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A complete axiom system for polygonal mereotopology of the real plane*

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Abstract

This paper presents a calculus for mereotopological reasoning in which two-dimensional spatial regions are treated as primitive entities. A first order predicate language $\mathcal{L}$ with a distinguished unary predicate $c(x)$, function-symbols $+$, $.$ and $-$ and constants 0 and 1 is defined. An interpretation $\mathfrak{R}$ for $\mathcal{L}$ is provided in which polygonal open subsets of the real plane serve as elements of the domain. Under this interpretation the predicate $c(x)$ is read as "region x is connected" and the function-symbols and constants are given their meaning in terms of a Boolean algebra of polygons. We give an alternative interpretation $\mathfrak{S}$ based on the real closed plane which turns out to be isomorphic to $\mathfrak{R}$. A set of axioms and a rule of inference are introduced. We prove the soundness and completeness of the calculus with respect to the given interpretation.

1 The problem

As anyone who has read a book on point-set topology knows, Euclidean spaces contain regions which could not possibly be useful for representing the shapes of everyday objects. Fractal dust, infinitely convoluted boundaries and other pathological constructions simply do not arise in the world of desks and chairs arranged in a room, or of plots of land drawn on a map, or of electronic components etched on a silicon chip. They are mere artifacts of a model of space according to which all spatial entities are sets of points, and all spatial properties are analysable in terms of the metric relations between points.

Perhaps, then, we can develop more efficient and parsimonious ways of representing and reasoning about space by taking regions, rather than points, as primitive. The best known theory of this kind is Tarski's [12] axiomatization of Euclidean geometry based on spheres. But the policy of taking regions as primitive is perhaps most attractive when considering problems involving mereological (part-whole) and other topological notions—that is, where no metric information is to hand. Recent interest in "mereotopology", much of it from within the AI community, dates from the work of Clarke [5],[6], following earlier work of Whitehead [13]. See, for example, Randall, Cui and Cohn [11], Gotts, Gooday and Cohn [7], and Borgo, Guarino and Masolo [4].

A *mereotopological calculus* is an axiomatic system in a formal language whose variables are to be thought of as ranging over spatial regions, and whose non-logical constants are to be thought of as expressing primitive topological properties and relations involving these regions. Mereotopological calculi vary as to which primitives they employ, and the axioms they propose. Clarke's calculus has a single binary relation of "connection" with the gloss that two regions are connected if they share a common point. Randall, Cui and Cohn also use a binary connection relation, but take two regions be connected if their *closures* share a common point. Borgo, Guarino and Masolo, by contrast, use a primitive "part-of" relation (the mereological component) together with a primitive property of "self-connectedness" (the topological component). However, all three approaches are motivated by the prospect of an adequate account of space in which regions are not identified with sets of points.

With any such calculus, the question arises as to whether the proposed axioms constitute a *good* theory of space—one that will lead to *correct* inferences about the everyday situations whose spatial features the calculus purports to model. More technically, we must ask whether physical space—at least approximately, and on everyday scales—is a model of the proposed axioms under the intended interpretation, and if so, how completely those axioms capture its features. This question has been answered for Clarke's system by Biacino and Gerla [2], who prove a completeness result guaranteeing that the regular sets of a Euclidean space are a model of Clarke's axioms (regular sets are explained below). This result is of interest because, whatever their faults, $\mathbb{R}^2$ and $\mathbb{R}^3$ are at least known to be workable models of physical space. Unfortunately, as Biacino and Gerla note, under the proposed interpretation, the language really allows only mereological relations to be expressed. In particular, Clarke's suggestions for capturing topological notions such as, for example, the relation of two objects' touching tangentially, do not have the desired effect. So although Biacino and Gerla's theorem is the kind of result needed to validate a mereotopological calculus, the particular interpretation to which it applies is too weak to be topologically interesting.

Another approach to mereotopology in which semantics plays an important part is taken by Asher and Vieu [1]. They present a calculus—also based on a binary relation of connection—together with a formal semantics in which the individuals are identified with certain subsets of a particular type of topological space. Soundness and completeness proofs are duly provided. Unfortunately, Asher and Vieu's topological spaces are strange objects, far removed from the standard Euclidean model of space. (In particular, as Asher and Vieu point out, these spaces are not dense.) While Asher and Vieu would reply that their aim is to present a mereotopological theory responsive to the demands of modelling cognition and natural language, rather than to reconstruct the mereotopology of Euclidean space, the radical nature of their models makes the axioms hard to assess.

In this paper, we present a mereotopological calculus for 2-dimensional spatial reasoning. Our axiom system, though considerably more complex than others that have been proposed, has the advantage of being sound and complete with respect to a familiar and topologically non-trivial spatial interpretation. By *familiar*, we mean a spatial interpretation based on an ontology known to provide a workable model of physical space. By *topologically non-trivial*, we mean one under which the formulae of the calculus express a wide range of topological (not just mereological) properties and relations. Thus, although regions are still regarded as primitive *within* the calculus, the axiom system is shown to characterize a familiar spatial ontology of proven utility.

2 A mereotopological calculus

2.1 The syntax of $\mathcal{L}$

Formally, $\mathcal{L}$ is a first-order language with equality, having the non-logical constants $c(x)$, 0, 1, $-$, $+$ and $.$, where $c(x)$ is a 1-place predicate, 0 and 1 are constant symbols, $-$ is a 1-place function symbol and $+$ and $.$ are 2-place function symbols. In other words, $\mathcal{L}$, is the language of Boolean algebra with a distinguished predicate $c(x)$.

Having defined our mereotopological language, we give a formal semantics in terms of familiar spatial constructions. We stress that the calculus itself cannot talk about these constructions: as far as it is concerned, spatial regions are primitive, and the interpretations of its non-logical constants are simply functions and relations defined over these primitives. But the familiar interpretation will guarantee that our mereotopological calculus really is an appropriate calculus for spatial reasoning.

In fact, we present two formal models for our mereotopological language $\mathcal{L}$, which we denote $\mathfrak{R}$ and $\mathfrak{S}$ and which turn out to be isomorphic. We begin with the more intuitive of the two.

2.2 The model $\mathfrak{R}$

Our first task is to establish our domain of interpretation, R. It is now fairly standard in treatments of mereotopology to confine attention to *regular* sets.

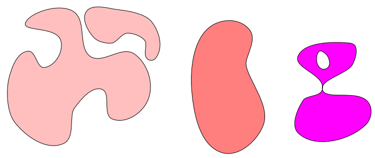
Definition 2.1 *Let X be a topological space and $x \subseteq X$. Then the set $\bigcup\{y \subseteq X | y \text{ open}, y \cap x = \emptyset\}$ is an open set in X called the* pseudocomplement *of x, written x'. We say that $x \subseteq X$ is* regular *if $x = x''$.*

The following well-known theorem underlies the importance of the regular sets to mereotopology. We state it here without proof. (See, e.g. Johnstone [8], chapter I, section 1.13.)

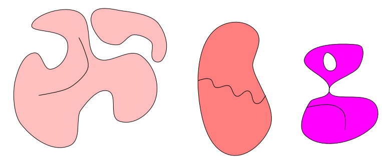
Theorem 2.1 *Let X be a topological space. Then the set of regular sets in X forms a Boolean algebra $M(X)$ with top and bottom defined by $1 = X$ and $0 = \emptyset$, and Boolean operations defined by $x.y = x \cap y$, $x + y = (x \cup y)''$ and $-x = x'$.*

Let $\mathbb{R}^2$ denote the real plane with the usual Euclidean topology. Our domain R will form a Boolean subalgebra of $M(\mathbb{R}^2)$.

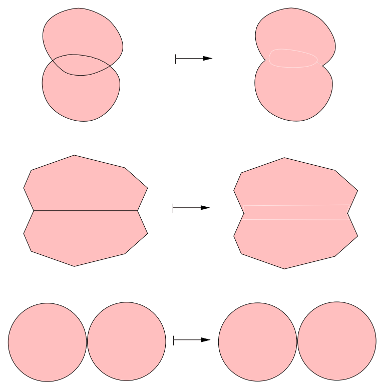
Before turning to its construction, let us pause to fix our intuitions about $M(\mathbb{R}^2)$. Basically, we can think of regular sets in $\mathbb{R}^2$ as open sets with no internal cracks or point-holes (compare fig. 1a with fig. 1b). The product, $x.y$, of two regular sets x and y is simply their intersection, which is guaranteed to be a regular set. The sum, $x+y$, of two regular sets x and y is a little more complicated; *very* roughly, it



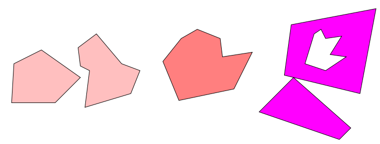
a) Three regular sets (each differently shaded)



b) Three non-regular sets



c) Three pairs of regular sets and their sums



d) Three polygons (each differently shaded)

Figure 1: Some regular sets of the plane and their Boolean combinations

is the union of x and y with any internal boundaries removed (fig. 1c). Finally, the pseudocomplement, $-x$, of a regular set x is simply that part of the plane not occupied by x or its boundary.

Any line in the plane cuts the plane into two connected, open sets, called *half-planes.* It is easy to see that these sets are regular, with each being the pseudocomplement of the other. Hence, we can speak about their sums, products and complements in $M(\mathbb{R}^2)$.

Definition 2.2 *A* basic polygon in $\mathbb{R}^2$ *is the intersection of finitely many half-planes in* $\mathbb{R}^2$*.*
A polygon in $\mathbb{R}^2$ *is the sum, in the Boolean algebra* $M(\mathbb{R}^2)$*, of any finite set of basic polygons in* $\mathbb{R}^2$*.*

We denote the set of polygons in $\mathbb{R}^2$ by R. Fig. 1d shows some polygons. Note that polygons, in this sense, need not be connected; nor need their complements be. Furthermore a polygon is not necessarily bounded. Note that $\emptyset$ and $\mathbb{R}^2$ also count as polygons. We have the following result:

Lemma 2.1 *R is a Boolean subalgebra of $M(\mathbb{R}^2)$.*

Proof: We need only show that R is closed under the Boolean operations. But this is obvious given the distribution laws for $M(\mathbb{R}^2)$. □

Idealizing slightly, R is the set of regions recognized by most computer systems specialized for handling plane spatial data, such as geographic information systems (*GIS*s). That is, such systems are limited to regions whose boundaries are made up of finitely many lines and line segments. Experience has shown that, for many practical purposes, no loss of useful expressive power results from limiting attention to polygons.

At last, then, we can define our familiar model $\mathfrak{R}$.

Definition 2.3 *The model $\mathfrak{R}$ has the domain R and the following interpretations of the predicate, constant and function symbols in $\mathcal{L}$:*

1. $[c(x)]^{\mathfrak{R}} = \{r \in R | r \text{ connected } \}$

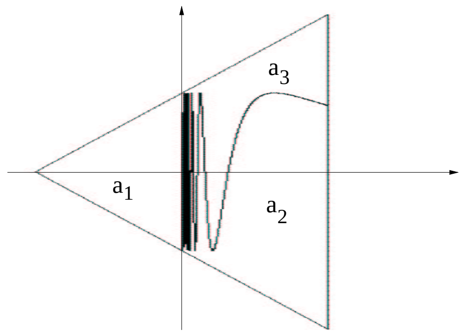


Figure 2: Some counterintuitive regions

2. $0^{\mathfrak{R}} = \emptyset$; $1^{\mathfrak{R}} = \mathbb{R}^2$

3. For all $r \in R$: $-^{\mathfrak{R}}(r) = -r$

4. For all $r_1, r_2 \in R$: $+^{\mathfrak{R}}(r_1, r_2) = r_1 + r_2$ and $.^{\mathfrak{R}}(r_1, r_2) = r_1.r_2$

Thus, the function symbols $+$, $.$ and $-$ have the obvious interpretation in terms of the Boolean algebra $M(\mathbb{R}^2)$, and a region satisfies the predicate $c(x)$ just in case it is connected in the usual sense of point-set topology. (Recall that an open set is connected if and only if it is not the union of any two disjoint, open, non-empty sets.) It turns out that, for disjoint, connected polygons r and s, $r+s$ is connected just in case r and s have a (non-trivial) edge in common; hence, the formula $c(x) \wedge c(y) \wedge x.y = 0 \wedge c(x+y)$ says (assuming x and y to be non-zero) that x and y are disjoint, connected regions touching externally along one or more edges. As a shorthand, if two regions satisfy the formula $c(x+y)$, we say that they are *connected to each other*.

To show that the pains we took to define our domain of interpretation were not in vain, consider the following formula of $\mathcal{L}$:

$$\forall x_1 \forall x_2 \forall x_3 \left(\left(\bigwedge_{1 \le i \le 3} c(x_i) \wedge c(x_1 + x_2 + x_3) \right) \rightarrow (c(x_1 + x_2) \vee c(x_1 + c_3)) \right). \tag{1}$$

This formula asserts that, if the sum of three connected regions is connected, then the first must be connected to at least one of the other two. Question: should this formula be a theorem of our mereotopological calculus?

The answer is that it depends on what, exactly, we count as regions. Formula (1) is true for the domain R (in fact, for any domain of reasonably well-behaved regions), but false for regular sets of the plane in general. For consider the regions a_1, a_2 and a_3 defined by

$$\begin{aligned} a_1 &= \{(x,y) | -1 < x < 0\ ;\ -1 - x < y < 1 + x\} \\ a_2 &= \{(x,y) | 0 < x < 1\ ;\ -1 - x < y < \sin(1/x)\} \\ a_3 &= \{(x,y) | 0 < x < 1\ ;\ \sin(1/x) < y < 1 + x\}\ , \end{aligned}$$

and depicted in fig. 2. It is not difficult to show that a_1, a_2 and a_3 are regular and connected, that $a_1 + a_2 + a_3$ is the interior of the large triangle in fig. 2 and so is connected, but that neither $a_1 + a_2$ nor $a_1 + a_3$ is connected.

In fact, for the purposes of *practical* mereotopological reasoning, (1) is quite a sensible formula to have as a theorem of our calculus, since—as it turns out—the only counterexamples are rather pathological and seem to be artifacts of the Euclidean model of space rather than anything that one could come across in real life. (Adopting such a stance is very much within the spirit of mereotopological research in AI, where the ability to reason efficiently in everyday situations is in focus.) But this case does demonstrate the importance of having a precise characterization of the regions our mereotopological calculus talks about: we ought to be clear, *if* (1) is a theorem of our calculus, what interpretations it is true in.

2.3 The model $\mathfrak{S}$

Before axiomatizing our mereotopological calculus, we present an alternative interpretation, based on the closed plane $\mathbb{R}^2 \cup \{\infty\}$, under the usual topology, hereinafter denoted Z^2. For technical reasons, we shall be working mostly in the closed plane.

Every line in the closed plane is taken to pass through the point ∞. It is easy to show (e.g. by considering the stereographic projection of the sphere onto the plane) that every line divides Z^2 into two regular connected sets, which we shall again refer to as half-planes. Since lines contain the point ∞, half-planes in the closed plane are literally the same sets of points as half-planes in the open plane.

Definition 2.4 *A* basic polygon *in Z^2 is the intersection of finitely many half-planes in Z^2. A* polygon *in Z^2 is the sum, in the Boolean algebra $M(Z^2)$, of any finite set of basic polygons in Z^2. We denote the set of polygons in Z^2 by S.*

Again, it is easy to show that S is a Boolean subalgebra of $M(Z^2)$, so we define the closed-plane model $\mathfrak{S}$ as follows.

Definition 2.5 *The model $\mathfrak{S}$ has the domain S and the following interpretations of the predicate, constant and function symbols in $\mathcal{L}$:*

1. $[c(x)]^{\mathfrak{S}} = \{r \in S | r \text{ connected }\}$
2. $0^{\mathfrak{S}} = \emptyset$; $1^{\mathfrak{S}} = Z^2$
3. For all $r \in S$: $-^{\mathfrak{S}}(r) = -r$
4. For all $r_1, r_2 \in S$: $+^{\mathfrak{S}}(r_1, r_2) = r_1 + r_2$ and $.^{\mathfrak{S}}(r_1, r_2) = r_1.r_2$.

2.4 The models $\mathfrak{R}$ and $\mathfrak{S}$ are isomorphic.

Denote the set of open sets in Z^2 (under the usual topology) by Ω. Let $\hat{x} = x \setminus \{\infty\}$ for all $x \in \Omega$ and let $\hat{\Omega} = \{\hat{x} | x \in \Omega\}$. Then $\langle \mathbb{R}^2, \hat{\Omega} \rangle$ is the usual topology on the open, real plane. Moreover, $\hat{\Omega} \subseteq \Omega$. If $u \in \hat{\Omega}$, then the pseudocomplement u^* of u in $\hat{\Omega}$ is in general different to the pseudocomplement u' of u in Ω; and the property of being regular in $\hat{\Omega}$ is different to the property of being regular in Ω. However, we have the following results:

Lemma 2.2 *If $x \in \Omega$, then $\hat{x}^* = \widehat{x'}$.*

Proof: Straightforward. □

Lemma 2.3 *If $x \in \Omega$ is regular in Ω, then $\hat{x}$ is regular in $\hat{\Omega}$.*

Proof: By two applications of lemma 2.2, $\hat{x}^{**} = \widehat{x''} = \hat{x}$. □

Lemma 2.4 *If $u \in \hat{\Omega}$ is regular in $\hat{\Omega}$, then there exists an $x \in \Omega$ regular in Ω such that $\hat{x} = u$.*

Proof: Recall that $\hat{\Omega} \subseteq \Omega$. It is easy to show that $u \in \Omega$ implies that u'' is regular in Ω. But again, by lemma 2.2, $\widehat{u''} = \hat{u}^{**} = u^{**} = u$. □

Thus, although $\mathfrak{R}$ and $\mathfrak{S}$ are based on non-homeomorphic topological spaces, we have:

Lemma 2.5 *The models $\mathfrak{R}$ and $\mathfrak{S}$ are isomorphic.*

Proof: Lemmas 2.3 and 2.4 show that $f : x \mapsto \hat{x}$ maps the set of regular sets in Ω onto the set of regular sets in $\hat{\Omega}$. And since no two regular sets in Ω differ by a single point, this mapping is 1–1. Moreover, given the fact that, for all $x, y \in \Omega$, $\hat{x} \cap \hat{y} = \widehat{x \cap y}$, lemma 2.2 further guarantees that f induces a Boolean algebra isomorphism taking $M(Z^2)$ to $M(\mathbb{R}^2)$. And since f maps half-planes in Z^2 to half-planes in $\mathbb{R}^2$, $f : S \to R$ is also a Boolean algebra isomorphism. Finally, we observe that $x \in \Omega$ is connected in Ω iff $\hat{x}$ is connected in $\hat{\Omega}$, so that $f : \mathfrak{S} \to \mathfrak{R}$ is a model isomorphism. □

It follows of course that $\mathrm{Th}(\mathfrak{R}) = \mathrm{Th}(\mathfrak{S})$.

3 Axiomatization

Our mereotopological calculus, \$, consists of a set of axioms and rules of inference stated in $\mathcal{L}$. The following abbreviations simplify the axioms.

1. Let $x \leq y$ stand for $x.y = x$. Intuitively, $x \leq y$ states that x is a subset of y.
2. Let $j(x)$ stand for $c(x) \wedge x \neq 0 \wedge c(-x) \wedge -x \neq 0$. In $\mathfrak{S}$ (but not in $\mathfrak{R}$) we can think of $j(x)$ as stating that x is a Jordan region. (See lemma 4.10.)
3. If $n \geq 1$, let $x_1 \oplus \ldots \oplus x_n = y$ stand for
$$x_1 + \ldots + x_n = y \wedge \bigwedge_{1 \leq i < j \leq n} x_i.x_j = 0 \wedge \bigwedge_{1 \leq i \leq n} (c(x_i) \wedge x_i \neq 0)$$
Intuitively, $x_1 \oplus \ldots \oplus x_n = y$ states that $x_1, \ldots, x_n$ form a partition of y each element of which is connected.
4. If $n \geq 1$, let $\beta_n(x)$ stand for
$$\exists z_1 \ldots \exists z_n \left(\bigwedge_{1 \leq i \leq n} c(z_i) \wedge \left(x = \sum_{1 \leq i \leq n} z_i \right) \right)$$
Intuitively, $\beta_n(x)$ states that x can be formed by summing n connected regions.

The axiom system \$ consists of the axioms and rules of inference of a complete system of first-order logic with equality, together with the following special axioms, axiom schemata and rules of inference.

1. The usual axioms of non-trivial Boolean algebra with Boolean operations $+$, $.$ and $-$, and (distinct) top and bottom elements 1 and 0.
2. $\forall x \forall y \forall z \Big(\big(c(x+y) \wedge c(y+z) \wedge y \neq 0 \big) \to c(x+y+z) \Big)$
3. Where $n > 1$, the axioms
$$\forall x_1 \ldots \forall x_n \left(\left(c \left(\sum_{1 \leq i \leq n} x_i \right) \wedge \bigwedge_{1 \leq i \leq n} c(x_i) \right) \to \bigvee_{2 \leq i \leq n} c(x_1 + x_i) \right)$$

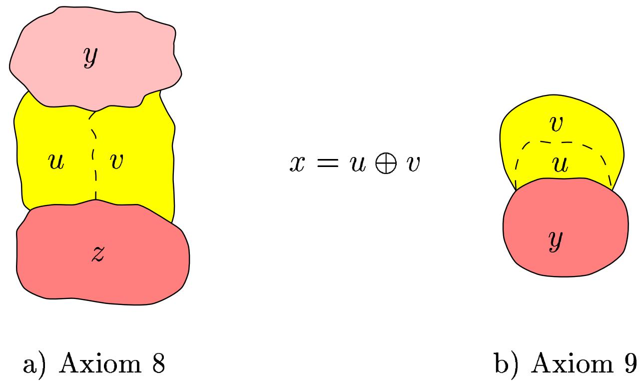


Figure 3: Instantiations of axioms 8 and 9

4. Where $n > 1$, the axioms

$$\forall x_1 \ldots \forall x_n \Big(\big(c \left(\sum_{1 \le i \le n} x_i \right) \wedge \bigwedge_{1 \le i \le n} c(x_i)\big) \rightarrow \bigvee_{1 \le i \le n} c \left(\sum_{1 \le j \le n}^{j \ne i} x_j \right) \Big)$$

5. $$\neg \exists x_1 \ldots \exists x_5 \left(\bigwedge_{1 \le i \le 5} (c(x_i) \wedge x_i \ne 0) \wedge \bigwedge_{1 \le i < j \le 5} (c(x_i + x_j) \wedge x_i . x_j = 0) \right)$$

6. $$\neg \exists x_1 \ldots \exists x_6 \left(\bigwedge_{1 \le i \le 6} (c(x_i) \wedge x_i \ne 0) \wedge \bigwedge_{1 \le i < j \le 6} x_i . x_j = 0 \wedge \bigwedge_{\substack{1 \le i \le 3 \\ 4 \le j \le 6}} c(x_i + x_j) \right)$$

7. $c(1)$

8. $$\begin{aligned} \forall x \forall y \forall z \Big(\big(& c(x) \wedge c(y) \wedge c(z) \wedge c(x+y) \wedge c(x+z) \wedge \\ & x.y = 0 \wedge x.z = 0 \wedge x \ne 0 \big) \rightarrow \\ & \exists u \exists v \big(u \oplus v = x \wedge c(u+y) \wedge c(u+z) \wedge c(v+y) \wedge c(v+z) \big) \Big) \end{aligned}$$

9. $$\begin{aligned} \forall x \forall y \Big(\big(& j(x) \wedge j(y) \wedge j(x+y) \wedge x.y = 0 \big) \rightarrow \\ & \exists u \exists v \big(u \oplus v = x \wedge c(u+y) \wedge \neg c(u + (-x).(-y)) \wedge \\ & c(v + (-x).(-y)) \wedge \neg c(v+y) \big) \Big) \end{aligned}$$

10. The infinitary rule of inference

$$\frac{\{\forall x(\beta_n(x) \rightarrow \phi(x)) | n \ge 1\}}{\forall x \phi(x)}$$

Axiom 2 ensures that two connected regions with a nonempty intersection have a connected sum. Axioms 3 and 4 impose restrictions on n-tuples ($n > 1$) of connected regions whose sum is connected. Specifically, axiom 3 states that every region in the n-tuple must be connected to some other region in the n-tuple;

axiom 4 states that we can always find a region in the n-tuple which, when removed, leaves an $(n-1)$-tuple whose sum is connected. Axioms 5 and 6 reflect the non-planarity of the graphs K_5 and $K_{3,3}$, respectively. Axiom 7 says that the entire space is connected. Axioms 8 and 9 guarantee the existence of enough regions in the model. Simple instantiations of axioms 8 and 9 are shown in figure 3 where region x is indicated by light grey areas. The precise content of the axioms is explained in the proof of theorem 5.2. Finally, the infinitary rule of inference serves to guarantee the existence of models in which every region is the sum of finitely many connected regions.

The formula $c(0)$ is a theorem of the system \$. (This fact will be useful in the sequel.) We give a proof here of this as an illustration of the infinitary rule of inference.

Lemma 3.1 $\vdash_{\$} c(0)$.

Proof: By the axioms for a non-trivial Boolean algebra, we have, for all $n \geq 1$:

$$\vdash_{\$} \forall x \forall x_1 \ldots \forall x_n \Big(\big(\bigwedge_{1 \leq i \leq n} c(x_i) \wedge (x = \sum_{1 \leq i \leq n} x_i) \wedge \bigwedge_{1 \leq i \leq n} (x_i = 0) \big) \rightarrow c(x) \Big)$$

$$\vdash_{\$} \forall x \forall x_1 \ldots \forall x_n \Big(\big(\bigwedge_{1 \leq i \leq n} c(x_i) \wedge (x = \sum_{1 \leq i \leq n} x_i) \wedge \neg \bigwedge_{1 \leq i \leq n} (x_i = 0) \big) \rightarrow (x \neq 0) \Big) .$$

Hence,

$$\vdash_{\$} \forall x \Big(\exists x_1 \ldots \exists x_n \big(\bigwedge_{1 \leq i \leq n} c(x_i) \wedge (x = \sum_{1 \leq i \leq n} x_i) \big) \rightarrow \big(c(x) \vee (x \neq 0) \big) \Big).$$

Using the abbreviation β_n gives us

$$\vdash_{\$} \forall x \Big(\beta_n(x) \rightarrow \big(c(x) \vee (x \neq 0) \big) \Big)$$

for all $n \geq 1$. By the infinitary rule of inference 10 we get

$$\vdash_{\$} \forall x \big(c(x) \vee (x \neq 0) \big)$$

and hence $\vdash_{\$} c(0)$. □

Let us denote the set of sentences (closed formulae) of $\mathcal{L}$ which are theorems of the system \$ by $T_{\$}$. The main technical result of this paper is to show that the axiom system \$ is sound and complete with respect to both of the familiar interpretations given above. In other words:

$$T_{\$} = \mathrm{Th}(\mathfrak{R}) = \mathrm{Th}(\mathfrak{S}) \ .$$

Further model theoretic aspects of the axiom system are investigated in [10].

4 The domain S and its properties

In this section, we establish some basic facts about S needed for the soundness and completeness proofs below. Many of these facts are obvious and some readers may therefore wish to skip to the next section. However, the notation introduced in definitions 4.1, 4.2, 4.3, 4.4 and 4.5 will be used in subsequent sections.

4.1 Results concerning regular sets

Definition 4.1 *If x is any set in a topological space, let $[x]$ denote the closure of x and $(x)^0$ the interior of x. We write $\mathcal{F}(x)$ to denote the* frontier of x, *namely* $[x] \setminus ([x])^0$.

We will use lemmas 4.1, 4.2 and 4.3 repeatedly in the sequel (sometimes without mention).

Lemma 4.1 *Let x be a subset of a topological space X. Then $x' = X \setminus [x]$ and $x'' = ([x])^0$.*

Proof: Straightforward. □

Hence, x is regular iff $x = ([x])^0$; and if x is any set, then $x'' \subseteq [x]$.

Lemma 4.2 *Let X, Y be topological spaces and ν a homeomorphism from X onto Y. Let a, b be regular sets in X. Then $\nu(a)$ and $\nu(b)$ are regular sets in Y with: (i) $\nu(a.b) = \nu(a).\nu(b)$; (ii) $\nu(-a) = -\nu(a)$; and (iii) $\nu(a+b) = \nu(a) + \nu(b)$.*

Proof: Straightforward. □

Lemma 4.3 *Let a_1, a_2, a_3 be regular sets of a topological space X with $a_1 + a_2, a_2 + a_3$ connected and $a_2 \neq 0$. Then $a_1 + a_2 + a_3$ is connected.*

Proof: Note that $a_1 + a_2 + a_3 = ((a_1 + a_2) \cup (a_2 + a_3))''$. The lemma then follows from the fact that $(a_1 + a_2) \cup (a_2 + a_3)$ is connected and the standard result that, if x is connected and $x \subseteq y \subseteq [x]$, then y is connected. □

We now introduce some concepts which will be used repeatedly in the proofs to come.

Definition 4.2 *Let X be a topological space. Let M be any Boolean subalgebra of $M(X)$. If A is a finite subset of M and the elements of A are pairwise disjoint and sum to $a \in M$, A is said to be a* partition of *a in M. If, in addition, every element of A is connected, we call A a* connected partition of *a in M. In the case $a = 1$, we refer to A, simply, as a* (connected) partition *in M.*

It is easy to see that, if x is regular, then $[x]$ is the disjoint union of x and $\mathcal{F}(x)$. The following (rather technical) lemma will be useful later.

Lemma 4.4 *Let X be a topological space and $a_1, \ldots, a_n$ a partition in $M(X)$. Let m be such that $1 \leq m \leq n$. Then*

$$a_1 + \ldots + a_m = a_1 \cup \ldots \cup a_m \cup \{p | \ p \in \mathcal{F}(a_i) \text{ for some } i \ (1 \leq i \leq m), \\ p \notin \mathcal{F}(a_j) \text{ for any } j \ (m < j \leq n)\}.$$

Proof: Denote the right hand side of the above equation by x. Suppose $p \in [a_j]$ for some j $(m < j \leq n)$. Then $p \in \mathcal{F}(a_j)$ or $p \in ([a_j])^0 = a_j$ since a_j is regular. If $p \in a_j$, then $p \notin [a_i]$ for any i $(1 \leq i \leq m)$ by the disjointness of $a_1, \ldots, a_n$. Either way, then, $p \notin x$.
Suppose $p \notin [a_j]$ for any j $(m < j \leq n)$. Then certainly $p \notin \mathcal{F}(a_j)$ for any j $(m < j \leq n)$. Moreover, $a_1, \ldots, a_n$ sum to 1, so $[a_1] \cup \ldots \cup [a_n] = 1$. Hence $p \in [a_i]$ for some i $(1 \leq i \leq m)$, so, again, $p \in \mathcal{F}(a_i)$ or $p \in ([a_i])^0 = a_i$ since a_i is regular. Either way, $p \in x$.
Hence $x = (1 \setminus [a_{m+1}]) \cap \ \ldots \ \cap (1 \setminus [a_n])$. That is, $x = (-a_{m+1}). \ \ldots \ .(-a_n) = a_1 + \ldots + a_m$. □

4.2 Basic properties of S

We begin with a lemma on which much of the subsequent analysis depends.

Lemma 4.5 *Any element of S is the sum of finitely many connected elements of S.*

Proof: Since half-planes in the open plane are convex sets, so are basic polygons. So every element of R is the sum of finitely many connected elements of R. The result then follows by lemma 2.5. □

It is easy to see that this property does not hold for all Boolean sub-algebras of $M(Z^2)$, even where the elements are relatively well-behaved. For example, if x and y are Jordan regions (i.e. topologically equivalent to the unit disk), the intersection $x.y$ can have infinitely many disconnected parts. It is precisely to prevent this possibility that we restrict ourselves to polygons.

As usual, we take a *component* of a set to be a maximal, nonempty, connected subset of that set.

Lemma 4.6 *Let $r \in S$ and let c be a component of r. Then $c \in S$. Moreover, r equals the sum of its components.*

Proof: By lemma 4.5, let $c_1, \ldots, c_n$ be connected elements of S such that $r = c_1 + \ldots + c_n$. For all i $(1 \leq i \leq n)$, if $c.c_i \neq 0$ then, by lemma 4.3, $c_i + c$ is connected. If, in addition, $(-c).c_i \neq 0$, then $c < c + c_i$, contradicting the maximality of c. Thus, if $c.c_i \neq 0$, then $(-c).c_i = 0$. Hence c can be expressed as the sum of various c_i $(1 \leq i \leq n)$, and $c \in S$. The rest of the lemma is trivial. □

Connected partitions play an important role in understanding S. In particular, we have:

Lemma 4.7 *Let $r_1, \ldots, r_n \in S$. Then there exists a connected partition C in S such that r_i is expressible as a sum of various $c \in C$ for each i $(1 \leq i \leq n)$.*

Proof: Let C be the set of all components of all non-zero products of the form $\pm r_1. \ \ldots \ . \pm r_n$. By lemma 4.6, these components are elements of S, and form a connected partition such that every r_i can be expressed as a sum of various C. □

4.3 Connected partitions and graphs

It will come as no surprise that we can picture connected partitions in S as the faces of piecewise linear graphs drawn in the closed plane.

Definition 4.3 *A* graph* *G is a plane graph in the closed real plane having no nodes of degree 0, together with a (possibly empty) set of* nodeless edges. *These nodeless edges are all Jordan curves intersecting no other edge of G (nodeless or otherwise).*
An edge of a graph is* piecewise linear *if it lies on finitely many lines. A graph* is piecewise linear if each of its edges is.*

Hence, all plane graphs in the closed plane are graphs*. Fig. 4 shows a graph* (where the page represents the whole closed plane). This particular graph* has two nodeless edges and no nodes of degree 2. Finally, we observe that Euler's formula for a k-component graph, namely $n - e + f = k + 1$, applies also to a k-component graph*, where nodeless edges do not count as components.

If G is a graph*, we denote by $|G|$ the set of points in the edges and vertices of G. We say that two graphs* G and G' are *topologically equivalent* if there exists a homeomorphism ν of the closed plane onto itself mapping $|G|$ to $|G'|$. A graph* is said to have an *isthmus* if it contains an edge whose removal would increase the number of its connected components.

The following theorem establishes the importance of piecewise linear graphs*.

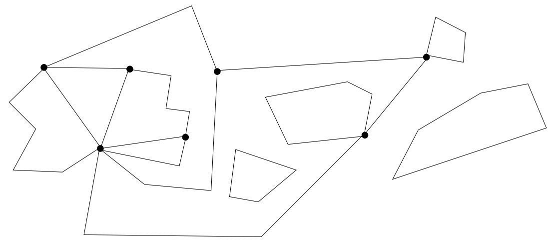


Figure 4: A graph* with two nodeless edges

Theorem 4.1 *Let $r_1, \ldots, r_n$ be a connected partition in S; then there exists a finite piecewise linear graph* with no isthmuses whose faces are precisely $r_1, \ldots, r_n$. Conversely, let G be a finite piecewise linear graph* with no isthmuses; then the faces of G form a finite connected partition in S.*

Proof: Consider all the half-planes involved in the construction of elements $r_1, \ldots, r_n$. The lines bounding these half-planes form a finite graph* G_0 in the obvious way, and the faces of G^* must form a connected partition of basic polygons, say, $b_1, \ldots, b_N$. Moreover, each r_i $(1 \leq i \leq n)$ can certainly be expressed as a sum of various b_j $(1 \leq j \leq N)$. By renumbering if necessary, let $r_1 = b_1 + \ldots + b_m$ for some m $(1 \leq m \leq N)$.
Now remove from G_0 all nodes p such that $p \notin \bigcup\{\mathcal{F}(b_k)|m < k \leq N\}$ and all edges e such that $e \not\subseteq \bigcup\{\mathcal{F}(b_k)|m < k \leq N\}$. The result will be a graph* G_1 in which the faces $b_1, \ldots, b_m$ are merged into a number of faces $f_1, \ldots, f_l$ for some l $(1 \leq l \leq m)$. The union of these faces will then be the set

$$b_1 \cup \ldots \cup b_m \cup \{p \in |G| : p \in \mathcal{F}(b_i) \text{ for some } i \ (1 \leq i \leq m), \\ p \notin \mathcal{F}(b_j) \text{ for any } j \ (m < j \leq N)\}.$$

By lemma 4.4 this set is just $b_1 + \ldots + b_m = r_1$. Since r_1 is connected, $l = 1$ and G_1 contains the face $f_1 = r_1$. Proceeding in the same way for $r_2, \ldots, r_n$ yields a graph* $G = G_n$ with faces $r_1, \ldots, r_n$. That G has no isthmuses follows from the fact that each face of G is regular.
Conversely, suppose that G is a finite piecewise linear graph*; then the edges of G lie on finitely many lines $l_1, \ldots, l_n$. Consider the graph G^* made up of all of these lines (extended in both directions). Each face of G^* is a basic polygon; hence each face f_i of G will be divided into a finite number of basic polygons, say, $b_{i,1}, \ldots, b_{i,m_i}$ by a finite number of lines. Since G has no isthmuses, f_i is a regular set, and it is easy to check that no smaller regular open set contains $b_{i,1}, \ldots, b_{i,m_i}$. In other words, $f_i = b_{i,1} + \ldots + b_{i,m_i} \in S$. □

Lemma 4.8 *Let $r_1, r_2 \in S$ be nonempty, disjoint and connected. Then $r_1 + r_2$ is connected iff some line-segment lies on the frontiers of both r_1 and r_2.*

Proof: By theorem 4.1 (p. 11) r_1 and r_2 are faces of some finite graph* G.

If α is a line segment s.t. $|\alpha| \subseteq \mathcal{F}(r_1) \cap \mathcal{F}(r_2)$, by the finiteness of G, we can find some line segment β with $|\beta| \subseteq |\alpha|$ such that β lies on the boundary of no other face of G. It follows that $|\beta| \subseteq r_1 + r_2$, so $r_1 \cup r_2 \cup |\beta|$ is path-connected, hence connected. Since $r_1 \cup r_2 \cup |\beta| \subseteq r_1 + r_2 \subseteq [r_1 \cup r_2 \cup |\beta|]$, $r_1 + r_2$ is connected.

Conversely, if $\mathcal{F}(r_1) \cap \mathcal{F}(r_2)$ contains no line segment, it is either a finite set of isolated points or is empty. It is easy to show that, in either case, $r_1 + r_2 = r_1 \cup r_2$ and so is not connected. □

Definition 4.4 *Let x be any open set in Z^2. An* end-cut *in x is a Jordan arc lying in x except for one of its endpoints. A* cross-cut *in x is a Jordan arc lying in x except for both of its (distinct) endpoints. We say that $\mathcal{F}(x)$ is* accessible *from x if, for any $p \in \mathcal{F}(x)$ and any $q \in x$, there is an end-cut in x from p to q.*

Lemma 4.9 *Let $r \in S$ be connected. Let $p \in \mathcal{F}(r)$ and $q \in r$. Then there exists a piecewise linear end-cut in r from p to q.*

Proof: By theorem 4.1, r is a face of some piecewise linear graph*. The lemma is then obvious. □

Hence, if $r \in S$ is connected, then $\mathcal{F}(r)$ is accessible from r.

Lemma 4.10 *Let $r \in S$. Then r is a Jordan region iff r is connected and nonzero with a connected and nonzero complement.*

Proof: Suppose r is connected and nonzero with a connected and nonzero complement. The converse of Jordan's theorem states that if a closed set has two complementary domains in the closed plane, from each of which it is accessible, then it is a Jordan curve. But $\mathcal{F}(r) = \mathcal{F}(-r)$ has r and $-r$ as its complementary domains, so must be a Jordan curve. The other direction is trivial. We remark that this lemma relies on the fact that our underlying topological space is the closed plane. □

Lemma 4.11 *Let $r, s \in S$ be disjoint Jordan regions. Then if $-(r+s)$ is connected, so is $\mathcal{F}(r) \cap \mathcal{F}(s)$.*

Proof: If $\mathcal{F}(r) \cap \mathcal{F}(s)$ has more than one component, lemma 4.9 guarantees that we can construct a Jordan curve in $[r+s]$ with points in $-(r+s)$ lying on either side of it, thus contradicting the connectedness of $-(r+s)$. □

Lemma 4.12 *If r_1, r_2 and r_3 are disjoint connected elements of S, then there exist at most two points lying on the frontiers of more than two of these regions.*

Proof: We suppose that p_1, p_2 and p_3 are distinct points all lying on the frontiers of r_1, r_2 and r_3 and derive a contradiction. Choose points q_1, q_2, q_3 such that $q_i \in r_i$ $(i = 1, 2, 3)$. Since r_1, r_2 and r_3 are polygons, it is clear that for $i = 1, 2, 3$, we can draw three end-cuts in r_i, say $\alpha_{i,1}$, $\alpha_{i,2}$ and $\alpha_{i,3}$ from the point q_i to the points p_1, p_2 and p_3, respectively. Since we can choose $\alpha_{i,1}$, $\alpha_{i,2}$ and $\alpha_{i,3}$ so that they intersect only at q_i, this gives us a planar embedding of the graph $K_{3,3}$, which is well-known to be non-planar (see, e.g. Bollobás [3], p.19). □

Lemma 4.13 *Let $r, s, t \in S$ be Jordan regions such that $r \oplus s \oplus t = 1$ and $r+s$ and $r+t$ are connected. Then $\mathcal{F}(r) \cap \mathcal{F}(s)$ and $\mathcal{F}(r) \cap \mathcal{F}(t)$ are Jordan arcs.*

Proof: By lemma 4.11, $\mathcal{F}(r) \cap \mathcal{F}(s)$ and $\mathcal{F}(r) \cap \mathcal{F}(t)$ are connected. We show that $\mathcal{F}(r) \cap \mathcal{F}(s) \cap \mathcal{F}(t)$ contains exactly two points, say p and q. These points divide the Jordan curve $\mathcal{F}(r)$ into two Jordan arcs. It is then easy to show using the connectedness of $\mathcal{F}(r) \cap \mathcal{F}(s)$ and $\mathcal{F}(r) \cap \mathcal{F}(t)$ that these Jordan arcs are exactly $\mathcal{F}(r) \cap \mathcal{F}(s)$ and $\mathcal{F}(r) \cap \mathcal{F}(t)$.

That $\mathcal{F}(r) \cap \mathcal{F}(s) \cap \mathcal{F}(t)$ consists of at most two points follows by lemma 4.12. That $\mathcal{F}(r) \cap \mathcal{F}(s) \cap \mathcal{F}(t)$ is not the empty set follows from the connectedness of $\mathcal{F}(r) = (\mathcal{F}(r) \cap \mathcal{F}(s)) \cup (\mathcal{F}(r) \cap \mathcal{F}(t))$. That $\mathcal{F}(r) \cap \mathcal{F}(s) \cap \mathcal{F}(t)$ is not a single point follows from the fact that the connectedness of $\mathcal{F}(r)$ is not destroyed by the removal of one point. □

The following general result on (abstract) graphs will be used in several places below.

Lemma 4.14 *Let G be a finite, connected graph. Then we can find a node of G which, when removed, still leaves a connected graph.*

Proof: Straightforward. □

Now let us apply this lemma to the analysis of S. Given connected, nonzero elements $r_1, \ldots, r_n \in S$, we can form the abstract graph whose nodes are $\{r_1, \ldots, r_n\}$ and whose edges are $\{(r_i, r_j) | 1 \leq i < j \leq n \text{ and } r_i + r_j \text{ is connected }\}$ (i.e. there are no multiple edges). This graph has the following useful property.

Lemma 4.15 *Let $n \geq 1$ and let $r_1, \ldots, r_n$ be nonzero, connected regions of S. Let G be the graph with nodes $\{r_1, \ldots, r_n\}$ and edges $\{(r_i, r_j) | 1 \leq i < j \leq n \text{ and } r_i + r_j \text{ is connected }\}$. Then $r_1 + \ldots + r_n$ is a connected element of S iff G is a connected graph.*

Proof: For the if-direction, we proceed by induction on n. If $n = 1$, the result is trivial. Otherwise, by lemma 4.14, we can suppose WLOG that the graph $G - \{r_1\}$ formed by removing r_1 and all its edges from G is connected. By inductive hypothesis, $r_2 + \ldots + r_n$ is connected. Since G is connected, there must be some i $(2 \leq i \leq n)$ such that $r_1 + r_i$ is connected. Since $r_i \neq 0$, $r_1 + r_2 + \ldots + r_n$ is connected by lemma 4.3.

For the only-if-direction, it suffices to show that, for all i,j $(1 \leq i < j \leq n)$, there is a sequence $i = i_1, \ldots, i_k = j$ such that $r_{i_h} + r_{i_{h+1}}$ is connected for all h $(1 \leq h < k)$. Let $p \in r_i$ and $q \in r_j$. By the connectedness of $r_1 + \ldots + r_n$, draw a Jordan arc α from p to q lying within $r_1 + \ldots + r_n$. Since $\mathcal{F}(r_i)$ is accessible from r_i, α can be chosen so as to visit each region only once. And since the frontier of each r_i lies on finitely many lines, we may assume that α can be chosen so that all points on α lie on the frontiers of at most two of the regions. Let the sequence of regions visited by α be $r_i = r_{i_1}, \ldots, r_{i_k} = r_j$. Then for all h $(1 \leq h < k)$, either $r_{i_h} \cap r_{i_{h+1}} \neq \emptyset$ or α visits a point p on a line segment shared by $\mathcal{F}(r_{i_h})$ and $\mathcal{F}(r_{i_{h+1}})$. Either way, $r_{i_h} + r_{i_{h+1}}$ is connected. □

The following lemmas are immediate consequences of lemma 4.15.

Lemma 4.16 *Let $r, s \in S$ be disjoint with r and $-s$ connected. Let t be a component of $-(r+s)$. Then $r + t$ is connected.*

Proof: If $r = 0$, the lemma is trivial. Otherwise, let the components of $-(r+s)$ be $t_1, \ldots, t_n$. Obviously, $t_i + t_j$ is not connected for all i,j $(1 \leq i < j \leq n)$. But $-s = t_1 + \ldots + t_n + r$ is connected. The lemma follows from lemma 4.15. □

Lemma 4.17 *Let $r_1, \ldots, r_n \in S$ be connected with $r_1 + \ldots + r_n$ connected. Then, by renumbering if necessary, $r_1 + \ldots + r_{n-1}$ is connected.*

Proof: From lemma 4.14 and lemma 4.15. □

A result related to lemma 4.17 applies to *Jordan* regions in S. The proof involves a slightly stronger version of lemma 4.14 but is otherwise similar. The details are routine and will be omitted.

Lemma 4.18 *Let $r_1, \ldots, r_n \in S$ be Jordan regions with $r_1 + \ldots + r_n$ a Jordan region. Then, by renumbering if necessary, $r_1 + \ldots + r_{n-1}$ is a Jordan region.*

4.4 Finiteness properties concerning S

The following lemmas are crucial to the completeness proof.

Lemma 4.19

There exists a function $e : \mathbb{N} \to \mathbb{N}$ such that, for all $n > 0$, if $r_1, \ldots, r_n$ are disjoint, connected elements of S, then there exist at most $e(n)$ points lying on the frontiers of more than two of these regions.

Proof: Since, by lemma 4.12 no more than two points can lie on the frontiers of any triple of regions, the lemma follows by putting $e(n) = n(n-1)(n-2)/3$. □

Lemma 4.20 *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that, for all $n > 0$, if A is any connected partition in S with n members and G is a piecewise linear graph* with no nodes of degree 2 whose faces in the closed plane are A, then the size of G is bounded by $f(n)$.*

Proof: It is easy to show that any node of degree greater than 2 of a plane graph with no isthmuses must lie on the frontier of at least 3 faces. Then, by lemma 4.19, the number of nodes in G is bounded by a function of n. The lemma then follows from Euler's formula. □

We then have:

Theorem 4.2 *There exists a function $g : \mathbb{N} \to \mathbb{N}$ such that, for all $n > 0$, there exist at most $g(n)$ n-element connected partitions in S up to topological equivalence.*

Proof: By theorem 4.1, page 11, any such partition is the set of faces of some piecewise linear graph* with no isthmuses, hence of some piecewise linear graph* with no isthmuses and no nodes of degree 2. By lemma 4.20, all such graphs* are of size bounded by $f(n)$. Since it can be shown that every abstract graph can be embedded in the closed plane in only finitely many ways up to topological equivalence, the result follows immediately. □

We note in passing that theorem 4.2 is false for Euclidean spaces of higher dimension than 2. It is also false for arbitrary partitions of $M(Z^2)$.

4.5 The homogeneity of S

The following lemmas are concerned with showing that S is, in a sense that will become clear below, topologically homogeneous.

It is well-known that every finite plane graph G in the closed plane can be continuously deformed into piecewise linear plane graph G'. (See, e.g. Bollobás [3], p.16.) Indeed, this can be done in such a way that piecewise linear edges in G are unaffected. In effect, finite plane graphs can have their edges 'straightened out' by a homeomorphism, without affecting any points in those faces whose frontiers involve only straight edges. These results can easily be extended to finite graphs*.

If ν is a homeomorphism of the closed plane onto itself and x a subset of the closed plane, we write $\nu|_x$ to denote the restriction of ν to x. Then we have:

Lemma 4.21 *Let r, s be connected elements of S such that there is a homeomorphism μ of the closed plane onto itself taking r to s. Let $r_1, \ldots, r_n$ be a connected partition of r in S. Then there exists a connected partition $s_1, \ldots, s_n$ of s in S and a homeomorphism ν of the closed plane onto itself such that $\nu|_{-r} = \mu|_{-r}$ and $\nu(r_i) = s_i$ for all i $(1 \leq i \leq n)$.*

Proof: Let the components of $-r$ be $t_1, \ldots, t_m$. Since $t_1, \ldots, t_m, r_1, \ldots, r_n$ is a connected partition, theorem 4.1 guarantees that we can find a piecewise linear graph* G with no isthmuses having these elements as faces. Now μ maps r to s, hence the components of $-r$ to the components to $-s$, hence G to a graph* G' with faces $u_1, \ldots, u_m, f_1, \ldots, f_n$, say, where $f_1 + \ldots + f_m = s$. But then we can continuously deform G' to a piecewise linear graph* G'' without affecting any points in $-s$ or its frontier. Hence, the faces of G'' will be $u_1, \ldots, u_m, s_1, \ldots, s_n$, say. Thus, there is a homeomorphism μ' of the closed plane onto itself which is the identity mapping outside s and which maps f_i to s_i, for all i $(1 \leq i \leq n)$. Since G'' clearly contains no isthmuses, theorem 4.1 guarantees that the faces of G'' will be in S, so that $\nu = \mu' \circ \mu$ is the required homeomorphism. □

Lemma 4.22 *Let r, s be connected elements of S such that there is a homeomorphism μ of the closed plane onto itself taking r to s. Let $r' \in S$ satisfy $r' \leq r$. Then there exists $s' \in S$ satisfying $s' \leq s$ and a homeomorphism ν of the closed plane onto itself such that $\nu|_{-r} = \mu|_{-r}$ and $\nu(r') = s'$.*

Proof: By lemma 4.7, we can find a finite connected partition of r in S some of whose elements sum to r'. The result then follows from lemma 4.21. □

Definition 4.5 *If $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ are regions of S such that there is a homeomorphism of the closed plane onto itself mapping r_i to s_i for all i $(1 \leq i \leq n)$, then we say that $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ are* topologically equivalent *and write $r_1, \ldots, r_n \sim s_1, \ldots, s_n$.*

Now we can state the lemma guaranteeing homogeneity of S:

Lemma 4.23 *Let $r_1, \ldots, r_n, s_1, \ldots, s_n, r \in S$ such that $r_1, \ldots, r_n \sim s_1, \ldots, s_n$. Then there exists $s \in S$ such that $r_1, \ldots, r_n, r \sim s_1, \ldots, s_n, s$.*

Proof: Let μ be a homeomorphism of the closed plane onto itself mapping $r_1, \ldots, r_n$ to $s_1, \ldots, s_n$. Let $c_1, \ldots, c_N$ be all the components of all products of the form $\pm r_1 . \ \ldots \ . \pm r_n$ and let $d_1, \ldots, d_N$ be all the components of all products of the form $\pm s_1 . \ \ldots \ . \pm s_n$. Then, by lemma 4.6, $c_1, \ldots, c_N$ and $d_1, \ldots, d_N$ are connected partitions in S, and by renumbering if necessary, μ maps $c_1, \ldots, c_N$ to $d_1, \ldots, d_N$. It suffices to find $s \in S$ such that $c_1, \ldots, c_N, r \sim d_1, \ldots, d_N, s$.
For all j $(1 \leq j \leq N)$, let $c'_j = r.c_j$. By lemma 4.22, there exists a $d'_j \in S$ and a homeomorphism ν_j of the closed plane onto itself mapping c'_j to d'_j and equal to μ outside c_j. Then the function

$$\nu = \bigcup \{\nu_j|_{c_j} : 1 \leq j \leq N\} \cup \mu|_{\mathcal{F}(c_1) \cup \ldots \cup \mathcal{F}(c_N)}$$

is a homeomorphism of the closed plane onto itself mapping c_j to d_j for all j $(1 \leq j \leq N)$ and mapping $r = c'_1 + \ldots + c'_N$ to $s = d'_1 + \ldots + d'_N \in S$ as required. □

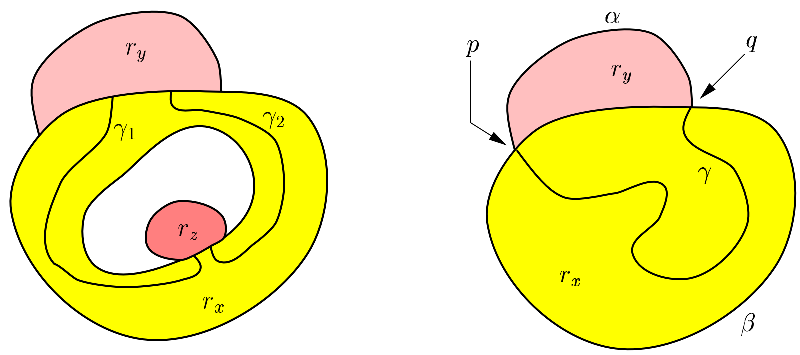
Lemma 4.23 has the immediate consequence that the model $\mathfrak{S}$ is "topological" in the following sense:

Lemma 4.24 *Let $r_1, \ldots, r_n, s_1, \ldots, s_n \in S$ such that $r_1, \ldots, r_n \sim s_1, \ldots, s_n$. Then $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ satisfy the same formulae in $\mathfrak{S}$.*

Proof: We prove by induction on the complexity of $\phi(x_1, \ldots, x_n)$ that, if $\mathfrak{S} \models \phi[r_1, \ldots, r_n]$, then $\mathfrak{S} \models \phi[s_1, \ldots, s_n]$.

If $\phi(x_1, \ldots, x_n)$ is $c(t)$, where t is some Boolean combination of the variables $x_1, \ldots, x_n$, then the result is guaranteed by lemma 4.2 and the fact that connectedness is a topological property.

The sole non-trivial recursive case is where $\phi(x_1, \ldots, x_n)$ is $\exists y \psi(x_1, \ldots, x_n, y)$. If $\mathfrak{S} \models \phi[r_1, \ldots, r_n]$, there exists $r \in S$ such that $\mathfrak{S} \models \psi[r_1, \ldots, r_n, r]$. By lemma 4.23, there exists $s \in S$ such that $r_1, \ldots, r_n, r \sim s_1, \ldots, s_n, s$. By inductive hypothesis, $\mathfrak{S} \models \psi[s_1, \ldots, s_n, s]$, hence $\mathfrak{S} \models \phi[s_1, \ldots, s_n]$. □



a) Axiom 8   b) Axiom 9

Figure 5: Illustration of two axioms

5 Correctness

5.1 Soundness

Theorem 5.1 (Soundness) *Let Φ be a set of sentences. If $\mathfrak{S} \models \Phi$ then Φ is $\$$-consistent.*

Proof: We show that all special axioms are true in $\mathfrak{S}$ and that the special rule of inference is truth-preserving in $\mathfrak{S}$.

Axioms 1: By lemma 2.1.

Axiom 2: By lemma 4.3.

Axiom schema 3: If some x_i is zero, then the conditional is trivial. If every x_i is nonzero, it follows by lemma 4.15.

Axiom schema 4: By lemma 4.17.

Axiom 5: Suppose $r_1, \ldots, r_5$ satisfied the condition inside the existential quantifiers. Then by lemma 4.9, we could construct a planar representation of the graph K_5, which is known to be non-planar (Bollobás [3], p.19).

Axiom 6: As for axiom 5, but with $K_{3,3}$ instead of K_5.

Axiom 7: The closed plane is connected.

Axiom 8: Refer to fig. 5a). Let r_x, r_y, r_z satisfy the antecedent of this axiom in $\mathfrak{S}$. We may assume that r_y and r_z are non-zero, since similar or easier arguments apply in the cases the cases where $r_y = 0$ or $r_z = 0$. By lemma 4.8, there exist line-segments α_y and α_z such that $|\alpha_y| \subseteq \mathcal{F}(r_x) \cap \mathcal{F}(r_y)$ and $|\alpha_z| \subseteq \mathcal{F}(r_x) \cap \mathcal{F}(r_z)$. Let $p_1, p_2 \in |\alpha_y|, q_1, q_2 \in |\alpha_z|$ be distinct from each other and from the endpoints of α_y and α_z. By lemma 4.9, let γ_1 be a piecewise linear cross-cut in r_x from p_1 to q_1. Either γ_1 partitions r_x into two connected regions r_u and r_v, or $r_x \setminus |\gamma_1|$ is still connected. In the former case, $r_u, r_v \in S$ by theorem 4.1, and it is easy to see that $r_x = r_u \oplus r_v$. In the latter case, construct a piecewise linear cross-cut γ_2 in $r_x \setminus |\gamma_1|$ joining p_2 and q_2. Now p_2 and q_2 lie in the same component of $Z^2 \setminus (r_x \setminus |\gamma_1|)$, since $p_2 \in [y]$, $q_2 \in [z]$ and γ_1 connects $[y]$ and $[z]$. It is a standard result (Newman [9], chapter V., theorem 11.7) that, if γ is a cross-cut in a nonempty open connected set U with endpoints in the same component of $Z^2 \setminus U$ then $U \setminus |\gamma|$ has two components. Hence γ_2 partitions $r_x \setminus |\gamma_1|$ into two connected regions. Together, then, γ_1 and γ_2 partition r_x into two connected regions, r_u and r_v. Again, by theorem 4.1 (p. 11) we have $r_u, r_v \in S$, and it is easy to see that $r_x = r_u \oplus r_v$. Since p_1, p_2, q_1 and q_2 are not endpoints of α_y or α_z, the pairs of regions $\{r_u, r_y\}$, $\{r_v, r_y\}$, $\{r_u, r_z\}$ and $\{r_v, r_z\}$ all have shared line segments on their frontiers. It follows from lemma 4.8 that r_x, r_y, r_z satisfy the consequent of this axiom in $\mathfrak{S}$.

Axiom 9: Refer to fig. 5b). Let r_x, r_y satisfy the antecedent of this axiom in $\mathfrak{S}$. Let $s = -(r_x + r_y)$. By lemma 4.10, r_x and r_y are Jordan regions such that $r_x + r_y$ and therefore s are also Jordan regions. Thus by lemma 4.13, $\mathcal{F}(r_x) \cap \mathcal{F}(r_y)$ is the locus of some Jordan arc α, and $\mathcal{F}(r_x) \cap \mathcal{F}(s)$ is the locus of some Jordan arc β, with the same end-points, say, p and q. By lemma 4.9, let γ be a piecewise linear cross-cut in r_x from p to q. Since r_x is a Jordan region, γ partitions r_x into two connected regions r_u, r_v. By theorem 4.1, $r_u, r_v \in S$, and it is easy to see that $r_x = r_u \oplus r_v$. Moreover, since $r_u + -r_x = -r_v$ and $r_v + -r_x = -r_u$ are connected and nonzero, r_u and r_v are Jordan regions. It is then easy to verify (exchanging r_u and r_v if necessary) that $\mathcal{F}(r_u) \cap \mathcal{F}(r_y) = |\alpha|$ and $\mathcal{F}(r_v) \cap \mathcal{F}(s) = |\beta|$, and that $\mathcal{F}(r_v) \cap \mathcal{F}(r_y) = \mathcal{F}(r_u) \cap \mathcal{F}(s) = \{p, q\}$. It follows from lemma 4.8 that r_x and r_y satisfy the consequent of this axiom in $\mathfrak{S}$.

Inference rule 10: Suppose that $\mathfrak{S} \models \forall x(\beta_n(x) \rightarrow \phi(x))$ for all $n \in \mathbb{N}$. Let $r \in S$. Then by lemma 4.5, there exist finitely many connected elements $r_1, \ldots, r_N \in S$ s.t. $r = r_1 + \ldots + r_N$. Hence, there is an N such that $\mathfrak{S} \models \beta_N[r]$, so that $\mathfrak{S} \models \phi[r]$. Hence $\mathfrak{S} \models \forall x \phi(x)$. □

5.2 Completeness

Theorem 5.2 (Completeness) *Let Φ be a set of \$-consistent sentences. Then $\mathfrak{S} \models \Phi$.*

Proof: The strategy is to construct a model $\mathfrak{A}$ of Φ respecting the axioms and rules of inference of \$, and then to embed its domain A into the closed plane in such a way that $\mathfrak{A} \subseteq \mathfrak{S}$. By proving a result on the way in which A is embedded in S, we then strengthen this relation to $\mathfrak{A} \preceq \mathfrak{S}$, from which it follows that $\mathfrak{S} \models \Phi$. For clarity, we break the proof up into four stages.

Stage 1: Let T be the set of \$-consequences of Φ. Since Φ is \$-consistent, T is consistent. Consider the set of formulae

$$\Sigma(x) = \{\neg\beta_N(x) | N \geq 1\} .$$

Suppose that $\theta(x)$ is any formula consistent with T. Since $T \not\models \forall x \neg\theta(x)$ and since T is \$-closed, the rule of inference 10 guarantees $T \not\models \forall x(\beta_N(x) \rightarrow \neg\theta(x))$ for some $N \geq 1$. Hence $\theta(x)$ consistent with T implies $T \not\models \forall x(\theta(x) \rightarrow \neg\beta_N(x))$ for some N—that is, T locally omits Σ. By the omitting types theorem, there exists a countable model $\mathfrak{A}$ of T omitting Σ.

When discussing the model $\mathfrak{A}$, we use the following conventions. If $a, b \in A$, we write $a + b$, $a.b$, and $-a$ to denote elements of A in the obvious way. If $a \in A$ and $\mathfrak{A} \models c[a]$ then we say that a is *connected*. In this context, then, the Boolean functions and the term "connected" do not have their normal senses, for the elements of A are not (necessarily) spatial regions. However, since we will be considering only the model $\mathfrak{A}$ in this stage of the proof, no confusion need arise. If $a_1, \ldots, a_n \in A$ are nonzero, connected and pairwise disjoint, we denote their sum by $a_1 \oplus \ldots \oplus a_n$. If $a_1 \oplus \ldots \oplus a_n = 1$, we say that $a_1, \ldots, a_n$ form a *connected partition*.

Having defined $\mathfrak{A}$, we now establish some of its basic properties. Since our objective is to embed $\mathfrak{A}$ as a submodel in $\mathfrak{S}$, we might as well assume that $|A| > 2$; otherwise this embedding is trivial.

Since $\mathfrak{A}$ is countable, let $A = \{a_1, a_2, \ldots\}$. And since $|A| > 2$, we may assume WLOG that $a_1 \notin \{0, 1\}$. We first show that, for any initial segment, $a_1, \ldots, a_n$, we can find $c_1, \ldots, c_N \in A$ satisfying the formula $c_1 \oplus \ldots \oplus c_N = 1$ in $\mathfrak{A}$ such that each a_i $(1 \le i \le n)$ can be expressed as a sum of some of the $c_1, \ldots, c_N$. For consider the M_n non-zero elements of the form:

$$e_j = \pm a_1. \ \ldots \ . \pm a_n \ .$$

where $\pm a_i$ is either a_i or $-a_i$. We call these e_j the *atoms generated by* $a_1, \ldots, a_n$. Since $\mathfrak{A}$ omits Σ, we must be able to find, for each j $(1 \le j \le M_n)$, a collection of connected elements $d_{j,1}, \ldots, d_{j,N_j}$ summing to e_j. We now take any pair of these elements $d_{j,k}$ and $d_{j,l}$ such that $d_{j,k} + d_{j,l}$ is connected and replace these elements by their sum $d_{j,k} + d_{j,l}$. By repeating this process sufficiently often, we obtain connected elements $e_{j,1}, \ldots, e_{j,N_j}$ summing to e_j such that no two of them have a connected sum. It follows from axiom 2 that these $e_{j,k}$ are pairwise disjoint. If we denote by $c_1, \ldots, c_N$ all the $e_{j,k}$ for the various atoms e_j (ignoring any zero elements), it is easy to see that:

1. $c_1, \ldots, c_N$ form a connected partition.
2. the atoms generated by $a_1, \ldots, a_n$, and hence $a_1, \ldots, a_n$ themselves, are expressible as sums of the $c_1, \ldots, c_N$;
3. if c_i and c_j $(1 \le i < j \le N)$ are contained within the same atom e_j generated by $a_1, \ldots, a_n$, then $c_i + c_j$ is not connected.

We call a collection $c_1, \ldots, c_N$ satisfying these three properties a *maximal connected partition* generated by $a_1, \ldots, a_n$.

So, given any initial segment $a_1, \ldots, a_n$ of A, let $c_1^{(n)}, \ldots, c_{N_n}^{(n)}$ be some maximal connected partition generated by $a_1, \ldots, a_n$. (The (n)-superscripts are for clarity when we consider maximal connected partitions corresponding to different initial segments of A.) We observe in passing that, since $a_1 \notin \{0, 1\}$, $M_n > 1$, so that $N_n > 1$ for all $n \ge 1$.

Claim 1 *If $m \le n$, then for each k $(1 \le k \le N_n)$, there exists j $(1 \le j \le N_m)$ such that $c_k^{(n)} \le c_j^{(m)}$.*

Proof: Write $d_1, \ldots, d_l$ for those $c_j^{(m)}$ such that $c_j^{(m)}.c_k^{(n)} \ne 0$; it suffices to show that $l = 1$. Since $c_1^{(m)}, \ldots, c_{N_m}^{(m)}$ form a partition, we have:

$$\sum_{1 \le h \le l} (c_k^{(n)} + d_h) = \sum_{1 \le h \le l} d_h \ .$$

By axiom 2, $c_k^{(n)} + d_h$ is connected for all h $(1 \le h \le l)$, since $c_k^{(n)}$ is connected and d_h is connected with $c_k^{(n)}.d_h \ne 0$. Then, by repeated applications of axiom 2, $\sum_{1 \le h \le l}(c_k^{(n)} + d_h)$ is connected, since $c_k^{(n)} \ne 0$. That is, $\sum_{1 \le h \le l} d_h$ is connected.

Suppose, then that $l > 1$. Then by axiom 3, there exists d_h, $(2 \le h \le l)$ such that $c(d_1 + d_h)$ is connected. But since d_1 and d_h have non-zero intersection with $c_k^{(n)}$, and since $c_k^{(n)}$ is contained in some atom generated by $a_1, \ldots, a_n$ with $m \le n$, it follows that d_1 and d_h are contained in the same atom generated by $a_1, \ldots, a_m$. But then it is impossible that $d_1 + d_h$ be connected by the fact that $c_1^{(m)}, \ldots, c_{N_m}^{(m)}$ is a maximal connected partition. Hence we cannot have $l > 1$. □

It follows from claim 1 that, if $m \le n$ each $c_j^{(m)}$ can be expressed as a sum of various $c_k^{(n)}$ and that, for each n, the $c_1^{(n)}, \ldots, c_{N_n}^{(n)}$ are unique. Hence we may speak of *the* maximal connected partition generated by the $a_1, \ldots, a_n$.

Stage 2: We now map each initial segment $a_1, \ldots, a_n$ of A into our standard domain S. Let n be a positive integer. We denote by $w^{(n)}$ the set of functions $g^{(n)} : \{c_1^{(n)}, \ldots, c_{N_n}^{(n)}\} \to S$ satisfying

G1: The regions $g^{(n)}(c_1^{(n)}), \ldots, g^{(n)}(c_{N_n}^{(n)})$ form a connected partition in S

G2: For all i, j $(1 \le i < j \le N_n)$, $g^{(n)}(c_i^{(n)}) + g^{(n)}(c_j^{(n)})$ is connected iff $c_i^{(n)} + c_j^{(n)}$ is connected.

We remark that, in G2, we have $g^{(n)}(c_i^{(n)}), g^{(n)}(c_j^{(n)}) \in S$ and $c_i^{(n)}, c_j^{(n)} \in A$. Hence, different senses of "+" and "connected" apply in the two cases. For $n = 0$ we define $w^{(0)} = \{\emptyset\}$.

Definition 5.1 *Let $b_1, \ldots, b_n \in A$. Form the graph G with nodes $\{b_1, \ldots, b_n\}$ and edges $\{(b_i, b_j) | 1 \le i < j \le n$ and $b_i + b_j$ is connected$\}$ (i.e. G has no multiple edges). We call G the* binary connection graph *on $b_1, \ldots, b_n$.*

Claim 2 *Let $b_1, \ldots, b_n \in A$ be nonzero and connected. Then $b_1 + \ldots + b_n$ is connected iff the binary connection graph on $b_1, \ldots, b_n$ is a connected graph.*

Proof: We proceed by induction on n for both directions. Let G be the binary connection graph on $b_1, \ldots, b_n$. If $n = 1$, the claim is trivial.

Suppose that $n > 1$ and G is a connected graph. By lemma 4.14, we can suppose WLOG that the graph $G - \{b_1\}$ formed by removing b_1 and all its edges from G is connected. By inductive hypothesis, $b_2 + \ldots + b_n$ is connected. Since G is connected, there must be some i $(2 \le i \le n)$ such that $b_1 + a_i$ is connected. Since b_i is nonzero, axiom 2 ensures that $b_1 + \ldots + b_n$ is connected.

Suppose that $n > 1$ and $b_1 + \ldots + b_n$ is connected. Axiom schema 4 ensures that, by renumbering if necessary, $b_2 + \ldots + b_n$ is connected. By inductive hypothesis, the graph $G - \{b_1\}$ is connected. Moreover, axiom schema 3 ensures that, for some i $(2 \le i \le n)$, $b_1 + b_i$ is connected. Hence G is connected. □

Claim 3 *The binary connection graph on a connected partition is planar.*

Proof: Let $c_1, \ldots, c_n$ be a connected partition, and let G be its binary connection graph. By a well-known theorem of Kuratowski, it suffices to show that G contains no subgraph identical to either K_5 or $K_{3,3}$ to within nodes of degree 2 (Bollobás [3], p.19). For definiteness, we concentrate on the case K_5. Let H be a subgraph of G identical to K_5 to within nodes of degree 2. If H contains nodes of degree 2, then re-number the nodes of G if necessary so that that $c_1, \ldots, c_5$ are the nodes of H of degree greater than 2, and $c_6, \ldots, c_{5+h}$ are the nodes of H of degree 2 lying between nodes c_4 and c_5, with $h > 0$. Then, by claim 2, $d = c_5 + c_6 + \ldots + c_{5+h}$ is connected, so that $c_1, \ldots, c_4, d, c_{6+h}, \ldots, c_n$ is a connected partition. Moreover, this new connected partition also contains a subgraph H' identical to K_5 to within nodes of degree 2, but having strictly fewer nodes of degree 2 than H. Proceeding in this way, we can find a connected partition with a binary connection graph G containing a subgraph isomorphic to K_5, which is impossible by axiom 5.
The case $K_{3,3}$ proceeds identically, except that we rely on axiom 6. □

Definition 5.2 *Let G be a plane graph. Its* geometric dual *G^* is obtained in the following way (cf. [14], p.72). A point v_i^* is chosen inside each face of G. These chosen points are the nodes of G^*. Corresponding to each edge e of G an edge e^* is drawn which crosses e but no other edge of G and joins the nodes v_i^* which lie in the faces adjoining e. These edges e^* are the edges of G^*.*

Claim 4 *For all $n \in \mathbb{N}$, $w^{(n)} \neq \emptyset$.*

Proof: If $n = 0$ the claim is trivial. Suppose $n \geq 1$. We show that, given a maximal connected partition $c_1^{(n)}, \ldots, c_{N_n}^{(n)}$ $(n \geq 1)$, there exists some $g^{(n)}$ satisfying G1 and G2. We observed above that $N_n > 1$. For the time being we shall drop the n-sub- and superscripts and write N for N_n and c_i for $c_i^{(n)}$. Let G be the binary connection graph on $c_1, \ldots, c_N$. By claim 3, G is planar. By axiom 7 and claim 2, G is connected. Let H be an embedding of G in the closed plane all of whose edges are piecewise linear. Since H is plane and connected, by a standard result ([14], p.73) it has a plane, connected geometric dual H^* which in turn has a geometric dual H^{**} isomorphic to H. Thus there exists a 1–1 function $h^{**} : \text{nodes}(G) \to \text{nodes}(H^{**})$ such that, for all i, j $(1 \leq i < j \leq N_n)$, (c_i, c_j) is a G-edge iff $(h^{**}(c_i), h^{**}(c_j))$ is an H^{**}-edge. By the construction of H^{**}, there exists a 1–1 function $h^* : \text{nodes}(H^{**}) \to \text{faces}(H^*)$ such that, for all $n_1, n_2 \in \text{nodes}(H^{**})$, there is an H^{**}-edge (n_1, n_2) iff $h^*(n_1)$ and $h^*(n_2)$ share an H^*-edge on their frontiers. Also by the construction of H^{**}, if H^* contains an isthmus, H^{**} and hence G contains a loop, which is impossible by definition, so H^* contains no isthmus. Moreover, H^* can obviously be constructed so that all its edges are piecewise linear. It follows from theorem 4.1 that the faces of H^* form a connected partition in S.

Now put $g^{(n)} = h^* \circ h^{**}$. Thus, $g^{(n)}$ is a function from $\{c_1, \ldots, c_{N_n}\}$ into S satisfying G1. To see that $g^{(n)}$ also satisfies G2, we note that, for all i, j $(1 \leq i < j \leq N)$, (c_i, c_j) is a G-edge iff $g^{(n)}(c_i)$ and $g^{(n)}(c_j)$ share an H^*-edge on their frontiers. By lemma 4.8, for all i, j $(1 \leq i < j \leq N_n)$, $c_i + c_j$ is connected iff $g^{(n)}(c_i) + g^{(n)}(c_j)$ is connected. Hence $g^{(n)}$ satisfies G2 and $w^{(n)} \neq \emptyset$ as required. □

We remark that, while the proof of claim 4 constructs an element of $w^{(n)}$, not all elements of $w^{(n)}$ can be constructed in this way.

We now proceed to establish some additional properties of the sets $w^{(n)}$ and their members. As usual, if $C \subseteq \{c_1^{(n)}, \ldots, c_{N_n}^{(n)}\}$, we write $g^{(n)}(C)$ to mean $\{g^{(n)}(c_i^{(n)}) | c_i^{(n)} \in C\}$.

Claim 5 *Let $C \subseteq \{c_1^{(n)}, \ldots, c_{N_n}^{(n)}\}$. Then $\sum C$ is connected iff $\sum g^{(n)}(C)$ is connected.*

(Note the two different uses of '$\sum$' and 'connected'.)

Proof: Suppose first that $C = \emptyset$. Then $\sum C = 0 \in A$ and $\sum g^{(n)}(C) = 0 \in S$. (Note the two different uses of '0'.) By lemma 3.1, $0 \in A$ is connected. Since $0 \in S$ is connected, the result holds.

Suppose next that $C \neq \emptyset$. By lemma 4.15, $\sum g^{(n)}(C)$ is connected iff the set of edges

$$\{(c_i, c_j) | c_i, c_j \in C, i \neq j, g^{(n)}(c_i) + g^{(n)}(c_j) \text{ connected}\}$$

forms a connected graph on C. By property G2, applied to $g^{(n)}$, this is true iff the set of edges

$$\{(c_i, c_j) | c_i, c_j \in C, i \neq j, c_i + c_j \text{ connected}\}$$

forms a connected graph on C. By claim 2, applied to C, this is true iff $\sum C$ is connected. □

Let $0 \leq m \leq n$. We now show how any mapping $g^{(n)} \in w^{(n)}$ can be used to construct a mapping in $w^{(m)}$. Since, for any i $(1 \leq i \leq N_m)$, $c_i^{(m)}$ can be expressed uniquely as a sum of various $c_j^{(n)}$, let us write

$$c_i^{(m)} = c_{i_1}^{(n)} + \ldots + c_{i_{M_i}}^{(n)} .$$

for all i $(1 \leq i \leq N_m)$. In addition, let $g^{(n)} \in w^{(n)}$. We define the *restriction* of $g^{(n)}$ to $c_1^{(m)}, \ldots, c_{N_m}^{(m)}$, written $g^{(n)}|_m$, as follows:

$$g^{(n)}|_m(c_i^{(m)}) = g^{(n)}(c_{i_1}^{(n)}) + \ldots + g^{(n)}(c_{i_{M_i}}^{(n)}) \text{ for } m > 0 \text{ and } g^{(n)}|_0 = \emptyset .$$

Claim 6 *Let $g^{(n)} \in w^{(n)}$ with $0 \leq m < n$. Then $g^{(n)}|_m \in w^{(m)}$.*

Proof: If $m = 0$ the claim is trivial. Suppose $m \geq 1$. We must prove that G1 and G2 hold of $g^{(n)}|_m$. G1 is trivial. For G2, we note that, by construction,

$$g^{(n)}|_m(c_i^{(m)}) + g^{(n)}|_m(c_j^{(m)}) = g^{(n)}(c_{i_1}^{(n)}) + \ldots + g^{(n)}(c_{i_{M_i}}^{(n)}) + g^{(n)}(c_{j_1}^{(n)}) + \ldots + g^{(n)}(c_{j_{M_j}}^{(n)}) \ .$$

By claim 5, this element of S is connected iff the element of A

$$c_{i_1}^{(n)} + \ldots + c_{i_{M_i}}^{(n)} + c_{j_1}^{(n)} + \ldots + c_{j_{M_j}}^{(n)} = c_i^{(m)} + c_j^{(m)}$$

is connected. Hence G2 holds as required. □

Stage 3: In stage 2, we showed how any initial segment of $\mathfrak{A}$ can be embedded in $\mathfrak{S}$. In this section, we show how these partial embeddings can be strung together into a single embedding of $\mathfrak{A}$ into $\mathfrak{S}$.

Suppose that $g_1^{(n)}, g_2^{(n)} \in w^{(n)}$. We say that $g_1^{(n)}$ is topologically equivalent to $g_2^{(n)}$, written $g_1^{(n)} \sim g_2^{(n)}$, if there exists a homeomorphism of the closed plane onto itself taking the elements of $g_1^{(n)}(c^{(n)})$ to the elements of $g_2^{(n)}(c^{(n)})$.

Clearly, $g_1^{(n)} \sim g_2^{(n)}$ is an equivalence relation on $w^{(n)}$. By theorem 4.2, there are only finitely many equivalence classes under $\sim$ contained in each $w^{(n)}$. Let us denote these by $w_1^{(n)}, \ldots, w_{k_n}^{(n)}$. If $m \leq n$ and there exists $g^{(n)} \in w_j^{(n)}$ such that $g^{(n)}|_m \in w_i^{(m)}$ then we write $w_i^{(m)} \preceq w_j^{(n)}$. If, in addition, $m < n$, we write $w_i^{(m)} \prec w_j^{(n)}$.

Now form the graph Ω whose nodes are all the $w_i^{(n)}$, and whose edges are

$$\{(w_i^{(n)}, w_j^{(n+1)}) | w_j^{(n)} \prec w_j^{(n+1)}\}$$

(i.e. Ω has no multiple edges). By claim 6, Ω is connected.

Claim 7 *Ω is a tree. That is, if $w_i^{(n)} \prec w_k^{(n+1)}$ and $w_j^{(n)} \prec w_k^{(n+1)}$ then $i = j$.*

Proof: Obvious. □

By claim 4, Ω is infinite. By theorem 4.2, Ω is locally finite. Then, by König's infinity lemma, for any node of Ω, there exists an infinite path in Ω starting at that node. Let π be an infinite path in Ω starting at the node $\{\emptyset\} = w_1^{(0)}$. Since Ω is a tree, this path gives us a sequence of equivalence classes $w_1^{(0)} \prec w_{i_1}^{(1)} \prec w_{i_2}^{(2)} \prec \ldots$.

Claim 8 *Let $g^{(n)} \in w_i^{(n)}$ and $(w_i^{(n)}, w_j^{(n+1)})$ be an edge of Ω. Then there exists a $g^{(n+1)} \in w_j^{(n+1)}$ such that $g^{(n+1)}|_n = g^{(n)}$.*

Proof: Since $(w_i^{(n)}, w_j^{(n+1)})$ is an edge of Ω, there exists an $h^{(n+1)} \in w_j^{(n+1)}$ and a homeomorphism μ of the closed plane onto itself such that $\mu \circ (h^{(n+1)}|_n) = g^{(n)}$. Denote those members of $c_1^{(n+1)}, \ldots, c_{N_n}^{(n+1)}$ which sum to $c_1^{(n)}$ by $d_1, \ldots, d_m$, and denote $h^{(n+1)}|_n(c_1^{(n)})$ by r. Then $h^{(n+1)}(d_1), \ldots, h^{(n+1)}(d_m)$ form a connected partition of r in S, so by lemma 4.21, there exists a connected partition $s_1, \ldots, s_m$ of $\mu(r)$ and a homeomorphism ν_1 of the closed plane onto itself, mapping $h^{(n+1)}(d_i)$ to s_i, for all i ($1 \leq i \leq m$) such that $\nu_1|_{-r} = \mu|_{-r}$. By repeating this step for $c_2^{(n)}, \ldots, c_{N_n}^{(n)}$, we can construct a homeomorphism ν of the closed plane onto itself mapping every $h^{(n+1)}(c^{(n+1)})$ to an element of S such that $\nu(h^{(n+1)}|_n(c_i^{(n)})) = \mu(h^{(n+1)}|_n(c_i^{(n)})) = g^{(n)}(c_i^{(n)})$ for all i ($1 \leq i \leq N_n$). Hence $(\nu \circ h^{(n+1)})|_n = \mu \circ (h^{(n+1)}|_n) = g^{(n)}$. Clearly

$\nu \circ h^{(n+1)} \in w_j^{(n+1)}$, since $h^{(n+1)} \in w_j^{(n+1)}$. Hence $g^{(n+1)} = \nu \circ h^{(n+1)}$ is our required element. $\square$

By claim 8, we can extract a sequence of embeddings:

$$\begin{array}{cccc} w_{i_0}^{(0)} \prec & w_{i_1}^{(1)} \prec & w_{i_2}^{(2)} \prec & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \emptyset = f^{(0)}, & f^{(1)}, & f^{(2)}, & \dots \end{array}$$

with the property that, for all m, n $(0 \leq m < n)$, $f^{(n)}|_m = f^{(m)}$.

Now let $a \in A$ be such that $a = c_{i_1}^{(n)} + \ldots + c_{i_k}^{(n)}$. Then we define

$$f(a) = f^{(n)}(c_{i_1}^{(n)}) + \ldots + f^{(n)}(c_{i_k}^{(n)}) .$$

(If $a = 0$, we take the right-hand side of this definition to denote $0 \in S$.)

The fact that $f^{(n)}|_m = f^{(m)}$ whenever $0 \leq m < n$ means that this mapping is well defined. It is easy to see that $f : A \to S$ is a Boolean algebra isomorphism; moreover, by claim 5, $f(a)$ is connected iff a is connected. Thus we might as well take A to be a subset of S; then the previously distinct uses of the Boolean functions and constants and the term "connected" become unambiguous. That is, we have proved:

Claim 9 $\mathfrak{A} \subseteq \mathfrak{S}$.

Stage 4: Having established claim 9, the next step is to prove that $\mathfrak{A}$ has *enough* elements to serve as a substitute for the whole of $\mathfrak{S}$. In the sequel, we shall forget our previous enumeration of A and just take $a_1, \ldots, a_n$ to be arbitrary elements of A.

Definition 5.3 *If $r_1, \ldots, r_n \in S$ form a partition and r_i is a Jordan region for all i $(1 \leq i \leq n)$, we say that $r_1, \ldots, r_n$ is a* Jordan *partition. If $r_i + r_j$ is connected, $i \neq j$ we say that r_i and r_j are* neighbours. *If $r_1, \ldots, r_n$ is a Jordan partition such that, for any neighbour r_i of r_1, $-(r_1 + r_i)$ is connected, then we say that the partition is* radial about *r_1.*

By lemma 4.13, if $r_1, \ldots, r_n$ is a Jordan partition radial about r_1 such that r_1 has at least 2 neighbours, then, for any neighbour r_i of r_1, $\mathcal{F}(r_1) \cap \mathcal{F}(r_i)$ is a Jordan arc. Recall that, if $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ are regions of S such that there is a homeomorphism of the closed plane onto itself mapping r_i to s_i for all i $(1 \leq i \leq n)$, then we say that $r_1, \ldots, r_n$ and $s_1, \ldots, s_n$ are *topologically equivalent* and write $r_1, \ldots, r_n \sim s_1, \ldots, s_n$.

Claim 10 *Let $a_1, \ldots, a_n \in A$ be a Jordan partition radial about a_1 such that a_1 has at least 3 neighbours. Let $b_1, b_2 \in S$ be Jordan regions with $a_1 = b_1 \oplus b_2$. Then there exist $c_1, c_2 \in A$ such that $a_1, \ldots, a_n, c_1, c_2 \sim a_1, \ldots, a_n, b_1, b_2$.*

Proof: Since a_1, b_1, b_2 are Jordan regions with $a_1 = b_1 \oplus b_2$, b_1 and b_2 must be separated by a cross-cut γ in a_1. For any neighbour a_i of a_1, $\mathcal{F}(a_1) \cap \mathcal{F}(a_i)$ is a Jordan arc. By inspection (fig. 6a), any point on $\mathcal{F}(a_1)$ lies on either one or two Jordan arcs of the form $\mathcal{F}(a_1) \cap \mathcal{F}(a_i)$ where a_i is a neighbour of a_1.

Let $p \in \mathcal{F}(a_1)$. We define the *character of p*, written $\chi(p)$ to be the set of those i $(2 \leq i \leq n)$ such that a_i is a neighbour of a_1 and $p \in \mathcal{F}(a_i)$. (See fig. 6a for examples.) Then, $\chi(p)$ has either 1 or 2 elements. If $\chi(p)$ has one element, then p lies on the Jordan arc $\mathcal{F}(a_1) \cap \mathcal{F}(a_i)$, but not at its endpoints. If $\chi(p)$ has two elements, then since a_1 has at least three neighbours, $\chi(p)$ determines p. Now let γ be a cross-cut in a_1. We define the *character of γ*, written $\chi(\gamma)$ to be the set of characters of its endpoints. (See fig. 6b and c for examples.) It is routine to show that, if γ_1 and γ_2 are two such cross-cuts and $\chi(\gamma_1) = \chi(\gamma_2)$, there is a homeomorphism of the closed plane onto itself taking a_i to itself for all i $(1 \leq i \leq n)$ and taking γ_1 to γ_2. So, to prove the lemma, it suffices to establish that, if γ_1 is any cross-cut in a_1, there

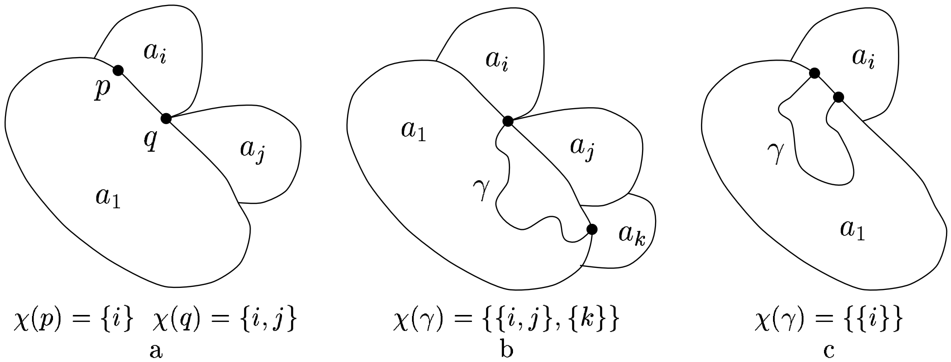


Figure 6: The hub a_1 of a radial partition

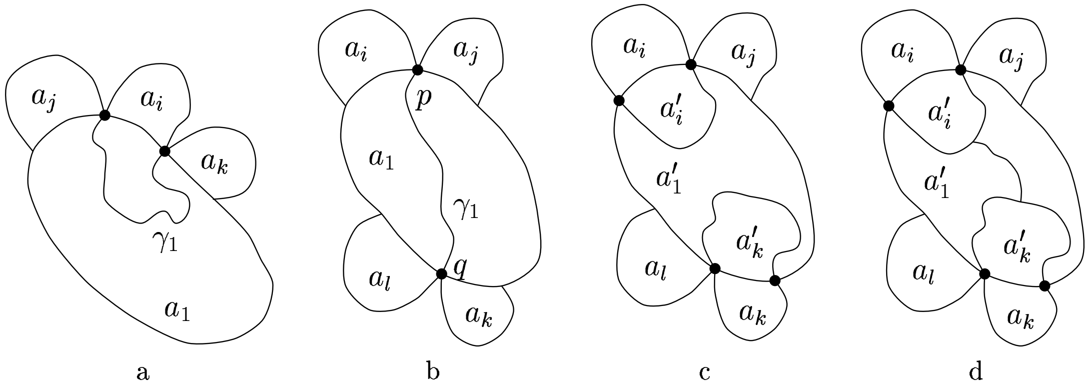


Figure 7: The construction of a cross-cut with a given character

exist Jordan regions $c_1, c_2 \in A$ with $a_1 = c_1 \oplus c_2$ such that the cross-cut γ_2 separating c_1 and c_2 in a_1 satisfies $\chi(\gamma_1) = \chi(\gamma_2)$.

Let the endpoints of γ_1 be p and q. We prove the result for the special case where $\chi(\gamma)$, $\chi(p)$ and $\chi(q)$ all contain two elements; the other cases are dealt with similarly. Fig. 7a shows the sub-case where $\chi(p)$ and $\chi(q)$ are non-disjoint; fig. 7b shows the sub-case where $\chi(p)$ and $\chi(q)$ are disjoint.

The sub-case of fig. 7a is trivial: the axiom 9 with a_1 substituted for x and a_j for y immediately guarantees the existence of $u, v \in A$ partitioning a_1, and hence separated by a cross-cut γ_2; moreover the connectivity conditions on u and v mean that γ_1 and γ_2 have the same endpoints, so that $\chi(\gamma_1) = \chi(\gamma_2)$.

The sub-case of fig. 7b requires a little more work. However, two applications of axiom 9 guarantee the existence in A of the regions a'_i, a'_k shown in fig. 7c. Axiom 8 then guarantees that the region labelled a'_1 in fig. 7c can be split into two regions as shown in fig. 7d. Summing together appropriate subdivisions of a_1 produces $c_1, c_2 \in A$ separated by an arc γ_2 satisfying $\chi(\gamma_1) = \chi(\gamma_2)$. □

The rest of this section is devoted to showing that we can relax the conditions of claim 10. First, we establish some results enabling us to decompose elements of A in various ways.

Claim 11 *Let $a \in A$. Then there exists $n \geq 0$ and Jordan regions $b_1, \ldots, b_n \in A$ such that $a = b_1 \oplus \ldots \oplus b_n$.*

(When $n = 0$, the right-hand side of this equation is taken to denote $0 \in S$).

Proof: We may as well assume that a has one component and $a \neq 1$, since extending the result to the other cases is trivial. We proceed by induction on the number k of components of $-a$. If $k = 1$, a is itself a Jordan region by lemma 4.10 and the result is certainly true. If $k > 1$, let c, d be distinct components of $-a$. By lemma 4.16 (setting $r = a$ and $s = 0$), $a + c$ and $a + d$ are connected. By axiom 8, substituting a for x, c for y and d for z, we are guaranteed the existence of connected regions $u, v \in A$, partitioning a such that $u + c$, $u + d$, $v + c$ and $v + d$ are all connected. Hence, both $-u$ and $-v$ have fewer than k components. By inductive hypothesis, u and v can be partitioned into finitely many Jordan regions in A. The result follows immediately. □

Claim 12 *Let $n \geq 1$ and let $a_1, \ldots, a_n \in A$. There exists a Jordan partition $c_1, \ldots, c_N \in A$ such that, for all i ($1 \leq i \leq n$), a_i can be expressed as the sum of various c_i.*

Proof: Immediate given claim 11. □

Claim 13 *Let $n > 1$ and let $a_1, \ldots, a_n \in A$ be a partition with a_1 a Jordan region. There exists a Jordan partition $a_1, c_2, \ldots, c_N \in A$ radial about a_1, such that, for all i ($2 \leq i \leq n$), a_i can be expressed as the sum of various c_j.*

Proof: By claim 12, we can find a Jordan partition $a_1, b_2, \ldots, b_M$ such that, for all i ($2 \leq i \leq n$) a_i can be expressed as the sum of various b_j. We now show that the b_j can be decomposed if necessary to form the required elements $c_2, \ldots, c_N$.

Suppose that b_i is a neighbour of a_1 such that $-(a_1 + b_i)$ is not connected. Then let $d \neq e$ be two components of $-(a_1 + b_i)$. By lemma 4.16, letting r be b_i, s be a_1 and, t be successively d and e, we know that both $b_i + d$ and $b_i + e$ are connected. In axiom 8, substitute b_i for x, d for y and e for z. Then there exist connected regions $u, v \in A$ partitioning b_i such that $u + d$, $u + e$, $v + d$ and $v + e$ are all connected. It follows that u and v are Jordan regions such that d and e belong to the same component of $-(a_1 + u)$ and also to the same component of $-(a_1 + v)$. Hence both $-(a_1 + u)$ and $-(a_1 + v)$ have fewer components than $-(a_1 + b_i)$. By replacing b_i with u and v and proceeding as before, we eventually reach a Jordan partition radial about a_1. □

Claim 14 *In claim 13, the $c_2, \ldots, c_N$ can be chosen so that a_1 has at least three neighbours.*

Proof: Immediate given claim 13 and axiom 8. □

Now let us return to the task of relaxing the conditions of claim 10.

Claim 15 *Let $n > 1$ and let $a_1, \ldots, a_n \in A$ be a partition such that a_1 is a Jordan region. Let $b_1, b_2 \in S$ be Jordan regions with $a_1 = b_1 \oplus b_2$. Then there exist $c_1, c_2 \in A$ such that $a_1, \ldots, a_n, c_1, c_2 \sim a_1, \ldots, a_n, b_1, b_2$.*

Proof: Immediate given claims 10 and 14. □

Claim 16 *Let $n > 1$ and let $a_1, \ldots, a_n \in A$ be a partition such that a_1 is a Jordan region. Let $b \in S$ be such that $b \leq a_1$. Then there exists $c \in A$ such that $a_1, \ldots, a_n, c \sim a_1, \ldots, a_n, b$.*

Proof: By claim 11, we can find a Jordan partition $b_1, \ldots, b_m$ of a_1 such that b can be expressed as the sum of various b_j. It suffices to show that there are $c_1, \ldots, c_m \in A$ such that

$$a_1, \ldots, a_n, b_1, \ldots, b_m \sim a_1, \ldots, a_n, c_1, \ldots, c_m$$

We proceed by induction on m. If $m = 1$, then $b_1 = a_1$ and we are done. If $m > 1$, by lemma 4.18, we can renumber the b_i if necessary so that b_1 and $b_2' = b_2 + \ldots + b_m$ are Jordan regions satisfying $a_1 = b_1 \oplus b_2'$. By claim 15, there exist $c_1, c_2' \in A$ such that $a_1, \ldots, a_n, b_1, b_2' \sim a_1, \ldots, a_n, c_1, c_2'$. Let ν be a homeomorphism of the closed plane onto itself mapping a_i to itself, b_1 to c_1 and b_2' to c_2'. It is easy to show that ν can be chosen so that $\nu(b_i) \in S$ for all i $(2 \leq i \leq m)$. But then the $\nu(b_i)$ form a Jordan partition of c_2' in the partition $c_2', c_1, a_2, \ldots, a_n$. By inductive hypothesis, there exist $c_2, \ldots, c_m \in A$ such that

$$c_2', c_1, a_2, \ldots, a_n, \nu(b_2), \ldots, \nu(b_m) \sim c_2', c_1, a_2, \ldots, a_n, c_2, \ldots, c_m \ .$$

The result then follows immediately. □

Claim 17 *Let $n > 1$ and let $a_1, \ldots, a_n \in A$ be a Jordan partition. Let $b \in S$. Then there exists $c \in A$ such that $a_1, \ldots, a_n, c \sim a_1, \ldots, a_n, b$.*

Proof: We let $b = b.a_1 + \ldots + b.a_n$ By considering these terms separately, we use claim 16 and an induction similar to that used in the proof of claim 16. The details are routine. □

Claim 18 *Let $n \geq 0$ and let $a_1, \ldots, a_n \in A$. Let $b \in S$. Then there exists $c \in A$ such that $a_1, \ldots, a_n, b \sim a_1, \ldots, a_n, c$.*

Proof: Immediate given claims 12 and 17. □

Thus, we have established that A forms a topologically homogeneous subset of S in the sense made precise by claim 18.

Stage 5: We now have all the important elements for our proof.

Claim 19 $\mathfrak{A} \preceq \mathfrak{S}$.

Proof: By claim 9, $\mathfrak{A} \subseteq \mathfrak{S}$. Let $n \geq 0$ and let $\phi(x_1, \ldots, x_n)$ be any formula of the form $\exists y \psi(x_1, \ldots, x_n, y)$. According to the Tarski-Vaught lemma, if we can show that for any $a_1, \ldots, a_n \in A$ such that $\mathfrak{S} \models \phi[a_1, \ldots, a_n]$, there exists $c \in A$ such that $\mathfrak{S} \models \psi[a_1, \ldots, a_n, c]$, then $\mathfrak{A} \preceq \mathfrak{S}$.

Let $a_1, \ldots, a_n$ and ϕ be as described. Then there exists $b \in S$ such that $\mathfrak{S} \models \psi[a_1, \ldots, a_n, b]$. By claim 18, there exists $c \in A$ such that $a_1, \ldots, a_n, b \sim a_1, \ldots, a_n, c$. By lemma 4.24, $\mathfrak{S} \models \psi[a_1, \ldots, a_n, c]$.
□

By the construction of A, $\mathfrak{A} \models \Phi$. By claim 19, $\mathfrak{S} \models \Phi$. This completes the proof of theorem 5.2. □

Finally, we have the result we want.

Corollary 1 *Let $T_\$$ denote the set of sentences which are \$-theorems. Then* $T_\$ = \mathrm{Th}(\mathfrak{S}) = \mathrm{Th}(\mathfrak{R})$.

Proof: Immediate by theorems 5.1, 5.2 and lemma 2.5. □

6 Conclusions

This paper has presented a calculus for mereotopological reasoning in which spatial regions are treated as primitive entities. We defined a language $\mathcal{L}$ with a one-place predicate $c(x)$, the function-symbols $+$, $\cdot$ and $-$ and the constants 0 and 1. We provided an interpretation $\mathfrak{R}$ for $\mathcal{L}$ in which regions are identified with polygonal regular sets of the real plane. Under this interpretation, the predicate $c(x)$ is read as "x is connected" (in the usual sense) and the Boolean function-symbols and constants are given their obvious meanings in terms of the appropriate regular Boolean algebra. We proved the soundness and completeness of our calculus with respect to an isomorphic model $\mathfrak{S}$ and therefore with respect to $\mathfrak{R}$ as well.

Thus, although our calculus takes regions to be primary, it is guaranteed an interpretation in terms of a model of the plane in which regions are identified with polygonal, regular subsets of $\mathbb{R}^2$. That is: the theorems of the calculus are precisely those formulae made true by this model. Hence, our mereotopological calculus really can claim to be a calculus of spatial regions. For there is good reason to suppose that the polygonal ontology assumed here constitutes an adequate model of 2-dimensional space for most practical purposes. (In particular, this ontology is the one employed by nearly all computer systems specialized for plane spatial representation such as geographic information systems.) In this respect, we claim, our calculus is superior to other mereotopological calculi that have been proposed in the literature.

The problem of axiomatizing less restricted Boolean subalgebras of $M(\mathbb{R}^2)$ than R—in particular, the whole of $M(\mathbb{R}^2)$—is open, as are the corresponding problems for Boolean subalgebras of $M(\mathbb{R}^3)$. While the three-dimensional case is certainly of interest, it is fair to say that, since a significant part of the motivation of mereotopology is to avoid bizarre, physically unrealizable regions, the axiomatization of $M(\mathbb{R}^2)$ is less pressing. However, there is no doubt that the domain R could be liberalized considerably without change to the resulting theory, although we at present lack a characterization of how this might be done.

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