \( n > 0 \). The operations + and \( \cdot \) remain associative and commutative on \( \mathbb{N}^* \), and obey the usual distributivity rules.

Extended natural numbers can be used to reason about countable collections of objects in familiar ways: if \( X,Y \) are disjoint sets of respective cardinalities \( x,y \in \mathbb{N}^* \), then \( X \cup Y \) has cardinality \( x + y \); similar remarks apply to multiplication.

Generally, speaking, results on systems of linear equations over \( \mathbb{N} \) can be extended unproblematically to results on the same systems of equations over \( \mathbb{N}^* \). Of particular interest here are Theorems 3.1 and Corollary 3.3. We have the following corollaries, the proofs of which we leave to the reader.

**Corollary 4.1** (of Theorem 3.1). Let \( E \) be a system of \( m \) equations and inequalities of the forms

\[
a_1x_1 + \cdots + a_nx_n \bowtie c,
\]

where the \( a_i \) and \( c \) are integers (not necessarily positive) and \( \bowtie \) is one of \( =, \leq \) or \( \geq \). If \( E \) has a solution over \( \mathbb{N}^* \), then it has a solution in which every finite value is bounded by a fixed exponential function of \( \|E\| \). Hence, the problem of determining the feasibility of such systems of linear equations and inequalities over \( \mathbb{N}^* \) is in \( \text{NPTime} \).

**Corollary 4.2** (of Theorem 3.3). Let \( E \) be a system of \( m \) equations and inequalities of the forms (3.7), and suppose that the absolute values of all the constants \( a_i \) and \( c \) occurring in this system are bounded above by some \( D > 1 \). If \( E \) has a solution over \( \mathbb{N}^* \), then it has a solution in which the number of non-zero values is at most \( \frac{5}{4}m \log(2mD + 1) \).
Lemma 4.1 allows us to restrict attention to $\mathcal{L}^2$-formulas of the form

$$\psi = \forall x \forall y (\alpha \lor x \equiv y) \land \bigwedge_{1 \leq h \leq m} \forall x \exists y C_h y (f_h(x,y) \land x \not\equiv y),$$

where $\alpha$ is a quantifier-free $\mathcal{L}^2$-formula, $f_1, \ldots, f_m$ are binary predicates and $m, C_1, \ldots, C_m$ are positive integers. In the analysis of models of such formulas, the binary predicates $f_1, \ldots, f_m$ play a special role. Accordingly, we employ special terminology to deal with them.

**Definition 4.2.** Let $\Sigma$ be a signature of unary and binary predicates, and $f_1, \ldots, f_m$ ($m > 0$) a list of distinct binary predicates in $\Sigma$. The pair $\langle \Sigma, (f_1, \ldots, f_m) \rangle$ is called a **classified signature**, and $f_1, \ldots, f_m$ are referred to as its **featured predicates**.

Where the featured predicates are not needed, we may treat classified signatures as ordinary signatures of unary and binary predicates. In particular, if $\langle \Sigma, \bar{f} \rangle$ is a classified signature with $s = |\Sigma|$ finite, we take the 1-types and 2-types over $\langle \Sigma, \bar{f} \rangle$ to be defined as for $\Sigma$. Furthermore, we shall always assume some standard enumeration $\pi_1, \ldots, \pi_L$ of the 1-types over $\langle \Sigma, \bar{f} \rangle$, where $L = 2^s$.

The following notions are specific to classified signatures.

**Definition 4.3.** Let $\langle \Sigma, \bar{f} \rangle$ be a classified signature, and let $\tau$ be a 2-type. We say that $\tau$ is a **message-type** (over $\langle \Sigma, \bar{f} \rangle$) if $f(x,y) \in \tau$ for some featured predicate $f$. If $\tau$ is a message-type such that $\tau^{-1}$ is also a message-type, we say that $\tau$ is **invertible**. On the other hand, if $\tau$ is a 2-type such that neither $\tau$ nor $\tau^{-1}$ is a message-type, we say that $\tau$ is a **silent** 2-type.

Thus, a 2-type $\tau$ is an invertible message-type if and only if there are featured predicates $f$ and $f'$ such that $f(x,y) \in \tau$ and $f'(y,x) \in \tau$. The terminology is meant to suggest the following imagery. Let $A$ be a structure interpreting the classified signature in question. If $\text{tp}^A[a,b]$ is a message-type $\mu$, then we may imagine that $a$ sends a message (of type $\mu$) to $b$. If $\mu$ is invertible, then $b$ replies by sending a message (of type $\mu^{-1}$) back to $a$. If $\text{tp}^A[a,b]$ is silent, then neither element sends a message to the other.

For any fixed classified signature $\langle \Sigma, \bar{f} \rangle$, let us enumerate the message-types over $\langle \Sigma, \bar{f} \rangle$, in some standard way, just as we did for the 1-types. Since the ordering is arbitrary, we shall insist, for notational convenience, that the invertible message-types always precede the non-invertible message-types. Thus the message-types over $\Sigma$ are enumerated as

$$\mu_1, \ldots, \mu_{M^*}, \mu_{M^*+1}, \ldots, \mu_M,$$

where $\mu_1, \ldots, \mu_{M^*}$ are all invertible, and $\mu_{M^*+1}, \ldots, \mu_M$ all non-invertible. It is easy to see that the number $M$ of message-types over $\Sigma$ satisfies $M \leq m^{2^{4s-1}}$.

The principal challenge in establishing the satisfiability of formulas of the form (4.1) consists in the very general nature of the structures we must work
4.2. CLASSIFIED SIGNATURES AND FEATURED PREDICATES

with. It is therefore useful to restrict attention to certain classes of structures that are guaranteed to include models of \( \varphi \), if any exist at all. We begin with a very simple such restriction.

**Definition 4.4.** Let \( \mathfrak{A} \) be a structure interpreting a classified signature \( \langle \Sigma, \bar{f} \rangle \), and \( C \) a positive integer. We say that \( \mathfrak{A} \) is \( C \)-bounded if, for all \( a \in A \) and all featured predicates \( f \) in \( \bar{f} \),

\[
1 \leq |\{b \in A \setminus \{a\} : \mathfrak{A} \models f[a, b]\}| \leq C.
\]

We say that \( \mathfrak{A} \) is bounded if it is \( C \)-bounded for some \( C \).

Thus, \( \mathfrak{A} \) is \( C \)-bounded just in case, for every featured predicate \( f \), every element of \( A \) is non-reflexively related to some element of \( A \) by \( f \), and no element of \( A \) is non-reflexively related to more than \( C \) elements of \( A \) by \( f \).

**Remark 4.2.** If \( \varphi \) is of the form (4.1), \( C \geq \max(C_1, \ldots, C_m) \) and \( \mathfrak{A} \models \varphi \), then \( \mathfrak{A} \) is \( C \)-bounded.

We now consider how to characterize the ‘local environment’ of an element in a bounded structure. The next definition makes use of our standard enumeration \( \mu_1, \ldots, \mu_{M^*}, \mu_{M^*+1}, \ldots, \mu_M \) of message types over \( \langle \Sigma, \bar{f} \rangle \).

**Definition 4.5.** Let \( \mathfrak{A} \) be a bounded structure interpreting a classified signature \( \langle \Sigma, \bar{f} \rangle \), and let \( a \) be an element of \( A \). The star-type of \( a \) in \( \mathfrak{A} \), denoted \( \text{st}_{\mathfrak{A}}[a] \), is the \( M \)-tuple \( \sigma = (v_1, \ldots, v_M) \) of natural numbers where, for all \( 1 \leq j \leq M \),

\[
v_j = |\{b \in A \setminus \{a\} : \text{tp}_\mathfrak{A}[a, b] = \mu_j\}|.
\]

Evidently, \( \sigma \) satisfies the condition

\[
v_i > 0 \text{ and } v_j > 0 \implies \text{tp}_1(\mu_i) = \text{tp}_1(\mu_j),
\]

for all \( i, j \) (\( 1 \leq i < j \leq M \)), with \( \text{tp}_\mathfrak{A}[a] \) the common value of \( \text{tp}_1(\mu_j) \). Accordingly, we take a star-type over \( \langle \Sigma, \bar{f} \rangle \) to be any \( M \)-tuple \( \sigma = (v_1, \ldots, v_M) \) satisfying (4.3). We denote the number \( v_j \) by \( \sigma[j] \), for all \( j \) (\( 1 \leq j \leq M \)). We say \( \sigma \) is \( C \)-bounded if, for all \( h \) (\( 1 \leq h \leq m \)),

\[
1 \leq \sum \{\sigma[j] : 1 \leq j \leq M, \ f_h(x, y) \in \mu_j\} \leq C.
\]

A bounded structure \( \mathfrak{A} \) is said to realize a star-type \( \sigma \) if, for some \( a \in A \), \( \text{st}_\mathfrak{A}[a] = \sigma \).

In general, if \( \mathfrak{A} \) is a structure and \( a \in A \), then the cardinalities \( |\{b \in A \setminus \{a\} : \text{tp}_\mathfrak{A}[a, b] = \mu_j\}| \) may be infinite. Since this certainly cannot happen in bounded structures, however, we restrict attention to bounded structures when talking about star-types of elements.

**Remark 4.3.** A bounded structure \( \mathfrak{A} \) is \( C \)-bounded if and only if every star-type realized in \( \mathfrak{A} \) is \( C \)-bounded.
Table 4.1: Notation for the 1-types, message-types and \( C \)-bounded star-types relative to a fixed classified signature \( \langle \Sigma, \bar{f} \rangle \) and positive integer \( C \).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>( \langle \Sigma, \bar{f} \rangle )</td>
<td>a classified signature</td>
</tr>
<tr>
<td>( s )</td>
<td>the number of symbols in ( \Sigma )</td>
</tr>
<tr>
<td>( L )</td>
<td>the number of 1-types over ( \Sigma )</td>
</tr>
<tr>
<td>( M^* )</td>
<td>the number of invertible message-types over ( \Sigma )</td>
</tr>
<tr>
<td>( M )</td>
<td>the number of message-types over ( \Sigma )</td>
</tr>
<tr>
<td>( \pi_1, \ldots, \pi_L )</td>
<td>the 1-types over ( \Sigma )</td>
</tr>
<tr>
<td>( \mu_1, \ldots, \mu_{M^*} )</td>
<td>the invertible message-types over ( \Sigma )</td>
</tr>
<tr>
<td>( \mu_{M^*+1}, \ldots, \mu_M )</td>
<td>the non-invertible message-types over ( \Sigma )</td>
</tr>
</tbody>
</table>

Given a classified signature \( \langle \Sigma, \bar{f} \rangle \) and a positive integer \( C \), there are only finitely many \( C \)-bounded star-types, and we may again enumerate them in some standard way as

\[
\sigma_1, \ldots, \sigma_N. \tag{4.4}
\]

Simple calculation shows that \( N \leq (C + 1)^M \), where, as we will recall, \( M \) is the number of message-types; thus, \( N \) is bounded above by doubly-exponential function of \( s = |\Sigma| \). It is inescapable that the number of \( C \)-bounded star-types grows as a doubly exponential function of \( s \). Note that the listing (4.4) depends not only on the classified signature, but also on the bound \( C \) of the star-types in question: both parameters are left implicit to reduce notational clutter.

The purpose of the notation introduced so far—which is summarized in Table 4.1—is to provide a mechanism for describing structures in terms of the distribution of star-types they realize. The following notion makes this idea precise.

**Definition 4.6.** Let \( \langle \Sigma, \bar{f} \rangle \) be a classified signature, \( C \) a positive integer, and \( \mathfrak{A} \) a \( C \)-bounded structure interpreting \( \langle \Sigma, \bar{f} \rangle \). Let \( \sigma_1, \ldots, \sigma_N \) be the standard enumeration of the \( C \)-bounded star-types. The \( C \)-histogram of \( \mathfrak{A} \) is the \( N \)-tuple \( \text{Hist}_C(\mathfrak{A}) = (w_1, \ldots, w_N) \) of cardinalities \( w_k = |\{ a \in A : \text{st}^C[a] = \sigma_k \}| \), for \( 1 \leq k \leq N \).

Note that, since we are confining attention to finite or countably infinite structures, the elements of \( \text{Hist}_C(\mathfrak{A}) \) will always be extended natural numbers.

**Remark 4.4.** Suppose \( \text{Hist}_C(\mathfrak{A}) = (w_1, \ldots, w_N) \). Then \( |A| = w_1 + \cdots + w_N \). In particular, \( \mathfrak{A} \) is finite if and only if every entry of \( \text{Hist}_C(\mathfrak{A}) \) is finite.

Although every \( C \)-bounded structure \( \mathfrak{A} \) interpreting a classified signature \( \langle \Sigma, \bar{f} \rangle \) has a \( C \)-histogram, not every tuple from \( \mathbb{N}^* \) (of the appropriate arity) is the \( C \)-histogram of a \( C \)-bounded structure. An important problem, addressed below, is to characterize those which are.

Much valuable information about structures can be extracted from their \( C \)-histograms. To do so, however, we need a little notation.
4.2. CLASSIFIED SIGNATURES AND FEATURED PREDICATES

**Definition 4.7.** Let \( \langle \Sigma, \vec{f} \rangle \) be a classified signature, and \( C \) a positive integer, with 1-types, message-types and \( C \)-bounded star-types enumerated as in Table 4.1. Where \( \Sigma, \vec{f} \) and \( C \) are clear from context, we write, for integers \( i, k \) in the ranges \( 1 \leq i \leq L, 1 \leq k \leq N \):

\[
o_{i,k} = \begin{cases} 1 & \text{if } \text{tp}(\sigma_k) = \pi_i \\ 0 & \text{otherwise}; \end{cases}
\]

\[
p_{i,k} = \begin{cases} 1 & \text{if, for all } j \ (1 \leq j \leq M), \ \text{tp}_2(\mu_j) = \pi_i \text{ implies } \sigma_k[j] = 0 \\ 0 & \text{otherwise}; \end{cases}
\]

\[
r_{i,k} = \sum_{j \in J} \sigma_k[j], \text{ where } J = \{ j \mid M^* + 1 \leq j \leq M \text{ and } \text{tp}_2(\mu_j) = \pi_i \};
\]

\[
s_{i,k} = \sum_{j \in J} \sigma_k[j], \text{ where } J = \{ j \mid 1 \leq j \leq M \text{ and } \text{tp}_2(\mu_j) = \pi_i \}.
\]

In addition, for integers \( i, j \) in the ranges \( 1 \leq i \leq L, 1 \leq j \leq M^* \), we write:

\[
q_{j,k} = \sigma_k[j].
\]

**Remark 4.5.** The constants \( q_{i,j} \) of Definition 4.7 are all bounded by \( C \); likewise, the constants \( r_{i,k} \) and \( s_{i,k} \) are all bounded by \( mC \).

**Remark 4.6.** Let \( \mathfrak{A} \) be a \( C \)-bounded structure interpreting \( \langle \Sigma, \vec{f} \rangle \). The constants of Definition 4.7 have the following intuitive interpretations, for all \( i, j, k \) in the appropriate ranges:

1. \( o_{i,k} = 1 \) just in case every element with star-type \( \sigma_k \) has 1-type \( \pi_i \);

2. \( p_{i,k} = 1 \) just in case no element with star-type \( \sigma_k \) sends a message to any element having 1-type \( \pi_i \);

3. \( q_{j,k} \) counts how many messages of (invertible) type \( \mu_j \) any element having star-type \( \sigma_k \) sends;

4. \( r_{i,k} \) is the total number of elements having 1-type \( \pi_i \) to which any element having star-type \( \sigma_k \) sends a non-invertible message; and

5. \( s_{i,k} \) is the total number of elements having 1-type \( \pi_i \) to which any element having star-type \( \sigma_k \) sends a message.

**Remark 4.7.** Let \( \mathfrak{A} \) be a \( C \)-bounded structure interpreting \( \langle \Sigma, \vec{f} \rangle \), with \( \text{Hist}_C(\mathfrak{A}) = \langle w_1, \ldots, w_N \rangle \). And let the constants \( o_{i,k}, p_{i,k} \) and \( q_{j,k} \) be as in Definition 4.7.
Writing

\begin{align*}
u_i &= \sum_{1 \leq k \leq N} o_{i,k} w_k \quad (4.5) \\
v_j &= \sum_{1 \leq k \leq N} q_{j,k} w_k \quad (4.6) \\
x_{i,i'} &= \sum_{1 \leq k \leq N} o_{i,k} p_{i,k} w_k, \quad (4.7)
\end{align*}

we have:

1. \( u_i \) is the number of elements \( a \in A \) such that \( tp^A[a] = \pi_i \);
2. \( v_j \) is the number of pairs \( \langle a, b \rangle \in A^2 \) such that \( a \neq b \) and \( tp^A[a, b] = \mu_j \);
3. \( x_{i,i'} \) is the number of elements \( a \in A \) such that \( tp^A[a] = \pi_i \) and \( a \) does not send a message to any element having 1-type \( \pi_i' \).

The constants \( r_{i,k} \) and \( s_{i,k} \) will be used later in this chapter.

4.3 Describing structures over classified signatures

We observed above that, in considering the satisfiability of formulas of the form (4.1), we can restrict attention to \( C \)-bounded structures, where \( C = \max(C_1, \ldots, C_m) \). The following three definitions encapsulate further such restrictions.

**Definition 4.8.** Let \( A \) be a structure interpreting a classified signature \( \langle \Sigma, \bar{f} \rangle \). We say that \( A \) is **chromatic** if the following two conditions are satisfied for all \( a, a', a'' \in A \).

1. If \( a \neq a' \) and \( tp^A[a, a'] \) is an invertible message-type, then \( tp^A[a] \neq tp^A[a'] \).
2. If \( a, a', a'' \) are all distinct and both \( tp^A[a, a'] \) and \( tp^A[a', a''] \) are invertible message-types, then \( tp^A[a] \neq tp^A[a''] \).

Thus, a structure is chromatic just in case distinct elements connected by a chain of 1 or 2 invertible message-types always have distinct 1-types.

**Remark 4.8.** Let \( A \) be a chromatic structure interpreting a classified signature \( \langle \Sigma, \bar{f} \rangle \), and let \( \pi' \) be a 1-type over \( \Sigma \). Let \( a \) be an element of \( A \). Then there is at most one element \( a' \in A \setminus \{a\} \) with 1-type \( \pi' \) such that \( a \) sends an invertible message to \( a' \). Furthermore, if \( tp^A[a] = \pi' \), then there is no such element \( a' \).

The next lemma ensures that we may restrict attention to chromatic, \( C \)-bounded structures.
Lemma 4.2. Let $\mathfrak{A}$ be a $C$-bounded structure interpreting a classified signature $\langle \Sigma, \bar{f} \rangle$, and $m = |\bar{f}|$. Then $\mathfrak{A}$ can be expanded to a chromatic structure $\mathfrak{A}'$ by interpreting $\lceil \log((mC)^2 + 1) \rceil$ new unary predicates.

Proof. Consider the (undirected) graph $G$ on $A$ whose edges are the pairs of distinct elements connected by a chain of 1 or 2 invertible message-types. That is, $G = (A, E_1 \cup E_2)$, where

$E_1 = \{(a, a') | a \neq a' \text{ and } \text{tp}_A[a, a'] \text{ is an invertible message-type}\}$

$E_2 = \{(a, a'') | a \neq a'' \text{ and for some } a' \in A, \text{ both } (a, a') \text{ and } (a', a'') \text{ are in } E_1\}$.

Since $\mathfrak{A}$ is $C$-bounded, the degree of $G$ (in the normal graph-theoretic sense) is at most $(mC)^2$. Now use the standard (greedy) algorithm to colour the nodes of $G$ with $(mC)^2 + 1$ colours in such a way that no edge joins two nodes of the same colour. By interpreting the $\lceil \log((mC)^2 + 1) \rceil$ new unary predicates to encode these colours, we obtain the desired chromatic expansion of $\mathfrak{A}$.

Definition 4.9. Let $\mathfrak{A}$ be a structure interpreting a signature $\Sigma$, and $Z$ a positive integer. We say that $\mathfrak{A}$ is $Z$-differentiated if, for every 1-type $\pi$ over $\Sigma$, the number $u$ of elements in $A$ having 1-type $\pi$ satisfies either $u \leq 1$ or $u > Z$.

Thus, in a $Z$-differentiated structure, every 1-type is realized either at most once or more than $Z$ times (possibly infinitely often). The next lemma ensures that we may restrict attention to chromatic, $C$-bounded, $Z$-differentiated structures, for any $Z > 0$.

Lemma 4.3. Let $\mathfrak{A}$ be a chromatic structure interpreting a classified signature $\langle \Sigma, \bar{f} \rangle$, and $Z$ a positive integer. Let $\Sigma'$ be the signature obtained by adding $\lceil \log Z \rceil$ new unary predicates to $\Sigma$. Then $\mathfrak{A}$ can be expanded to a chromatic, $Z$-differentiated structure interpreting the classified signature $\langle \Sigma', \bar{f} \rangle$.

Proof. For each 1-type $\pi$ realized more than once but no more than $Z$ times, colour the elements having 1-type $\pi$ using $Z$ different colours. For each 1-type $\pi'$ realized either once or more than $Z$ times, colour the elements having 1-type $\pi'$ using a single colour. At most $Z$ colours are required for this process. By interpreting the $\lceil \log Z \rceil$ new unary predicates to encode these colours, we obtain the desired chromatic expansion $\mathfrak{A}'$. This process obviously preserves chromaticity.

If $U$ is any set, by an unordered pair from $U$, we simply mean a subset $\{u, v\} \subseteq U$, where $u$ and $v$ are not necessarily distinct.

Definition 4.10. Let $\langle \Sigma, \bar{f} \rangle$ be a classified signature. A regulator over $\langle \Sigma, \bar{f} \rangle$ is a partial function $\theta$ mapping unordered pairs of 1-types to silent 2-types such that, whenever $\theta(\{\pi, \pi'\}) = \tau$ is defined,

$\{\text{tp}_1(\tau), \text{tp}_2(\tau)\} = \{\pi, \pi'\}$.

A structure $\mathfrak{A}$ interpreting $\langle \Sigma, \bar{f} \rangle$ is regulated if there exists a regulator $\theta$ satisfying the following condition. For any distinct $a, b \in A$ such that $\text{tp}_A[a] = \pi$, $\text{tp}_A[b] = \pi'$.
Proof. Let \( B \) be a set of elements having 1-type \( \pi \), with \( |B| = (mC)^2 + mC + 1 \), and let \( B' \) be a set of elements having 1-type \( \pi' \), disjoint from \( B \), with \( |B'| = mC + 1 \). (Note that, since \( |B| + |B'| = (mC + 1)^2 + 1 \), and since \( \pi \) and \( \pi' \) are realized in \( A \) at least \( (mC + 1)^2 + 1 \) times, such sets can be found even if \( \pi = \pi' \).) Let

\[
B_0 = \{ b \in B \mid \text{for some } b' \in B', \ b' \text{ sends a message to } b \}.
\]

Since \( A \) is \( C \)-bounded, and \( |B'| = mC + 1 \), we have \( |B_0| \leq mC(mC + 1) \). So let \( b \in B \setminus B_0 \). But again, since \( b \) can send a message to at most \( mC \) elements of \( B' \), there exists \( b' \in B' \) such that \( b \) does not send a message to \( b' \). \( \square \)

4.4 Finite characterizations of structures

With these preliminaries on structures interpreting classified signatures behind us, the next step is to introduce data-structures to describe such structures in terms of the patterns of local configurations they exhibit. While the models
4.4. FINITE CHARACTERIZATIONS OF STRUCTURES

themselves may be very large or even infinite, the data-structures describing them will in general be much smaller. Furthermore, these data structures contain all the information needed to determine whether the structures they describe are models of $\varphi$.

**Definition 4.11.** Let $\langle \Sigma, \bar{f} \rangle$ be a classified signature, $C$ a positive integer, and $\sigma_1, \ldots, \sigma_N$ the standard enumeration of $C$-bounded star-types over $\langle \Sigma, \bar{f} \rangle$. A frame over $\langle \Sigma, \bar{f} \rangle$ is a triple $\mathcal{F} = \langle C, K, \theta \rangle$, where $K$ is a non-empty subset of $\{1, \ldots, N\}$, and $\theta$ is a regulator over $\langle \Sigma, \bar{f} \rangle$. If $\mathfrak{A}$ is a ($C$-bounded, regulated) structure over $\langle \Sigma, \bar{f} \rangle$, we say that $\mathcal{F}$ describes $\mathfrak{A}$ just in case $\{\sigma_k \mid k \in K\}$ is exactly the set of star-types realized in $\mathfrak{A}$, and $\theta$ is a regulator for $\mathfrak{A}$.

**Remark 4.9.** Let $\mathfrak{A}$ be a $C$-bounded, regulated structure over a classified signature $\langle \Sigma, \bar{f} \rangle$. Then there exists a frame $\mathcal{F} = \langle C, K, \theta \rangle$ over $\langle \Sigma, \bar{f} \rangle$ such that $\mathcal{F}$ describes $\mathfrak{A}$.

For the next definition, recall Condition (4.3) in our definition of a star-type $(v_1, \ldots, v_M)$: the non-zero entries $v_j$ all have a common value for the 1-type $tp_1(\mu_j)$. Of particular interest here are the 1-types $tp_2(\mu_j)$ in the case where $\mu_j$ is an invertible message type.

**Definition 4.12.** Let $\langle \Sigma, \bar{f} \rangle$ be a classified signature, $C$ a positive integer, and $\sigma$ a $C$-bounded star-type over $\langle \Sigma, \bar{f} \rangle$. Using the notation of Table 4.1, we say that $\sigma$ is chromatic if, for all $j$ ($1 \leq j \leq M*$), the 1-types $tp_2(\mu_j)$ for which $\sigma[j] > 0$ are distinct from each other and also distinct from the common 1-type $tp_1(\mu_j)$. Let $\mathcal{F} = \langle C, K, \theta \rangle$ be a frame over $\langle \Sigma, \bar{f} \rangle$. We say that $\mathcal{F}$ is chromatic if $\sigma_k$ is chromatic for every $k \in K$.

**Lemma 4.6.** Suppose $\mathcal{F}$ is a frame over a classified signature describing some structure $\mathfrak{A}$. Then $\mathcal{F}$ is chromatic if and only if $\mathfrak{A}$ is chromatic.

**Proof.** Suppose $\mathfrak{A}$ is not chromatic. If $tp^\mathfrak{A}[a, b]$ is an invertible message type but $tp^\mathfrak{A}[a] = tp^\mathfrak{A}[b]$, then neither $st^\mathfrak{A}[a]$ nor $st^\mathfrak{A}[b]$ is chromatic. On the other hand, if $tp^\mathfrak{A}[a, b]$ and $tp^\mathfrak{A}[b, c]$ are invertible message types with $a \neq c$ but $tp^\mathfrak{A}[a] = tp^\mathfrak{A}[c]$, then $st^\mathfrak{A}[b]$ is not chromatic. Either way, $\mathcal{F}$ is not chromatic. The reverse implication follows similarly. □

Frames also contain enough information to determine whether any structure they describe is a model of a formula $\varphi$ of the form (4.1).

**Definition 4.13.** Let $\varphi$ be any formula of the form (4.1) over a signature $\Sigma$, let $\bar{f} = (f_1, \ldots, f_m)$, and let $\mathcal{F} = \langle C, K, \theta \rangle$ be a frame over $\langle \Sigma, \bar{f} \rangle$, where $C \geq \max(C_1, \ldots, C_m)$. We write $\mathcal{F} \models \varphi$ if the following conditions are satisfied.

1. For all $k \in K$ and all $j$ ($1 \leq j \leq M$), if $\sigma_k[j] > 0$ then
   $$\models \bigwedge \mu_j \rightarrow \alpha(x, y) \land \alpha(y, x).$$
2. For all \( \{\pi, \pi'\} \in \text{dom}(\theta) \),
\[
\models \theta(\{\pi, \pi'\}) \rightarrow (\alpha(x, y) \land \alpha(y, x)).
\]

3. For all \( k \in K \) and all \( h \ (1 \leq h \leq m) \),
\[
\sum \{\sigma_k[j] \mid 1 \leq j \leq M \text{ and } f_k(x, y) \in \mu_j\} = C_h.
\]

**Lemma 4.7.** Let \( \varphi, F \) be as in Definition 4.13, and suppose \( \mathfrak{A} \) is a bounded, regulated structure over \( \langle \Sigma, f \rangle \) such that \( F \) describes \( \mathfrak{A} \). Then \( \mathfrak{A} \models \varphi \) if and only if \( F \models \varphi \).

**Proof.** Immediate once the definitions are unravelled.

We now return to the question posed above: when is a tuple from \( \mathbb{N}^* \) (of the appropriate arity) the \( C \)-histogram of a structure \( \mathfrak{A} \) interpreting a classified signature \( \langle \Sigma, f \rangle \)?

**Definition 4.14.** Let \( F = (C, K, \theta) \) be a frame over a classified signature \( \langle \Sigma, f \rangle \), with \( m = |f| \), let the number of \( C \)-bounded star-types over \( \langle \Sigma, f \rangle \) be \( N \), let \( \bar{w} = (w_1, \ldots, w_N) \) be an \( N \)-tuple of elements of \( \mathbb{N}^* \), and let the variables \( u_i \), \( v_j \) and \( x_{i,i'} \) be defined as in (4.5)–(4.7). We say that \( \bar{w} \) is a solution of \( F \) if \( K \) is exactly the set of indices \( k \) such that \( w_k > 0 \), and the following conditions are satisfied for all \( i \ (1 \leq i \leq L) \), all \( i' \ (1 \leq i' \leq L) \), and all \( j \ (1 \leq j \leq M^*) \):

(C1) \( v_j = v_{j'} \), where \( j' \) is such that \( \mu_j^{-1} = \mu_{j'} \);

(C2) if \( u_i = 0 \), then \( \sum \{w_k \mid s_{i,k} > 0\} = 0 \); if \( u_i = 1 \), then \( \sum \{w_k \mid s_{i,k} > 1\} = 0 \);

(C3) \( u_i \leq 1 \) or \( u_i > (mC + 1)^2 \);

(C4) if \( u_i \leq 1 \), then for all positive integers \( D \leq mC \), we have either \( x_{i,i'} \geq D \) or \( \sum \{w_k \mid o_{i,k} = 1 \text{ and } r_{i,k} \leq D\} = 0 \);

(C5) if \( \{\pi_i, \pi_{i'}\} \notin \text{dom}(\theta) \), then either \( u_i \leq 1 \) or \( u_{i'} \leq 1 \);

(C6) if \( \{\pi_i, \pi_{i'}\} \notin \text{dom}(\theta) \), then for all positive integers \( D \leq mC \), we have either \( x_{i,i'} \leq D \) or \( \sum \{w_k \mid o_{i,k} = 1 \text{ and } r_{i,k} \leq D\} = 0 \).

We call \( \bar{w} \) finite if each of its elements is in \( \mathbb{N} \). If \( F \) has a (finite) solution, we say that \( F \) is (finitely) solvable.
The two main lemmas of this chapter may now be stated. They tell us that we may treat (finitely) solvable, chromatic frames as substitutes for (finite) bounded, \((mC + 1)^2\)-differentiated chromatic structures.

**Lemma 4.8.** Let \((\Sigma, \bar{\sigma})\) be a classified signature, and write \(m = |\bar{\sigma}|\). Let \(\mathfrak{A}\) be a \(C\)-bounded, \((mC + 1)^2\)-differentiated structure interpreting \((\Sigma, \bar{\sigma})\), for some \(C > 0\). If \(\mathcal{F} = (C, K, \theta)\) is a frame describing \(\mathfrak{A}\), then \(\text{Hist}_C(\mathfrak{A})\) is a solution of \(\mathcal{F}\).

**Proof.** We employ the notation of Table 4.1, write \(\text{Hist}_C(\mathfrak{A}) = \bar{w} = (w_1, \ldots, w_N)\), and make free use of Remarks 4.6 and 4.7. Since \(\mathcal{F}\) describes \(\mathfrak{A}\), \(K\) is certainly the set of indices \(k\) for which \(w_k > 0\). We show that \(\bar{w}\) satisfies Conditions C1–C6.

C1: If \(\mu_j^{-1} = \mu_j'\), then the sets \(\{(a, b) \mid a \neq b \text{ and } \text{tp}_j^a[a, b] = \mu_j\}\) and \(\{(a, b) \mid a \neq b \text{ and } \text{tp}_j^a[a, b] = \mu_j'\}\) can obviously be put in 1–1 correspondence, namely: \((a, b) \mapsto (b, a)\). But the cardinalities of these sets are \(v_j\) and \(v_j'\), respectively.

C2: If \(\sigma_k\) is realized in \(\mathfrak{A}\), any element of \(A\) having this star-type sends a message to exactly \(s_{i,k}\) elements having 1-type \(\pi_i\). But \(u_i\) is the number of elements of \(A\) having 1-type \(\pi_i\). Hence, \(s_{i,k} \leq u_i\).

C3: Immediate given that \(\mathfrak{A}\) is \((mC + 1)^2\)-differentiated.

C4: Suppose that, for some \(k (1 \leq k \leq N), w_k > 0, a_{i,k} = 1\) and \(r_{\ell,k} \geq D > x_{\ell,i};\) we must show that \(u_k > 1\). Since \(w_k > 0\), let \(a \in A\) have star-type \(\sigma_k\); thus, \(a\) sends a non-invertible message to \(r_{\ell,k}\) elements having 1-type \(\pi_{\ell}\). Since there are only \(x_{\ell,i} < r_{\ell,k}\) elements having 1-type \(\pi_{\ell}\) and not sending a message to any element of 1-type \(\pi_i\), we must be able to find elements, say \(b, c\), such that \(a\) sends a non-invertible message to \(b\), \(b\) sends a message to \(c\), and \(c\) has 1-type \(\pi_i\). Note that \(c \neq a\), since otherwise the message from \(a\) to \(b\) would be invertible. On the other hand, since \(a_{i,k} = 1\), \(a\) is itself of 1-type \(\pi_i\), whence \(u_i > 1\), as required.

C5: Suppose \(u_i > 1\) and \(u_{i'} > 1\); we must show that \(\{\pi_i, \pi_{i'}\} \in \text{dom}(\theta)\). Since \(\mathfrak{A}\) is \((mC + 1)^2\)-differentiated, \(u_i > (mC + 1)^2\), and \(u_{i'} > (mC + 1)^2\). Since \(\mathfrak{A}\) is also \(C\)-bounded, by Lemma 4.5 we may pick distinct \(b, b' \in A\) such that \(\text{tp}_j^a[b] = \pi_i\), \(\text{tp}_j^a[b'] = \pi_{i'}\) and \(\text{tp}_j^a[b, b'] = \pi_i\) silent. Since \(\mathcal{F}\) describes \(\mathfrak{A}\), \(\{\pi_i, \pi_{i'}\} \in \text{dom}(\theta)\), as required.

C6: Suppose that, for some \(k (1 \leq k \leq N), w_k > 0, a_{i,k} = 1\) and \(r_{\ell,k} \leq D < x_{\ell,i};\) we must show that \(\{\pi_i, \pi_{i'}\} \in \text{dom}(\theta)\). Since \(w_k > 0\), let \(a \in A\) have star-type \(\sigma_k\); thus, \(a\) sends a non-invertible message to only \(r_{\ell,k}\) elements having 1-type \(\pi_{\ell}\). Since there are \(x_{\ell,i} > r_{\ell,k}\) elements having 1-type \(\pi_{\ell}\) and not sending a message to any element of 1-type \(\pi_i\), we must be able to find one of these elements, say \(b\), such that \(a\) does not send a non-invertible message to \(b\). But since \(a_{i,k} = 1\), \(a\) is itself of 1-type \(\pi_i\), and so \(b\) sends no message—invertible or non-invertible—to \(a\). That is, \(\text{tp}[a, b] = \pi_i\) silent. Since \(\mathcal{F}\) describes \(\mathfrak{A}\), \(\{\pi_i, \pi_{i'}\} \in \text{dom}(\theta)\), as required. \(\square\)
Lemma 4.9. Let \((\Sigma, \bar{f})\) be a classified signature, and \(\mathcal{F} = \langle C, K, \theta \rangle\) a chromatic frame over \((\Sigma, \bar{f})\). If \(\bar{w}\) is a solution of \(\mathcal{F}\), then there exists a \(C\)-bounded structure \(\mathfrak{A}\) over \((\Sigma, \bar{f})\) such that \(\mathcal{F}\) describes \(\mathfrak{A}\); moreover, \(\text{Hist}_{C}(\mathfrak{A}) = \bar{w}\).

Proof. Let \(\bar{w} = (w_1, \ldots, w_N)\) be a solution of \(\mathcal{F}\). We employ the constants \(o_{i,k}, p_{i,k}, q_{j,k}, r_{i,k}\) and \(s_{i,k}\) (with indices in the appropriate ranges) of Definition 4.7, bearing in mind their intuitive interpretations given in Remark 4.6.

This permits us to define the quantities

\[
\begin{align*}
    u_i &= \sum_{1 \leq k \leq N} o_{i,k} w_k \\
    v_j &= \sum_{1 \leq k \leq N} q_{j,k} w_k \\
    x_{i,i'} &= \sum_{1 \leq k \leq N} o_{i,k} p_{i',k} w_k,
\end{align*}
\]

(again, with indices in the appropriate ranges) exactly as we did in equations (4.5)–(4.7). Of course, in the present context, these quantities are simply extended natural numbers; however, it will transpire that, once our structure \(\mathfrak{A}\) is built, they will have exactly the intuitive interpretations given in Remark 4.7.

For every \(k\) (1 \(\leq k \leq N\)), let \(A_k\) be a set of cardinality \(w_k\), and let \(A\) be the disjoint union of the \(A_k\). Think of \(A_k\) as the set of elements which ‘want’ to have star-type \(\sigma_k\). In addition, we define, for all \(i\) (1 \(\leq i \leq L\), all \(i'\) (1 \(\leq i' \leq L\)) and all \(j\) (1 \(\leq j \leq M^*\)):

\[
\begin{align*}
    U_i &= \bigcup \{A_k \mid 1 \leq k \leq N \text{ and } o_{i,k} = 1\} \\
    V_j &= \bigcup \{A_k \mid 1 \leq k \leq N \text{ and } q_{j,k} = 1\} \\
    X_{i,i'} &= \bigcup \{A_k \mid 1 \leq k \leq N \text{ and } o_{i,k} p_{i',k} = 1\}.
\end{align*}
\]

Since \(\mathcal{F}\) is chromatic, \(q_{j,k} \leq 1\) for all \(j\) (1 \(\leq j \leq M^*\)) and all \(k\) (1 \(\leq k \leq N\)). Thus, for all \(i\), \(i'\) and \(j\) in the appropriate ranges:

\[
\begin{align*}
    |U_i| &= u_i; & |V_j| &= v_j; & |X_{i,i'}| &= x_{i,i'}.
\end{align*}
\]

Think of \(U_i\) as the set of elements which ‘want’ to have 1-type \(\pi_i\), \(V_j\) as the set of elements which ‘want’ to send an (invertible) message of type \(\mu_j\) to some other element, and \(X_{i,i'}\) as the set of elements in \(U_i\) which do not ‘want’ to send a message to any element having 1-type \(\pi_{i'}\). We remark that \(A_k \subseteq U_i\) if and only if \(\text{tp}(\sigma_k) = \pi_i\). Moreover, for all \(j\) (1 \(\leq j \leq M^*\)), if \(V_j \neq \emptyset\), there exists a unique \(i\) (1 \(\leq i \leq L\)) such that \(V_j \subseteq U_i\)—namely, that \(i\) such that \(\text{tp}(\mu_j) = \pi_i\). We now convert the domain \(A\) into a structure \(\mathfrak{A}\) in four steps.

Step 1 (Interpreting the unary predicates and diagonals of binary predicates): For every \(k\) (1 \(\leq k \leq N\)) and every \(a \in A_k\), set \(\text{tp}_{\mathfrak{A}}(a) = \text{tp}(\sigma_k)\). At the end of this step, we have, for every \(i\) (1 \(\leq i \leq L\)) and every \(a \in U_i\), \(\text{tp}_{\mathfrak{A}}(a) = \pi_i\).
Step 2 (Fixing the invertible message-types): Consider any \( j \) (\( 1 \leq j \leq M^* \)), and let \( j' \) be such that \( \mu_{j'}^{-1} = \mu_j \). We fix all the invertible messages sent, in either direction, between elements of \( V_j \) and those of \( V_{j'} \). More specifically, for any \( a \in A_k \subseteq V_j \), if \( \sigma_k \) requires to send an invertible message of type \( \mu_j \), we find an element of \( V_{j'} \) to receive this message, and hence to send an invertible message of type \( \mu_{j'} \) back to \( a \). By C1, \( V_j \) and \( V_{j'} \) are equinumerous. Let \( i \) and \( i' \) be such that \( \text{tp}_1(\mu_j) = \pi_i \) and \( \text{tp}_1(\mu_{j'}) = \text{tp}_2(\mu_j) = \pi_{i'} \); thus, \( V_j \subseteq U_i \) and \( V_{j'} \subseteq U_{i'} \). If \( V_j = V_{j'} = \emptyset \), there are no elements to deal with, so we need take no action. So assume that \( V_j \neq \emptyset \). Thus, there exists \( k \in K \) such that \( q_{j,k} = \mu_j[k] = 1 \), whence, since \( F \) is chromatic, \( \text{tp}_1(\mu_j) \neq \text{tp}_2(\mu_j) = \text{tp}_1(\mu_{j'}) \). That is \( i \neq i' \), so that \( V_j \subseteq U_i \) and \( V_{j'} \subseteq U_{i'} \) are disjoint. In particular, \( j \neq j' \); so let us then suppose, without loss of generality, that \( j' > j \). Pick some 1–1 correspondence between the equinumerous sets \( V_j \) and \( V_{j'} \); and for every \( a \in V_j \), set \( \text{tp}^3[a, a'] = \mu_j \), where \( a' \) is the element of \( V_{j'} \) corresponding to \( a \in V_j \). Since \( V_j \) and \( V_{j'} \) are disjoint, we have \( a \neq a' \), so that these assignments are meaningful. We must also show that they are consistent with Step 1, and do not clash with each other. Certainly, the assignments are consistent with Step 1, since all elements of \( U_i \supseteq V_j \) were assigned the 1-type \( \pi_i = \text{tp}_1(\mu_j) \), and all elements of \( U_{i'} \supseteq V_{j'} \) were assigned the 1-type \( \pi_{i'} = \text{tp}_2(\mu_j) \). To show that these assignments do not clash with each other, we demonstrate that if, in addition, \( a \in V_h \) and \( a' \in V_{i'} \) for some \( h \) (\( 1 \leq h \leq M^* \)) with \( \mu_h^{-1} = \mu_{i'} \), then \( h = j \) and \( h' = j' \). For consider the star-type \( \sigma_k \), where \( a \in A_k \); since \( a \in V_j \cap V_{i'} \), we have \( \sigma_k[j] \) and \( \sigma_k[j'] \) both positive. Yet, since also \( a' \in V_h \cap V_{i'} \), we have \( \text{tp}_1(\mu_{j'}) = \text{tp}_1(\mu_{i'}) \), i.e. \( \text{tp}_2(\mu_j) = \text{tp}_2(\mu_{i'}) \), which, if \( j \neq h \), contradicts the supposition that \( F \) is a chromatic frame. Symmetrically, \( h' = j' \). Therefore, no 2-type \( \text{tp}^3[a, a'] \) can be assigned twice in this step. We make one further observation before proceeding. Suppose that \( \text{tp}^3[a, a'] \) is assigned in this step and that \( a \in U_i \); we claim that \( a' \not\in X_{i'} \) for any \( i' \). To see this, suppose \( a \in V_{j'} \subseteq U_i \) and \( a' \in A_{k'} \subseteq V_{j'} \); with \( \mu_{j'}^{-1} = \mu_j \). Then \( \text{tp}_1(\mu_j) = \text{tp}_2(\mu_{j'}) = \pi_j \). But then \( \pi_{i'}[j'] > 0 \), whence \( p_{i', k'} = 0 \), whence \( a' \not\in X_{i'} \). This observation will be useful in Step 3.

Step 3 (Fixing the non-invertible message-types): Let \( i \) and \( i' \) be such that \( 1 \leq i \leq i' \leq L \). We fix all the non-invertible messages sent, in either direction, between \( U_i \) and \( U_{i'} \). More specifically, for any \( a \in A_k \subseteq U_i \), if \( \sigma_k \) requires various non-invertible messages to be sent to elements with 1-type \( \pi_{i'} \), we find elements of \( U_{i'} \) to receive these messages; in the other direction, for any \( a' \in A_{k'} \subseteq U_{i'} \), if \( \sigma_{k'} \) requires various non-invertible messages to be sent to elements with 1-type \( \pi_i \), we find elements of \( U_i \) to receive these messages. Let \( Z = (mC + 1)^2 \); thus, \( Z > 3mC \). By C3, either \( u_i \leq 1 \) or \( u_i > Z \); similarly, either \( u_{i'} \leq 1 \) or \( u_{i'} > Z \). We consider five cases.

Case 1: \( u_i = 0 \). In this case, there are no elements of \( U_i \) and hence no 2-type assignments to be made for elements of \( U_i \). In the other direction, by C2, \( s_{1,k} = 0 \) for all \( k \in K \), whence \( \sigma_k[j] = 0 \) for all \( k \in K \) and for all \( j \) (\( 1 \leq j \leq M \)) such that \( \text{tp}_2(\mu_j) = \pi_i \). Thus, no element of \( A \) in particular of \( U_{i'} \).
'wants' to send a message to an element with 1-type $\pi_i$, so we need take no action.

Case 2: $u_i = 1$. The situation is illustrated in the left-hand diagram of Fig. 4.2. Let $a$ be the sole element of $U_i$, and let $k$ be such that $a \in A_k$. Thus, $A_k = U_i = \{a\}$. We first deal with the non-invertible messages sent from $a$ to elements of $U_{i'}$. Since $a_{i,k} = 1$, C4, ensures that $x_{i',a} \geq r_{i',k}$, so that we may select a subset $R \subseteq X_{i',i}$ satisfying $|R| = r_{i',k}$. For each $j$ ($M^* + 1 \leq j \leq M$), if $t_{p_2}(\mu_j) = \pi_{i'}$, select $\sigma_k[j]$ elements $a'$ of $R$, and make the assignment $t_{p_3}[a, a'] = \mu_j$. (There are enough such elements by the definition of $r_{i',k}$.) These assignments are clearly consistent with those made in Step 1. We observed above that $t_{p_3}[a, a']$ is assigned in Step 2 only if $a' \notin X_{i',i}$; hence the assignments just made cannot clash with those of Step 2. We deal next with the assignment of non-invertible messages sent from elements of $U_{i'}$ to $a$. Consider any $a' \in A_{i'} \subseteq U_{i'}$. By C2, $s_{i',k'} \leq 1$. Hence, there is at most one value of $j$ in the range $1 \leq j \leq M$ such that $\sigma_{i'}[j] > 0$ and $t_{p_2}(\mu_j) = \pi_i$; moreover, this value $\sigma_{i'}[j]$ cannot exceed 1. That is: any element $a' \in U_{i'}$ requires to send at most one message—say, of type $\mu_j$—to any element of 1-type $\pi_i$. We could therefore satisfy this requirement by setting—if we may—$t_{p_3}[a', a] = \mu_j$. To see that we may, it suffices to establish: (i) $a$ and $a'$ are distinct; (ii) the assignment is consistent with those made in Step 1; (iii) the assignment does not clash with those made in Step 2, and those made above in this step. (i) To see that $a \neq a'$, observe first that, if $\sigma_i[j] = 1$ and $t_{p_2}(\mu_j) = \pi_i$, then $\mu_j$ must be non-invertible, by the chromaticity of $F$; in other words, $M^* + 1 \leq j \leq M$. Observe also that $p_{i,k} = 0$, and hence, bearing in mind that $a$ is the sole element of $U_i$, that $x_{i,a} = 0$. It follows from C4 that $r_{i,k} = 0$. But then, from the definition of $r_{i,k}$, we have $\sigma_k[j] = 0$ for all $j$ ($M^* + 1 \leq j \leq M$)—a contradiction. Hence, $a \neq a'$, as required. (ii) By the selection of $j$, it is obvious that the assignment $t_{p_3}[a', a] = \mu_j$ is consistent with
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those made in Step 1. (iii) To see that this assignment does not clash with the assignments made in Step 2, we observe that, since $s_{i,k'} \leq 1$, $\sigma_{k'}$ cannot require to send both invertible and non-invertible messages to elements of 1-type $\pi_i$. To see that this assignment does not clash with the assignments made above in this step, we simply note that, if $\sigma_{k'}$ requires to send a message to an element of 1-type $\pi_i$, then $a' \not\in X_{i,k'}$.

Case 3: $u_{i'} = 0$ and $u_i > Z$. Symmetrical to Case 1.

Case 4: $u_{i'} = 1$ and $u_i > Z$. Symmetrical to Case 2.

Case 5: $u_i > Z$ and $u_{i'} > Z$. Since $Z > 3mC$, partition $U_i$ into three sets $U_{i,0}, U_{i,1}, U_{i,2}$, each containing at least $mC$ elements; and similarly for $U_{i'}$. Suppose $a \not\in U_i$. Then for some $h$ ($0 \leq h < 3$), $a \in U_{i,h}$. Let $k$ be such that $a \in A_k$, and let $h' = h + 1$ (mod 3). For all $j$ ($M^* + 1 \leq j \leq M$), select $\sigma_{k}[j]$ fresh elements $a'$ of $U_{i',k'}$, such that $tp^\sigma[a,a']$ was not assigned in Step 2, and set $tp^\sigma[a,a'] = \mu_j$. Since $\sigma_k$ is $C$-bounded, it is obvious that $\sigma_k[1] + \cdots + \sigma_k[M] \leq mC \leq |U_{i',k'}|$, so that we never run out of fresh elements to select. In this way, we deal with all messages sent from $U_i$ to $U_{i'}$; the messages sent from $U_{i'}$ to $U_i$ are dealt with symmetrically. Since $tp_1(\mu_j) = \pi_i$ and $tp_2(\mu_j) = \pi_{i'}$, these assignments are consistent with Step 1; and they do not clash with any assignments made in Step 2, because of the way in which the elements $a'$ were selected. Further, the fact that $h' = h + 1$ (mod 3), ensures that these assignments do not clash with each other (even if $i = i'$), as is evident from the right-hand diagram of Fig. 4.2.

Performing these assignments for all pairs $i, i'$ such that $1 \leq i \leq i' \leq L$, completes Step 3. At the end of Step 3, then, for all $k$ ($1 \leq k \leq N$), all $a \in A_k$, and all $j$ ($1 \leq j \leq M$), there are exactly $\sigma_k[j]$ elements $a' \in A$ such that $a \neq a'$ and $tp^\sigma[a,a'] = \mu_j$.

Step 4 (Fixing the silent 2-types): Finally, we use the regulator, $\theta$, of $F = (C, K, \theta)$ to deal with all the remaining 2-types in $\mathfrak{A}$. Let $a, a'$ be distinct elements of $A$ such that $tp^\sigma[a,a']$ has not yet been assigned. Let $i, i', k, k'$ be such that $a \in A_k \subseteq U_i$ and $a' \in A_{k'} \subseteq U_{i'}$, and assume, without loss of generality, that $i \leq i'$. We claim that $\{\pi_i, \pi_{i'}\} \in \text{dom}(\theta)$. For suppose otherwise.

By C5, we have either $u_i = 1$ or $u_{i'} = 1$. Assume the former. If $p_{i,k'} = 0$, then there is some $j' (1 \leq j' \leq M)$ such that $\sigma_{k'}[j'] > 0$ and $tp_2(\mu_{j'}) = \pi_{i'}$, whence—bearing in mind that $a$ is the unique element of $U_i$—$tp^\sigma[a,a']$ will certainly have been assigned in Step 2 (if $\mu_j$ is an invertible message-type) or in Step 3 Case 2 (if $\mu_{j'}$ is a non-invertible message-type), contradicting the fact that $tp^\sigma[a,a']$ is unassigned. Thus, $p_{i,k'} = 1$, and hence $\sigma_{k',k} = 1$. That is: $a' \in X_{i,k}$. But $|X_{i,k}| = x_{i,k}$. Noting that $w_k > 0$ and $\alpha_{i,k} = 1$, and continuing to assume that $\{\pi_i, \pi_{i'}\} \in \text{dom}(\theta)$, it follows from C6 that $x_{i,k} \leq r_{i,k}$. Yet in Step 3 (Case 2), $r_{i,k}$ elements of $X_{i,k}$ were chosen to receive messages from $a$. Hence $a'$ must be among these elements, again contradicting the fact that $tp^\sigma[a,a']$ is unassigned. The case where $u_{i'} \leq 1$ proceeds symmetrically. Thus, we have established that, if $tp^\sigma[a,a']$ has not yet been assigned, then $\{\pi_i, \pi_{i'}\} \in \text{dom}(\theta)$, so that we can make the assignment $tp^\sigma[a,a'] = \theta(\{\pi_i, \pi_{i'}\})$. Since $tp_1(\theta(\{\pi_i, \pi_{i'}\})) = \pi_i$
and $\text{tp}_2(\theta(\{\pi_i, \pi_{\ell}\})) = \pi_{\ell}$, these assignments are consistent with Step 1. By construction, they cannot over-write any assignments in Steps 2 or 3. Evidently, we can proceed in this way until all remaining 2-types have been assigned. Moreover, since each $\theta(\{\pi_i, \pi_{\ell}\})$ is silent, this step does not spoil the work of Steps 2–3: we still have that, for all $k$ ($1 \leq k \leq N$), all $a \in A_k$, and all $\ell$ ($1 \leq \ell \leq M$), there are exactly $\sigma_k[\ell]$ elements $a' \in A$ such that $a \neq a'$ and $\text{tp}^A[a, a'] = \mu_\ell$.

This completes the construction of $\mathfrak{A}$. It is easy to see that $\mathcal{F}$ describes $\mathfrak{A}$, and that $\text{Hist}_{C}(\mathfrak{A}) = \bar{w}$.

The next lemma tells us that, if solvability is what interests us, we may restrict attention to small frames:

**Lemma 4.10.** Let $\mathcal{F}' = (C, K', \theta)$ be a (finitely) solvable frame over a classified signature $(\Sigma, \bar{f})$, $m = |f|$ and $s = |\Sigma|$. Then there exists a non-empty $K \subseteq K'$ such that the frame $\mathcal{F} = (C, K, \theta)$ is also (finitely) solvable, and $|K| \leq p(mC)2^{p(s)}$, where $p$ is a fixed polynomial.

**Proof.** There are fixed polynomials $p', q'$ such that $p'(mC)2^{p'(s)}$ bounds the number of equations and inequalities mentioned in (C1)–(C6) in Definition 4.14. If any such system of equations and inequalities has a (finite) solution, by Corollary 4.2 (or by Corollary 3.3), it has a (finite) solution $w_1, \ldots, w_N$ with at most $p(mC)2^{p(s)}$ non-zero values for some fixed polynomial $p$. Now let $K = \{k \in K' \mid w_k \neq 0\}$.

We can now use Corollaries 4.1 and 4.2 to achieve the main goal of this chapter.

**Theorem 4.1.** The problems $\text{Sat-C}^2$ and $\text{Fin-Sat-C}^2$ are in NExpTime.

**Proof.** Let a $C^2$-formula $\psi$ be given. By Lemma 4.1, we may compute a formula $\varphi$ of the form (4.1) in polynomial time, such that $\varphi$ and $\psi$ are satisfiable over the same domains of size greater than $C = \max(C_1, \ldots, C_m)$. Let $\Sigma$ be the signature of $\varphi$ together with $2[\log((mC + 1)^2)]$ new unary predicates, and let $\bar{f} = (f_1, \ldots, f_m)$. Thus $(\Sigma, \bar{f})$ is a classified signature. Write $s = |\Sigma|$.

We claim that $\varphi$ is (finitely) satisfiable if and only if there exists a chromatic (finitely) solvable frame $\mathcal{F} = (C, K, \theta)$ over $(\Sigma, \bar{f})$ such that $\mathcal{F} \models \varphi$, and $|K| \leq p(mC)2^{p(s)}$, where $p$ is some fixed polynomial, independent of $\varphi$. Suppose first that $\varphi$ has a (finite) model $\mathfrak{A}'$. Evidently, $\mathfrak{A}'$ is $C$-bounded. By Lemmas 4.2, 4.3 and 4.4, $\varphi$ has a (finite) $C$-bounded, chromatic, $(mC + 1)^2$-differentiated, regulated model $\mathfrak{A}$ over $(\Sigma, \bar{f})$. Let $\mathcal{F} = (C, K, \theta)$ be a frame describing $\mathfrak{A}$. By Lemma 4.6, $\mathcal{F}$ is chromatic; by Lemma 4.7, $\mathcal{F} \models \varphi$; by Lemma 4.8, $\mathcal{F}$ has a (finite) solution; and by Lemma 4.10 we may assume without loss of generality that $|K|$ is bounded by $p(mC)2^{p(s)}$, where $p$ is a fixed polynomial. By ignoring all zero-terms, Corollary 4.1 then ensures that there exists a solution all of whose values are bounded by $2^{p'(mC2^{p(s)})}$, where $p'$ is again a fixed polynomial. Conversely, suppose that $\mathcal{F} = (K, C, \theta)$ is a chromatic frame such that $\mathcal{F} \models \varphi$,...
and $F$ has a (finite) solution. By Lemma 4.9, there exists a (finite) structure $A$ such that $F$ describes $A$, and by Lemma 4.7, $A \models \varphi$.

Consider the following non-deterministic procedure, where $q_1$ and $q_2$ are fixed polynomials, and $n = \|\varphi\|$. 

1. Guess a chromatic frame $F = (C, K, \theta)$ with $|K| \leq 2^{q_1(n)}$ and check that $F \models \varphi$;
2. Guess a tuple $\vec{w} = (w_1, \ldots, w_N)$ of $\mathbb{N}^*$ such that, for all $k \ (1 \leq k \leq N)$,
   - if $k \notin K$, then $k = 0$, and either $w_k \leq 2^{q_2(n)}$ or $w_k = \aleph_0$.
3. If $\vec{w}$ is a solution of $F$, succeed; else fail.

Note that, even though $N$ may be doubly exponentially large as a function of $\|\varphi\|$, by recording only the non-zero values in $\vec{w}$, we can check in singly-exponential time whether $\vec{w}$ is a solution of $F$. Thus, for all polynomials $q_1$ and $q_2$, this procedure runs in time bounded by an exponential function of $\|\varphi\|$. But the claim of the previous paragraph shows that, for suitable $q_1$ and $q_2$, it has a successfully terminating run if and only $\varphi$ is satisfiable. This proves that Sat-$C^2$ is in NExpTime. To do the same for Fin-Sat-$C^2$, we simply modify line 3 to insist that $\vec{w}$ be a tuple of natural numbers.

It is well known that the satisfiability (= finite satisfiability) problem for the two-variable fragment of first-order logic without counting quantifiers is already NExpTime-hard. Thus, the NExpTime bound of Theorem 4.1 is tight.

**Corollary 4.3.** Let $\varphi$ be a formula of $C^2$. If $\varphi$ is finitely satisfiable, then it is satisfiable in a structure of size bounded by a doubly exponential function of $\|\varphi\|$.

**Proof.** Immediate from Lemmas 4.8 and 4.9, Corollary 4.1 and Remark 4.4.

We saw in Example 4.2 that there exists a sequence of finitely satisfiable $C^2$-formulas $\{\varphi_n\}_{n \geq 0}$ such that $\|\varphi_n\|$ is bounded above by a polynomial function of $n$, while the smallest satisfying model of $\varphi_n$ has cardinality bounded below by a doubly exponential function of $n$. Thus, the doubly-exponential bound of Corollary 4.3 is tight.

### 4.5 Bibliographic notes

The decidability of Sat-$C^2$ and Fin-Sat-$C^2$ was proved, without complexity bounds, by Grädel, Otto and Rosen [11] (who are also responsible for Example 4.2). The former result is established there by showing that we may confine attention to a recursively enumerable collection of structures, which establishes that Sat-$C^2$ is in r.e. Since $C^2$ is a fragment of first-order logic, Sat-$C^2$ is certainly in co-r.e., thus establishing decidability. Decidability of Fin-Sat-$C^2$ is established by reduction to a system of integer linear equations, along similar lines to the treatment here. Using the result of Lemma 3.5 (unavailable at the time), those
equations would yield a 2-NEXPTIME upper complexity bound. A complexity bound of 2-NEXPTIME for Sat-$C^k$ was reported by Pacholski, Szwast and Tendera [20, 21] using (in effect) the technique of constructing periodic models directly. The first NEXPTIME upper-bounds for Sat-$C^2$ and Fin-Sat-$C^2$, as well as the ‘small’ model property of Corollary 4.3 for the latter, were given by Pratt-Hartmann [24]. The proof there employs an elaborate construction to show that, if a formula of the form (4.1) has a model at all, then it has a model in a structure realizing a number of star-types bounded by a singly exponential function of $\|\varphi\|$. Lemma 3.5 obviates the need for this construction.

Exercises

1. Give detailed proofs of Corollaries 4.1 and 4.2.