Chapter 1

FIRST-ORDER MEREOTOPOLOGY

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1. Introduction

One of the many achievements of coordinate geometry has been to provide a conceptually elegant and unifying account of the nature of geometrical entities. According to this account, the one primitive spatial entity is the point, and the one primitive geometrical property of points is coordinate position. All other geometrical entities—lines, curves, surfaces and bodies—are nothing but collections of points; and all properties and relations involving these entities may be defined in terms of the relative positions of the points which make them up. The success and power of this reduction is so great that the identification of spatial regions with the sets of points they contain has come to seem virtually axiomatic.

Over the years, however, various authors have expressed disquiet with this conceptual régime. The primary source of the disquiet is the conviction that our theory of space should use only those resources absolutely necessary to systematize the data of spatial experience. For points are such remote abstractions from the objects with which we daily interact, and co-ordinate position such a distant relative of the spatial properties and relations which we directly perceive, that the question arises as to whether alternative mathematical models of space are not possible—in particular, models in which the primitive spatial entities are not points, but regions, and in which the primitive spatial properties and relations are qualitative rather than quantitative.
An example will help to make these worries more concrete. Consider any stable, medium-sized physical object, for example, a coffee cup. We all agree that this cup has a particular shape, which we may take to correspond to the region of space which it occupies at some instant. On the familiar point-based model of space, this region is a set of points. But suppose we now ask: is this set topologically open, semi-open or closed? That is: does it include none, some, or all of its boundary points? It is hard to see how we could answer this question. Not by microscopic analysis, since physical objects lose their definition on very small scales. And not by mathematical argument, since a world in which—say—cups are closed and saucers open is surely as logically possible as one where these topological characteristics are reversed. But if space really is made up of points as (modern) textbooks tell us, any assignment of a region of space to the coffee cup must answer the question. Perhaps then this model postulates too much.

This chapter addresses the question: what region-based accounts of the topological structure of space are possible? What can we say about them? How do they relate to each other and to the point-based models with which we are so familiar?

2. Mereotopologies

The purpose of section is to outline the conceptual framework for region-based theories of space adopted in this chapter. Specifically, we introduce the concept of a mereotopology over a topological space, we discuss the role of mereotopologies as interpretations of signatures of topological primitives, and we list some key mathematical questions concerning them.

We assume familiarity with fundamental concepts and standard facts of point-set topology and Boolean algebra: for details, see, e.g. Kelley, 1955 and Koppelberg, 1989, Ch. 1, respectively. In the context of point-set topology, if \( u \) is any subset of a topological space \( X \), we denote the interior of \( u \) by \( u^0 \) and the closure of \( u \) by \( u^- \). (The more usual notations of \( \mathring{u} \) and \( \bar{u} \) for the closure of \( u \) are reserved for other purposes.) We write \( \mathcal{F}(u) \) to denote the frontier of \( u \), namely \( u^- \setminus u^0 \).

2.1 Regular open sets

How might we go about building a region-based model of the space we inhabit? The example of the coffee cup suggests that any such model should resolve the issue of frontier points. The following technical details are well-suited to this purpose.
Definition 1.1 Let \( u \) be a subset of some topological space \( X \). We say that \( u \) is regular open (in \( X \)) if \( u \) is equal to the interior of its closure. We denote the set of regular open subsets of \( X \) by \( \text{RO}(X) \).

To fix our intuitions, consider the space \( X = \mathbb{R}^2 \). The elements of \( \text{RO}(\mathbb{R}^2) \) are the open subsets of \( \mathbb{R}^2 \) having no 'cracks' or 'pin-holes' (Fig. 1.1). Corresponding remarks apply to the case \( X = \mathbb{R}^3 \). Taking

![Regular and Non-regular open sets](image)

Figure 1.1. Some regular and non-regular open sets of the Euclidean plane.

regions of space to be regular open subsets of \( \mathbb{R}^3 \) finesses the issues encountered above concerning frontier points: regions are open by fiat. At the same time, however, it provides us with satisfying formal reconstructions of the intuitive notions of intersecting, merging and complementing regions, by means of the following standard theorem (see, for example, Koppelberg, 1989, pp. 25–27).

Proposition 1.2 Let \( X \) be a topological space. Then \( \text{RO}(X) \) is a Boolean algebra under the order \( \subseteq \). In this Boolean algebra, top and bottom are defined by \( 1 = X \) and \( 0 = \emptyset \), and Boolean operations are defined by \( x \cdot y = x \cap y \), \( x + y = (x \cup y)^\circ \) and \( -x = X \setminus x^- \).

Again, we can fix our intuitions regarding Proposition 1.2 by considering the case \( X = \mathbb{R}^2 \). The product, \( x \cdot y \), of two regular open sets \( x \) and \( y \) is simply their intersection, which is guaranteed to be a regular open set. The sum, \( x + y \), of two regular open sets \( x \) and \( y \) is a little more complicated; very roughly, it is the union of \( x \) and \( y \) with any internal boundaries removed (Fig. 1.2). Finally, the complement, \( -x \), of a regular open set \( x \) in \( \text{RO}(\mathbb{R}^2) \) is simply that part of the plane not occupied by \( x \) or its frontier. Corresponding remarks apply to the case \( X = \mathbb{R}^3 \).

It sometimes helps to reformulate the definition of regular open sets as follows. If \( u \subseteq X \), then \( \bigcup \{ o \subseteq X \mid o \text{ open}, o \cap u = \emptyset \} \) is the largest open subset of \( X \) disjoint from \( u \). We call this set the pseudo-complement of \( u \), denoted \( u^* \). From the above definitions, \( u^* = X \setminus u^- \) and \( u^{**} = (u^-)^\circ \). Hence, \( u \) is regular if and only if \( u = u^{**} \); and, if \( u \) is regular open, \( u^* \) is simply \( -u \). The following lemma shows that every subset of \( X \) is 'close' to a regular open subset.
Lemma 1.3 Let $X$ be a topological space. For every $u \subseteq X$, the set $r = (u^-)^0$ is an element of $\text{RO}(X)$ such that $u^0 \subseteq r \subseteq u^-$. If $u$ is open, then $r$ is unique.

Proof Obviously $u^0 \subseteq r \subseteq u^-$. To show that $(u^-)^0 \in \text{RO}(X)$, it suffices to show that $u^{****} = u^{**}$. If $v$ is any set at all, then $v^{**} \cap u^* = \emptyset$, whence $v^* \subseteq v^{***}$. Moreover, if $o$ is any open set, then $o^{***}$ is an open set disjoint from $o^{**}$ and hence disjoint from every open set disjoint from $o^*$ and hence disjoint from $o$ itself, whence $o^{***} \subseteq o^*$. Thus, for any open set $o$, $o^{***} = o^*$. Since $u^*$ is open, we have $u^{****} = u^{**}$. For the final statement, if $s \in \text{RO}(X)$ also satisfies $u \subseteq s \subseteq u^-$, then the (regular) open sets $s \cdot -r$ and $r \cdot -s$ are both in $u^- \setminus u$ and so are empty. $\Box$

Figure 1.2. Three pairs of regions and their sums in $\text{RO}({\mathbb{R}}^2)$.

For the above reasons, it has become common practice in discussions of mereotopology to model regions of space as regular open subsets of $\mathbb{R}^3$; and that is the approach we shall take here. In the sequel, we shall always use the letters $r, s, t$ to range over regular open sets; when we are concerned only with regular open sets, we write $r \leq s$ in preference to $r \subseteq s$, $0$ in preference to $\emptyset$ and $r \cdot s$ in preference to $r \cap s$. Resorting to regular open sets is of course not the only way of dealing with boundary disputes. One obvious alternative is to use regular closed sets (sets equal to the closures of their interiors), since the regular closed sets of any topological space also form a Boolean algebra, which is in fact isomorphic to the Boolean algebra of regular open sets. Thus, in
modelling regions as regular open sets of \( \mathbb{R}^3 \), it is understood that it is the resulting structure that is important, not the precise constitution of its elements. Understanding what this idea means in detail forms a central theme of this chapter.

We conclude our discussion of regular open sets by proving some technical results which will be useful below. Recall in this context that, if \( u, v \) are connected subsets of a topological space, with \( u \cap v \neq \emptyset \), then \( u \cup v \) is connected. Moreover, if \( u \) is connected and \( u \subseteq v \subseteq u^- \), then \( v \) is connected.

**Lemma 1.4** Let \( X \) be a topological space, let \( u, v \subseteq X \) and let \( r, s \in \text{RO}(X) \). We have:

(i) \((u \cup v)^{-0} = u^{-0} + v^{-0}\);

(ii) \((r \cup s \subseteq r + s \subseteq r \cup s \cup (r^- \cap s^-) \subseteq (r \cup s)^-\);

(iii) \((r + s)^- = r^- \cup s^- = (r \cup s)^-\);

(iv) if \( r \) and \( s \) are connected with \( r \cdot s > 0 \), then \( r + s \) is connected.

**Proof** (i) By Lemma 1.3, \((u \cup v)^{-0}\) is a regular open set which evidently contains the regular open sets \( u^{-0} \) and \( v^{-0} \). Certainly, then \( u^{-0} + v^{-0} \subseteq (u \cup v)^{-0} \). For the reverse inclusion, \((X \setminus u)^0 \cap (X \setminus v)^0 \cap (u \cup v)^{-0} = \emptyset\), whence \((X \setminus u)^0 \cap (X \setminus v)^0 \cap (u \cup v)^{-0} = \emptyset\), whence \((X \setminus u)^0 \cap (X \setminus v)^0 \cap (u \cup v)^{-0} = \emptyset\), whence \((X \setminus u)^0 \cap (X \setminus v)^0 \cap (u \cup v)^{0-} = \emptyset\), whence \((X \setminus u)^0 \cap (X \setminus v)^0 \cap (u \cup v)^{-0} = \emptyset\). That is: \((u \cup v)^{-0} \subseteq (u^{-0} \cup v^{-0})^{-0}\). But by Proposition 1.2, \((u^{-0} \cup v^{-0})^{-0} = u^{-0} + v^{-0}\).

(ii) The only non-trivial inclusion is \( r + s \subseteq r \cup s \cup (r^- \cap s^-) \). So suppose \( p \not\in s \) and \( p \not\in r^- \). That is, \( p \in (-s)^- \) and \( p \in -r \). But then, for all open \( o \) with \( p \in o \), \( o \cap -r \) is also open with \( p \in o \cap -r \), whence \( (o \cap -r) \cap -s \neq \emptyset \) — that is, \( o \cap (-r \cdot -s) \neq \emptyset \). Hence \( p \in (-r \cdot -s)^- \) so \( p \not\in (-r \cdot -s) = r + s \). A similar argument applies if \( p \not\in r \) and \( p \not\in s^- \).

(iii) \((r+s)^- = X \setminus -(r+s) = X \setminus (-r \cdot -s) = (X \setminus -r) \cup (X \setminus -s) = r^- \cup s^- \).

(iv) Certainly, \( r \cup s \) is connected, and by (ii), \( r \cup s \subseteq r + s \subseteq (r \cup s)^- \), whence \( r + s \) is connected.

We note in passing that determining the validity of statements such as those of Lemma 1.4 is actually a decidable problem. See, e.g. Cantone and Cutello, 1994, Nutt, 1999, Pratt-Hartmann, 2002 and, for a fuller discussion, Ch. ??.
2.2 Mereotopologies

We have argued, provisionally, that, for a subset of $\mathbb{R}^3$ to count as a region, it should be regular open. However, it would be hasty to assume that all regular open subsets of $\mathbb{R}^3$ should count as regions, at least if spatial regions are supposed to be parts of space occupied (or left unoccupied) by physical objects. Consider, for example, the bizarre region commonly known as Alexander’s horned sphere and depicted in Fig. 1.3. (The reader is referred to Alexander, 1924a for details of the construction.) The interior of Alexander’s horned sphere is certainly regular open, yet it is a poor candidate to represent the space occupied by a physical object. In fact, this region is a “ball” so twisted in space that its complement in $\text{RO}(\mathbb{R}^3)$ is not simply connected! Nor are such pathological objects to be found only in three-dimensional space, as we shall see below. And such examples suggest that we should at least be open to the possibility of region-based models of space in which only some regular open subsets of $\mathbb{R}^3$ qualify as regions. This immediately presents us with the question: if not all subsets of $\mathbb{R}^3$ qualify as bona fide regions, which do? As we shall see, the answers available and the issues which hinge on them require detailed analysis.

In view of these uncertainties, we adopt a very general notion of a region-based model of space—just sufficiently constrained that we can sensibly confine attention to the structure of regions in question without worrying about the points of which they are composed. In the context of point-set topology, a topological space is commonly said to be semi-regular if it has a basis of regular open sets, and locally connected if it has a basis of connected sets. It easy to see that, in a locally connected space, every component of an open set is open. Recall also, in the context of Boolean algebras, that, if $B$ is a Boolean algebra and $B'$ a Boolean subalgebra of $B$, then $B'$ is said to be dense (in $B$) if, for every $b \in B$ with $0 < b$, there exists $\theta \in B'$ with $0 < \theta \leq b$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure1.png}
\caption{Alexander's horned sphere}
\end{figure}
Definition 1.5 Let $X$ be a topological space. A mereotopology over $X$ is a Boolean sub-algebra $M$ of RO($X$) such that, if $\alpha$ is an open subset of $X$ and $p \in \alpha$, there exists $r \in M$ such that $p \in r \subseteq \alpha$. We refer to the elements of $M$ as regions. If $M$ is a mereotopology such that any component of a region in $M$ is also a region in $M$, then we say that $M$ respects components.

Note that a mereotopology over $X$ is always a dense subalgebra of RO($X$). Our first task is to check that RO($X$) is a mereotopology, for a suitable class of topological spaces.

Lemma 1.6 Let $X$ be a semi-regular space. Then RO($X$) is a mereotopology over $\mathbb{R}^n$; if $X$ is also locally connected, then RO($X$) respects components.

Proof The first part of the lemma is instant from the relevant definitions. For the second part, let $r \in$ RO($X$), and let $s$ be a component of $r$. Since $X$ is locally connected, $s$ is open, whence, by Lemma 1.3, $(s^{-})^0$ is regular open with $s \subseteq (s^{-})^0 \subseteq s^{-}$. Then, $s^{-0}$ is a connected subset of $r$ including $s$, whence $s^{-0} = s$ by the maximality of $s$.

QED

Some etymological explanation is in order here. The term mereology was first introduced by Lesniewski, and denotes the logic of the part-whole relationship. (For a survey, see, e.g. Simons, 1987.) The term mereotopology is a much more recent coinage, and standardly denotes the study of topological relationships in which regions, rather than points, are the primitive objects. (It is unclear where the word first appeared in print.) The employment of the word as a count-noun in Definition 1.5, to denote a certain class of mathematical structures, is new here, and prompted by analogy with the parallel usage of the word topology.

The foregoing discussion suggests that our search for a region-based model of space should begin with an examination of mereotopologies over $\mathbb{R}^3$. This approach may at first seem dissatisfying, because it depends for its formulation on the very point-based model of space we are trying to escape. As we shall see, however, it is the structure of the resulting collection of regions that will interest us—and the characterization of that structure in purely intrinsic terms form one of the main themes of this chapter. But before we can seek such intrinsic characterizations, we must first clarify what it is we want to characterize.

2.3 Geometric mereotopologies

The question before us is to identify the regular open subsets of $\mathbb{R}^3$ which we are prepared to count as ‘sensible’ regions of space. Here is a
standard answer from the mathematical literature. Let $L'$ be the first-order language with the arithmetic signature $(\lt, +, \cdot, 0, 1)$, interpreted over $\mathbb{R}$ in the usual way. (This interpretation is of course completely separate from our use of the same symbols to denote Boolean operations on regular open sets!) For the purposes of this chapter, we may say that a set $u \subseteq \mathbb{R}^n$ is semi-algebraic if there exists an $L'$-formula $\phi(\vec{x}, \vec{y})$ in $n + m$ variables $\vec{x}, \vec{y}$ and an $m$-tuple of real numbers $\vec{b}$ such that

$$u = \{ \vec{a} \in \mathbb{R}^n \mid \text{the } (n + m) \text{-tuple } \vec{a}, \vec{b} \text{ satisfies the formula } \phi(\vec{x}, \vec{y}) \}.$$ 

For a detailed discussion of semi-algebraic sets, see, e.g., van den Dries, 1998, Bodnack et al., 1998 and also Ch. ?? (The more standard definition of semi-algebraic sets is equivalent to ours, and makes the name less puzzling.) For merotopological purposes, we are exclusively interested in those semi-algebraic subsets of $\mathbb{R}^n$ which are regular open.

**Definition 1.7** For $n > 0$, we denote the set of regular open, semi-algebraic sets in $\mathbb{R}^n$ by $\text{ROS}(\mathbb{R}^n)$.

**Lemma 1.8** For $n > 0$, $\text{ROS}(\mathbb{R}^n)$ is a merotopology over $\mathbb{R}^n$.

**Proof** We first show that $\text{ROS}(\mathbb{R}^n)$ is a Boolean subalgebra of $\text{RO}(\mathbb{R}^n)$. Evidently, $0, 1 \in \text{ROS}(\mathbb{R}^n)$. Moreover, if a set $u$ is definable by a first-order formula in the language of arithmetic, then so are its closure $\overline{u}$ and its interior $u^0$. Hence, if $r, s \in \text{ROS}(\mathbb{R}^n)$, then so are $r \cdot s = r \cap s$, $r + s = (r \cup s)^0$ and $-r = \mathbb{R}^n \setminus r^\circ$. We must establish that, for $p \in o$ with $o \subseteq \mathbb{R}^n$ open, there exists $r \in \text{ROS}(\mathbb{R}^n)$ such that $p \in r \subseteq o$. But this is obvious since any open ball is an element of $\text{ROS}(\mathbb{R}^n)$. 

The structure of regular open semi-algebraic subsets of $\mathbb{R}^3$ might have a better claim to count as a region-based model of space than the whole of $\text{RO}(\mathbb{R}^3)$, because it does a good job of ruling out pathological regular open sets. For example, the horned sphere of Fig. 1.3 is not semi-algebraic.

More generally, semi-algebraic sets count as well-behaved. One of their fundamental properties is that they admit of ‘cell decompositions’. If $d > 0$, a $d$-cell in $\mathbb{R}^n$ is any semi-algebraic subset of $\mathbb{R}^n$ homeomorphic to the open $d$-dimensional ball; a $0$-cell in $\mathbb{R}^n$ is a singleton; and a cell is a $d$-cell for some $d$ ($0 \leq d \leq n$). The following result is standard (van den Dries, 1998, Ch. 3, Theorem 2.11).

**Proposition 1.9 (Cell Decomposition Theorem)** If $u$ is a semi-algebraic subset of $\mathbb{R}^n$, then $u$ is the union of a finite collection of pairwise disjoint, semi-algebraic cells.
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For regular open semi-algebraic sets, this yields:

**Lemma 1.10** Every $r \in \text{ROS}(\mathbb{R}^n)$ is the sum of finitely many pairwise disjoint $n$-cells in $\text{ROS}(\mathbb{R}^n)$.

**Proof** By Proposition 1.9, let $r = u_1 \cup \ldots \cup u_m$ where the $u_i$ are pairwise disjoint, semi-algebraic cells. Since $r$ is regular, $r = r^{-0} = (u_1 \cup \ldots \cup u_m)^{-0} = u_1^{-0} + \ldots + u_m^{-0}$, by Lemma 1.4 (i). If $u_i$ is a $d$-cell for $d < n$, then $u_i^{-0} = 0$; if $u_i$ is an $n$-cell, $u_i^{-0} = u_i$. \(\text{QED}\)

The following notion will play an important part in the ensuing discussion.

**Definition 1.11** A mereotopology $M$ is finitely decomposable if every region in $M$ is the sum of finitely many connected regions in $M$.

**Lemma 1.12** $\text{ROS}(\mathbb{R}^n)$ is finitely decomposable.

**Proof** By Lemma 1.10, since cells are connected. \(\text{QED}\)

**Lemma 1.13** Every finitely decomposable mereotopology $M$ over a locally connected space $X$ respects components; moreover, every region in $M$ is the sum of its components.

**Proof** Suppose $r \in M$, and $s$ is a component of $r$. By Lemma 1.6, $s \in \text{RO}(X)$. Let $r_1, \ldots, r_n$ be connected elements of $M$ such that $r = r_1 + \ldots + r_n$. By the maximality of $s$ and Lemma 1.4 (iv), either $r_i \leq s$ or $r_i \cdot s = 0$ for all $i$ ($1 \leq i \leq n$). Thus, $s$ is the sum of those $r_i$ such that $r_i \leq s$. \(\text{QED}\)

Of course, the converse of Lemma 1.13 is false: although $\text{RO}(X)$ respects components for any locally connected space $X$, it is easy to see that, for example, $\text{RO}(\mathbb{R}^n)$ is not finitely decomposable for any $n > 0$.

The mereotopology $\text{ROS}(\mathbb{R}^n)$ is thus at least a plausible region-based model of the space we inhabit. But it is not the only candidate for this job. Observe that any $(n-1)$-dimensional hyperplane of $\mathbb{R}^n$ cuts $\mathbb{R}^n$ into two residual domains, which we shall call half-spaces. It is easy to see that these half-spaces are regular open, with each being the pseudo-complement of the other. Hence, we can speak about the sums, products and complements of half-spaces in $\text{RO}(\mathbb{R}^n)$.

**Definition 1.14** A basic polytope in $\mathbb{R}^n$ is the product, in $\text{RO}(\mathbb{R}^n)$, of finitely many half-spaces. A polytope in $\mathbb{R}^n$ is the sum, in $\text{RO}(\mathbb{R}^n)$, of any finite set of basic polytopes. We denote the set of polytopes in $\mathbb{R}^n$
by \( \text{ROP}(\mathbb{R}^n) \); we call the polytopes in \( \text{ROP}(\mathbb{R}^2) \) polygons and those in \( \text{ROP}(\mathbb{R}^3) \) polyhedra.

Thus, polytopes (in our sense) may be unbounded, disconnected, and may have disconnected complements. Fig. 1.4 shows a selection of polygons. (In alternative parlance, the elements of \( \text{ROP}(\mathbb{R}^n) \) are the regular open semi-linear sets.) Evidently, the polyhedra constitute a more parsimonious region-based model of space than does \( \text{ROS}(\mathbb{R}^3) \).

Indeed, the following construction gives us a more parsimonious spatial ontology still. If an \((n-1)\)-dimensional hyperplane in \( \mathbb{R}^n \) is defined by an equation \( a_0 + a_1 x_1 + \cdots + a_n x_n = 0 \), where the \( a_i \) (0 \( \leq i \leq n \)), are rational numbers, we call it a rational hyperplane; and if a half-space is bounded by a rational hyperplane, we call it a rational half-space. Now we define:

**Definition 1.15** A basic rational polytope in \( \mathbb{R}^n \) is the product, in \( \text{RO}(\mathbb{R}^n) \), of finitely many rational half-spaces. A rational polytope in \( \mathbb{R}^n \) is the sum, in \( \text{RO}(\mathbb{R}^n) \), of any finite set of basic rational polytopes. We denote the set of rational polytopes in \( \mathbb{R}^n \) by \( \text{ROQ}(\mathbb{R}^n) \); we call the elements of \( \text{ROQ}(\mathbb{R}^2) \) rational polygons and those of \( \text{ROQ}(\mathbb{R}^3) \) rational polyhedra.

Evidently, \( \text{ROQ}(\mathbb{R}^n) \subset \text{ROP}(\mathbb{R}^n) \subset \text{ROS}(\mathbb{R}^n) \subset \text{RO}(\mathbb{R}^n) \). Note that \( \text{ROQ}(\mathbb{R}^n) \) is countable.

**Lemma 1.16** The collections \( \text{ROP}(\mathbb{R}^n) \) and \( \text{ROQ}(\mathbb{R}^n) \) are finitely decomposable mereotopologies over \( \mathbb{R}^n \).

**Proof** Basic polytopes are convex, and hence connected. QED

As models of the space we inhabit, \( \text{ROP}(\mathbb{R}^3) \) and \( \text{ROQ}(\mathbb{R}^3) \) may seem overly austere—for they contain no regions with curved boundaries. However, their study turns out to be instructive, as we shall see below.

### 2.4 Interpretations

So far, we have discussed various ways of selecting a collection of ‘regions’ from among the subsets of \( \mathbb{R}^n \). But this selection process only
really becomes interesting when we consider formal languages whose variables range over these collections, and whose non-logical constants belong to a limited repertoire of spatial primitives.

We assume familiarity with basic first-order logic: for details, see Hodges, 1993, Ch 1. In this context, we employ the following standard notation and terminology. Let $\Sigma$ be a signature consisting of (zero or more) predicates, function-symbols and individual constants; we denote the first-order language with signature $\Sigma$ by $L_\Sigma$. An $L_\Sigma$-formula with no free variables is called an $L_\Sigma$-sentence. Let $\mathfrak{A}$ be a structure interpreting the symbols in $\Sigma$ over some domain $A$ (assumed non-empty). For any $L_\Sigma$-formula $\phi(\bar{x})$, with $n > 0$ free-variables $\bar{x}$ and any $n$-tuple $\bar{a}$ from $A$, we write $\mathfrak{A} \models \phi[\bar{a}]$ if $\bar{a}$ satisfies $\phi(\bar{x})$ in $\mathfrak{A}$; similarly, for any $L_\Sigma$-sentence $\phi$, we write $\mathfrak{A} \models \phi$ if $\phi$ is true in $\mathfrak{A}$. We call $\{ \psi \mid \psi \text{ an } L_\Sigma\text{-sentence and } \mathfrak{A} \models \psi \}$ the $L_\Sigma$-theory of $\mathfrak{A}$, denoted $\text{Th}_\Sigma(\mathfrak{A})$. Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent (for $\Sigma$), written $\mathfrak{A} \equiv_\Sigma \mathfrak{B}$, if $\text{Th}_\Sigma(\mathfrak{A}) = \text{Th}_\Sigma(\mathfrak{B})$. We write $f : \mathfrak{A} \simeq_\Sigma \mathfrak{B}$ if $f$ is a $\Sigma$-structure isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$ (and $\mathfrak{A} \simeq_\Sigma \mathfrak{B}$ if such an $f$ exists). It is a simple result that if $f : \mathfrak{A} \simeq_\Sigma \mathfrak{B}$ and $\phi(\bar{x})$ is an $L_\Sigma$-sentence, then $\mathfrak{A} \models \phi[\bar{a}]$ implies $\mathfrak{B} \models \phi[f(\bar{a})]$ for every tuple $\bar{a}$ from $A$; in particular, $\mathfrak{A} \simeq_\Sigma \mathfrak{B}$ implies $\mathfrak{A} \equiv_\Sigma \mathfrak{B}$. We write $\mathfrak{A} \subseteq_\Sigma \mathfrak{B}$, if $\mathfrak{A}$ is a submodel of $\mathfrak{B}$ (i.e. $A \subseteq B$ and $\mathfrak{A}$ is the restriction of $\mathfrak{B}$ to $A$), and $\mathfrak{A} \preceq_\Sigma \mathfrak{B}$ if $\mathfrak{A}$ is an elementary submodel of $\mathfrak{B}$ (i.e. $A \subseteq B$ and and every tuple $\bar{a}$ of $A$ satisfies the same $L_\Sigma$-formulas in both $\mathfrak{A}$ and $\mathfrak{B}$). We say that $\mathfrak{A}$ is elementarily embeddable in $\mathfrak{B}$ if $\mathfrak{A}$ is isomorphic to an elementary submodel of $\mathfrak{B}$. Trivially, $\mathfrak{A} \preceq_\Sigma \mathfrak{B}$ implies $\mathfrak{A} \equiv_\Sigma \mathfrak{B}$. Reference to the signature $\Sigma$, and the associated subscripts, is suppressed when clear from context.

Let $M$ be a mereotopology over some topological space $X$. If $\Sigma$ is a signature whose symbols conventionally denote familiar mereological or topological concepts, then $M$ can always be regarded as a $\Sigma$-structure by interpreting the symbols of $\Sigma$ in the familiar way. In particular, we take the symbols $0, 1, +, \cdot, -$ and $\leq$ to have the obvious (Boolean algebra) interpretations over $M$; similarly, we take the unary predicate $c$ to denote the property of being connected, and the binary predicate $C$ to denote the relation which holds between two regions if and only if their topological closures intersect. Table 1.1 gives a formal summary. Under these interpretations, we may regard any mereotopology $M$ as an interpretation for the signature $\Sigma = (0, 1, +, \cdot, -, \leq, c, C)$, or any subset thereof. That is: any $L_\Sigma$-sentence has a truth-value in $M$, and any $L_\Sigma$-formula $\phi(\bar{x})$ with $n > 0$ free-variables defines an $n$-ary relation over $M$, namely, the set of $n$-tuples from $M$ satisfying $\phi(\bar{x})$. We remark that our interpretation of $C$ is intended as a rational reconstruction of
Table 1.1. Interpretations of common mereotopological primitives, where \( M \) is a mereotopology over a topological space \( X \).

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Type</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>individual constant</td>
<td>( 0^M = \emptyset )</td>
</tr>
<tr>
<td>1</td>
<td>individual constant</td>
<td>( 1^M = X )</td>
</tr>
<tr>
<td>+</td>
<td>binary function</td>
<td>( +^M (r, s) = ((r \cup s)^-)^0 )</td>
</tr>
<tr>
<td>\cdot</td>
<td>binary function</td>
<td>( \cdot^M (r, s) = r \cap s )</td>
</tr>
<tr>
<td>-</td>
<td>unary function</td>
<td>( -^M (r) = X \setminus r^\circ )</td>
</tr>
<tr>
<td>\leq</td>
<td>binary predicate</td>
<td>( \leq^M = { (r, s) \in M^2 \mid r \subseteq s } )</td>
</tr>
<tr>
<td>c</td>
<td>unary predicate</td>
<td>( c^M = { r \in M \mid r \text{ connected} } )</td>
</tr>
<tr>
<td>C</td>
<td>binary predicate</td>
<td>( C^M = { (r, s) \in M^2 \mid r^\circ \cap s^\circ \neq \emptyset } )</td>
</tr>
</tbody>
</table>

The relation which Whitehead, 1929 called “extensive connection”, and which has historically played a prominent role in region-based theories of space. Since Whitehead’s term risks confusion with the standard topological notion of connectedness, we follow more recent usage and read \( C(x, y) \) as “\( x \) contacts \( y \)”. Some examples will help to clarify the issues that arise concerning first-order languages interpreted over mereotopologies.

**Example 1.17** Let \( \Sigma = (C, c, \leq) \), and let \( \psi_{\text{inf}} \) be the \( L_{\Sigma} \)-sentence

\[
\forall x \forall y (C(x, y) \rightarrow \exists z (c(z) \land z \leq y \land C(x, z))).
\]

This sentence ‘says’ that, if a region contacts another region, then it contacts some connected part of it. Let \( M \) be any finitely decomposable mereotopology; then \( M \models \psi_{\text{inf}} \). For suppose \( M \models C[r, s] \), and let \( s_1, \ldots, s_m \), be connected regions of \( M \) summing to \( s \). By Lemma 1.4(iii), \( s^\circ = s_1^\circ \cup \cdots \cup s_m^\circ \), whence \( M \models C[r, s_i] \) for some \( i \). On the other hand, it is not difficult to see that \( \text{RO}(\mathbb{R}^2) \nvdash \psi_{\text{inf}} \). Fig. 1.5 shows two regular open regions \( r, s \) in the plane, where \( r \) has infinitely many components, and \( s \) touches the closure of \( r \) but is separated from each of its components.

Example 1.5 shows, in particular, that the differences between the region-based models of space \( \text{RO}(\mathbb{R}^2) \) and \( \text{ROS}(\mathbb{R}^3) \) are ‘visible’ to certain first-order languages with signatures of topological primitives. In fact, the existence of regions with infinitely many components is not the only difference between these mereotopologies, as the next example shows.

**Example 1.18** Let \( \Sigma = (c, +) \), and let \( \psi_{\text{sum}} \) be the \( L_{\Sigma} \)-sentence

\[
\forall x_1 \forall x_2 \forall x_3 (c(x_1) \land c(x_2) \land c(x_3) \land c(x_1 + x_2 + x_3) \rightarrow (c(x_1 + x_2) \lor c(x_1 + x_3))).
\]
Figure 1.5. Two elements in $\text{RO}(\mathbb{R}^2)$, one with infinitely many components.

Figure 1.6. Three elements in $\text{RO}(\mathbb{R}^2)$.

This sentence ‘says’ that if three connected regions have a connected sum, then the first must form a connected sum with one of the other two. We show in Lemma 1.56 below that, if $M$ is any of $\text{ROS}(\mathbb{R}^2)$, $\text{ROP}(\mathbb{R}^2)$ or $\text{ROQ}(\mathbb{R}^2)$, then $M \models \psi_{\text{sum}}$. However, it turns out that $\text{RO}(\mathbb{R}^2) \not\models \psi_{\text{sum}}$.

For let

\[
\begin{align*}
  r_1 &= \{(x,y)| -1 < x < 0; \ -1 - x < y < 1 + x\} \\
  r_2 &= \{(x,y)| 0 < x < 1; \ -1 - x < y < \sin(1/x)\} \\
  r_3 &= \{(x,y)| 0 < x < 1; \ \sin(1/x) < y < 1 + x\}
\end{align*}
\]

as depicted in Fig. 1.6. It is easy to check that $r_1 + r_2 + r_3$ is the large triangle, and so is certainly connected, but that neither $r_1 + r_2$ nor $r_1 + r_3$ is connected.

We shall see in Section 5 that, in some sense, Examples 1.17 and 1.18 represent the only differences between $\text{RO}(\mathbb{R}^2)$ and $\text{ROS}(\mathbb{R}^2)$. 
Our final example illustrate a rather different set of issues concerning first-order mereotopological theories. We require the following fact about the topology of Euclidean spaces (Newman, 1964, p. 137).

**Proposition 1.19** If $d_1$ and $d_2$ are non-intersecting closed sets in $\mathbb{R}^n$, and points $p$ and $q$ are connected in $\mathbb{R}^n \setminus d_1$ and also in $\mathbb{R}^n \setminus d_2$, then $p$ and $q$ are connected in $\mathbb{R}^n \setminus (d_1 \cup d_2)$.

**Example 1.20** Let $\Sigma = (C, c, \cdot, \lnot)$, and let $\psi_{\text{sep}}$ be the $L_\Sigma$-sentence

$$\forall x \forall y(c(x) \land c(y) \rightarrow (c(x \cdot y) \lor C(\lnot x, \lnot y))).$$

This sentence ‘says’ that the closures of the complements of any two connected regions whose product is not connected intersect. Suppose that $r, s \in \text{RO}(\mathbb{R}^n)$ are connected, with $r \cdot s$ not connected. Putting $d_1 = \mathbb{R}^n \setminus r$ and $d_2 = \mathbb{R}^n \setminus s$, we have $d_1 \cup d_2 = \mathbb{R}^n \setminus (r \cdot s)$, whence, by Proposition 1.19, $(-r)^- \cap (-s)^- \neq \emptyset$. Thus, if $M$ is a mereotopology over any of the spaces $\mathbb{R}^n$, $M \models \psi_{\text{sep}}$. However, $\psi_{\text{sep}}$ is not true for all mereotopologies. For example, let $X$ be the surface of a torus, let $M$ be RO$(X)$, and let $r, s \in M$ be such that $-r$ and $-s$ are as illustrated in Fig. 1.7. By inspection, $r$ and $s$ are connected, $r \cdot s$ is not connected, and $-r$ does not contact $-s$. Hence, $M \models \lnot \psi_{\text{sep}}$.

Thus the regular open algebra of the torus and the Euclidean plane have different first-order mereotopological theories over the signature \{\(C, c, \cdot, \lnot\)\}.

There is nothing privileged about the above collection of primitives: in principle, we could employ any signature whose symbols can be given fixed interpretations over the structures we choose to confine our attention to. Since this chapter deals with topological notions, we consider only signatures with fixed topological interpretations—that is, signatures whose interpretations are preserved by homeomorphisms of the underlying topological space. For brevity, we speak of a 'signature of
topological primitives'. For investigations of region-based theories with non-topological signatures, see, e.g. Davis et al., 1999, Pratt, 1999.

Given a mereotopology $M$ and a signature $\Sigma$ of topological primitives, three salient issues present themselves. The first concerns the *expressive power* of a first-order topological language $L_\Sigma$ over a mereotopology $M$. Any $L_\Sigma$-formula $\phi(\vec{x})$ with free variables $\vec{x} = x_1, \ldots, x_n$ defines an $n$-ary relation over $M$—namely, the set of $n$-tuples $\vec{r}$ satisfying $\phi(\vec{x})$ in $M$. And it is therefore natural to ask which relations can be so defined, and in particular, which primitives can be defined in terms of which others. Of particular interest in this regard is the property of being topologically indistinguishable from a specific object or tuple of objects. That is, given a tuple $\vec{r}$ from $M$, we would particularly like to know whether $L_\Sigma$ is expressive enough to give a topologically complete characterization of $\vec{r}$. The answers to these questions depends heavily on the mereotopology $M$: Sections 3 and 4 analyse the expressive power of various first-order topological languages for well-behaved mereotopologies over the Euclidean plane. Section 6 analyses the much more difficult case of well-behaved mereotopologies over $\mathbb{R}^3$.

The second salient issue concerns the $L_\Sigma$-theory of $M$. Examples 1.17 and 1.18 show that restricting regions to be *semi-algebraic* (regular open) sets does affect the resulting first-order theory over some signatures of topological primitives. And the question therefore arises as to what other restrictions might be sensible, and what effect, if any, these restrictions have on the resulting first-order mereotopological theories. Most ambitiously, perhaps, we might ask whether the set of first-order sentences in various mereotopologies can be axiomatically characterized. Section 5 provides an example of such an axiomatic characterization. As a by-product of this analysis, we show that a wide range of plane mereotopologies share the same $L_\Sigma$-theory for (most) topological signatures $\Sigma$, and we venture to take that theory as the *standard* first-order $L_\Sigma$-theory of plane mereotopology. In this sense, the choice of what, exactly, counts as a region is much less critical than we might at first have supposed.

The third salient issue concerns the ontological commitments entailed by first-order mereotopological theories. To understand this issue, recall that a mereotopology $M$ is a collection of subsets of some topological space, which we have chosen to regard as a $\Sigma$-structure, for some signature $\Sigma$ of topological primitives. Any such mereotopology $M$ thus defines an $L_\Sigma$-theory $\text{Th}_\Sigma(M)$. But of course, *any* $\Sigma$ structure $\mathfrak{A}$ with $\text{Th}_\Sigma(\mathfrak{A}) = \text{Th}_\Sigma(M)$ can be thought of as a (region-based) model of space which, from the point of view of $L_\Sigma$, makes exactly the same predictions as $M$. It is therefore natural to ask which structures these are, and what,
if anything, we can say about their relationship to $M$. Notice that the elements of such $\Sigma$-structures need not be regions of topological spaces at all; as such they are genuinely region-based theories of space. In particular, we may ask whether mereotopologies in general admit of intrinsic characterizations making no reference to the topological spaces whose regions they make up. And we may further ask—particularly in the light of Example 1.20—what information those intrinsic characterizations yield about the topological spaces in question. Section 7 answers these, and related, questions.

The above three issues constitute the primary agenda of mereotopology, as conceived here.

3. **Defining topological relations**

Our task in this section is to compare the relative expressiveness of first-order languages having different signatures of topological primitives. Our main result is that $L_C$ is at least as expressive as $L_{c,\leq}$ over all sensible mereotopologies. We also show that over some mereotopologies of interest, $L_{c,\leq}$ is also at least as expressive as $L_C$.

We assume familiarity with the standard ($T_i$-) separation properties of topological spaces. Terminology varies here; we adopt the convention according to which $T_i$-separation for $i > 2$ does not by definition imply $T_1$-separation; and we say that a space $X$ is *Hausdorff* if it satisfies $T_2$-separation, *regular* if it satisfies both $T_3$- and $T_1$-separation, and *normal* if it satisfies both $T_4$- and $T_1$-separation.) In addition, we occasionally employ the following less familiar separation property (Düntsch and Winter, 2005).

**Definition 1.21** A topological space is weakly regular if it is semi-regular and, for any non-empty open set $u$, there exists a non-empty open set $v$ with $v^- \subseteq u$.

We have

$X$ is normal $\Rightarrow$ $X$ is regular $\Rightarrow$ $X$ is weakly regular $\Rightarrow$ $X$ is semi-regular.

The reverse implications all fail (see Düntsch and Winter, 2005 regarding weak regularity, and Steen and Seebach, 1995 for the other cases).

3.1 **Contact**

We begin by defining the part-of relation in $L_C$.

**Lemma 1.22** Let $M$ be a mereotopology over a weakly regular space $X$, and let $r_1, r_2 \in M$. Then $r_1 \leq r_2$ if and only if $M \models \phi_{\leq}[r_1, r_2]$, where $\phi_{\leq}(x_1, x_2)$ is the $L_C$-formula $\forall z(C(x_1, z) \rightarrow C(x_2, z))$. 
First-Order Mereology

Proof. If \( r_1 \leq r_2 \) then \( r_1^- \subseteq r_2^- \), so \( s^- \cap r_1^- \neq \emptyset \) implies \( s^- \cap r_2^- \neq \emptyset \) for any \( s \). Conversely, if \( r_1 \not\leq r_2 \), by weak regularity, let \( u \) be a non-empty, open set such that \( u^- \subseteq r_1 \cdot (-r_2) \). Since \( M \) is a mereotopology, let \( s \in M \) be such that \( 0 \neq s \subseteq u \). Then \( s^- \cap r_1^- \neq \emptyset \), but \( s^- \cap r_2^- = \emptyset \). \( \square \)

In dealing with mereotopologies over weakly regular spaces, we may therefore write the expression \( u \leq v \) in \( L_C \)-formulas, as a shorthand for \( \phi_{\leq}(u,v) \). It follows that the Boolean constants and functions 0, 1, +, and \( \cdot \) and \( \neg \) are also \( L_C \)-definable for mereotopologies over weakly regular spaces, and we again freely employ these symbols in \( L_C \)-formulas as a shorthand for their definitions.

We now turn to defining the property of connectedness in \( L_C \). We need some technical lemmas.

Lemma 1.23 Let \( M \) be a mereotopology over a regular topological space \( X \). If \( d \subseteq X \) is closed and \( p \not\in d \), there exists \( r \in M \) such that \( p \in r \) and \( d \subseteq r^- \). In fact, there exist \( r, s \in M \) such that \( p \in r \), \( d \subseteq s \) and \( r^- \cap s^- = \emptyset \).

Proof. For the first statement, by \( T_3 \)-separation, let \( u, v \) be disjoint open subsets of \( X \) such that \( p \in u \) and \( d \subseteq v \). Since \( M \) is a mereotopology, there exists \( r \in M \) such that \( p \in r \subseteq u \), whence \( d \subseteq v \subseteq X \setminus r^- = -r \). The second statement follows by two applications of the first: choose \( s \in M \) such that \( p \in -s \) and \( d \subseteq s \); now choose \( r \in M \) such that \( p \in r \) and \( s^- \subseteq r^- \). \( \square \)

Lemma 1.24 Let \( r, s \in \text{RO}(X) \) for some topological space \( X \). If \( p \in r^- \) and \( p \in s \), then \( p \in (r \cdot s)^- \).

Proof. Let \( u \) be any open set containing \( p \). Then \( u \cap s \) is also an open set containing \( p \), whence \( u \cap s \cap r \neq \emptyset \), since \( p \in r^- \). That is, \( u \cap (s \cdot r) \neq \emptyset \). \( \square \)

Lemma 1.25 Let \( M \) be a mereotopology over a regular topological space. For all \( r_1, r_2 \in M \), \( r_1^- \cap r_2^- \cap (r_1 + r_2) \neq \emptyset \) if and only if there exist \( r_1', r_2' \in M \) such that \( r_1' \leq r_1, r_2' \leq r_2, r_1'^- \cap r_2'^- \neq \emptyset \) and \( (r_1' + r_2')^- \cap (-r_1 + r_2))^\neg = \emptyset \).

Proof. The if-direction is immediate. For the only-if-direction, suppose \( p \in r_1^- \cap r_2^- \cap (r_1 + r_2) \). By Lemma 1.23, let \( s \in M \) be such that \( p \in s \) and \( (-r_1 + r_2)^- \subseteq -s \); and let \( r_1' = r_1 \cdot s \) and \( r_2' = r_2 \cdot s \). By Lemma 1.24, \( p \in r_1'^- \cap r_2'^- \), whence \( r_1' \) and \( r_2' \) have the required properties. \( \square \)
Lemma 1.26  Let $M$ be a mereotopology which respects components. Then $r \in M$ is connected if and only if $r_1^- \cap r_2^- \cap r \neq \emptyset$ for all nonempty, disjoint $r_1, r_2 \in M$ such that $r_1 + r_2 = r$.

Proof  Suppose $r_1$ and $r_2$ are non-empty, disjoint elements of $M$ such that $r_1 + r_2 = r$ and $r_1^- \cap r_2^- \cap r = \emptyset$. By Lemma 1.4 (ii), $r = r_1 \cup r_2$, so that $r$ is not connected. Conversely, suppose $r$ is not connected. Let $r_1$ be a component of $r$ and let $r_2 = r \setminus r_1$. Since $M$ respects components, $r_1 \in M$. Since $r_1 \subseteq r_1 \cup (r_1^- \cap r_2) \subseteq r_1^-$, $r_1 \cup (r_1^- \cap r_2)$ is connected, whence $r_1^- \cap r_2 = \emptyset$ by maximality of components. Thus, $r_2 = r \setminus r_1^- = r \cdot (-r_1)$. Moreover, since $r_1$ is open and $r_1 \cap r_2 = \emptyset$, we have $r_1 \cap r_2^- = \emptyset$. Therefore $\emptyset = r_1^- \cap r_2^- \cap (r_1 \cup r_2) = r_1^- \cap r_2^- \cap r$ as required. \qed

Lemma 1.27  Let $M$ be a mereotopology over a regular topological space $X$ such that $M$ respects components, and let $r \in M$. Then $r$ is connected if and only if $M \models \phi(c)[r]$, where $\phi(c)$ is the $L$-formula

$$\forall x_1 \forall x_2 (x_1 > 0 \land x_2 > 0 \land x_1 \cdot x_2 = 0 \land x_1 + x_2 = x \rightarrow \exists x_1' \exists x_2' (x_1' \leq x_1 \land x_2' \leq x_2 \land C(x_1', x_2') \land -C(x_1' + x_2', -x))).$$

Proof  Lemmas 1.25 and 1.26. \qed

Together, Lemmas 1.22 and 1.27 guarantee that, for all mereotopologies over regular topological spaces which respect components, the language $L_C$ is at least as expressive as $L_{c \leq}$. We take it that all mereotopologies of interest fulfill these conditions: that is, the above reconstructions of the part-whole relation and the property of connectedness in $L_C$ are very robust.

We present a further—and more surprising—demonstration of the expressive power of $L_C$ in mereotopologies defined over $\mathbb{R}^2$. We require the following fact about the topology of Euclidean spaces (Newman, 1964, p. 112, c.f. Proposition 1.19).

Proposition 1.28  Let $d_1$ and $d_2$ be closed sets in $\mathbb{R}^2$, at least one of which is bounded. If $\mathbb{R}^2 \setminus d_1$, $\mathbb{R}^2 \setminus d_2$ and $d_1 \cap d_2$ are all connected, then so is $\mathbb{R}^2 \setminus (d_1 \cup d_2)$.

Lemma 1.29  Let $s_1, s_2, t \in RO(\mathbb{R}^2)$ such that: (i) either $s_1$ is bounded or $s_2$ is bounded; (ii) $-(s_1 + t)$, $-(s_2 + t)$ and $t$ are all connected; and (iii) $s_1^- \cap s_2^- = \emptyset$. Then $-(s_1 + s_2 + t)$ is also connected.

Proof  Set $d_i = (s_i + t)^-$ (for $i = 1, 2$). Thus, the complement of $d_i$ is $-(s_i + t)$ (for $i = 1, 2$), and the complement of $d_1 \cup d_2$ is $-(s_1 + s_2 + t)$.
Moreover, since \( t \) is connected, so is \( t^- \), whence \( d_1 \cap d_2 = (s_1 + t)^- \cap (s_2 + t)^- = (s_1^- \cup t^-) \cap (s_2^- \cup t^-) = (s_1^- \cap s_2^-) \cup t^- = t^- \) is connected. The result follows by Proposition 1.28. QED

Let \( \phi_c \) be as defined in Lemma 1.27, and let \( \phi_{ub}(y_1, y_2) \) be the \( LC \)-formula

\[ \exists z(\phi_c(-(y_1 + z)) \land \phi_c(-(y_2 + z)) \land \phi_c(z) \land \neg \phi_c(-(y_1 + y_2 + z))) \].

**Lemma 1.30** Let \( M \) be a mereotopology over \( \mathbb{R}^2 \) such that \( M \) respects components and every unbounded element in \( M \) includes regions \( s_1, s_2 \) and \( t \) situated as in Fig. 1.8. Then for all \( r \in M, r \) is bounded if and only if \( M \models \phi_{ub}(r) \), where \( \phi_{ub}(x) \) is the \( LC \)-formula:

\[ \forall y_1 \forall y_2 (y_1 \leq x \land y_2 \leq x \land \phi_{ub}(y_1, y_2) \rightarrow C(y_1, y_2)) \].

(The superscript 2 in \( \phi_{ub} \) refers to the fact that this formula works for mereotopologies over \( \mathbb{R}^2 \), and not, for example \( \mathbb{R}^3 \).)

**Proof** If \( r \) does not satisfy \( \phi_{ub}(x) \) then, by Lemma 1.29, \( r \) contains two unbounded regions, so is certainly itself unbounded. Conversely, if \( r \) is unbounded, let \( s_1, s_2, t \in M \) be subsets of \( r \) situated as in Fig. 1.8. Thus, \( s_1 \leq r, s_2 \leq r \) and \( s_1^- \cap s_2^- = \emptyset \), but at the same time, \( s_1, s_2 \) satisfies \( \phi_{ub}(y_1, y_2) \), with \( t \) a witness for the existentially quantified \( z \). Hence \( r \) does not satisfy \( \phi_{ub}(x) \). QED

It is simple to verify that the mereotopologies \( RO(\mathbb{R}^2) \), \( ROS(\mathbb{R}^2) \), \( ROP(\mathbb{R}^2) \) and \( ROQ(\mathbb{R}^2) \) satisfy the conditions of Lemma 1.30. Hence, the property of boundedness is \( LC \)-definable in all these mereotopologies. Nevertheless, Lemma 1.30, unlike Lemmas 1.22 and 1.27, has a fragile character, in that it depends on a very specific feature of the topological space \( \mathbb{R}^2 \); in particular, it fails to define boundedness for the corresponding mereotopologies over \( \mathbb{R}^3 \). We will see in Section 6 that boundedness is also \( LC \)-definable in well-behaved mereotopologies over \( \mathbb{R}^3 \), but we have to go to much more trouble.
3.2 Reconstruction of points

In mereotopologies, the primitive objects—that is, the entities over which variables range—are regions, rather than points; but it is often simple to ‘construct’ points from regions, and ‘simulate’ statements about points using statements about regions. One way to construct the point \( p \) is as a pair of regions whose closures intersect in the singleton \{p\}, as we now proceed to show. (There are also more sophisticated ways, described in Section 7.1.)

**Lemma 1.31** Let \( M \) be a mereotopology over a regular topological space, and let \( r, s \in M \). Then \( r^- \cap s^- \) is a singleton if and only if \( M \models \phi_{\text{se}}[r, s] \), where \( \phi_{\text{se}}(x_1, x_2) \) is the formula

\[
C(x_1, x_2) \wedge \\
\forall y_1 \forall y_2 (y_1 \leq x_1 \wedge y_2 \leq x_2 \wedge C(y_1, x_2) \wedge C(y_2, x_1) \rightarrow C(y_1, y_2)).
\]

Furthermore, if \( r^- \cap s^- = \{p\} \) and \( t \in M \), then \( p \in t \) if and only if \( M \models \phi_{\text{se}}[r, s, t] \), where \( \phi_{\text{se}}(x_1, x_2, x_3) \) is the formula

\[
\exists y_1 (y_1 \leq x_1 \wedge C(y_1, x_2) \wedge \neg C(y_1, -x_3));
\]

likewise, \( p \in t^- \) if and only if \( M \models \phi_{\text{se}}[r, s, t] \), where \( \phi_{\text{se}}(x_1, x_2, x_3) \) is the formula

\[
\forall y_1 (y_1 \leq x_1 \wedge C(y_1, x_2) \rightarrow C(y_1, x_3)).
\]

**Proof** Routine by Lemmas 1.23 and 1.24. \( \Box \)

If \( M \) is a mereotopology over a topological space \( X \), let us say that \( M \) is complete if every point in \( X \) is the singleton intersection of some pair regions in \( M \). For example, the mereotopologies \( \text{ROP}(\mathbb{R}^n) \), \( \text{ROS}(\mathbb{R}^n) \) evidently possess this property; by contrast, \( \text{ROQ}(\mathbb{R}^n) \) does not. We might say that, in a complete mereotopology, points can be ‘simulated’ by pairs of regions satisfying the formula \( \phi_{\text{se}} \). If \( M \) is a complete mereotopology over a regular space, Lemma 1.31 gives us the right to include expressions such as, for example, \( x_1 \cap \overline{x_2} \neq \emptyset \) or \( \mathcal{F}(x_1) \cap \mathcal{F}(x_2) \subseteq \mathcal{F}(x_3) \cap \mathcal{F}(x_4) \) etc. in \( L_C \)-formulas with the obvious interpretation, since such expressions can evidently be replaced by \textit{bona fide} \( L_C \)-formulas with the appropriate extension over \( M \).

The following lemma illustrates how easily we can express various topological relations in \( L_C \):

**Lemma 1.32** Let \( r, s \in \text{ROP}(\mathbb{R}^n) \). Then \( r^- \cap s^- \) is connected if and only if \( \text{ROP}(\mathbb{R}^n) \models \phi_3[r, s] \), where \( \phi_3(x, y) \) is the formula

\[
\forall z (x^- \cap y^- \cap z \neq \emptyset \wedge x^- \cap y^- \cap -z \neq \emptyset \wedge x^- \cap y^- \subseteq z \cup -z).
\]
First-Order Mereology

Proof. The only-if direction is immediate. So suppose $r^- \cap s^-$ is not connected; we must find a witness for $z$ to show that $ROP(\mathbb{R}^n) \models \neg \phi_{z}(r, s)$. But, by construction of $ROP(\mathbb{R}^n)$, both $r^-$ and $s^-$ are expressible as finite unions of closed, convex sets; and so, therefore, is $r^- \cap s^-$. Since this latter set is not connected, it can be written as $d \cup e$, such that $d \cap e = \emptyset$ and $d$ and $e$ are both finite unions of non-empty, closed, convex sets—say, $d = d_1 \cup \cdots \cup d_l$, $e = e_1 \cup \cdots \cup e_m$. Given that any pair of disjoint, closed, convex sets in $\mathbb{R}^n$ can be separated by a hyperplane, we have half-spaces $h_{i,j}$ such that $d_i \subseteq h_{i,j}$ and $e_j \subseteq -h_{i,j}$ for all $i, j$ ($1 \leq i \leq l, 1 \leq j \leq m$). Then the required witness is

$$t = \sum_{1 \leq i \leq l, 1 \leq j \leq m} \prod h_{i,j}.$$

QED

3.3 Compactifications

Before discussing the expressive power of $L_{c, \leq}$, we introduce some additional technical material that will be useful throughout this chapter. Recall that a topological space is said to be locally compact if every point has a compact neighbourhood. This property ‘transfers’, for Hausdorff spaces, to mereotopologies defined over them:

**Lemma 1.33** Let $M$ be a mereotopology over a locally compact, Hausdorff space $X$, and let $p \in X$. Then $p$ is contained within some $r \in M$ such that $r^-$ is compact.

Proof. Let $p \in X$. Assuming $X$ is locally compact, let $d \subseteq X$ be compact and $o \subseteq d$ be open such that $p \in o$. Now let $r \in M$ such that $p \in r \subseteq o \subseteq d$. But a closed subset of a compact set is always compact, and, in a Hausdorff space, every compact set is closed. Therefore $r^- \subseteq d^- = d$ is compact, as required. QED

Let $X$ be a topological space, and let $\tau$ denote the collection of open sets of $X$. Now set $\hat{X} = X \cup \{\infty\}$, where $\infty$ is some object not in $X$. For $o \in \tau$, denote by $\hat{o}$ the set

$$\hat{o} = \begin{cases} o \cup \{\infty\} & \text{if } X \setminus o \text{ is compact;} \\ o & \text{otherwise,} \end{cases}$$

and denote by $\hat{\tau}$ the set $\tau \cup \{\hat{o} \mid o \in \tau\}$. Then we can take $\hat{X}$ to be a topological space whose collection of open sets is $\hat{\tau}$. Under this topology...
(which we always assume), we call $\hat{X}$ the one-point (or Alexandroff) compactification of $X$. The object $\infty$ is called the point at infinity. The space $\hat{X}$ is always compact. If $X$ is locally compact and Hausdorff, then $\hat{X}$ is also Hausdorff, and hence normal.

**Notation 1.34** In this chapter, we denote spheres, open balls and closed balls in Euclidean spaces as follows

\[ S^n = \{ (a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1} \mid a_1^2 + \cdots + a_{n+1}^2 = 1 \} \]

\[ B^n = \{ (a_1, \ldots, a_n) \in \mathbb{R}^n \mid a_1^2 + \cdots + a_n^2 < 1 \} \]

\[ D^n = \{ (a_1, \ldots, a_n) \in \mathbb{R}^n \mid a_1^2 + \cdots + a_n^2 \leq 1 \}; \]

and we assume the usual topology on these sets.

(Recall that, by a $d$-cell, we mean any set homeomorphic to the open $d$-dimensional ball $B^d$.) In the special cases $X = \mathbb{R}^n$, it is well-known that $\hat{X}$ is homeomorphic to $S^n$ via the mapping:

\[ \infty \mapsto (0, \ldots, 0, 1) \]

\[ (a_1, \ldots, a_n) \mapsto (a'_1, \ldots, a'_{n+1}), \]

where

\[ a'_i = 4a_i/(a_1^2 + \cdots + a_n^2 + 4) \quad \text{for } 1 \leq i \leq n \]

\[ a'_{n+1} = (a_1^2 + \cdots + a_n^2 - 4)/(a_1^2 + \cdots + a_n^2 + 4). \]

This mapping may be regarded as a stereographic projection by embedding $\mathbb{R}^n$ in the hyperplane of $\mathbb{R}^{n+1}$ defined in Cartesian geometry by the equation $x_{n+1} = -1$. This projection is depicted for the case $n = 2$ in Fig. 1.9. By way of allusion to this homeomorphism:

**Notation 1.35** Let $S^n$ denote the 1-point compactification of $\mathbb{R}^n$.

**Lemma 1.36** Let $X$ be a non-compact topological space. Then the mapping $r \mapsto \hat{r}$ is a Boolean algebra isomorphism from $\text{RO}(X)$ to $\text{RO}(\hat{X})$.

*Proof* The function $o \mapsto \hat{o}$ is monotone, because a closed subset of a compact set is compact. Let $o_1$ and $o_2$ be open subsets of $X$, with $o = o_1 \cap o_2$. Since $X \setminus o = (X \setminus o_1) \cup (X \setminus o_2)$ is compact if and only if both $(X \setminus o_1)$ and $(X \setminus o_2)$ are compact, we have $\infty \in \hat{o}$ if and only if $\infty \in \hat{o_1} \cap \hat{o_2}$, whence $\hat{o} = \hat{o_1} \cap \hat{o_2}$.
Figure 1.9. Stereographic projection of $S^2$ onto the 1-point compactification of $\mathbb{R}^2$.

If $u$ is open in $X$, let $u^*$ denote the pseudo-complement of $u$ in $X$, and let $(\hat{u})^*$ denote the pseudo-complement of $\hat{u}$ in $\hat{X}$. We claim that, for any open set $u$ of $X$, with $v = u^*$, $(\hat{u})^* = \hat{v}$. By definition, $\hat{v}$ is open in $\hat{X}$, and we have just shown that $\hat{u} \cap \hat{v} = \emptyset = \emptyset$. Moreover, if $w$ is any open set in $\hat{X}$ disjoint from $\hat{u}$, then for some open subset $w'$ of $X$, we have either $w = w'$ or $w = w'$. Either way $u \cap w' = \emptyset$, whence $w' \subseteq v$ and $w' \subseteq \hat{v}$ by monotonicity. Hence $\hat{v}$ is the largest open subset of $\hat{X}$ disjoint from $\hat{u}$, i.e. $(\hat{u})^* = \hat{v}$.

Note that if $r \in RO(X)$, we have $r = r^{**}$ and $-r = r^*$. Hence $r^{**} = r$, so that $r \in RO(\hat{X})$. Conversely, if $u' \in RO(\hat{X})$, then $u' = \hat{u}$, for some open $u \subseteq X$. But then, we have $\hat{u} = \hat{u}^{**} = \hat{x}$, where $x = u^{**}$. Since the function $o \mapsto \hat{o}$ is injective, $u = u^{**}$. That is, $u \in RO(X)$. QED

**Lemma 1.37** Let $X$ be a topological space and $o \subseteq X$ open. If $o$ is connected in $X$, then $\hat{o}$ is connected in $\hat{X}$. Conversely, suppose $X$ is noncompact, and for any closed subsets $d_1$ and $d_2$ of $X$ with $X = d_1 \cup d_2$ and $d_1 \cap d_2$ compact, either $d_1$ is compact or $d_2$ is compact. If $\hat{o}$ is connected in $\hat{X}$, then $o$ is connected in $X$.

**Proof** Suppose $o$ is open in $X$. If $\hat{o}$ is not connected in $\hat{X}$, let $\hat{o}_1$, $\hat{o}_2$ be non-empty open subsets of $X$ such that $\hat{o} = \hat{o}_1 \cup \hat{o}_2$ and $\hat{o}_1 \cap \hat{o}_2 = \emptyset$. Then $o = o_1 \cup o_2$ and $o_1 \cap o_2 = \emptyset$, so $o$ is not connected in $X$. Conversely, suppose $o$ is not connected in $X$, so let $o_1$, $o_2$ be nonempty open subsets of $X$ such that $o = o_1 \cup o_2$ and $o_1 \cap o_2 = \emptyset$. If $X \setminus o$ is not compact, then neither $X \setminus o_1$ nor $X \setminus o_2$ is compact, so that $\hat{o} = o = o_1 \cup o_2 = \hat{o}_1 \cup \hat{o}_2$ and $\hat{o}_1 \cap \hat{o}_2 = \emptyset$, whence $\hat{o}$ is not connected. If, on the other hand, $X \setminus o$ is compact, by the condition of the lemma, either $X \setminus o_1$ or $X \setminus o_2$ is compact, whence $\hat{o} = o \cup \{\infty\} = o_1 \cup o_2 \cup \{\infty\} = \hat{o}_1 \cup \hat{o}_2$. Moreover, by repeating the first paragraph of the proof of Lemma 1.36, we have $\hat{o}_1 \cap \hat{o}_2 = \emptyset = \emptyset$. It follows that $\hat{o}$ is not connected. QED
The well-known Heine-Borel theorem states that, in $\mathbb{R}^n$, a set is compact if and only if it is closed and bounded. It is therefore easy to see that $\mathbb{R}^n$ satisfies the condition of Lemma 1.37.

**Lemma 1.38** Let $n > 0$ and let $M$ be any mereotopology over $\mathbb{R}^n$. Then the mapping $r \mapsto \tilde{r}$ defines a structure isomorphism from $M$ to $\tilde{M}$ for the signature $(c, \leq)$: that is, $M \simeq_{c, \leq} \tilde{M}$.

**Proof** Lemmas 1.36 and 1.37. \[\Box\]

**Lemma 1.39** Let $X$ be a locally compact, non-compact topological space and $M$ a mereotopology over $X$. Define $\tilde{M} = \{\tilde{r} \mid r \in M\}$. Then $\tilde{M}$ is a mereotopology over $X$. We call $\tilde{M}$ the 1-point compactification of $M$. If $M$ is finitely decomposable, then so is $\tilde{M}$.

**Proof** Suppose that $\infty \in \partial$ with $\partial$ open in $X$; we show that there exists some $r \in \tilde{M}$ such that $\infty \in r \subseteq \partial$. Since $M$ is a mereotopology over $X$ and $X$ is locally compact, Lemma 1.33 gives us a cover of $X \setminus \partial$ by elements of $M$ whose closures are compact. Since $\infty \in \partial$, $X \setminus \partial$ is compact, so that this cover has a finite sub-cover, say $r_1, \ldots, r_n$. Let $r = -(r_1 + \cdots + r_n)$. Thus, $X \setminus r = r_1^\sim \cup \cdots \cup r_n^\sim$ is compact and includes $\partial$, whence $r$ has the required properties. The rest of the Lemma follows from Lemma 1.37. \[\Box\]

Suppose now that $X = \mathbb{R}^n$ for some $n > 0$, and let $M$ be a mereotopology over $\mathbb{R}^n$ respecting components. Then $X$ satisfies the condition of Lemma 1.37, so by Lemma 1.39, $\tilde{M}$ is a mereotopology over $\mathbb{R}^n$ respecting components. Since $\mathbb{S}^n$ denotes the 1-point compactification of $\mathbb{R}^n$, the 1-point compactification of $\text{RO}(\mathbb{R}^n)$ is thus $\text{RO}(\mathbb{S}^n)$.

**Notation 1.40** Let $\text{ROS}(\mathbb{S}^n)$ denote the 1-point compactification of $\text{ROS}(\mathbb{R}^n)$, and similarly for $\text{ROP}(\mathbb{S}^n)$, $\text{ROQ}(\mathbb{S}^n)$.

It is often more convenient to work with $\mathbb{S}^2$ and $\mathbb{S}^3$ rather than $\mathbb{R}^2$ and $\mathbb{R}^3$. When we need to make the distinction explicit, we refer to elements of $\text{ROP}(\mathbb{R}^n)$ as polytopes (polyhedra, polygons) in open space and those of $\text{ROP}(\mathbb{S}^n)$ as polytopes (polyhedra, polygons) in closed space. Note that, by Lemma 1.38, the mereotopologies $\text{RO}(\mathbb{R}^n)$, $\text{ROP}(\mathbb{R}^n)$, $\text{ROQ}(\mathbb{R}^n)$ and $\text{ROS}(\mathbb{R}^n)$ certainly all have the same $L_{c, \leq}$-theories as their respective 1-point compactifications.

### 3.4 Connectedness: the closed plane

We have seen that, over most mereotopologies of interest, the language $L_C$ is as expressive as the language $L_{c, \leq}$. The question therefore arises
as to whether a converse reduction is possible. In this section, we show that, for well-behaved mereotopologies over $\mathbb{S}^2$, the answer is positive.

We assume familiarity with basic geometric topology in the plane; for details, see Newman, 1964. Recall in this context that a Jordan arc in a topological space $X$ is a homeomorphism from the unit interval $[0, 1]$ into $X$, and a Jordan curve in $X$, a homeomorphism from the unit circle $S^1$ into $X$. The Jordan curve, Theorem states that the locus of a Jordan curve in $\mathbb{R}^2$ separates $\mathbb{R}^2$ into two residual domains, exactly one of which is bounded. If we regard $S^1$ as the intersection of the plane $x_1 = 0$ with $\mathbb{S}^2$, the Schönflies Theorem states that a Jordan curve $\gamma : S^1 \to \mathbb{S}^2$ may be extended to a homeomorphism $S^2 \to \mathbb{S}^2$. Thus, if $\gamma$ is a Jordan curve in $\mathbb{S}^2$, the residual domains of $|\gamma|$ are 2-cells in $\mathbb{S}^2$; and if $\gamma$ is a Jordan curve in $\mathbb{R}^2$, the bounded residual domain of $\gamma$ is a 2-cell in $\mathbb{R}^2$.

The following concepts are important in understanding the good behaviour of the mereotopologies $\text{ROS}(\mathbb{R}^2)$, $\text{ROP}(\mathbb{R}^2)$ and $\text{ROQ}(\mathbb{R}^2)$.

**Definition 1.41** Let $X$ be a topological space, $u \subseteq X$ and $p, q \in F(u)$. An end-cut to $p$ in $u$ is a Jordan arc in $X$ such that $f(1) = p$ and $f([0, 1]) \subseteq u$. Likewise, a cross-cut from $p$ to $q$ in $u$ is a Jordan arc in $X$ such that $f(0) = p$, $f(1) = q$ and $f([0, 1]) \subseteq u$. Let $M$ be a mereotopology over $X$. We say that $M$ has curve-selection if, for all $r \in M$ and all $p \in F(r)$, there exists an end-cut in $r$ to $p$.

The existence of end-cuts is by no means a universal property of regular open sets in $\mathbb{R}^n$ (for $n > 1$). However, the regions in $\text{ROS}(\mathbb{R}^2)$, $\text{ROP}(\mathbb{R}^2)$ and $\text{ROQ}(\mathbb{R}^2)$ are well-behaved in this regard, as the following results show.

**Lemma 1.42** Let $r \in \text{ROP}(\mathbb{R}^n)$ and $p \in r^-$. Then there exists a linear function $f : [0, 1] \to \mathbb{R}^n$ such that $f(1) = p$ and $f([0, 1]) \subseteq r$. If $p$ has rational coordinates, we may choose $f$ so that it has parameters from $\mathbb{Q}$.

**Proof** The proposition holds for basic polytopes because their closures are convex. It holds for all polytopes because if $r = r_1 + \cdots r_n$, $r^- = r_1^- \cup \cdots \cup r_n^-$ by Lemma 1.4 (iii). \qed

The semi-algebraic case is much more involved. However, we have the following Theorem (van den Dries, 1998, Ch. 6, Corollary 1.5; Bodnack et al., 1998 Theorem 2.5.5).

**Proposition 1.43** (Curve-selection lemma) Let $S$ be a semi-algebraic subset of $\mathbb{R}^n$ and $p \in S^-$. Then there exists a continuous semi-algebraic function $f : [0, 1] \to \mathbb{R}^n$ such that $f(1) = p$ and $f([0, 1]) \subseteq S$. 


Thus, the mereotopologies \( \text{ROS}(\mathbb{R}^2) \), \( \text{ROP}(\mathbb{R}^2) \) and \( \text{ROQ}(\mathbb{R}^2) \) all certainly have curve-selection. Moreover, by making only minor modifications to the relevant arguments, it can be shown that \( \text{ROS}(\mathbb{S}^2) \), \( \text{ROP}(\mathbb{S}^2) \) and \( \text{ROQ}(\mathbb{S}^2) \) all have curve-selection too.

With these preliminaries behind us, we can turn to the expressive power of \( L_{c,\leq} \). We note in passing that, since \( \leq \) is a primitive of \( L_{c,\leq} \), we may write the Boolean operators and constants \( +, \cdot, - \), 0 and 1 in \( L_{c,\leq} \)-formulas, assuming them to be replaced by their usual definitions. In mereotopologies over the closed plane having curve-selection, we can express the property of being a 2-cell using an \( L_{c,\leq} \)-formula. To see this, we recall that the Jordan Curve Theorem has the following converse (see Newman, 1964 Chapter VI, Theorem 16.1).

**Proposition 1.44 (Converse of Jordan’s Theorem)** Let \( d \) be a closed subset of \( \mathbb{S}^2 \) such that \( \mathbb{S}^2 \setminus d \) has two components, and suppose that, for each \( p \in d \), and each component \( o \) of \( \mathbb{S}^2 \setminus d \), there is an end-cut to \( p \) in \( o \). Then \( d \) is the locus of a Jordan curve.

Then we have:

**Lemma 1.45** Let \( M \) be any mereotopology over \( \mathbb{S}^2 \) having curve-selection. Then, for all \( r \in M \), \( r \) is a 2-cell if and only if \( r \) is non-zero and connected with non-zero connected complement—that is, if and only if \( M \models \psi_1[r] \), where \( \psi_1(x) \) is the \( L_{c,\leq} \)-formula

\[
c(x) \land x > 0 \land c(-x) \land -x > 0.
\]

**Proof** If \( M \models \psi_1[r] \), then \( d = \mathcal{F}(r) \) satisfies the conditions of Proposition 1.44, since \( M \) has curve-selection. The other direction is immediate. \( \square \)

Furthermore:

**Lemma 1.46** Let \( M \) be a mereotopology over \( \mathbb{R}^2 \) having curve-selection and also satisfying the conditions of Lemma 1.30. Then \( r \in M \) is a 2-cell if and only if \( r \) satisfies the \( L_C \)-formula

\[
\phi_c(x) \land x > 0 \land \phi_{c_2}(-x) \land -x > 0 \land \phi_{c_3}(x),
\]

where \( \phi_c(x) \) and \( \phi_{c_2}(x) \) are as defined in Lemmas 1.27 and 1.30, respectively.

**Proof** If \( r \) satisfies the formula, then the bounded set \( \mathcal{F}(r) \) is the locus of a Jordan curve in \( \mathbb{S}^2 \) and hence in \( \mathbb{R}^2 \) by the same reasoning as for
Lemma 1.45, and since $r$ is the bounded residual domain of this set, it is a 2-cell. The other direction is again immediate. \qedsymbol

We now proceed to a direct comparison between $L_{C, \leq}$ and $L_C$. Proposition 1.28 has a closed-plane variant, in which the condition that one of $d_1$ and $d_2$ is bounded may be dropped.

**Proposition 1.47** Let $d_1$ and $d_2$ be closed sets in $\mathbb{S}^2$. If $\mathbb{S}^2 \setminus d_1$, $\mathbb{S}^2 \setminus d_2$ and $d_1 \cap d_2$ are all connected, then so is $\mathbb{S}^2 \setminus (d_1 \cup d_2)$.

This leads to a closed-plane variant of Lemma 1.29:

**Lemma 1.48** Let $s_1, s_2, t \in \text{RO}(\mathbb{S}^2)$ such that: (i) $-(s_1 + t), -(s_2 + t)$ and $t$ are all connected; and (ii) $s_1^- \cap s_2^- = \emptyset$. Then $-(s_1 + s_2 + t)$ is also connected.

**Proof** As for Lemma 1.29, using Proposition 1.47 in place of Proposition 1.28. \qedsymbol

**Lemma 1.49** Let $M$ be any finitely decomposable mereotopology over $\mathbb{S}^2$ having curve-selection, let $\psi_{ab}(y_1, y_2)$ be the $L_{C, \leq}$-formula

$$\exists z(c(-(y_1 + z)) \wedge c(-(y_2 + z)) \wedge c(z) \wedge \neg c(-(y_1 + y_2 + z))),$$

and let $\psi_C(x_1, x_2)$ be the $L_{C, \leq}$-formula

$$\exists y_1 \exists y_2 (y_1 \leq x_1 \wedge y_2 \leq x_2 \wedge \psi_{ab}(y_1, y_2)).$$

Then, for all $r_1, r_2 \in M$, $r_1^- \cap r_2^- \neq \emptyset$ if and only if $M \models \psi_C[r_1, r_2]$.

**Proof** The if-direction follows from Lemma 1.48. The only-if direction is left as a (fiddly) exercise. \qedsymbol

Putting together Lemmas 1.22, 1.27 and 1.49, we see that $L_{C, \leq}$ is exactly as expressive as $L_C$ in well-behaved mereotopologies over the closed plane $\mathbb{S}^2$.

As a final example of the expressiveness of the language $L_C$, we observe that it can distinguish between $\mathbb{R}^2$ and its 1-point compactification.

**Theorem 1.50** Let $M$ be any of ROS($\mathbb{R}^2$), ROP($\mathbb{R}^2$) or ROQ($\mathbb{R}^2$). Then $M \not\models C \hat{M}$.

**Proof** Recall the $L_C$-formula $\phi_{b^2}(x)$ defined in Lemma 1.30, and expressing the property of being bounded over $M$. Evidently, $M \not\models \forall x \phi_{b^2}(x)$. But it is an easy consequence of Lemma 1.48 that $M \models \forall x \phi_{b^2}(x)$. \qedsymbol

Theorem 1.50 stands in sharp contrast to the situation with the signature $\{e, \leq\}$ reported in Lemma 1.38.
4. Expressiveness of first-order languages in plane mereotopologies

In the previous section, we examined the relative expressive power of the languages $L_C$ and $L_{C,<}$ for various mereotopologies, in particular those defined over $\mathbb{R}^2$ and $\mathbb{S}^2$. This section characterizes that expressive power in a more ‘absolute’ way. We employ the following terminology:

**Definition 1.5.1** Let $X$ be a topological space and let $\bar{u} = u_1, \ldots, u_n$, $\bar{v} = v_1, \ldots, v_n$ be $n$-tuples of subsets of $X$. We say that $\bar{u}$ and $\bar{v}$ are similarly situated (in $X$), and write $\bar{u} \sim_X \bar{v}$, if there is a homeomorphism of $X$ onto itself mapping $\bar{u}$ to $\bar{v}$. If $X$ is clear from context, we omit reference to it, and simply write $\bar{u} \sim \bar{v}$. Now let $M$ be a mereotopology over $X$ and $\Sigma$ a signature of topological primitives. For any $L_\Sigma$-formula $\phi$ with free-variables $\bar{x}$, we say that $\phi$ is topologically complete (in $M$ over $X$) if any pair of tuples of the appropriate arity satisfying $\phi(\bar{x})$ in $M$ are similarly situated in $X$.

Readers familiar with basic geometric topology will recognize that the mereotopologies ROS($\mathbb{S}^2$), ROP($\mathbb{S}^2$) and ROQ($\mathbb{S}^2$) are all (finitely) ‘triangular’ (in the sense of van den Dries, 1998). Moreover, the observations of Section 3.2 strongly suggest that triangulations in these mereotopologies can be combinatorially described using first-order formulas with $C$ as their only primitive. And since combinatorially isomorphic triangulations are similarly situated, it should be entirely unsurprising that every tuple in these mereotopologies satisfies a topologically complete $L_C$-formula (and hence also a topologically complete $L_{C,<}$-formula). That is: every tuple of regions in any of the mereotopologies ROS($\mathbb{S}^2$), ROP($\mathbb{S}^2$) and ROQ($\mathbb{S}^2$) can be completely topologically described by an $L_C$-formula (or by an $L_{C,<}$-formula). Results of this general kind were proved, independently, by Kuijpers et al., 1995, Papadimitriou et al., 1999 and Pratt and Schoop, 2000, by a variety of methods. Our objective here is a systematic and general investigation of this topic, using an approach which will prove useful in Sections 5 and 7.

4.1 Connected partitions

We have seen that, given a collection $\Sigma$ of topological primitives, any mereotopology can be regarded as a $\Sigma$-structure by interpreting the symbols in $\Sigma$ in the standard way. And the question then naturally arises as to whether we can obtain a converse to this observation. That is: under what conditions is a given $\Sigma$-structure isomorphic to some mereotopology—or perhaps, to some mereotopology belonging to a certain class? Since this question will preoccupy us in the sequel, some of
the results below will be presented at a higher level of generality than their immediate applications warrant.

Accordingly, throughout Sections 4.1 and 4.2, \( \mathfrak{A} \) shall denote an arbitrary structure interpreting the signature \( \{ 0, 1, +, \cdot, -, c \} \), such that the reduct of \( \mathfrak{A} \) to the signature \( \{ 0, 1, +, \cdot, - \} \), is a Boolean algebra. To avoid notational clutter, if \( a, b \in A \), we write \( 0, -a, a + b \) etc., rather than the more correct \( \overline{0}^\mathfrak{A}, -^\mathfrak{A}(a), +^\mathfrak{A}(a, b) \) etc. In addition, abusing terminology slightly, we call an element \( a \in A \) connected if \( \mathfrak{A} \models c[a] \); and we say that \( \mathfrak{A} \) is \textit{finitely decomposable} if, for every \( a \in A \), there exist connected elements \( a_1, \ldots, a_n \) of \( \mathfrak{A} \) such that \( a = a_1 + \cdots + a_n \). Of course, in case \( \mathfrak{A} \) is a meretopology \( M \), this usage is consistent with that adopted above. As usual in the context of Boolean algebras, we take a \textit{partition} in \( \mathfrak{A} \) to be a tuple of non-zero, pairwise disjoint elements summing to 1. If \( \bar{a} \) is any tuple from \( \mathfrak{A} \) (not necessarily a partition), and \( \bar{b} \) a partition in \( \mathfrak{A} \), we say that \( \bar{a} \) can be refined to \( \bar{b} \) if every element of \( \bar{a} \) can be written as the sum of (zero or more) elements of \( \bar{b} \).

**Definition 1.52** A partition \( \bar{a} = a_1, \ldots, a_n \) in \( \mathfrak{A} \) such that \( a_i \) is connected for all \( i \) (\( 1 \leq i \leq n \)) is called a connected partition.

Let \( \psi_{\text{con}} \) denote the \( L_{c, \leq} \)-sentence

\[
\forall x \forall y (c(x) \land c(y) \land x \cdot y \neq 0 \rightarrow c(x + y)).
\]

Thus, \( \psi_{\text{con}} \) 'says' that the sum of two overlapping connected regions is connected.

**Lemma 1.53** Let \( M \) any meretopology. Then \( M \models \psi_{\text{con}} \).

**Proof** A restatement of Lemma 1.4 (iv). \( \text{QED} \)

**Claim 1.54** Suppose \( \mathfrak{A} \) is finitely decomposable, and \( \mathfrak{A} \models \psi_{\text{con}} \). Then every tuple in \( \mathfrak{A} \) can be refined to a connected partition.

**Proof** Given elements \( a_1, \ldots, a_n \), collect all the non-zero products \( b_1, \ldots, b_N \) of the form: \( \pm a_1 \cdot \cdots \cdot \pm a_n \). For each \( j \) (\( 1 \leq j \leq N \)), let \( b_{j,1}, \ldots, b_{j,N_j} \) be connected elements of \( \mathfrak{A} \) summing to \( b_j \). If, for any two of these elements, say \( b_{jk} \) and \( b_{j'k} \), we have \( b_{jk} \cdot b_{j'k} > 0 \), then we can replace them by their sum \( b_{jk} + b_{j'k} \) (which is connected, because \( M \models \psi_{\text{con}} \)). Proceeding in this way, we obtain the desired refinement. \( \text{QED} \)

Note that, in particular, every tuple in any finitely decomposable meretopology can be refined to a connected partition.
Let us restrict attention now to finitely decomposable merotopologies over $\mathbb{S}^2$ having curve-selection.

**Lemma 1.55** Let $M$ be a merotopology over $\mathbb{R}^2$ or $\mathbb{S}^2$ having curve-selection. If $r_1, r_2$ and $r_3$ are pairwise disjoint, connected elements of $M$, then there exist at most two points lying on the frontiers of all three regions.

**Proof** We suppose that $p_1, p_2$ and $p_3$ are distinct points all lying on the frontiers of $r_1, r_2$ and $r_3$ and derive a contradiction. Choose points $q_1, q_2, q_3$ such that $q_i \in r_i$ ($i = 1, 2, 3$). By curve-selection, draw three end-cuts in $r_i$, say $\gamma_{i,1}, \gamma_{i,2}$ and $\gamma_{i,3}$ from $q_i$ to $p_1, p_2$ and $p_3$, respectively. It is easy to see that, within each $r_j$ ($1 \leq j \leq 3$), the $\gamma_{i,j}$ can be chosen so that they intersect only at $q_i$. But since the $r_j$ are disjoint, each $\gamma_{i,j}$ intersects any other $\gamma_{i',j'}$ only in $p_i$ or $q_i$. And it is well known that this is impossible (see the right-hand graph in Fig. 1.11). Q.E.D.

For $n > 2$, let $\psi^n_{\text{sum}}$ denote the $L_{c,\leq}$-formula

$$\forall x_1 \ldots \forall x_n \left(c(x_1 + \cdots + x_n) \land \bigwedge_{1 \leq i \leq n} c(x_i) \rightarrow \bigvee_{2 \leq i \leq n} c(x_1 + x_i)\right).$$

(The formula $\psi_{\text{sum}}$ of Example 1.18 is just $\psi^3_{\text{sum}}$.) Thus, $\psi^n_{\text{sum}}$ ‘says’ that, if $n$ connected regions have a connected sum, the first must form a connected sum with at least one of the others.

**Lemma 1.56** Let $M$ be a finitely decomposable merotopology over $\mathbb{S}^2$ having curve-selection. Then $M \models \psi^n_{\text{sum}}$ for all $n > 1$.

**Proof** Let $r_1, \ldots, r_n$ be connected with $r_1 + \cdots + r_n$ also connected. Assume first that the $r_i$ are pairwise disjoint. Let $p \in r_1$ and $q \in r_2 + \cdots + r_n$. By the connectedness of $r_1 + \cdots + r_n$, draw a Jordan arc $\gamma$ from $p$ to $q$ lying within $r_1 + \cdots + r_n$. By Lemma 1.55, only finitely many points can lie on the frontiers of more than two of the $r_i$, and we may certainly ensure that $\gamma$ avoids all such points. By renumbering if necessary, we may assume that $\gamma$ visits a point $p \in r_1 \cap r_2 \cap (r_1 + \cdots + r_n)$. But by the construction of $\gamma$, $p \notin r_i$ for all $i > 2$, whence $p \in -r_i$ for all such $r$. Therefore, $p \in r_1 \cap r_2 \cap (r_1 + r_2)$, whence $r_1 + r_2$ is connected.

Finally, we relax the assumption that the $r_i$ are pairwise disjoint. Since $M$ is finitely decomposable, we have that each element of $\bar{r}$ is the sum of zero or more members of a tuple $\bar{s}$ of pairwise disjoint connected elements with the same sum. The result then follows easily by repeated applications of Lemma 1.53. Q.E.D.

In the sequel, we abbreviate the formula

$$x_1 + x_2 = x \land x_1 > 0 \land x_2 > 0 \land x_1 \cdot x_2 = 0 \land c(x_1) \land c(x_2)$$
by $x_1 \oplus x_2 = x$; thus, $x_1 \oplus x_2 = x$ ‘says’ that $x$ can be partitioned into non-empty, disjoint connected regions $x_1$ and $x_2$. Now let $\psi_{\text{break}}$ denote the $L_{CL}$-formula

$$
\forall x \forall y_1 \forall y_2 \left( (c(x) \land c(y_1) \land c(y_2) \land c(x + y_1) \land c(x + y_2) \land x \cdot y_1 = 0 \land x \cdot y_2 = 0 \land x \neq 0) \rightarrow \exists x_1 \exists x_2 (x_1 \oplus x_2 = x \land c(x_1 + y_1) \land c(x_1 + y_2) \land c(x_2 + y_1) \land c(x_2 + y_2)) \right).
$$

Thus, $\psi_{\text{break}}$ ‘says’ that, if $r$, $s_1$, $s_2$ are connected regions such that $r$ is non-zero, disjoint from $s_1$ and $s_2$, and forms a connected sum with both $s_1$ and $s_2$, then $r$ can be partitioned into connected, non-zero regions $r_1$, $r_2$ such that each of $r_1$ and $r_2$ forms a connected sum with each of $s_1$ and $s_2$. Fig. 1.10a illustrates this configuration; note that $\neg r$ need not be connected.

**Lemma 1.57** Let $M$ be a finitely decomposable mereotopology over $\mathbb{S}^2$ having curve-selection. Then $M \models \psi_{\text{break}}$.

**Proof** Let $r$, $s_1$, $s_2$ be as above. We may assume that $s_1$ and $s_2$ are nonzero, since otherwise, similar or easier arguments apply. Refer to Fig. 1.10b. For $i = 1, 2$, since $r + s_i$ is connected, by Lemma 1.4 (ii), $r^- \cap s_i^- \cap (r + s_i) \neq \emptyset$. In fact, since the removal of finitely many points
from the open set \( r+s_i \) does not disconnect it, we can choose four distinct
points \( p_i, q_i \ (i = 1, 2) \) such that \( p_i, q_i \in r^- \cap s_i^- \cap (r+s_i) \). Since \( M \) has
curve-selection and \( r \) is connected it is easy to see that, by exchanging
\( q_1 \) and \( q_2 \) if necessary, we can draw cross-cuts \( \gamma \) from \( p_1 \) to \( p_2 \) and \( \delta \) from
\( q_1 \) to \( q_2 \) such that \( |\gamma| \) and \( |\delta| \) are disjoint. Since \( S^2 \) is normal and \( M \) a
mereotopology, we can cover \( |\delta| \) with elements of \( M \) whose closures are
disjoint from \( |\gamma| \). By compactness of \( |\delta| \), this cover has a finite subcover,
t_1, \ldots, t_N, \) say. Let \( t = r \cdot (t_1 + \cdots + t_N) \); evidently, \( q_1 \) and \( q_2 \) lie on
the frontier of the same component \( t' \) of \( t \). Likewise, \( p_1 \) and \( p_2 \) lie on
the frontier of the same component of \( r - t' \); call this component \( r_1 \in M \),
and let \( r_2 = r - r_1 \). It is easy to check that \( r_1 \) and \( r_2 \) have the required
properties. \( \Box \)

4.2 Neighbourhood graphs

As before, \( \mathcal{A} \) shall denote an arbitrary structure interpreting the
signature \( \{ 0, 1, +, \cdot, -, c \} \), such that the reduct of \( \mathcal{A} \) to \( \{ 0, 1, +, \cdot, -, c \} \), is a
Boolean algebra. Recall the notion of connected partition introduced in
Definition 1.52.

**Definition 1.58** Let \( \bar{a} = a_1, \ldots, a_n \) be a connected partition in \( \mathcal{A} \). A
connected partition \( \bar{a} \) is called a \( c^1 \)-partition if, for every \( I \subseteq \{ 1, \ldots, n \} \)
such that \( |I| < h \), the element \( (\sum_{i \in I} a_i) \) is connected.

If \( \mathcal{A} \models c(1) \), then a \( c^1 \)-partition in \( \mathcal{A} \) is the same thing as a connected
partition. Furthermore, if \( \mathcal{A} \) is in fact a mereotopology over \( S^2 \) having
curve-selection, then, by Lemma 1.45, a \( c^2 \)-partition in \( \mathcal{A} \) is the same
thing as a partition consisting entirely of 2-cells. It is \( c^3 \)-partitions,
however, that will mainly preoccupy us in the sequel.

We assume familiarity with basic graph theory: for details, see Diestel,
1991 Chapter 1. Recall in this context that a graph is a pair \( G = (V, E) \)
where \( V \) is a set (called vertices) and \( E \) is a set of 2-element subsets of
\( V \) (called edges). We denote \( V \) by \( V(G) \) and \( E \) by \( E(G) \). Note that,
on this definition, graphs have no ‘loops’ or ‘multiple edges’. If \( G \) is a
graph and \( U \) is a proper subset of \( V(G) \), we denote by \( G \setminus U \) the result of
deleting all the nodes in \( U \) from \( G \); and if \( e = (v, v') \in E(G) \), we denote
by the \( G/e \) the result of contracting \( G \) by merging \( v \) and \( v' \) into a single
(new) node \( v'' \), such that \((v'', w)\) is an edge of \( G/e \) just in case either
\((v, w)\) or \((v', w)\) is an edge of \( G \). If a graph \( H \) can be obtained from \( G \) by
a sequence of deletions and contractions, then \( H \) is said to be a minor
of \( G \). Finally we take the terms path, cycle, connected, component to be
defined in the standard way. In particular, recall that, for \( h > 0 \), \( G \) is said
to be \( h \)-connected if \( G \setminus U \) is connected for every \( U \subseteq G \) such that
\(|U| < h \).
Definition 1.59 Let $\vec{a} = a_1, \ldots, a_n$ be a tuple from $\mathcal{A}$. If $a_i + a_j$ is connected for $1 \leq i < j \leq n$, we say that $a_i$ and $a_j$ are neighbours. The neighbourhood, graph of $\vec{a}$, denoted $N_\vec{a}$, is the graph with nodes $\{a_1, \ldots, a_n\}$ and edges $\{(a_i, a_j) \mid a_i$ and $a_j$ are neighbours}\).

Claim 1.60 Suppose $\mathcal{A} \models \psi_{\text{con}}$ and $\mathcal{A} \models \psi_{\text{sum}}^n$ for all $n > 2$. Let $\vec{a} = a_1, \ldots, a_n$ be a tuple of connected elements of $\mathcal{A}$, such that $a_{n-1} + a_n$ is connected. Let $\vec{a}' = a_1, \ldots, a_{n-2}, (a_{n-1} + a_n)$. Then $N_{\vec{a}'} = N_\vec{a}/(n-1, n)$.

Proof For $1 \leq j < n-1$, $a_j + (a_{n-1} + a_n)$ is connected if and only if $a_j + a_{n-1}$ is connected or $a_j + a_n$ is connected. \(\Box\)

Claim 1.61 Suppose $\mathcal{A} \models \psi_{\text{con}}$ and $\mathcal{A} \models \psi_{\text{sum}}^n$ for all $n > 2$. Let $\vec{a} = a_1, \ldots, a_n$ be a tuple of connected elements of $\mathcal{A}$, with $a = a_1 + \ldots + a_n$. Then $a$ is connected if and only if $N_\vec{a}$ is a connected graph.

Proof The if-direction follows easily from the fact that $\mathcal{A} \models \psi_{\text{con}}$. For the only-if direction, note that the claim is trivial if $n = 1$, so assume $n > 1$, and that the claim holds for tuples of fewer than $n$ elements. Since $\mathcal{A} \models \psi_{\text{sum}}$ there exists $i$ ($1 \leq i < n$) such that $a_i$ and $a_n$ are neighbours. By renumbering if necessary, assume $i = n-1$, and let $\vec{a}'$ be as in Claim 1.60, so that $N_{\vec{a}'} = N_\vec{a}/(a_{n-1}, a_n)$. But $N_{\vec{a}'}$ is connected by inductive hypothesis, whence $N_\vec{a}$ is connected too. \(\Box\)

Claim 1.62 Suppose $\mathcal{A} \models \psi_{\text{con}}$ and $\mathcal{A} \models \psi_{\text{sum}}^n$ for all $n > 2$. Let $\vec{a}$ be a connected partition in $\mathcal{A}$, and let $h \geq 1$. Then $\vec{a}$ is a $c^h$-partition if and only if $N_\vec{a}$ is an $h$-connected graph.

Proof Immediate by Claim 1.61. \(\Box\)

Claim 1.63 Suppose $\mathcal{A} \models c(1)$, $\mathcal{A} \models \psi_{\text{con}}$, $\mathcal{A} \models \psi_{\text{sum}}^n$ for all $n > 2$, and $\mathcal{A} \models \psi_{\text{break}}$. Then every connected partition in $\mathcal{A}$ can be refined to a $c^3$-partition.

Proof We make free use of Claim 1.61. Let $\vec{a}$ be a connected partition. We show first that $\vec{a}$ can be refined to a $c^2$-partition. Choose an element $a$ of $\vec{a}$ such that the number $k$ of components of the graph $N_\vec{a} \setminus \{a\}$ is maximal. And let there be $m > 0$ elements $a$ for which this maximum value is attained. If $\vec{a}$ is not already a $c^2$-partition, then $k > 1$. Let $H_1, H_2$ be distinct components of $N_\vec{a} \setminus \{a\}$. Since $N_\vec{a}$ is connected, there exist $b_1 \in H_1, b_2 \in H_2$ such that $a + b_1$ and $a + b_2$ are connected.
Since $\mathcal{A} \models \psi_{\mathrm{break}}$, let $a_1, a_2$ be non-empty, connected, disjoint elements summing to $a$ with $a_1 + b_1$, $a_1 + b_2$, $a_2 + b_1$ and $a_2 + b_2$ all connected; and let $\vec{b}$ be the connected partition which results from replacing $a$ by $a_1$ and $a_2$. Evidently, for $i = 1, 2$, $N_{\vec{a}} \setminus \{ a_i \}$ has strictly fewer than $k$ components. That is, the number of elements $b$ in $\vec{b}$ such that $N_{\vec{a}} \setminus \{ b \}$ has $k$ components is strictly less than $m$. Proceeding in this way, we eventually obtain a $c^2$-partition.

Now let $\vec{a}$ be a $c^2$-partition. We show that $\vec{a}$ can be refined to a $c^3$-partition. If $\vec{a}$ is not a $c^3$-partition, choose a pair of distinct elements $a$ and $a'$ such that the number $k$ of components of the graph $N_{\vec{a}} \setminus \{ a, a' \}$ is maximal; and let there be $m > 0$ unordered pairs $(a, a')$ for which this maximum value is attained. Let $H_1, H_2$ be distinct components of $N_{\vec{a}} \setminus \{ a, a' \}$. Since $\vec{a}$ is a $c^2$-partition, there exist $b_1 \in H_1, b_2 \in H_2$ such that $a + b_1$ and $a + b_2$ are connected. And since $\mathcal{A} \models \psi_{\mathrm{break}}$, let $a_1, a_2$ be non-empty, connected, disjoint elements summing to $a$ with $a_1 + b_1$, $a_1 + b_2$, $a_2 + b_1$ and $a_2 + b_2$ all connected; and let $\vec{b}$ be the connected partition which results from replacing $a$ by $a_1$ and $a_2$. Evidently, for $i = 1, 2$, $N_{\vec{a}} \setminus \{ a, a' \}$ has strictly fewer than $k$ components. Moreover, suppose $a''$ is any other element of $\vec{a}$ (distinct from $a$ and $a'$) such that $N_{\vec{a}} \setminus \{ a, a', a'' \}$ also has $k$ components. We claim that $N_{\vec{a}} \setminus \{ a_1, a'' \}$ and $N_{\vec{a}} \setminus \{ a_2, a'' \}$ cannot both have $k$ components. Working for the moment on this assumption, we see that the number of pairs $b, b'$ in $\vec{b}$ such that $N_{\vec{a}} \setminus \{ b, b' \}$ has $k$ components is strictly less than $m$. Proceeding in this way, we eventually obtain a $c^3$-partition.

It remains only to verify that the graphs $N_{\vec{a}} \setminus \{ a_1, a'' \}$ and $N_{\vec{a}} \setminus \{ a_2, a'' \}$ encountered above do not both have $k$ components. If $a \in A$ and $B \subseteq A$, let us say that $a$ is a neighbour of $B$ if $a$ is a neighbour of some element of $B$. Let the components of $N_{\vec{a}} \setminus \{ a, a'' \}$ be $H_1, \ldots, H_k$. Since $\vec{b}$ is a $c^2$-partition, we have that, for all $i$ ($1 \leq i \leq k$), $a$ is a neighbour of $H_i$, and therefore either $a_1$ or $a_2$ is a neighbour of $H_i$. Hence, we can reorder the $H_i$ if necessary so that, for some $p, q$ with $0 \leq p < q \leq k + 1$, $a_1$ is a neighbour of $H_i$ if and only if $i < q$ and $a_2$ is a neighbour of $H_i$ if and only if $p < i$. Thus, the components of $N_{\vec{a}} \setminus \{ a_1, a'' \}$ are $H_1, \ldots, H_p, (\{ a_2 \} \cup H_{p+1} \cup \cdots \cup H_k)$, and the components of $N_{\vec{a}} \setminus \{ a_2, a'' \}$ are $(\{ a_1 \} \cup H_1 \cup \cdots \cup H_{q-1}), H_q, \ldots, H_k$. If these number $k$ in each case, we have $p = k - 1$ and $q = 2$. But $a'$ lies in one of the $H_i$, and $a_1$ and $a_2$ were chosen so that they are both neighbours of this $a'$. Hence $a_1$ and $a_2$ are both neighbours of $H_i$, whence $p < q - 1$. This yields $k \leq 1$, contradicting our assumption that $\vec{a}$ is not a $c^3$-partition.

We finish with a technical result which will be required later.
First-Order Mereotopology

Definition 1.64 If $\tilde{a} = a_1, \ldots, a_N$ is a connected partition in $\mathcal{A}$ such that, for any neighbour $a_j$ of $a_i$, $-(a_i + a_j)$ is connected, we say that $\tilde{a}$ is radial about $a_i$.

Note incidentally that a $c^2$-partition is radial about each of its members.

Claim 1.65 Suppose $\mathcal{A} \models c(1)$, $\mathcal{A} \models \psi_{\text{con}}$, $\mathcal{A} \models \psi_{\text{sum}}$ for all $n > 2$, and $\mathcal{A} \models \psi_{\text{break}}$. Let $n > 1$ and let $\tilde{a} = a_1, \ldots, a_n$ be a connected partition in $\mathcal{A}$ with $-a_1$ connected. Then $\tilde{a}$ can be refined to a $c^2$-partition $a_1, b_2, \ldots, b_N$, radial about $a_1$, in which $a_1$ has at least three neighbours.

Proof Similar to the above. QED

We conclude with a further corollary of Claim 1.61. We employ the following fact from graph theory, whose proof we leave to the reader.

Proposition 1.66 If $G$ is a finite 2-connected graph of order at least 2, and $v \in V(G)$, then there exists a $w \in V(G)$ such that $\{v, w\} \in E(G)$, and the removal of both $v$ and $w$ from $G$ leaves a connected graph.

Corollary 1.67 Let $M$ be a finitely decomposable mereotopology over $\mathbb{S}^2$ having curve-selection, and let $\bar{r} = r_1, \ldots, r_n$ be a partition in $M$ consisting entirely of 2-cells. Then, by re-numbering if necessary, we have, for all $k$ ($1 \leq k < n$), $r_1 + \cdots + r_k$ is a 2-cell.

That is: partitions of the closed plane into 2-balls are always `shellable'. The analogous result for three-dimensional space fails (Rudin, 1958).

4.3 Partition Graphs

We now prove that, if $\bar{r}$ is a $c^2$-partition in a finitely decomposable mereotopology over $\mathbb{S}^2$ having curve-selection, then the neighbourhood graph of $\bar{r}$ fixes its topological properties completely.

We assume familiarity with the basic theory of plane graphs; for details, see Diestel, 1991 Chapter 4. In this context, suppose that $e \subseteq \mathbb{S}^2$ is the locus of a Jordan arc. Then $e$ has two endpoints; all other points are called interior points, and we denote the set of these interior points by $(e)$. (Of course, $e$ is not the topological interior of the set $e$ in $\mathbb{S}^2$; but no confusion should arise in this regard.) A plane graph is a pair $G = (V, E)$, where $V$ is a finite subset of $\mathbb{S}^2$ and $E$ is a collection of sets $e \subseteq \mathbb{S}^2$ such that $e$ is the locus of a Jordan arc, satisfying the following conditions for all $v \in V$ and all $e, e' \in E$:

1. if $e \in E$ and $p$ is an endpoint of $e$, then $p \in V$;
2. $v \neq (e)$, and if $e \neq e'$ then $(e) \cap (e') = \emptyset$;
3 if \( e \neq e' \), then \( e \) and \( e' \) do not join the same pair of endpoints.

The elements of \( V \) are called *vertices* of \( G \), and the elements of \( E \), the *edges* of \( G \); an edge \( e \in E \) is said to *join* the vertices at its endpoints. We denote \( V \) by \( V(G) \), \( E \) by \( E(G) \) and \( V \cup E \) by \( |G| \). The components of \( \mathbb{S}^2 \setminus |G| \) are called the *faces* of \( G \), and we denote the set of these faces by \( F(G) \). A plane graph is *semi-algebraic* if its edges are the loci of semi-algebraic Jordan arcs; similarly for the terms *piecewise linear* and *rational piecewise linear*. Notice that, on our definition, plane graphs have no 'loops' or 'multiple edges'. (Some authors prefer the term *simple graph.*) A plane graph will be regarded as an abstract graph in the obvious way, and we carry over notation and terminology accordingly. Conversely, if \( G = (V, E) \) is an abstract graph, a *drawing* of \( G \) is a plane graph \( G' = (V', E') \) for which there exists a function \( \epsilon \) mapping \( V \) 1–1 onto \( V' \) and \( E \) 1–1 onto \( E' \) such that for all \((v, v') \in E\), \( \epsilon((v, v')) \) joins \( \epsilon(v) \) and \( \epsilon(v') \). We call \( \epsilon \) an *embedding*. If \( G \) has a drawing, \( G \) is *planar*. Not all abstract graphs are planar, of course: the graphs \( K^5 \) and \( K_{3,3} \) illustrated in Fig. 1.11 are familiar non-planar graphs. Indeed, this fact has a converse:

**Proposition 1.68 (Kuratowski, Wagner)** A graph is planar if and only if it has no minor isomorphic to either \( K^5 \) or \( K_{3,3} \).

We further assume familiarity with the notion of *duality* for plane graphs. Let \( G \) and \( G' \) be plane graphs. We say that \( G' \) is a *geometrical dual* of \( G \) if there are bijections \( f_E : F(G) \to V(G') \) and \( f_E : E(G) \to E(G') \) such that, for all \( f \in F(G) \) and \( e \in E(G) \):

1. \( f_E(f) \in f' \);
2. \( f_E(e) \cap e' \) is a single point interior to both \( f_E(e) \) and \( e' \), and \( f_E(e) \cap e' = \emptyset \) for all \( e' \neq e \).
In our terminology, not every plane graph has a dual, because we do not allow graphs to contain loops or multiple edges. However, we rely below on the following sufficient condition (Wilson, 1979, p. 76).

**Proposition 1.69** Every 3-connected plane graph has a dual.

The following fact is also well-known.

**Lemma 1.70** Let $G$ and $G'$ be connected plane graphs such that $G'$ is a geometrical dual of $G$. Then there is a bijection $f_v : V(G) \rightarrow F(G')$ such that, for all $v \in V(G)$, $v \in f_v(v)$. Hence, $G$ is a dual of $G'$.

**Proof** Every face of $G'$ contains at least one vertex of $G$ by construction; it contains at most one by Euler’s formula $|F(G)| - |E(G)| + |V(G)| = 2$ applied to $G$ and $G'$.

Finally, duals are unique, in the following sense (Diestel, 1991, p. 88).

**Proposition 1.71** Let $G$ be a plane graph and let $G'$ and $G''$ be plane graphs which are both geometric duals of $G$. Then there is a homomorphism $h : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ mapping $G'$ to $G''$. In fact, $h$ can be chosen such that, for all $v \in G$, if $f'$ and $f''$ are the faces of $G'$ and $G''$, respectively, containing $v$, then $h$ maps $f'$ to $f''$.

Now let us apply these ideas to the graphs whose faces are $c^3$-partitions in well-behaved, closed-plane mereotopologies.

**Lemma 1.72** Let $X$ be a topological space, and let $r$, $s$ be disjoint elements of RO($X$) with $p \in \mathcal{F}(r) \setminus \mathcal{F}(s)$. Then $p \in \mathcal{F}(-(r + s))$.

**Proof** By Lemma 1.4 (ii), $p \not\in r + s$.

**Lemma 1.73** Let $M$ be a mereotopology over $\mathbb{S}^2$ having curve-selection, and let $\mathcal{r} = r_1, \ldots, r_n$ be a $c^3$-partition in $M$. For all $i, j$ ($1 \leq i < j \leq n$), $\mathcal{F}(r_i) \cap \mathcal{F}(r_j)$ is connected.

**Proof** We may assume that $n \geq 3$. Since $\mathcal{r}$ is certainly a $c^3$-partition, every $\mathcal{F}(r_i)$ ($1 \leq i \leq n$) is a Jordan curve by Lemma 1.45. Suppose, for contradiction, that $\mathcal{F}(r_i) \cap \mathcal{F}(r_j)$ is not connected, and let $p, q \in \mathcal{F}(r_i) \cap \mathcal{F}(r_j)$ be separated in $\mathcal{F}(r_i)$ by $\{p', q'\} \subseteq \mathcal{F}(r_i) \setminus \mathcal{F}(r_j)$. By Lemma 1.72, $p', q' \in \mathcal{F}(-(r_i + r_j))$, so that, by the connectedness of $-(r_i + r_j)$, we can draw a cross-cut $\gamma'$ (Definition 1.41) from $p'$ to $q'$ in $-(r_i + r_j) \subseteq -r_i$. By the connectedness of $r_j$, we can likewise draw a cross-cut $\gamma$ from $p$ to $q$ in $r_j \subseteq -r_i$. But $-r_i$ is a 2-cell, whence $\gamma$ and $\gamma'$ are easily seen to intersect at an interior point, which is impossible, since $r_j \cap -(r_i + r_j)$ is empty. 

**QED**
Lemma 1.74 Let $M$ be a meager topology over $\mathbb{S}^2$ having curve-selection, and let $\vec{r} = r_1, \ldots, r_n$ $(n \geq 4)$ be a $c^3$-partition in $M$. Then there exists a unique plane graph $G$ drawn in $\mathbb{S}^2$ such that the collection of sets $\{r_1, \ldots, r_n\}$ are exactly $F(G)$ and the collection of sets $\{F(r_i) \cap F(r_j) \mid 1 \leq i < j \leq n, r_i + r_j$ is connected $\}$ are exactly $E(G)$.

Proof Let $i, j, k$ be distinct integers in the range $[1, n]$. Since $\vec{r}$ is a $c^3$-partition, $r_j^+ \cup r_k^- = \mathbb{S}^2 \setminus (r_j + r_k)$ does not separate the non-empty sets $r_j$ and $-(r_i + r_j + r_k)$, whence $F(r_i) \cap (F(r_j) \cup F(r_k))$ is not the whole of the Jordan curve $F(r_i)$. And since, by Lemma 1.73, $F(r_i) \cap F(r_j)$ is a connected subset of $F(r_i)$, $F(r_i) \cap F(r_j)$ is either a point or the locus of a Jordan arc. Indeed, $F(r_i)$ must include at least three Jordan arcs of the form $F(r_i) \cap F(r_j)$ for various $j$ distinct from $i$. Let the vertices of $G$ be the end points of all Jordan arcs of the form $F(r_i) \cap F(r_j)$, and let the edges of $G$ be the segments of the various $F(r_i)$ connecting them. To show that $G$ is a plane graph, we must establish that if $F(r_i) \cap F(r_j)$ is a Jordan arc $\gamma$, then for all $k$ $(1 \leq k \leq n)$ with $k \neq i, j$, $F(r_k)$ contains no interior points of $\gamma$. For otherwise, let $p' \in F(r_k)$ be an interior point of $\gamma$, and pick any $q' \in F(r_i) \setminus F(r_j)$. Then $p' \in F(-(r_i + r_j))$ and also, by Lemma 1.72, $q' \in F(-(r_i + r_j))$. If we now choose $p$ and $q$ in $F(r_i) \cap F(r_j)$ separating $p'$ and $q'$ on the Jordan curve $F(r_i)$, the derivation of a contradiction proceeds as in Lemma 1.73. Hence no point of $F(r_k)$ is an interior point of $\gamma$, as required. Moreover, no two Jordan arcs in $E$ can have the same end-points, since $\vec{r}$ is a $c^3$-partition. It follows that $G$ is a plane graph as required. Evidently, $F(G) = \{r_1, \ldots, r_n\}$ and $E(G')$ is the collection of sets $F(r_i) \cap F(r_j)$ for $1 \leq i < j \leq n$ which are Jordan arcs.

It therefore remains only to show that $F(r_i) \cap F(r_j)$ is a Jordan arc if and only if $r_i + r_j$ is connected. Note that $r_1 \cup r_2$ is trivially not connected. By Lemma 1.4 (ii), $r_i \cup r_j \subseteq r_i + r_j \subseteq r_i \cup r_j \cup (F(r_i) \cap F(r_j))$, and the removal of a single point from a connected, open set does not render it disconnected. Hence, if $r_i + r_j$ is connected, $F(r_i) \cap F(r_j)$ is neither empty nor a singleton, and hence is a Jordan arc. Conversely, suppose $F(r_i) \cap F(r_j)$ is a Jordan arc. We have already shown that, if $p$ is an interior point of this arc, $p \notin \cup_{k \neq i, j} F_k = (\sum_{k \neq i, j} r_k)^-$. That is, $p \in -\sum_{k \neq i, j} r_k = r_i + r_j$. Hence $F(r_i) \cap F(r_j) \cap (r_i + r_j)$ is non-empty, whence $r_i + r_j$ is connected. QED

Definition 1.75 Let $M$ be a meager topology over $\mathbb{S}^2$ having curve-selection, and let $\vec{r} = r_1, \ldots, r_n$ $(n \geq 4)$ be a $c^3$-partition in $M$. We call the unique plane graph $G$ satisfying the conditions of Lemma 1.74 the the partition graph of $\vec{r}$.
Warning: the neighbourhood graph and the partition graph of a \( c^2 \)-partition are not the same sort of thing. The former is an abstract graph whose nodes are regions and whose edges are pairs of regions; the latter is a plane graph, whose nodes are points and whose edges are the loci of Jordan arcs.

**Lemma 1.76** Let \( M \) be a mereotopology over \( S^2 \) having curve-selection, let \( \bar{r} = r_1, \ldots, r_n \) (\( n \geq 4 \)) be a \( c^2 \)-partition in \( M \), and let \( G \) be its partition graph. Then there is a plane embedding \( \epsilon \) of \( N_r \) such that \( \epsilon(N_r) \) is a geometrical dual of \( G \) and, for all \( i, \ (1 \leq i \leq n) \), \( \epsilon(r_i) \in r_i \).

**Proof** Almost immediate from the definition of partition graph. \( \Box \)

From Claim 1.62, \( c^2 \)-partitions have 3-connected neighbourhood graphs. But 3-connected graphs have the crucial property that all their drawings are topologically the same.

**Proposition 1.77** (Whitney) Let \( G \) and \( G' \) be 3-connected plane graphs and \( f : G \rightarrow G' \) a graph isomorphism. Then \( f \) can be extended to a homeomorphism \( h : S^2 \rightarrow S^2 \).

Let \( M \) be a finitely decomposable mereotopology over \( S^2 \), and let \( \bar{r} = r_1, \ldots, r_n \) and \( \bar{s} = s_1, \ldots, s_n \) be \( n \)-tuples from \( M \). We are interested in the case where the mapping \( r_i \rightarrow s_i \) is a graph isomorphism from \( N_r \) to \( N_s \)—that is, where, for all \( i, j, \ (1 \leq i < j \leq n) \), \( r_i + r_j \) is connected if and only if \( s_i + s_j \) is connected. We say in this case that \( \bar{r} \) and \( \bar{s} \) have the same neighbourhood structure.

**Theorem 1.78** Let \( M \) be a finitely decomposable mereotopology over \( S^2 \) having curve-selection. Then any two \( c^2 \)-partitions in \( M \) having the same neighbourhood structure are similarly situated in \( S^2 \).

**Proof** It is straightforward to verify that, if \( n \leq 3 \), all \( n \)-element \( c^2 \)-partitions in \( M \) are similarly situated in \( S^2 \). Thus, we may assume that \( n \geq 4 \). Let \( \bar{r} = r_1, \ldots, r_n \) and \( \bar{s} = s_1, \ldots, s_n \) be \( c^2 \)-partitions with the same neighbourhood structure, and let \( G \) and \( H \) be their respective partition graphs. By Lemma 1.76, let \( G^* \) and \( H^* \) be embeddings of \( N_r \) and \( N_s \), geometrically dual to \( G \) and \( H \), respectively, let \( p_i \) be the vertex of \( G^* \) contained in \( r_i \) and let \( q_i \) be the vertex of \( H^* \) contained in \( s_i \) for all \( i, \ (1 \leq i \leq n) \). Hence, there is a graph isomorphism \( f : G^* \rightarrow H^* \) mapping \( p_i \) to \( q_i \). Since \( G^* \) and \( H^* \) are 3-connected, Proposition 1.77 guarantees that \( f \) can be extended to a homeomorphism \( h : S^2 \rightarrow S^2 \). Then \( h(G) \) and \( H \) are both geometrical duals of the plane graph \( h(G^*) = H^* \), such that, for all \( i, \ (1 \leq i \leq n) \) the faces \( h(r_i) \) and \( s_i \) contain the vertex \( h(p_i) = q_i \). By Proposition 1.71, let \( h' \) be a homeomorphism mapping
Figure 1.12. Only the left-hand graph defines a connected partition in \( \text{RO}(S^2) \).

\( h(G) \) to \( H \) such that \( h'(h(r_i)) = s_i \). Thus, \( \mathcal{F} \) and \( \mathfrak{s} \) are similarly situated.

QED

We finish this discussion of partition graphs with some ‘obvious’ lemmas concerning connected partitions in \( \text{ROP}(S^2) \) and related nceotopologies. Readers irritated by proofs of such evident truths may skip to Theorem 1.82.

**Lemma 1.79** Let \( G \) be a plane graph such that \( G \) has no isolated vertices, and every edge of \( G \) lies on the boundary of (at least) 2 faces of \( G \). Then the members of \( F(G) \) are regular open, and form a connected partition in \( \text{RO}(S^2) \). Moreover, if \( G' \) is another such plane graph, with \( |G| \subseteq |G'| \), then, for every \( f \in F(G) \), \( f = \sum \{ f' \in F(G') \mid f' \subseteq f \} \).

**Proof** Let \( G = (V,E) \), and suppose \( f \in F(G) \) and \( p \in \mathcal{F}(f) \). Since \( G \) has no isolated vertices, there exists \( e \in E \) such that \( p \in e \) and hence some \( f' \in F(G) \), distinct from \( f \), such that \( e \subseteq \mathcal{F}(f') \). Since \( f' \) is disjoint from \( f^- \), \( p \in (S^2 \setminus f^-)^- = S^2 \setminus (f^-)^0 \), i.e. \( p \not\in (f^-)^0 \). Thus, the open set \( f \) satisfies \( (f^-)^0 \subseteq f \), and so is regular open. By Lemma 1.4 (ii), \( \bigcup F(G) \subseteq \sum F(G) \subseteq (\bigcup F(G))^- = \bigcup \{ f^- \mid f \in F(G) \} = S^2 \). But by Lemma 1.3, \( \sum F(G) \) is the unique regular open set lying between \( \bigcup F(G) \) and its closure; i.e. \( \sum F(G) = 1 \). Hence, the elements of \( F(G) \) form a connected partition in \( \text{RO}(S^2) \). The last part of the lemma then follows from Lemma 1.3, since, if \( f \in F(G) \), then both \( f \) and \( \sum \{ f' \in F(G') \mid f' \subseteq f \} \) are regular open sets sandwiched between \( \bigcup \{ f' \in F(G') \mid f' \subseteq f \} \) and its closure.

QED

Of course, the converse of Lemma 1.79 is false: the configuration of Example 1.18 shows that not every connected partition in \( \text{RO}(S^2) \) is the set of faces of some plane graph.

**Lemma 1.80** If \( G \) is a piecewise linear plane graph such that \( G \) has no isolated vertices and every edge of \( G \) lies on the boundary of exactly 2 faces, then the faces of \( G \) form a connected partition in \( \text{ROP}(S^2) \).
Proof. Let $L_1, \ldots, L_m$ be straight lines extending (in both directions) each of the line segments making up $G$. Let $G'$ be the graph whose nodes are the points of intersection of the $L_i$ (including $\infty$) and whose edges are the segments of the $L_i$ joining them; and let $P$ be the set of non-zero products $+s_1 \cdot \cdots \cdot +s_m$, where $s_i$ is one of the residual half-planes of $L_i$ for $1 \leq i \leq m$. By simple set-algebra, $\bigcup P = \bigcup F(G')$; and since every $r \in P$ is connected, and every $f \in F(G')$ is a maximal connected subset of $\mathbb{S}^2 \setminus |G'|$, $r \cap f \neq \emptyset$ implies $r \subseteq f$. Hence every $f \in F(G')$ is a union of elements of $P$. But since these elements are non-empty open and disjoint and $f$ is connected, $f$ simply is some element of $P$, and hence is an element of $ROP(\mathbb{S}^2)$. Since $|G| \subseteq |G'|$, the result follows by the last part of Lemma 1.79. \hfill QED

Lemma 1.80 does have a converse:

**Lemma 1.81** If $\bar{\tau}$ is a connected partition in $ROP(\mathbb{S}^2)$, then $\bar{\tau}$ is the set of faces of some piecewise linear plane graph $G$; moreover, for any such plane graph $G$, $G$ has no isolated vertices, and every edge of $G$ lies on the boundary of exactly 2 faces.

Proof. By Claim 1.63, refine $\bar{\tau} = r_1, \ldots, r_n$ to a $c^3$-partition $\bar{\ell} = \{t_1, \ldots, t_N\}$, and let $G_0$ be the partition graph of $\bar{\ell}$. Suppose, by renumbering if necessary, that $r_1 = t_1 + \cdots + t_m$. Note that, if $e \in E(G_0)$, we have, for all $j \ (1 \leq j \leq N)$, $(e) \subseteq t_j^e$ or $(e) \cap t_j^e = \emptyset$. Hence if $(e) \not\subseteq r_1$, then $(e) \cap \bigcup_{m<j\leq N} t_j^e = \emptyset$, whence $e \subseteq t_j^e$ for some $j \ (m<j\leq N)$.

Let $G_1$ be the graph obtained from $G_0$ by first removing any edge $e$ such that $(e) \subseteq r_1$, and then removing any vertex $v$ such that $v \in r_1$. Since $r_1$ is open, the endpoints of every remaining arc are among the remaining vertices, so $G_1$ really is a plane graph. Moreover, if $m<j\leq N$, then $t_j^e \cap r_1 = \emptyset$, so that none of the vertices and edges removed from $G_0$ intersects $t_j^e$; hence $t_j$ is a face of $G_1$. Therefore, the set of points

$$S = \{ t_j \in F(G_0) \mid 1 \leq j \leq m \} \cup \{ e \in E(G_0) \mid (e) \subseteq r_1 \} \cup \{ v \in V(G_0) \mid v \in r_1 \}$$

must be the union of some faces of $G_1$. Trivially, $S \subseteq r_1$. We claim that $r_1 \subseteq S$. For if $p \in S^2$, exactly one of the following three cases holds: (i) $p \in t_j$ for some $j$; (ii) $p \in V(G_0)$; or (iii) $p \in e$ for some $e \in E(G_0)$. In case (i), either $p \in S$ or $p \not\in r_1$, according as $j \leq m$. In case (ii), trivially, either $p \in S$ or $p \not\in r_1$. In case (iii), if $p \not\in S$, then $p \in (e) \not\subseteq r_1$, whence $p \in e \subseteq t_j^e$ for some $j \ (m<j\leq N)$, whence $p \in \bigcup_{m<j\leq N} t_j^e = S^2 \setminus r_1$. This proves that $r_1 \subseteq S$. Thus, $r_1 = S$ is
the union of a number of faces of \( G_1 \). But \( r_1 \) is by assumption connected, so \( r_1 \) is a face of \( G_1 \). Proceeding in the same way for \( r_2, \ldots, r_n \), we obtain the desired graph \( G = G_n \).

Lemmas 1.80 and 1.81 concern the mereotopology \( \text{ROP}(\mathbb{S}^2) \), but almost exactly similar arguments can be given for \( \text{ROS}(\mathbb{S}^2) \) and \( \text{ROQ}(\mathbb{S}^2) \). We omit the details, which are routine. Summarizing, we have:

**Theorem 1.82** A tuple \( \bar{u} \) of subsets of \( \mathbb{S}^2 \) is a connected partition in \( \text{ROS}(\mathbb{S}^2) \) (alternatively: \( \text{ROP}(\mathbb{S}^2) \), \( \text{ROQ}(\mathbb{S}^2) \)) if and only if it is the set of faces of a semi-algebraic (respectively: piecewise linear, rational piece-wise linear) graph with no isolated vertices and every edge lying on the boundary of two faces.

### 4.4 Expressive power of first-order languages in plane mereotopologies

We are now in a position to give an absolute characterization of the expressive power of the languages \( L_{\leq} \) and \( L_C \) over certain mereotopologies of interest. Recall the concept of topologically complete formula given in Definition 1.51. The following notation will be useful in constructing topologically complete formulas.

**Notation 1.83** Given a fixed Boolean algebra, a Boolean matrix is a rectangular matrix whose entries are the elements 1 and 0. If \( \bar{r} \) is an \( n \)-tuple, \( \bar{z} \) an \( N \)-tuple, and \( A \) a Boolean matrix with \( N \) rows and \( n \) columns, we write \( \bar{r} = \bar{z}A \) to indicate that each element of \( \bar{r} \) is the sum of certain elements of \( \bar{z} \) as indicated by the elements of \( A \) via normal matrix multiplication. Similarly, we write \( \bar{x} = \bar{z}A \) in first-order formulas to abbreviate the obvious conjunction of Boolean algebra equations.

**Theorem 1.84** Let \( M \) be any finitely decomposable mereotopology over \( \mathbb{S}^2 \) having curve-selection, and let \( \Sigma \) be the signature \((e, \leq, +, \cdot, \neg)\). Every tuple from \( M \) satisfies some (purely existential) \( L_{\Sigma} \)-formula which is topologically complete in \( M \) over \( \mathbb{S}^2 \).

**Proof** Writing \( \bar{z} \) for \( z_1, \ldots, z_N \), let \( \psi^N_{\Sigma}(\bar{z}) \) be the formula:

\[
\bigwedge_{1 \leq i \leq N} (c(z_i) \land z_i > 0) \land \\
\bigwedge_{1 \leq i \leq j \leq N} (c(-(z_i + z_j)) \land z_i \cdot z_j = 0) \land \sum_{1 \leq i \leq N} z_i = 1.
\]
Thus, $M \models \psi^N_{c^3}[\vec{s}]$ if and only if $\vec{s}$ is an $N$-element $c^3$-partition. If $\vec{s} = s_1, \ldots, s_N$ is a $c^3$-partition in $M$, let $\psi^\vec{s}_r(\vec{z})$ be the formula:

$$\land \{ c(z_i + z_j) \mid 1 \leq i < j \leq N \text{ and } s_i + s_j \text{ is connected} \} \land$$

$$\land \{ \neg c(z_i + z_j) \mid 1 \leq i < j \leq N \text{ and } s_i + s_j \text{ is not connected} \},$$

where $\vec{z}$ is the tuple of variables $z_1, \ldots, z_n$. Thus, $\psi^\vec{s}_r(\vec{z})$ encodes the neighbourhood structure of $\vec{s}$. Now let $\vec{r} = r_1, \ldots, r_n$ be any tuple of elements of $M$. By Claim 1.63, there exists a $c^3$-partition $\vec{s} = s_1, \ldots, s_N$ in $M$ and a Boolean matrix $A$ such that $\vec{r} = \vec{s} A$. Writing $\vec{x}$ for $x_1, \ldots, x_n$, let $\psi_{\vec{r}}(\vec{x})$ be the formula

$$\exists \vec{x} (\psi^N_{c^3}(\vec{z}) \land \psi^\vec{s}_r(\vec{z}) \land \vec{x} = \vec{z} A).$$

Certainly, $M \models \psi_{\vec{r}}[\vec{x}]$. And if $\vec{r}'$ is a tuple from $M$ such that $M \models \psi_{\vec{r}'}[\vec{x}]$, let $\vec{s}' = s'_1, \ldots, s'_N$ be corresponding witnesses for the existentially quantified variables $\vec{z}$. Then $s_1, \ldots, s_N$ and $s'_1, \ldots, s'_N$ are $c^3$-partitions in $\mathbb{S}^2$ which have the same neighbourhood structure, and hence which are similarly situated in $\mathbb{S}^2$, by Theorem 1.78. It follows that $\vec{r}$ and $\vec{r}'$ are similarly situated in $\mathbb{S}^2$ too. Thus, $\psi_{\vec{r}}(\vec{x})$ is topologically complete in $M$ over $\mathbb{S}^2$.

**Corollary 1.85** Let $M$ be any finitely decomposable mereotopology over $\mathbb{S}^2$ having curve-selection. Every tuple from $M$ satisfies some $L_{C^r}$-formula which is topologically complete in $M$ over $\mathbb{S}^2$.

**Proof** Theorem 1.84 and Lemmas 1.22, 1.27 and 1.49.

Thus, for certain well-behaved mereotopologies over $\mathbb{S}^2$, both $L_{C^r}$ and $L_{C^r, \bigtriangleup}$ are, as we might put it, ‘topologically fully descriptive’.

We now turn to the question of expressive power in mereotopologies over $\mathbb{R}^2$. We need some auxiliary lemmas.

**Lemma 1.86** Let $\vec{r} = r_1, \ldots, r_n$ be a $c^3$-partition in any mereotopology $M$ over $\mathbb{S}^2$ having curve-selection. Let $p, p' \in \mathbb{S}^2$ such that, for all $i (1 \leq i \leq n)$, $p \in r_i^-$ if and only if $p' \in r_i^-$. Then there is a homeomorphism $h : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ mapping $p$ to $p'$ and fixing each $r_i$.

**Proof** Obvious, by viewing $\vec{r}$ as $F(G)$ for some plane graph $G$.

**Lemma 1.87** Let $\vec{r} = r_1, \ldots, r_n$ and $\vec{r}' = r'_1, \ldots, r'_n$ be similarly situated $c^3$-partitions in any mereotopology $M$ over $\mathbb{S}^2$ having curve-selection. Let $p \in \mathbb{S}^2$ such that, for all $i (1 \leq i \leq n)$, $p \in r_i^-$ if and only if $p \in r'_i^-$. 

**Proof**
Then there is a homeomorphism \( h : \mathbb{S}^2 \to \mathbb{S}^2 \) fixing \( p \) and mapping \( \vec{r} \) to \( \vec{r}' \).

**Proof**  Let \( h' : \mathbb{S}^2 \to \mathbb{S}^2 \) be some homeomorphism mapping \( \vec{r} \) to \( \vec{r}' \). Then, for all \( i \) (\( 1 \leq i \leq N \)), \( h'(p) \in r_i' \) if and only if \( p \in r_i \). By Lemma 1.86, let \( h'' : \mathbb{S}^2 \to \mathbb{S}^2 \) be a homeomorphism fixing each \( r_i' \), and mapping \( h'(p) \) to \( p \). Then \( h := h'' \circ h' \) has the required properties. \( \Box \)

**Theorem 1.88**  Let \( M \) be any finitely decomposable mereotopology over \( \mathbb{R}^2 \) such that \( M \) has curve-selection. Every tuple from \( M \) satisfies some \( L_C \)-formula which is topologically complete in \( M \) over \( \mathbb{R}^2 \).

**Proof**  Given any tuple \( s_1, \ldots, s_N \) from \( M \), let \( \phi_{\pm}^N(\bar{z}) \) be the \( L_C \)-formula

\[
\bigwedge \{ \phi_{\pm}^n(z_i) \mid 1 \leq i \leq N \text{ and } s_i \text{ is bounded} \} \land \\
\bigwedge \{ \neg\phi_{\pm}^n(z_i) \mid 1 \leq i \leq N \text{ and } s_i \text{ is not bounded} \},
\]

where \( \bar{z} \) is the tuple of variables \( z_1, \ldots, z_N \), and \( \phi_{\pm} \) is as in Lemma 1.30. Thus, \( \phi_{\pm}^N(\bar{z}) \) encodes the pattern of boundedness in the tuple \( \bar{s} \). Now, given a tuple \( \vec{r} \), let \( \bar{s} \) be an \( N \)-element \( c^3 \)-partition in \( M \) refining \( \vec{r} \), and let \( A \) be a Boolean matrix satisfying \( \vec{r} = \bar{s}A \). Using the translation from \( L_{c<} \) to \( L_C \) established by Lemmas 1.22 and 1.27, let \( \phi_{\pm}^N(\bar{z}) \) and \( \phi_{\pm}^N(\bar{z}) \) be the \( L_C \)-formulas corresponding to \( \psi_{\pm}^N(\bar{z}) \) and \( \psi_{\pm}^N(\bar{z}) \) in the proof of Theorem 1.84. Writing \( \bar{x} \) for \( x_1, \ldots, x_n \), let \( \phi_{\vec{r}}(\bar{x}) \) be the formula

\[
\exists \bar{z} (\phi_{\pm}^N(\bar{z}) \land \phi_{\pm}^N(\bar{x}) \land \phi_{\pm}^{\infty}(\bar{z}) \land \bar{z} = \bar{z}A).
\]

Certainly, \( M \models \psi_{\vec{r}}[\vec{r}] \); and if \( \vec{r}' \) is a tuple from \( M \) such that \( M \models \psi_{\vec{r}}[\vec{r}'] \), let \( \bar{s}' = s'_1, \ldots, s'_N \) again be a corresponding witnesses for the existentially quantified variables \( \bar{z} \). Then \( s_1, \ldots, s_N \) and \( s'_1, \ldots, s'_N \) are \( c^3 \)-partitions in \( \mathbb{S}^2 \) which have the same neighbourhood structure, so that by Theorem 1.78 and Lemma 1.87, there is a homeomorphism \( h : \mathbb{S}^2 \to \mathbb{S}^2 \) fixing \( \infty \) and mapping each \( s_i \) to \( s'_i \). Hence \( \bar{s} \) and \( \bar{s}' \) are similarly situated in \( \mathbb{R}^2 \), whence \( \vec{r} \) and \( \vec{r}' \) are similarly situated in \( \mathbb{R}^2 \) too. \( \Box \)

Thus, for well-behaved mereotopologies over \( \mathbb{R}^2 \), \( L_C \) is, as we might put it, topologically fully descriptive.

### 4.5 Homogeneous mereotopologies

Up to this point, we have been concerned only to show that certain relations can be defined by first-order formulas with signatures of topological primitives. We turn now briefly to the question of which relations cannot be so defined.
At first glance, one might assume that languages with purely topological primitives can express only topological concepts in mereotopologies over which they are interpreted. However, this assumption is correct only if the mereotopologies in question have a certain property. Recall that, for a fixed topological space \( X \), we write \( \bar{u} \sim \bar{v} \) to mean that the tuples of subsets \( \bar{u} \) and \( \bar{v} \) are similarly situated in \( X \) (Definition 1.51).

**Definition 1.89** Let \( M \) be a mereotopology over \( X \). We say \( M \) is homogeneous (over \( X \)) if, given any tuples \( \bar{r}, \bar{s} \) from \( M \) with \( \bar{r} \sim \bar{s} \) and any element \( r \in M \), there exists an element \( s \in M \) with \( \bar{r}, r \sim \bar{s}, s \). Let \( M' \) also be a mereotopology over \( X \), with \( M' \subseteq M \). We say \( M' \) is homogeneously embedded in \( M \) (over \( X \)) if, given any tuple \( \bar{r} \) from \( M' \), and any \( r \in M \), there exists \( s \in M' \) with \( \bar{r}, r \sim \bar{s}, s \).

**Lemma 1.90** Let \( X \) be either \( \mathbb{R}^2 \) or \( \mathbb{S}^2 \), and let \( M \) be any of \( \text{ROS}(X) \), \( \text{ROP}(X) \) or \( \text{ROQ}(X) \). Then \( M \) is homogeneous.

**Proof** Assume \( M = \text{ROS}(\mathbb{S}^2) \); the other cases are identical. Let \( \bar{r}, \bar{s} \) be tuples from \( M \), and let \( r \in M \). Let \( \bar{h} \) be a connected partition refining \( \bar{r}, r \) and so by Theorem 1.82 is the set of faces of some semi-algebraic plane graph \( G \). If \( h : \mathbb{S}^2 \rightarrow \mathbb{S}^2 \) is a homeomorphism mapping \( \bar{r} \) to \( \bar{s} \), then \( h \) maps \( G \) to a plane graph \( H \). But then it is not difficult to show that the edges of \( H \) can be deformed into a semi-algebraic plane graph \( H' \), and moreover, that this may be done in such a way that existing semi-algebraic edges are unaffected. By Theorem 1.82, the faces of the resulting graph are elements of \( M \); hence we have a homeomorphism mapping \( \bar{r} \) to \( \bar{s} \) and taking \( r \) to some element \( s \) of \( M \).

Homogeneity and homogeneous embedding are important because of the following facts.

**Lemma 1.91** Let \( M \) be a homogeneous mereotopology over a topological space \( X \), and fix a signature \( \Sigma \) of topological primitives. If \( \bar{r} \) and \( \bar{s} \) are tuples of \( M \) which are similarly situated in \( X \), then \( \bar{r} \) and \( \bar{s} \) satisfy the same \( L_\Sigma \)-formulas in \( M \).

**Proof** We show by induction on the complexity of \( \phi[\bar{x}] \in L_\Sigma \) that, if \( \bar{r} \) and \( \bar{s} \) are tuples of the appropriate arity which are similarly situated in \( X \), then \( M \models \phi[\bar{r}] \) implies \( M \models \phi[\bar{s}] \). The base case follows from the fact that the primitives in \( \Sigma \) have topological interpretations. The only non-trivial recursive case is where \( \phi[\bar{x}] = \exists y \psi(\bar{x}, y) \). If \( M \models \phi[\bar{r}] \), there exists \( r \in M \) such that \( M \models \psi[\bar{r}, r] \), and by homogeneity, if \( \bar{r} \sim \bar{s} \), there exists \( s \in M \) such that \( \bar{r}, r \sim \bar{s}, s \), whence \( M \models \psi[\bar{s}, s] \) by inductive hypothesis, so that \( M \models \phi[\bar{s}] \) as required.
Lemma 1.91 gives an upper bound on the expressive power of first-order languages with signatures of topological primitives interpreted over homogeneous mereotopologies: such languages cannot distinguish between similarly situated tuples. It thus provides a partial converse to Theorems 1.84 and 1.88. It also yields an easy proof that, over well-behaved open-plane mereotopologies, \( L_{c,\leq} \) cannot express the property of being bounded:

**Theorem 1.92** Let \( M \) be a mereotopology over \( \mathbb{R}^2 \) such that \( \hat{M} \) is homogeneous, and suppose \( M \) has curve-selection and contains a region \( r \) similarly situated in \( \mathbb{R}^2 \) to the open unit disc \( B^2 \). Then there exists no formula \( \psi(x) \) of \( L_{c,\leq} \) such that, for all \( r \in M \), \( r \) is bounded if and only if \( M \models \psi[r] \).

**Proof** Suppose such a formula \( \psi(x) \) exists. Then \( M \models \psi[r] \), and by Lemma 1.38, \( \hat{M} \models \psi[\hat{r}] \). Since \( M \) has curve-selection, by Proposition 1.44 both \( \hat{r} \) and its complement \( \neg(\hat{r}) \) in \( \hat{M} \) are 2-cells in \( S^2 \), and hence are similarly situated. By Lemma 1.91, \( \hat{M} \models \psi[\neg(\hat{r})] \), and so by Lemma 1.38, \( M \models \psi[\neg r] \). This contradicts the fact that \( \neg r \) is unbounded. \( \Box \)

Finally, we return to the relationship between ROS\((X)\), ROP\((X)\) and ROQ\((X)\).

**Lemma 1.93** Let \( X \) be either \( \mathbb{R}^2 \) or \( S^2 \). Then ROQ\((X)\) is homogeneously embedded in ROP\((X)\), which is in turn homogeneously embedded in ROS\((X)\).

**Proof** Virtually identical to the proof of Lemma 1.90. \( \Box \)

The following result is well-known (see, for example, Hodges, 1993 p. 55).

**Proposition 1.94 (Tarski-Vaught)** Let \( \mathfrak{A} \), \( \mathfrak{B} \) be structures with \( \mathfrak{A} \subseteq \mathfrak{B} \), and suppose that, for any \( n \)-tuple \( \bar{a} \) from \( A \) and any formula \( \phi(x) \) of the form \( \exists y \psi(x, y) \) such that \( \mathfrak{B} \models \phi[\bar{a}] \), there exists \( a \in A \) such that \( \mathfrak{B} \models \psi[\bar{a}, a] \). Then \( \mathfrak{A} \preceq \mathfrak{B} \).

**Lemma 1.95** Let \( M, M' \) be mereotopologies over a topological space \( X \), with \( M \) homogeneous and \( M' \) homogeneously embedded in \( M \). Fix a signature of topological primitives. Then \( M' \preceq M \).

**Proof** By assumption, \( M' \subseteq M \). Let \( \bar{r} \) be an \( n \)-tuple of elements of \( M' \), and let \( \phi(\bar{x}) \) be any formula of \( L_\Sigma \) of the form \( \exists y \psi(\bar{x}, y) \) such that \( M \models \phi[\bar{a}] \). Then there exists \( r \in M \) such that \( M \models \psi[\bar{r}, r] \). Since \( M' \) is homogeneously embedded in \( M \), there exists \( s \in M' \) such that
\( \varphi, r \sim s, s \). Since \( M \) is homogeneous, \( M \models \psi[\varphi, s] \) by Lemma 1.91. The result then follows by Proposition 1.94.

Hence, for \( X \) either \( \mathbb{R}^2 \) or \( \mathbb{S}^2 \), and over any signature \( \Sigma \) of topological primitives, we have \( \text{ROQ}(X) \preceq \text{ROP}(X) \preceq \text{ROS}(X) \). In particular, these three structures have identical \( L_\Sigma \)-theories. We show in the sequel that this is no accident: almost any ‘reasonable’ mereotopology over \( \mathbb{S}^2 \) has the same \( L_\Sigma \)-theory. Anticipating these results, we employ the following notation and terminology.

**Definition 1.96** Let \( \Sigma \) be a signature of topological primitives. We call the theory \( \text{Th}_\Sigma(\text{ROS}(\mathbb{S}^2)) \) the standard \( L_\Sigma \)-theory (of closed plane mereotopology), and denote it \( T_\Sigma \).

## 5. Axiomatization

In this section, we provide an axiomatic characterization of \( T_{\langle \leq, \leq \rangle} \), the standard \( L_{\langle \leq, \leq \rangle} \)-theory of closed plane mereotopology. The material is essentially that of Pratt and Schoop, 1998. The axiom system in question will help us to identify mereotopologies over \( \mathbb{S}^2 \) having the standard \( L_{\langle \leq, \leq \rangle} \)-theory.

As before, we write \( \psi_3(x) \) for the \( L_{\langle \leq, \leq \rangle} \)-formula stating that \( x \) forms \( n \)-element \( c_3 \)-partition, and \( x = u \oplus v \) for the \( L_{\langle \leq, \leq \rangle} \)-formula stating that \( u \) and \( v \) are disjoint, non-zero, connected regions summing to \( x \). Let \( M \) be a mereotopology over \( \mathbb{S}^2 \) having curve-selection. Consider a triple \( r, s, t \) from \( M \) satisfying the formula \( \psi_3(x, y, z) \). By Lemma 1.45, each of these regions is a 2-cell, and it is easy to see that the closures of any two of these intersect in a Jordan arc. (Formally, this follows by Lemma 1.73.) Now let \( \psi_{\text{split}} \) denote the \( L_{\langle \leq, \leq \rangle} \)-formula

\[
\forall x \forall y \forall z (\psi_3(x, y, z) \rightarrow \\
\exists u \exists v (u \oplus v = x \land c(u + y) \land \neg c(u + z) \land c(v + z) \land \neg c(v + y))).
\]

Informally, \( \psi_{\text{split}} \) ‘says’ that, given two 2-cells \( r \) and \( s \) whose frontiers intersect in a Jordan arc, \( r \) can be partitioned into two connected regions using a cross-cut whose end-points are the end-points of that Jordan arc (Fig. 1.13a).

**Definition 1.97** A mereotopology \( M \) is splittable if \( M \models \psi_{\text{split}} \).

The following lemma is unsurprising.

**Lemma 1.98** The mereotopologies \( \text{ROS}(\mathbb{S}^2) \), \( \text{ROP}(\mathbb{S}^2) \) and \( \text{ROQ}(\mathbb{S}^2) \) are splittable.

**Proof** Almost immediate from Theorem 1.84, Lemma 1.42 and Proposition 1.43. \( \Box \)
However, not all finitely decomposable mereotopologies over $S^2$ having curve-selection are splittable. If an $(n - 1)$-dimensional hyperplane in $\mathbb{R}^n$ is defined by an equation $x_i = 0$, where $0 \leq i \leq n$, we call it an axis-oriented hyperplane; and if a half-space is bounded by an axis-oriented hyperplane, we call it an axis-oriented half-space.

**Example 1.99** Define ROX($S^n$) to be the Boolean sub-algebra of RO($S^n$) generated by the axis-oriented half-spaces. It is easy to see that ROX($S^n$) is a finitely decomposable mereotopology over $S^n$ having curve-selection. However, ROX($S^n$) $\neq \psi_{\text{split}}$, as is clear in the case $n = 2$ by inspection of Fig. 1.13b).

Thus, whereas RO($S^2$) has, as it were, too many regions for the standard theory, ROX($S^2$) has too few. As we have observed, RO($S^2$) is not finitely decomposable, and lacks curve-selection, while ROX($S^2$) is not splittable. It transpires that these represent the only ways of failing to exhibit the standard theory of closed plane mereotopology. Specifically, we show in this section that all splittable, finitely decomposable mereotopologies over $S^2$ having curve-selection have the same $L_{c, \leq}$-theory. Our strategy is to pick one splittable, finitely decomposable mereotopologies over $S^2$ having curve-selection—ROP($S^2$) will do—and characterize its theory axiomatically. We then merely need to check that our axiom system is correct for all such mereotopologies.
5.1 The axioms

Our axiom system comprises three parts: a general inference system, a set of proper axioms and an \( \omega \)-rule. (i) The general inference system is simply any complete Hilbert system for first-order logic, restricted to the signature \( \{+, \cdot, -, \leq, c\} \). (ii) The proper axioms are as follows:

1. the usual axioms of Boolean algebra, and the axiom \( 0 \neq 1 \);
2. the axiom \( \psi_{\text{con}} \) (Lemma 1.53);
3. where \( n > 2 \), the axioms \( \psi^n_{\text{num}} \) (Lemma 1.56);
4. the axiom

\[
\neg \exists x_1 \ldots \exists x_5 \left( \bigwedge_{1 \leq i \leq 5} (c(x_i) \land x_i \neq 0) \land \\
\bigwedge_{1 \leq i < j \leq 5} (c(x_i + x_j) \land x_i \cdot x_j = 0) \right);
\]

5. the axiom

\[
\neg \exists x_1 \ldots \exists x_6 \left( \bigwedge_{1 \leq i \leq 6} (c(x_i) \land x_i \neq 0) \land \\
\bigwedge_{1 \leq i < j \leq 6} x_i \cdot x_j = 0 \land \bigwedge_{1 \leq i \leq 3} c(x_i + x_j) \right);
\]

6. the axioms \( c(0) \) and \( c(1) \);
7. the axiom \( \phi_{\text{break}} \) (Lemma 1.57);
8. the axiom \( \phi_{\text{split}} \) (Definition 1.97).

(iii) The final component of our axiom system is the \( \omega \)-rule. If \( n \geq 1 \), we let \( \psi^n_c(x) \) stand for the formula

\[
\exists z_1 \ldots \exists z_n \left( \bigwedge_{1 \leq i \leq n} c(z_i) \land (x = z_1 + \cdots + z_n) \right).
\]

Thus, \( \psi^n_c(x) \) ‘says’ that \( x \) can be formed by summing \( n \) connected regions. The \( \omega \)-rule is then the (infinitary) rule of inference:

\[
\frac{\forall x(\psi^n_c(x) \rightarrow \phi(x)) \mid n \geq 1}{\forall x \phi(x)}.
\]
Let $\Phi$ be a set of $L_{c,\leq}$-sentences. A proof with premises $\Phi$ in the above system is a sequence of $L_{c,\leq}$-formulas $\{\phi_{\alpha}\}_{\alpha<\beta}$, for some ordinal $\beta$ (not necessarily finite) such that every $\phi_{\alpha}$ is either (i) an element of $\Phi$ or (ii) an axiom or (iii) the result of applying a rule of inference to some formulas $\phi_\gamma$ with $\gamma < \alpha$. If $\psi$ is the last line of some such proof, we write $\Phi \vdash \psi$. If $\Phi = \{\phi\}$ we write $\phi \vdash \psi$, and if $\Phi = \emptyset$ we write $\vdash \psi$ and call $\psi$ a theorem. Let us denote the set of theorems by $T_{AX}$. The main result of Section 5 is:

**Theorem 1.100** $T_{AX}$ is the complete $L_{c,\leq}$-theory of any finitely decomposable, splittable meretopology over $\mathbb{S}^2$ having curve-selection.

**Proof** Lemmas 1.102 and 1.104, below. \hfill QED

Of course, this entails that all such meretopologies, considered as $\{c,\leq\}$-structures are elementarily equivalent.

The $\omega$-rule is less unfamiliar than one might at first think. Essentially, it says that if a property holds of every region which is the sum of finitely many connected regions, then it simply holds of every region. This conditional is obviously true in a finitely decomposable meretopology. Thus, a proof involving the $\omega$-rule is analogous to an argument of the kind encountered in elementary algebra textbooks in which one proves a property of all polynomials by showing that it holds of all polynomials of some arbitrary degree $n$. Nevertheless, the inclusion of an infinitary proof rule does mean that we ought to check the deduction theorem.

**Lemma 1.101** Let $\phi$ be an $L_{c,\leq}$-sentence and $\psi$ an $L_{c,\leq}$-formula such that $\phi \vdash \psi$. Then $\vdash \phi \rightarrow \psi$.

**Proof** By assumption, there is a proof $\{\phi_{\alpha}\}_{\alpha<\beta+1}$ with premises $\{\phi\}$ and last line $\phi_\beta = \psi$. Without loss of generality, we may assume that the first (actually, zeroth) line of the proof $\psi_0$ is $\phi$. We proceed by induction on $\beta$. The case $\beta = 0$ is trivial, since $\vdash \phi \rightarrow \phi$. If $\beta > 0$, then either $\phi_\beta$ is an axiom or is derived from applying a rule of inference to earlier lines of the proof. The only interesting case is where $\phi = \forall x \pi$ is derived by the $\omega$-rule from the formulas $\forall x (\psi^n_{\phi}(x) \rightarrow \pi)$ occurring earlier in the proof. But the inductive hypothesis then yields $\vdash \phi \rightarrow \forall x (\psi^n_{\phi}(x) \rightarrow \pi)$, for each $n$, whence $\vdash \forall x (\psi^n_{\phi}(x) \rightarrow (\phi \rightarrow \pi))$. The $\omega$-rule then yields $\vdash \forall x (\phi \rightarrow \pi)$, whence $\vdash \phi \rightarrow \forall x \pi$ (note that $\phi$ is a sentence), as required. \hfill QED

We remark in passing that the axiom $c(0)$ is actually redundant: it can be derived from the other axioms and proof rules.

### 5.2 Correctness

In this section, we establish the easy half of Theorem 1.100.
Lemma 1.102 If $M$ is a splittable, finitely decomposable mereotopology over $\mathbb{S}^2$ having curve-selection, then $M \models T_{\text{Ax}}$.

Proof We follow the enumeration in Section 5.1, showing that the proper axioms are all true in $M$ and that the $\omega$-rule is truth-preserving.

1. $M$ is a mereotopology.
2. Lemma 1.53.
3. Lemma 1.56.
4. Suppose $r_1, \ldots, r_5$ are connected, non-empty and pairwise disjoint, and that any pair of them have a connected sum. By Lemma 1.26, choose points $p_i \in r_i$ and $q_{i,j} \in F(r_i) \cap F(r_j) \cap (r_i + r_j)$ ($1 \leq i < j \leq 5$). For each $i$ ($1 \leq i \leq 5$), draw end-cuts in $r_i$ from $p_i$ to all the points $q_{i,j}$ and $q_{i,j}$. It is easy to see that these can be chosen so that any pair of these end-cuts intersect only in the point $p_i$. Ignoring the points $q_{i,j}$, we have a plane drawing of the graph $K^5$, which is known to be non-planar (Fig. 1.11).
5. As for axiom 4, but with $K_{3,3}$ instead of $K^5$.
6. Trivial.
7. Lemma 1.57.
8. $M$ is splittable.

The $\omega$-rule is obviously truth-preserving, because $M$ is finitely decomposable. QED

5.3 Completeness

In this section, we establish the difficult half of Theorem 1.100. We make use of the omitting types theorem: for details, see, e.g., Hodges, 1993, pp 333. Let $\mathfrak{A}$ be a structure, $\Phi(x)$ a set of formulas with free variable $x$, and $T$ a set of sentences. We say that $\mathfrak{A}$ omits $\Phi(x)$ if, for all $a \in A$, $\mathfrak{A} \not\models \Phi[a]$. We say that $T$ locally omits $\Phi(x)$ if, for every formula $\theta(x)$ with free variable $x$ such that $\theta$ is consistent with $T$, there exists $\phi(x) \in \Phi(x)$ such that $T \not\models \forall x(\theta(x) \rightarrow \phi(x))$. The following theorem is a well-known strengthening of the completeness theorem for first-order logic.

Proposition 1.103 (Omitting Types Theorem) If a consistent theory $T$ locally omits a set of formulas $\Phi(x)$, then $T$ has a countable model omitting $\Phi(x)$. 
With these preliminaries behind us, we can proceed with our completeness proof.

**Lemma 1.104** If $\phi$ is an $L_{c, \le}$-sentence, and $\text{ROP}(S^2) \models \phi$, then $\phi \in T_{Ax}$.

**Proof** Suppose that $\phi \not\in T_{Ax}$. We are required to prove that $\text{ROP}(S^2) \models \neg\phi$. Let $T$ be the set of all and only those $L_{c, \le}$-sentences $\psi$ such that $\neg\psi \vdash \psi$. By Lemma 1.101, $T$ is a consistent set of sentences, and from the $\omega$-rule, $T$ locally omits the type $\{\neg\psi^n_c(x) \mid n > 0\}$. By Proposition 1.103, there exists a countable model $\mathfrak{A} \models T$ omitting that type. Fix the structure $\mathfrak{A}$ for the remainder of this proof.

We now proceed in three stages. Stage 1 establishes some basic facts about $\mathfrak{A}$; Stage 2 shows that $\mathfrak{A}$ can be embedded in the $\{c, \le\}$-structure $\text{ROP}(S^2)$; Stage 3 shows that the embedding we have chosen is in fact elementary.

**Stage 1:** Axioms 1 ensure that the reduct of $\mathfrak{A}$ to the signature $\{+, \cdot, -, \le\}$ is a Boolean algebra. Such structures were discussed in Section 4.2, where various terminology and notational conventions were introduced. We carry these over to the present proof. Using that terminology, another way of saying that $\mathfrak{A}$ omits the type $\{\neg\psi^n_c(x) \mid n > 0\}$ is to say that $\mathfrak{A}$ is finitely decomposable.

By Axioms 2, 3, 6 and 7, all the claims in Section 4.2 hold of $\mathfrak{A}$. In particular, every tuple can be refined to a connected partition, and hence to a $c^2$- and a $c^3$-partition. Furthermore, we have

**Claim 1.105** Let $\bar{b} = b_1, \ldots, b_n$ be a connected partition in $\mathfrak{A}$. Then the neighbourhood graph of $\bar{b}$ is planar.

**Proof** By Proposition 1.68, if the neighbourhood graph $G$ of $\bar{b}$ is not planar, it contains either $K_5$ or $K_{3,3}$ as a minor. But then there is a sequence of contractions of $G$ resulting in a graph $H$ which has either $K_5$ or $K_{3,3}$ as a sub-graph. By repeated applications of Claim 1.60 (re-numbering the $b_i$ as necessary), there is a connected partition $\bar{s}$ in $\mathfrak{A}$ whose neighbourhood graph contains $K_5$ or $K_{3,3}$ as a sub-graph. But this is impossible by Axioms 4 and 5. \(\text{QED}\)

**Stage 2:** Since $\mathfrak{A}$ is countable, let $A = \{a_1, a_2, \ldots\}$. Let $N_0 = 1$ and let $\bar{c}^0$ be the 1-tuple whose element is the unit of the Boolean algebra $\mathfrak{A}$. Trivially, $c^0$ is a $c^3$-partition. For $n \geq 0$, suppose that the $c^3$-partition $\bar{c}^{(n)} = c_{1}^{(n)}, \ldots, c_{N_n}^{(n)}$ in $\mathfrak{A}$ has been defined; then, by Claim 1.63, let $\bar{c}^{(n+1)} = c_{1}^{(n+1)}, \ldots, c_{N_{n+1}}^{(n+1)}$ be a $c^3$-partition in $\mathfrak{A}$ refining the tuple
$c_1^{(n)}, \ldots, c_{N_n}^{(n)}, a_{n+1}$. It is then obvious that, for each $n > 0$, $c^{(n)}$ refines the tuple $a_1, \ldots, a_n$ and also every tuple $\hat{c}^{(m)}$ for all $m$ ($0 < m \leq n$). We fix the enumerations $a_0, a_1, \ldots$ and $\hat{c}^{(0)}, \hat{c}^{(1)}, \ldots$ for the remainder of Stage 2.

For brevity, denote ROP($\mathbb{S}^2$) by $S$. We now map each initial segment $a_1, \ldots, a_n$ of $A$ into $S$. Let $w^{(n)}$ be the set of functions $g^{(n)} : \{c_1^{(n)}, \ldots, c_N^{(n)}\} \rightarrow S$ satisfying the conditions:

G1: the regions $g^{(n)}(c_1^{(n)}), \ldots, g^{(n)}(c_N^{(n)})$ form a connected partition;

G2: for all $i, j$ ($1 \leq i < j \leq N_n$), $g^{(n)}(c_i^{(n)}) + g^{(n)}(c_j^{(n)})$ is connected if and only if $c_i^{(n)} + c_j^{(n)}$ is connected.

We remark that, in G2, we have $g^{(n)}(c_i^{(n)}), g^{(n)}(c_j^{(n)}) \in S$ and $c_i^{(n)}, c_j^{(n)} \in A$. Hence, different senses of “+” and “connected” apply in the two cases.

**Claim 1.106** For all $n \in \mathbb{N}$, $w^{(n)} \neq \emptyset$.

**Proof** For the proof of this claim, we shall drop the $n$-sub- and superscripts and write $N$ for $N_n$ and $c_i$ for $c_i^{(n)}$. Let $G$ be the neighbourhood graph on $c_1, \ldots, c_N$. By Claim 1.105, $G$ is planar. By Axioms 6 and Claim 1.61, $G$ is connected. Let $H$ be a drawing of $G$ in $\mathbb{S}^2$ (under some mapping $\epsilon : V(G) \rightarrow V(H)$); we may assume that $H$ is piecewise linear. By Proposition 1.69, let $H^*$ be a geometric dual of $H$, which we may likewise assume to be piecewise linear. By Lemma 1.70, every vertex of $H$ lies in exactly one face of $H^*$. It follows that every edge of $H^*$ is on the boundary of two faces; moreover, $H^*$ by construction contains no isolated nodes. By Theorem 1.82, the faces of $H^*$ form a connected partition in $S$. So define $g(c_i)$ to be the face of $H^*$ containing the $H$-vertex $\epsilon(c_i)$. Properties G1 and G2 are then almost immediate. QED

**Claim 1.107** Let $I \subseteq \{1, \ldots, N_n\}$, and let $g^{(n)} \in w^{(n)}$. Then $\sum_{i \in I} c_i$ is connected if and only if $\sum_{i \in I} g^{(n)}(c_i)$ is connected.

**Proof** Claim 1.61 and property G2. QED

Suppose $n > m \geq 0$, so that $c^{(n)}$ refines $c^{(m)}$. For all $i$ ($1 \leq i \leq N_n$), let $c_{i,1}, \ldots, c_{i,M_i}$ be the collection of elements of $c^{(n)}$ which sum to $c_i^{(m)}$. If $g^{(n)} \in w^{(n)}$, then, we may define the restriction of $g^{(n)}$ to $c^{(m)}$, written $g^{(n)}|_m$, as follows:

$$g^{(n)}|_m(c_i^{(m)}) = g^{(n)}(c_{i,1}^{(n)}) + \ldots + g^{(n)}(c_{i,M_i}^{(n)})$$
Claim 1.108 Let $g^{(n)} \in w^{(n)}$ with $0 \leq m < n$. Then $g^{(n)}|_m \in w^{(m)}$.

Proof We must prove that G1 and G2 hold of $g^{(n)}|_m$. G1 is trivial. For G2, we note that, by construction,

$$g^{(n)}|_m (c_{i_j}^{(m)}) + g^{(n)}|_m (c_{j_i}^{(m)}) = g^{(n)}(c_{i_j}^{(m)}[1, i]) + \ldots + g^{(n)}(c_{i_j}^{(m)}[j, M_i]) + g^{(n)}(c_{j_i}^{(m)}[1, i]) + \ldots + g^{(n)}(c_{j_i}^{(m)}[j, M_j]).$$

By Claim 1.107, this element of $S$ is connected if and only if the element of $A$

$$c_{i_j}^{(n)}[1, i] + \ldots + c_{i_j}^{(n)}[j, M_i] + c_{j_i}^{(n)}[1, i] + \ldots + c_{j_i}^{(n)}[j, M_j] = c_{i_j}^{(m)} + c_{j_i}^{(m)}$$

is connected. Hence G2 holds as required. QED

Claim 1.109 Let $g \in w^{(n)}$. Then there exists a $g' \in w^{(n+1)}$ such that $g'|_m = g$.

Proof Choose any $g'' \in w^{(n+1)}$. By Claim 1.108, $g''|_m \in w^{(n)}$. Letting $\vec{r} = g(c_1), \ldots, g(c_{N_n})$ and $\vec{z} = g''|_m (c_1), \ldots, g''|_m (c_{N_n})$, we see that $\vec{r}$ and $\vec{z}$ are $c^3$-partitions in $S$ with the same neighbourhood graphs—namely, the neighbourhood graph of $c_1, \ldots, c_{N_n}$. By Theorem 1.78, let $h : \mathbb{S}^2 \to \mathbb{S}^2$ be a homeomorphism taking $\vec{z}$ to $\vec{r}$. Thus, $h \circ g''$ maps $\vec{z}^{(n+1)}$ to the faces of a plane graph $G$ in $\mathbb{S}^2$ whose edges include the frontiers of the elements $\vec{r}$. Now let $h' : \mathbb{S}^2 \to \mathbb{S}^2$ be a deformation making $G$ piecewise linear, while leaving any already piecewise linear edges unaffected. By Theorem 1.82, $g' = h' \circ h \circ g'' \in w^{(n+1)}$ maps $\vec{z}^{(n+1)}$ to an $N_{n+1}$-tuple in $S$ and it is easy to see that $g'$ satisfies the conditions of the claim. QED

By Claim 1.109, there exists a sequence of embeddings:

$$\emptyset = g^{[0]}, g^{[1]}, g^{[2]}, \ldots$$

such that, for all $n$ ($0 < n$), $g^{(n)}$ maps $\vec{e}^{(n)}$ to $S$, and, for all $m, n$ ($0 \leq m < n$), $g^{(n)}|_m = g^{(m)}$.

Now let $a \in A$ be such that $a = c_{i_1}^{(n)} + \ldots + c_{i_k}^{(n)}$. Then we define

$$g(a) = g^{(n)}(c_{i_1}^{(n)}) + \ldots + g^{(n)}(c_{i_k}^{(n)}).$$

The fact that $g^{(n)}|_m = g^{(m)}$ whenever $0 \leq m < n$ means that this mapping is well defined. It is easy to see that $g : A \to S$ is a Boolean algebra isomorphism; moreover, by Claim 1.107, $g(a)$ is connected if and only if $a$ is connected. That is, we have proved:
Claim 1.110 $\mathfrak{A}$ can be isomorphically embedded in ROP($S^2$), regarded as a $\{c, \leq\}$-structure.

In view of Claim 1.110, and in order to simplify notation, we might as well take $\mathfrak{A}$ to be a substructure of ROP($S^2$). Note that the previously distinct uses of the Boolean functions and the term “connected” become unambiguous, as do “connected partition”, “$c^h$-partition”, “neighbour”, and so on. Moreover, since $A \subseteq S$, we may meaningfully talk about the frontier $\mathcal{F}(a)$ of any $a \in A$, and apply all the results established previously about elements of ROP($S^2$). For example, by Lemma 1.73, if $r_1, \ldots, r_n$ is a $c^2$-partition in $A$ radial about $r_1$ such that $r_1$ has at least 2 neighbours, then, for any neighbour $r_i$ of $r_1$, $\mathcal{F}(r_1) \cap \mathcal{F}(r_i)$ is a Jordan arc. Recall that, for tuples $\vec{r}$ and $\vec{s}$ from ROP($S^2$), we write $\vec{r} \sim \vec{s}$ if $\vec{r}$ and $\vec{s}$ are similarly situated (in $S^2$).

Stage 3: In the previous stage, we established that $\mathfrak{A}$ can be chosen to be a substructure of ROP($S^2$). In this stage, we show that, in that case, $\mathfrak{A}$ is in fact an elementary substructure of ROP($S^2$).

Claim 1.111 Let $a_1, \ldots, a_n \in A$ be a $c^2$-partition radial about $a_1$ such that $a_1$ has at least 3 neighbours. Let $r_1, r_2 \in S$ be disjoint 2-cells with $a_1 = r_1 + r_2$. Then there exist $c_1, c_2 \in A$ such that $a_1, \ldots, a_n, c_1, c_2 \sim a_1, \ldots, a_n, r_1, r_2$.

Proof. Since $a_1, r_1, r_2$ are 2-cells with $a_1$ equal to the disjoint sum of $r_1$ and $r_2$, $r_1$ and $r_2$ must be separated by a cross-cut $\gamma$ in $a_1$. For any neighbour $a_i$ of $a_1$, $\mathcal{F}(a_1) \cap \mathcal{F}(a_i)$ is a Jordan arc. Let $p \in \mathcal{F}(a_1)$. By inspection, $p$ lies on either one or two Jordan arcs of the form $\mathcal{F}(a_i) \cap \mathcal{F}(a_1)$ where $a_i$ is a neighbour of $a_1$. We define the character of $p$, written $\chi(p)$ to be the set of those $i$ $(2 \leq i \leq n)$ such that $a_i$ is a neighbour of $a_1$.
and \( p \in \mathcal{F}(a_i) \) (Fig. 1.14a). Note that \( \chi(p) \) has either 1 or 2 elements. If \( \chi(p) \) has one element, then \( p \) lies on some Jordan arc \( \mathcal{F}(a_1) \cap \mathcal{F}(a_i) \), but not at its endpoints. If \( \chi(p) \) has two elements, then since \( a_1 \) has at least three neighbours, \( \chi(p) \) determines \( p \). Now let \( \gamma \) be a cross-cut in \( a_1 \). We define the \textit{character of} \( \gamma \), written \( \chi(\gamma) \) to be the set of characters of its endpoints. (See Fig. 1.14b and Fig. 1.14c for examples.) It is routine to show that, if \( \gamma_1 \) and \( \gamma_2 \) are two such cross-cuts and \( \chi(\gamma_1) = \chi(\gamma_2) \), there is a homeomorphism of the closed plane onto itself taking \( a_i \) to itself for all \( i \) \((1 \leq i \leq n)\) and taking \( \gamma_1 \) to \( \gamma_2 \). So, to prove the lemma, it suffices to establish that, if \( \gamma_1 \) is any cross-cut in \( a_1 \), there exist disjoint 2-cells \( c_1, c_2 \in A \) with \( a_1 = c_1 + c_2 \) such that the cross-cut \( \gamma_2 \) separating \( c_1 \) and \( c_2 \) in \( a_1 \) satisfies \( \chi(\gamma_1) = \chi(\gamma_2) \).

Let the endpoints of \( \gamma_1 \) be \( p \) and \( q \). We prove the result for the special case where \( \chi(\gamma) \), \( \chi(p) \) and \( \chi(q) \) all contain two elements; the other cases are dealt with similarly. Fig. 1.15a shows the sub-case where \( \chi(p) \) and \( \chi(q) \) are non-disjoint; Fig. 1.15b shows the sub-case where \( \chi(p) \) and \( \chi(q) \) are disjoint.

The sub-case of Fig. 1.15a is trivial: Axiom 8 with \( a_1 \) substituted for \( x \) and \( a_i \) for \( y \) immediately guarantees the existence of \( c_1, c_2 \in A \) partitioning \( a_1 \), and hence separated by a cross-cut \( \gamma_2 \); moreover the connectivity conditions on \( c_1 \) and \( c_2 \) mean that \( \gamma_1 \) and \( \gamma_2 \) have the same endpoints, so that \( \chi(\gamma_1) = \chi(\gamma_2) \).

The sub-case of Fig. 1.15b requires a little more work. However, two applications of Axiom 8 guarantee the existence in \( A \) of the regions \( a'_1, a'_k \) as in Fig. 1.15c. Axiom 7 then guarantees that the region labelled \( a'_1 \) in Fig. 1.15c can be split into two regions as shown in Fig. 1.15d.
Summing together appropriate subdivisions of \(a_1\) produces \(c_1, c_2 \in A\) separated by an arc \(\gamma_2\) satisfying \(\chi(\gamma_1) = \chi(\gamma_2)\).

The rest of this section is devoted to showing that we can relax the conditions of Claim 1.111.

**Claim 1.112** Let \(n > 1\) and let \(a_1, \ldots, a_n \in A\) be a partition such that \(a_1\) is a 2-cell. Let \(r_1, r_2 \in S\) be disjoint 2-cells with \(a_1 = r_1 + r_2\). Then there exist \(c_1, c_2 \in A\) such that \(a_1, \ldots, a_n, c_1, c_2 \sim a_1, \ldots, a_n, r_1, r_2\).

**Proof** Immediate given claims 1.65 and 1.111.

**Claim 1.113** Let \(n > 1\) and let \(a_1, \ldots, a_n \in A\) be a partition such that \(a_1\) is a 2-cell. Let \(r \in S\) be such that \(r \leq a_1\). Then there exists \(c \in A\) such that \(a_1, \ldots, a_n, c \sim a_1, \ldots, a_n, r\).

**Proof** By the construction of \(S = \text{ROP}(S^2)\), we can partition \(a_1\) into 2-cells \(r_1, \ldots, r_m\) such that \(r\) can be expressed as the sum of various \(r_j\). It suffices to show that there are \(c_1, \ldots, c_m \in A\) such that

\[
a_1, \ldots, a_n, r_1, \ldots, r_m \sim a_1, \ldots, a_n, c_1, \ldots, c_m.
\]

We proceed by induction on \(m\). If \(m = 1\), then \(r_1 = a_1\) and we are done. If \(m > 1\), by Corollary 1.67, we can renumber the \(r_i\) if necessary so that \(r_1\) and \(r'_2 = r_2 + \ldots + r_m\) are 2-cells. By Claim 1.112, there exist \(c_1, c'_2 \in A\) such that \(a_1, \ldots, a_n, r_1, r'_2 \sim a_1, \ldots, a_n, c_1, c'_2\). Let \(h\) be a homeomorphism of the closed plane onto itself mapping \(a_1\) to itself, \(r_1\) to \(c_1\) and \(r'_2\) to \(c'_2\). By exactly the same argument as for Lemma 1.90, \(h\) can be chosen so that \(h(r_i) \in S\) for all \(i\) (\(2 \leq i \leq m\)). But then the \(h(r_i)\) partition the 2-cell \(c'_2\) into 2-cells. So consider the partition \(c'_2, c_1, a_2, \ldots, a_n\). By inductive hypothesis, there exist \(c_2, \ldots, c_m \in A\) such that

\[
c'_2, c_1, a_2, \ldots, a_n, h(r_2), \ldots, h(r_m) \sim c'_2, c_1, a_2, \ldots, a_n, c_2, \ldots, c_m.
\]

The result then follows.

**Claim 1.114** Let \(n > 1\) and let \(a_1, \ldots, a_n \in A\) be a \(c^2\)-partition. Let \(r \in S\). Then there exists \(c \in A\) such that \(a_1, \ldots, a_n, c \sim a_1, \ldots, a_n, r\).

**Proof** Write \(r\) as the sum \(r \cdot a_1 + \ldots + r \cdot a_n\). By considering these terms separately, we use Claim 1.113 and an induction similar to that used in the proof of Claim 1.113. The details are routine.
Claim 1.115 Let \( n \geq 0 \) and let \( a_1, \ldots, a_n \in A \). Let \( r \in S \). Then there exists \( c \in A \) such that \( a_1, \ldots, a_n, r \sim a_1, \ldots, a_n, c \).

Proof Immediate given Claims 1.54, 1.63 and 1.114.

Claim 1.116 \( \mathfrak{A} \preceq \text{ROP}(S^2) \).

Proof We certainly have \( \mathfrak{A} \subseteq \text{ROP}(S^2) \). Let \( n \geq 0 \) and let \( \phi(x_1, \ldots, x_n) \) be any formula of the form \( \exists y \psi(x_1, \ldots, x_n, y) \). Let \( a_1, \ldots, a_n \in A \) such that \( \text{ROP}(S^2) \models \phi[a_1, \ldots, a_n] \). Then there exists \( r \in S \) such that \( \text{ROP}(S^2) \models \psi[a_1, \ldots, a_n, r] \). By Claim 1.115, there exists \( c \in A \) such that \( a_1, \ldots, a_n, r \sim a_1, \ldots, a_n, c \). By Lemmas 1.90 and 1.91, \( \text{ROP}(S^2) \models \psi[a_1, \ldots, a_n, c] \). The claim then follows by Proposition 1.94. \( \text{QED} \)

By Claim 1.116, \( \mathfrak{A} \) and \( \text{ROP}(S^2) \) have the same theory. But by construction, \( \mathfrak{A} \models \neg \phi \), whence \( \text{ROP}(S^2) \models \neg \phi \), which completes the proof of Lemma 1.104. \( \text{QED} \)

Corollary 1.117 All splittable, finitely decomposable mereotopologies over \( S^2 \) with curve-selection have the same \( L_{c, \leq} \)-theory, and hence also the same \( L_C \)-theory.

Thus, while Examples 1.17, 1.18 and 1.99 show that there certainly are elementarily inequivalent mereotopologies over \( \mathbb{R}^2 \) and \( S^2 \), Corollary 1.117 indicates that there is nothing like the free-for-all one might initially expect. At least for the signatures \( \{ c, \leq \} \) and \( \{ C \} \), the reference to \( T_\mathbb{R} \) as the standard first-order mereotopological theory of the closed plane is justified. Corollary 1.173 generalizes this result to apply to any signature of topological primitives.

For reasons of simplicity (which we trust the reader will appreciate) we have provided an axiomatization of well-behaved plane mereotopologies only for the language \( L_{c, \leq} \). It should be clear from the foregoing discussion, however, that an analogous result could be obtained for the language \( L_C \), which as we noted, is more expressive over \( \text{ROP}(\mathbb{R}^2) \). Such an axiomatization was developed in Schoop, 1999.

Of course, it is one thing to have an axiomatic characterization of the \( L_{c, \leq} \)-theory of \( \text{ROP}(S^2) \)—quite another to determine whether a given \( L_{c, \leq} \)-sentence is a member of it. The question therefore arises as to the computational characteristics of this problem. Dornheim, 1998 showed (in effect) that this theory is undecidable and hence (since it is a complete theory), not r.e. It follows that the \( \omega \)-rule (or some equivalent mechanism) is indispensable in this axiomatization. In fact, Schaefer and Štefankovič, 2004 showed (in effect) that the decision problem for \( \text{Th}_{c, \leq} \text{ROP}(S^2) \) is at least as hard as that of second-order arithmetic.
Specifically, Schaefer and Štefankovič effectively encode second-order arithmetic in a first-order language with variables ranging over 2-cells in \( \mathbb{R}^2 \) and primitive predicates expressing the so-called RCC-relations (see Randell et al., 1992; Egenhofer, 1991; but it is easy to see that that theory can in turn be effectively encoded in \( \text{Th}_{\leq} \text{ROP}(\mathbb{S}^2) \). Schaefer and Štefankovič also consider the complexity of the quantifier-free fragment of their logic, a problem closely related to the well-known problem of recognizing so-called string-graphs (see e.g., Erdély et al., 1976; Kratochvíl, 1988), and show that it is in NEXPTIME. In Schaefer et al., 2003, this bound is improved to \( \text{NP} \)—a very surprising result.

6. Spatial mereotopology

In this section, we extend the main results of Section 4 to the spatial mereotopology \( \text{ROP}(\mathbb{R}^3) \). This material is a tidied up version of Pratt and Schoop, 2002.

6.1 Facts about \( \text{ROP}(\mathbb{R}^3) \) and \( \text{ROP}(\mathbb{S}^3) \)

Recall that a 2-manifold is a Hausdorff space locally homeomorphic at every point to the open disc \( \mathbb{B}^2 \), and that a surface is a connected 2-manifold.

**Lemma 1.118** Let \( X \) be either \( \mathbb{R}^n \) or \( \mathbb{S}^n \), and let \( M \) be a mereotopology over \( X \) having curve-selection. If \( r \in M \) with \( r \) and \( \lnot r \) both connected, then \( \mathcal{F}(r) \) is connected.

*Proof* Consider the case \( X = \mathbb{R}^n \). Let \( r \in M \) be connected and non-empty with connected, non-empty complement, and suppose the closed set \( \mathcal{F}(r) \) is not connected. Let \( d_1 \) and \( d_2 \) be closed sets partitioning \( \mathcal{F}(r) \), and let \( p \in r \), \( q \in \lnot r \). Since \( r \) is connected with connected complement, it is easy to see that the conditions of Proposition 1.19 are fulfilled, so that \( p \) and \( q \) are connected in \( \mathbb{R}^n \setminus (d_1 \cup d_2) \). But this is absurd given that \( d_1 \cup d_2 = \mathcal{F}(r) \). The case \( X = \mathbb{S}^n \) follows easily.

**QED**

**Lemma 1.119** Let \( r \in \text{ROP}(\mathbb{S}^3) \) such that \( r \) and \( \lnot r \) are non-empty and connected, and \( \mathcal{F}(r) \) is not a surface. Then the graph \( K^5 \) can be drawn in \( \mathcal{F}(r) \).

*Proof* It is easy to see that \( \mathcal{F}(r) \) can be finitely triangulated. Call any point where \( \mathcal{F}(r) \) is not locally homeomorphic to \( \mathbb{B}^2 \) a bad point; and call any edge of the triangulation all of whose points are bad a bad edge. By the properties of triangulations, any bad point either occurs on a bad edge or else is an isolated bad point at a vertex of the triangulation.
If there is a bad edge, then more than two triangles must share this edge, and the embedding of $K^3$ in $\mathcal{F}(r)$ proceeds as shown in Fig. 1.16a. Assume, then, that there are no bad edges, but that some vertex $p$ of the triangulation is an isolated bad point. Call two triangles with $p$ as a vertex *neighbours* if they share an edge having $p$ as a vertex. Since all edges are good, these triangles can clearly be arranged into disjoint cycles such that each triangle belongs to the same cycle as its two neighbours. Choose one such cycle. By applying a homeomorphism if necessary, we may assume that this triangle-cycle forms a cone with vertex $p$ as shown in Fig. 1.16b. Since there are only finitely many triangles in the triangulation, we can ensure that we choose a triangle-cycle such that the points inside the tip of the cone either all belong to $r$ or all belong to $-r$. Let $s$ be either $r$ or $-r$ depending on which of these possibilities is realized. Note that, since $r$ is non-empty and connected with non-empty, connected complement, so is $s$.

Let $t \in \text{ROP}(\mathbb{S}^3)$ be a small element representing the tip of the cone, indicated by the light dotted lines in Fig. 1.16b. Removing $t$ from $s$ visibly does not disconnect $s$, so that $s \cdot -t$ is connected; moreover, $t$ shares some face with $-s$, so that $t + -s = -(s \cdot -t)$ is also connected. Thus, $s \cdot -t$ is non-empty and connected with non-empty connected complement, whence, by Lemma 1.118, $\mathcal{F}(s \cdot -t)$ is connected. Moreover, since $p$ is bad, there must be at least two triangle-cycles with $p$ as vertex; whence $p \in \mathcal{F}(s \cdot -t)$. Thus we may choose a point $q$ on the base rim of $t$ and connect it to $p$ by a Jordan arc $\alpha$ in $\mathcal{F}(s)$ such that the locus of $\alpha$ is disjoint from $\mathcal{F}(t)$ except for its endpoints, as shown in Fig. 1.16c. The embedding of $K^3$ in $\mathcal{F}(s) = \mathcal{F}(r)$ then proceeds as depicted. QED

One notable difference between $\mathbb{S}^2$ and $\mathbb{S}^3$ is that the Schönflies Theorem, which holds in the former, fails in the latter. In fact, the patho-
logical ‘region’ known as Alexander’s horned sphere, and depicted in
Fig. 1.3 is the best-known counterexample: the frontier of this region
is homeomorphic to $\mathbb{S}^2$, but its exterior is not simply connected, and is
certainly therefore not homeomorphic to $\mathbb{B}^3$. Nevertheless, Alexander,
1924b also proved a Schönflies-type result for polyhedra, which, in our
notation, can be written as follows. (See also Moise, 1977, Ch. 17.)

**Proposition 1.120** Let $r \in \text{ROP}(\mathbb{S}^3)$ be such that $\mathcal{F}(r)$ is homeomorphic to $\mathbb{S}^2$. Then both $r^-$ and $(-r)^-$ are homeomorphic to $\mathbb{D}^3$.

To avoid cumbersome locations in the sequel, we define:

**Definition 1.121** Let $X$ be either $\mathbb{R}^n$ or $\mathbb{S}^n$. A ball in $X$ is a subset of $X$ similarly situated in $X$ to the unit ball $\mathbb{B}^3$. A polyhedral ball in $X$ is a ball which is an element of $\text{ROP}(X)$.

Thus, if $r \in \text{ROP}(\mathbb{S}^3)$ with $\mathcal{F}(r)$ homeomorphic to $\mathbb{S}^2$, then $r$ and $-r$ are both balls in $\mathbb{S}^3$. Furthermore, if $r \in \text{ROP}(\mathbb{R}^3)$ with $\mathcal{F}(r)$ homeomorphic to $\mathbb{S}^2$ (and hence bounded), then then exactly one of $r$ and $-r$ is a ball in $\mathbb{R}^3$. We note in passing:

**Lemma 1.122** If $r \in \text{RO}(\mathbb{S}^3)$ is a (polyhedral) ball in $\mathbb{S}^3$, then so is $-r$.

**Proof** By definition, $r$ is similarly situated in $\mathbb{S}^3$ to $u = \mathbb{B}^3(0,1)$. By considering a spherical inversion, $u$ is similarly situated in $\mathbb{S}^3$ to $-u$.

QED

The following well-known theorem will also prove useful in the sequel
(see, e.g. Massey, 1967, p. 10).

**Proposition 1.123** (Classification Theorem for Surfaces) Every compact surface is homeomorphic to either (i) $\mathbb{S}^2$ or (ii) the sum of finitely many connected tori or (iii) the sum of finitely many projective planes.

### 6.2 Expressing familiar spatial concepts in $L_C$

Our next task is to show that certain familiar concepts defined on
the mereotopology $\text{ROP}(\mathbb{R}^3)$ can be expressed using $L_C$-formulas. As a preliminary, recall the discussion of Section 3.2, which showed that: (i)
expressions such as $x^- \cap y^- \cap z \neq \emptyset$ etc. can be regarded as $L_C$-formulas;
and (ii) there is an $L_C$-formula $\phi_{\{i\}}(x, y)$ which we may read as “$x^- \cap y^-$ is connected”.

Now suppose $r$ and $s$ are elements of $\text{ROP}(\mathbb{R}^3)$, and consider, for example, the set $\mathcal{F}(r) \setminus \mathcal{F}(s)$. Evidently, this set is connected if and only
if it is piecewise-linear arc-connected, and therefore if and only if any two points in it are contained within some connected set of the form \( r^- \cap t^- \subseteq \mathcal{F}(r) \setminus \mathcal{F}(s) \) with \( t \in \text{ROP}(\mathbb{R}^3) \). It follows from the discussion of Section 3.2 that there is an \( L_C \)-formula satisfied by a pair of regions \( r, s \) if and only if \( \mathcal{F}(r) \setminus \mathcal{F}(s) \) is connected. In the sequel, then, we write, without further commentary, expressions such as \( c(\mathcal{F}(x) \setminus \mathcal{F}(y)) \) etc. as \( L_C \)-formulas having the obvious interpretations.

**Lemma 1.124** There exists an \( L_C \)-formula \( \phi_{K^5}(x) \) such that, for all \( r \in \text{ROP}(\mathbb{R}^3) \), \( \text{ROP}(\mathbb{R}^3) \models \phi_{K^5}(r) \) if and only if \( K^5 \) is embeddable in \( \mathcal{F}(r) \).

**Proof** The graph \( K^5 \) is evidently embeddable in \( \mathcal{F}(r) \) if and only if there exist polyhedra \( v_i (1 \leq i \leq 5) \) and \( e_{i,j} (1 \leq i < j \leq 5) \), all disjoint from \( r \) and from each other, satisfying the following conditions:

1. For all \( i (1 \leq i \leq 5) \), \( v_i^- \cap r^- \) is a singleton
2. For all \( i, j (1 \leq i < j \leq 5) \), \( e_{i,j}^- \cap r^- \) is connected
3. For all \( i, j, i', j' (1 \leq i < j \leq 5, 1 \leq i' < j' \leq 5) \), \( \{i, j\} \cap \{i', j'\} = \emptyset \) implies \( e_{i,j}^- \cap e_{i', j'}^- \cap r^- = \emptyset \), and \( \{i, j\} \cap \{i', j'\} = \{k\} \) implies \( e_{i,j}^- \cap e_{i', j'}^- \cap r^- = v_k^\circ \cap r^- \).

(Note incidentally that the polyhedra \( e_{i,j} \) are not themselves required to be connected—only the sets \( e_{i,j}^- \cap r^- = \mathcal{F}(e_{i,j}) \cap \mathcal{F}(r) \).) But the above conditions are expressible in \( L_C \) over \( \text{ROP}(\mathbb{R}^3) \).

**QED**

**Lemma 1.125** There exists an \( L_C \)-formula \( \phi_{\mathcal{F}^r}(x) \) such that, for all \( r \in \text{ROP}(\mathbb{R}^3) \):

1. if \( \mathcal{F}(r) \) is connected and unbounded, then \( \text{ROP}(\mathbb{R}^3) \models \phi_{\mathcal{F}^r}(r) \);
2. if \( \mathcal{F}(r) \) is homeomorphic to \( \mathbb{S}^2 \), then \( \text{ROP}(\mathbb{R}^3) \not\models \phi_{\mathcal{F}^r}(r) \).

**Proof** Let \( \phi_{\mathcal{F}^r}(x) \) be

\[
\exists y_1 \exists y_2 (y_1 \cdot x = 0 \land y_2 \cdot x = 0 \land c(\mathcal{F}(x) \cap \mathcal{F}(y_1) \cap \mathcal{F}(y_2)) \land \\
c(\mathcal{F}(x) \setminus \mathcal{F}(y_1)) \land c(\mathcal{F}(x) \setminus \mathcal{F}(y_2)) \land \neg c(\mathcal{F}(x) \setminus (\mathcal{F}(y_1) \cup \mathcal{F}(y_2))).
\]

Thus, \( \phi_{\mathcal{F}^r}(x) \) ‘says’ that there exist polyhedra \( y_1 \) and \( y_2 \), disjoint from \( x \), such that the sets \( \mathcal{F}(x) \cap \mathcal{F}(y_1) \cap \mathcal{F}(y_2) \), \( \mathcal{F}(x) \setminus \mathcal{F}(y_1) \) and \( \mathcal{F}(x) \setminus \mathcal{F}(y_2) \) are all connected, but the set \( \mathcal{F}(x) \setminus (\mathcal{F}(y_1) \cup \mathcal{F}(y_2)) \) is not.
Suppose $\mathcal{F}(r)$ is connected and unbounded. Let $r$ be a Boolean combination of finitely many half-spaces, corresponding to a finite set of planes, say, $P_1, \ldots, P_m$; it is then easy to see that $\mathcal{F}(r) \subseteq P_1 \cup \cdots \cup P_m$.

Since $\mathcal{F}(r)$ is unbounded, we can draw in $\mathcal{F}(r)$ a rectangular figure $G$, unbounded on one side (dotted lines in Fig. 1.17), such that $G$ intersects only one of the $P_i$. Let $s_1, s_2 \in \text{ROP}(\mathbb{R}^3)$ be laminas, infinitely extended in one direction, and placed on $G$ (on the outside of $r$) so that $\mathcal{F}(r) \cap \mathcal{F}(s_1)$ and $\mathcal{F}(r) \cap \mathcal{F}(s_2)$ are arranged as shown. Since $\mathcal{F}(r)$ is connected, $\mathcal{F}(r) \setminus \mathcal{F}(s_1)$ and $\mathcal{F}(r) \setminus \mathcal{F}(s_2)$ are also connected; and since $G$ lies on just one of the $P_i$, $\mathcal{F}(r) \setminus (\mathcal{F}(s_1) \cup \mathcal{F}(s_2))$ is not connected. Thus $\text{ROP}(\mathbb{R}^3) \models \gamma[r]$. The second part of the Lemma follows by Proposition 1.47.

Let $\phi_k(x)$ be the $L_C$-formula defined in Lemma 1.27 and satisfied by $r \in \text{ROP}(\mathbb{R}^3)$ if and only if $r$ is connected, and let $\phi_1(x)$ abbreviate the formula $x \neq 0 \land x \neq 1 \land \phi_k(x) \land \phi_k(-x)$.

**Lemma 1.126** For all $r \in \text{ROP}(\mathbb{R}^3)$, $r$ satisfies $\phi_1(x) \land \neg \phi_{K^3}(x) \land \neg \phi_{K^3}(x)$ if and only if $\mathcal{F}(r)$ is homeomorphic to $\mathbb{S}^2$.

**Proof** Suppose $\mathcal{F}(r)$ is homeomorphic to $\mathbb{S}^2$. Certainly, by Proposition 1.120, $\text{ROP}(\mathbb{R}^3) \models \phi_1[r]$; by Lemma 1.124, $\text{ROP}(\mathbb{R}^3) \models \neg \phi_{K^3}[r]$; by Lemma 1.125, $\text{ROP}(\mathbb{R}^3) \models \neg \phi_{K^3}[r]$. Conversely, suppose that $r$ satisfies $\phi_1(x) \land \neg \phi_{K^3}(x) \land \neg \phi_{K^3}(x)$. By Lemma 1.118, $\mathcal{F}(r)$ is connected,
and by the first part of Lemma 1.125, \( \mathcal{F}(r) \) is bounded. Moreover, \( K^5 \) cannot be embedded in \( \mathcal{F}(r) \), by Lemma 1.124. Hence \( \mathcal{F}(r) = F(\hat{r}) \) is a compact surface, by Lemma 1.119. The result then follows from Proposition 1.123. 

**LEMMA 1.127** Let \( r \in \text{ROP}(\mathbb{R}^3) \) satisfy \( \phi_1(x) \land \neg \phi_{K^5}(x) \land \phi_{\nu^+}(x) \). Then \( r \) is unbounded.

**Proof** Suppose for contradiction that \( r \) is bounded, so that we also have \( r \in \text{ROP}(\mathbb{S}^3) \). By Lemma 1.119, \( \mathcal{F}(r) \) is a surface. Moreover, since \( r \) is bounded, \( \mathcal{F}(r) \) is compact, and since \( K^5 \) cannot be drawn in \( \mathcal{F}(s) \), \( \mathcal{F}(r) \) is homeomorphic to \( \mathbb{S}^2 \) by Proposition 1.123. But since \( \text{ROP}(\mathbb{R}^3) \models \phi_{\nu^+}[r] \), this contradicts the second part of Lemma 1.125. Hence, \( r \) is unbounded. \( \quad \Box \)

**LEMMA 1.128** There exists an LC-formula \( \phi_{\nu^3}(x) \) such that, for all \( r \in \text{ROP}(\mathbb{R}^3) \), \( \text{ROP}(\mathbb{R}^3) \models \phi_{\nu^3}[r] \) if and only if \( r \) is bounded.

**Proof** Let \( \phi_{\nu^3}(x) \) be the formula

\[
\exists y \exists z(x \leq y \land y \cdot z = 0 \land \\
\phi_1(y) \land \neg \phi_{K^5}(y) \land \neg \phi_{\nu^+}(y) \land \phi_1(z) \land \neg \phi_{K^5}(z) \land \phi_{\nu^+}(z)).
\]

If \( r \) is bounded, let \( s \in \text{ROP}(\mathbb{R}^3) \) be a ball in \( \mathbb{R}^3 \) such that \( r \leq s \); and let \( t \in \text{ROP}(\mathbb{R}^3) \) be a half-space disjoint from \( s \). By Lemma 1.125, \( \text{ROP}(\mathbb{R}^3) \models \neg \phi_{\nu^3}[s] \) and \( \text{ROP}(\mathbb{R}^3) \models \phi_{\nu^3}[t] \). Thus, \( s \) and \( t \) are suitable witnesses for \( y \) and \( z \) in \( \phi_{\nu^3}(x) \), so that \( \text{ROP}(\mathbb{R}^3) \models \phi_{\nu^3}[r] \).

Conversely, suppose that \( \text{ROP}(\mathbb{R}^3) \models \phi_{\nu^3}[r] \). Let \( s \) and \( t \) be witnesses for \( y \) and \( z \). By Lemma 1.126, \( \mathcal{F}(s) \) is homeomorphic to \( \mathbb{S}^2 \), whence, by Proposition 1.120, exactly one of \( s \) and \( -s \) is a ball in \( \mathbb{R}^3 \). By Lemma 1.127, \( t \) is unbounded, and so intersects the complement of every ball in \( \mathbb{R}^3 \). Therefore \( -s \) is not a ball in \( \mathbb{R}^3 \), so \( s \) is. Hence, \( r \) is bounded. \( \quad \Box \)

**THEOREM 1.129** There exists a formula \( \phi_3(x) \) such that, for all \( r \in \text{ROP}(\mathbb{R}^3) \), \( \text{ROP}(\mathbb{R}^3) \models \phi_3[r] \) if and only if \( r \) is a polyhedral ball in \( \mathbb{R}^3 \).

**Proof** Let \( \phi_3(x) \) be

\[
\phi_1(x) \land \neg \phi_{K^5}(x) \land \neg \phi_{\nu^+}(x) \land \phi_{\nu^3}(x),
\]

and apply Lemmas 1.126 and 1.128. \( \quad \Box \)
Thus, with a little effort, we can define certain familiar topological notions over \( \text{ROP}(\mathbb{R}^3) \) using \( L_C \)-formulas. The following technical material, which is devoted to defining some decidedly unfamiliar topological notions over \( \text{ROP}(\mathbb{R}^3) \), will be used in the sequel. We recall the discussion of compactifications in Section 3.3, and consider the mapping \( r \mapsto \hat{r} \) from \( \text{ROP}(\mathbb{R}^3) \) to its 1-point compactification \( \text{ROP}(\mathbb{S}^3) \). By Lemmas 1.36 and 1.37, this mapping is a Boolean algebra isomorphism and preserves the properties of connectedness and non-connectedness. For technical reasons, we will occasionally need to consider properties of elements in \( \text{ROP}(\mathbb{R}^3) \) whose defining conditions make reference to their counterparts in \( \text{ROP}(\mathbb{S}^3) \).

For all \( r \in \text{ROP}(\mathbb{R}^3) \), \( \in \hat{r} \) if and only if \( \overline{r} \) is bounded, and \( \in \hat{r} \) if and only if \( r \) is unbounded (where the closure operator \( \overline{\cdot} \) refers to the topology on \( \mathbb{S}^3 \)). By Lemma 1.128 then, it is harmless to employ the expression \( \in \hat{x} \) in \( L_C \)-formulas, since we can take it as a mnemonic for \( \phi_{B^5}(\overline{x}) \); and similarly for expressions such as \( \in \hat{\overline{x}} \), \( \in \hat{\mathcal{F}}(\hat{x}) \), etc.

**Lemma 1.130** There exists a formula \( \phi_{\hat{K}^5}(x) \) satisfied by \( r \in \text{ROP}(\mathbb{R}^3) \) if and only if \( K^5 \) is embeddable in \( \mathcal{F}(\hat{r}) \).

**Proof** As for Lemma 1.124, making the obvious adjustments to accommodate the point at infinity. \( \quad \text{QED} \)

**Lemma 1.131** There exists a formula \( \phi_{\hat{B}}(x) \) such that, for all \( r \in \text{ROP}(\mathbb{R}^3) \), \( \text{ROP}(\mathbb{R}^3) \models \phi_{\hat{B}}[r] \) if and only if \( \hat{r} \) is a ball in \( \mathbb{S}^3 \).

**Proof** Let \( \phi_{B}(x) \) be \( \phi_{B}(x) \land -\phi_{\hat{K}^5}(x) \). If \( \hat{r} \) is a ball in \( \mathbb{S}^3 \), it is evident that \( \text{ROP}(\mathbb{R}^3) \models \phi_{B}[\hat{r}] \). Conversely, suppose \( \text{ROP}(\mathbb{R}^3) \models \phi_{B}[\hat{r}] \). By Lemmas 1.119 and 1.130, \( \mathcal{F}(\hat{r}) \) is a surface in \( \mathbb{S}^3 \). Furthermore, by Proposition 1.123, \( \mathcal{F}(\hat{r}) \) is homeomorphic to \( \mathbb{S}^2 \). The result then follows by Proposition 1.120. \( \quad \text{QED} \)

### 6.3 Characterizing triangulations in \( L_C \)

In Section 4, we showed that every tuple in \( \text{ROP}(\mathbb{R}^2) \) satisfies a topologically complete \( L_C \)-formula—that is, an \( L_C \)-formula with the property that all tuples satisfying it are similarly situated. Our proof exploited Whitney’s theorem on 3-connected graphs in the plane to show that any \( c^3 \)-partition in \( \text{ROP}(\mathbb{S}^2) \) is determined up to similar situation by its neighbourhood graph. However, Whitney’s theorem is not available for \( \mathbb{S}^3 \), and so we must adopt an alternative approach to analysing the expressive power of \( L_C \) over \( \text{ROP}(\mathbb{R}^3) \). This approach has the advantage of
being, in some ways, more straightforward than that of Section 4, though
the topologically complete formulas it constructs are more complicated.

We assume familiarity with the basic theory of triangulations and
PL-complexes: for details, see, e.g., Moise, 1977, Ch. 7. We also re-
quire the following ‘obvious’ result about balls in $\mathbb{S}^3$ (Pratt and Schoop,

**Proposition 1.132** Let $r, s \in \text{RO}(\mathbb{S}^3)$ be disjoint balls in $\mathbb{S}^3$ such that
$r + s$ is also a ball in $\mathbb{S}^3$. Then $\mathcal{F}(r) \cap \mathcal{F}(s) \cap \mathcal{F}(r + s)$ is the locus of a
Jordan curve, and $\mathcal{F}(r) \cap \mathcal{F}(s)$ is homeomorphic to the closed disc $D^2$.

The situation is illustrated in Fig. 1.18.

**Definition 1.133** A quadruple $q = \langle r_1, r_2, r_3, r_4 \rangle$ of pairwise disjoint
elements of $\text{ROP}(\mathbb{S}^3)$ is a $q$-cell in $\mathbb{S}^3$ if, for all non-empty $J \subseteq \{1, 2, 3, 4\}$,
the polyhedron $\sum_{j \in J} r_j$ is a ball in $\mathbb{S}^3$.

The reference to the containing space $\mathbb{S}^3$ is significant: in the sequel, we
introduce $q$-cells in $\mathbb{R}^3$. However, we sometimes speak simply of $q$-cells if
it is clear from context which space we are talking about (or if it makes
no difference).

**Example 1.134** Consider the regular open tetrahedron $t_0$ with vertices
$v_1 = (0, 0, 0), v_2 = (1, 0, 0), v_3 = (0, 1, 0), v_4 = (0, 0, 1)$. Let $t_1, t_2, t_3, t_4$
be the four regular open tetrahedra (taken in some fixed order) each hav-
ing three vertices from $\{v_1, \ldots, v_4\}$ and the point $(1/4, 1/4, 1/4)$ as the
fourth vertex (Fig. 1.19). Evidently, the quadruple $q_0 = \langle t_1, t_2, t_3, t_4 \rangle$ is
a $q$-cell.

**Theorem 1.135** All $q$-cells in $\mathbb{S}^3$ are similarly situated in $\mathbb{S}^3$. 
**First-Order Mereotopology**

![Diagram](image)

*Figure 1.19*  The q-cell $q_0$

![Diagram](image)

*Figure 1.20*  Possible arrangements of $\mathcal{F}(a) \cap S$, $\mathcal{F}(b) \cap S$, $\mathcal{F}(c) \cap S$ and $\mathcal{F}(d) \cap S$, where $S = \mathcal{F}(a + b + c)$ (Proof of Theorem 1.133).

**Proof**  Let $\langle a, b, c, d \rangle$ be a q-cell. Since $a$, $b$, $c$, $a + b$, $b + c$, $a + c$ and $a + b + c$ are balls, by Proposition 1.132, the sets $\mathcal{F}(a) \cap \mathcal{F}(b)$, $\mathcal{F}(a) \cap \mathcal{F}(c)$, $\mathcal{F}(b) \cap \mathcal{F}(c)$ and $\mathcal{F}(a + b) \cap \mathcal{F}(e)$ are all closed discs. Letting $S = \mathcal{F}(a + b + c)$, it is then easy to show that the sets $\mathcal{F}(a) \cap S$, $\mathcal{F}(b) \cap S$ and $\mathcal{F}(c) \cap S$ must be arranged on $S$ as shown in Fig. 1.20a, up to similar situation. Let $e = -(a + b + c + d)$; then, by Lemma 1.122, $\sum (B \cup \{e\})$ is a ball for any proper subset $B \subset \{a, b, c, d\}$. Thus, all of the sets $a + b + c$, $d$, $e$, $a + b + c + d$ and $a + b + c + e$ are balls. By Proposition 1.132 again, $\mathcal{F}(d) \cap S$ and $\mathcal{F}(e) \cap S$ are both closed discs, whose common frontier in the space $S$ is the locus of some Jordan curve $\gamma$, say.

Consider how $\gamma$ might be drawn on $S$. Since $a + d$ and $a + e$ are balls, by Proposition 1.132, $\mathcal{F}(a) \cap \mathcal{F}(d)$ and $\mathcal{F}(a) \cap \mathcal{F}(e)$ are closed discs. Similarly, $\mathcal{F}(b) \cap \mathcal{F}(d)$, $\mathcal{F}(b) \cap \mathcal{F}(e)$, $\mathcal{F}(c) \cap \mathcal{F}(d)$ and $\mathcal{F}(c) \cap \mathcal{F}(e)$ are closed discs. Hence $\gamma$ divides each of the three sets $\mathcal{F}(a) \cap S$, $\mathcal{F}(b) \cap S$ and
\[ F(c) \cap S \] into two residual domains. Moreover, \( \gamma \) cannot pass through either of the points \( X \) or \( Y \) in Fig. 1.20a; for otherwise, one of the sets \( F(a) \cap F(d), F(b) \cap F(d), F(c) \cap F(d) \), \( F(a) \cap F(e), F(b) \cap F(e) \) or \( F(c) \cap F(e) \) would fail to be a closed disc. It is then easy to see that \( \gamma \) and the region \( F(d) \cap S \) it encloses must lie in \( S \) as shown in Fig. 1.20b or Fig. 1.20c, up to similar situation. But these two arrangements of \( a, b, c, d \) are mirror images.

QED

**Notation 1.136** If \( q = \langle t_1, \ldots, t_4 \rangle \) is a \( q \)-cell, denote the component polyhedron \( t_i \) by \( q[i] \) for all \( i \) \((1 \leq i \leq 4)\). Denote the polyhedron \( t_1 + \cdots + t_4 \) by \( \hat{q} \).

In Example 1.134, \( q_0 \) is the interior of the convex hull of the points \( V = \{v_1, \ldots, v_4\} \). We employ familiar terms from discussions of simplicial complexes: a face of \( q_0 \) is the convex closure of any non-empty subset of \( V \); a face of \( q_0 \) is proper if it is not the whole of \( \hat{q}_0 \); a vertex of \( q_0 \) is an element of \( V \).

**Definition 1.137** Let \( q \) be any \( q \)-cell, and \( h \) a homeomorphism of \( S^3 \) onto itself taking \( q_0 \) to \( q \). A (proper) face of \( q \) is a set of points \( h(F) \), where \( F \) is a (proper) face of \( q_0 \). A vertex of \( q \) is a point \( h(v) \), where \( v \) is a vertex of \( q_0 \).

We remark that, in Definition 1.137, a suitable homeomorphism \( h \) can always be found, by Theorem 1.135; moreover, since the faces of \( q_0 \) are expressible as set-algebraic combinations of the polyhedra \( t_1, \ldots, t_4 \) and their topological closures, the precise choice of \( h \) does not matter. Thus, \( q \)-cells are simply homeomorphic images of the \( q \)-cell \( q_0 \) of Example 1.134, with the notions of face and vertex transferred in the obvious way.

**Definition 1.138** A \( q \)-cell partition (in \( ROP(S^3) \)) is a sequence \( \tilde{q} = q_1, \ldots, q_n \) of \( q \)-cells in \( S^3 \) such that (i) \( \tilde{q}_1, \ldots, \tilde{q}_n \) is a partition in \( ROP(S^3) \); and (ii) for all \( i, j \) \((1 \leq i < j \leq n)\), if \( F \) is a face of \( q_i \) and \( G \) a face of \( q_j \), then \( F \cap G \) is either empty or a face of both \( q_i \) and \( q_j \). A vertex of a \( q \)-cell partition is a vertex of one of its elements.

Thus, \( q \)-cell partitions define (finite) PL-complexes in the obvious way: each \( q \)-cell in the partition corresponds to a PL 3-simplex, and its proper faces to PL \( d \)-simplices for \( d < 3 \).

**Definition 1.139** Let \( q = q_1, \ldots, q_N \) and \( q' = q'_1, \ldots, q'_N \) be \( q \)-cell partitions in \( ROP(S^3) \). We say that \( \tilde{q} \) and \( \tilde{q}' \) are isomorphic if there is a bijection between the vertices of \( \tilde{q} \) and the vertices of \( \tilde{q}' \) such that, for all
\[i, j \ (1 \leq i \leq N, \ 1 \leq j \leq 4), \text{ the vertices of } q_i \text{ lying on the frontier of } q_i[j] \text{ are mapped to the vertices of } q'_i \text{ lying on the frontier of } q'_i[j].\]

**Lemma 1.140** Isomorphic q-cell partitions in \(\text{ROP}(S^3)\) are similarly situated in \(S^3\).

**Proof** Isomorphic q-cell partitions define isomorphic PL-complexes. \(\Box\)

We conclude this sub-section by extending the notions of q-cell and q-cell partition to the open space \(\mathbb{R}^3\).

**Definition 1.141** A quadruple \(q = (r_1, r_2, r_3, r_4)\) of elements of \(\text{ROP}(\mathbb{R}^3)\) is a q-cell in \(\mathbb{R}^3\) if \(\tilde{q} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3, \tilde{r}_4)\) is a q-cell in \(S^3\). A sequence \(\tilde{q} = q_1, \ldots, q_n\) of q-cells in \(\mathbb{R}^3\) is a q-cell-partition in \(\text{ROP}(\mathbb{R}^3)\) if \(\tilde{q}_1, \ldots, \tilde{q}_n\) is a q-cell partition in \(\text{ROP}(S^3)\).

**Definition 1.142** Let \(\tilde{q} = q_1, \ldots, q_n\) and \(\tilde{q}' = q'_1, \ldots, q'_n\) be q-cell partitions in \(\text{ROP}(\mathbb{R}^3)\). We say that \(\tilde{q}\) and \(\tilde{q}'\) are isomorphic if: (i) the corresponding q-cell partitions \(\tilde{q}_1, \ldots, \tilde{q}_n\) and \(\tilde{q}'_1, \ldots, \tilde{q}'_n\) in \(\text{ROP}(S^3)\) are isomorphic; and (ii) for all \(i, j \ (1 \leq i \leq n, \ 1 \leq j \leq 4), q_i[j] \text{ is bounded if and only if } q'_i[j] \text{ is bounded.}\)

Intuitively, knowing which \(q_i[j]\) are bounded for a q-cell partition \(q_1, \ldots, q_n\) in \(\text{ROP}(\mathbb{R}^3)\) amounts to knowing, up to homeomorphism, where the point at infinity is in the corresponding q-cell partition in \(\text{ROP}(S^3)\). More precisely, we have:

**Lemma 1.143** Let \(\tilde{q} = q_1, \ldots, q_n\) and \(\tilde{q}' = q'_1, \ldots, q'_n\) be similarly situated q-cell partitions in \(\text{ROP}(S^3)\). Let \(p \in S^3\) such that, for all \(i, j \ (1 \leq i \leq n, \ 1 \leq j \leq 4), p \in (q_i[j])^\pm\) if and only if \(p \in (q'_i[j])^\pm\). Then there is a homeomorphism \(h : S^3 \to S^3\) fixing \(p\) and mapping \(\tilde{q}\) to \(\tilde{q}'\).

**Proof** Parallel to the proof of Lemma 1.87. \(\Box\)

**Theorem 1.144** Isomorphic q-cell partitions in \(\text{ROP}(\mathbb{R}^3)\) are similarly situated in \(\mathbb{R}^3\).

**Proof** Let \(q_1, \ldots, q_n\) and \(q'_1, \ldots, q'_n\) be isomorphic q-cell partitions in \(\text{ROP}(\mathbb{R}^3)\). Then \(\tilde{q}_1, \ldots, \tilde{q}_n\) and \(\tilde{q}'_1, \ldots, \tilde{q}'_n\) are isomorphic q-cell partitions such that, for all \(i, j \ (1 \leq i \leq n, \ 1 \leq j \leq 4), \infty \in (\tilde{q}_i[j])^\pm\) if and only if \(\infty \in (\tilde{q}'_i[j])^\pm\). By Lemmas 1.140 and 1.143, there exists a homeomorphism \(h : S^3 \to S^3\) onto itself mapping \(\tilde{q}_1, \ldots, \tilde{q}_n\) to \(\tilde{q}'_1, \ldots, \tilde{q}'_n\), and fixing \(\infty\). Thus, \(h' = h \setminus \{\infty, \infty\}\) is a homeomorphism of \(\mathbb{R}^3\) onto itself mapping \(q_1, \ldots, q_n\) to \(q'_1, \ldots, q'_n\). \(\Box\)
6.4 Expressive power of $L_C$ in ROP($\mathbb{R}^3$)

We are now ready to show that every tuple in ROP($\mathbb{R}^3$) satisfies a formula which is topologically complete in ROP($\mathbb{R}^3$) over $\mathbb{R}^3$.

**Lemma 1.145** For all $N > 0$, there exists a formula $\phi^N_q(\vec{z})$ such that, for any $4N$-tuple $\vec{t}$ from ROP($\mathbb{R}^3$), ROP($\mathbb{R}^3$) $\models \phi^N_q(\vec{p})$ if and only if $\vec{t}$ is a $q$-cell partition in ROP($\mathbb{R}^3$).

**Proof** Let $\phi_B(x)$ be as defined in Lemma 1.131, and suppose $s_1, \ldots, s_4 \in$ ROP($\mathbb{R}^3$). Then the quadruple $\langle s_1, \ldots, s_4 \rangle$ is a $q$-cell in $\mathbb{S}^3$ if and only if ROP($\mathbb{R}^3$) $\models \phi_q[s_1, \ldots, s_4]$, where $\phi_q(y_1, \ldots, y_4)$ is the formula

$$\bigwedge \left\{ \phi_B \left( \sum_{j \in J} y_j \right) \mid \emptyset \neq J \subseteq \{1, 2, 3, 4\} \right\}.$$ 

The result then follows easily. \qed

**Lemma 1.146** Let $\vec{t}$ be a $4N$-tuple forming an $N$-element $q$-cell partition in ROP($\mathbb{R}^3$). Then we can find a formula $\gamma(\vec{z})$ such that, for any $4N$-tuple $\vec{p}$ of ROP($\mathbb{R}^3$), ROP($\mathbb{R}^3$) $\models \gamma(\vec{p})$ if and only if $\vec{p}$ is an $N$-element $q$-cell partition isomorphic to $\vec{t}$.

**Proof** Almost immediate from Lemmas 1.128 and 1.145 and the discussion of Section 3.2. \qed

**Lemma 1.147** Every $q$-cell partition in ROP($\mathbb{R}^3$) satisfies a $L_C$-formula which is topologically complete in ROP($\mathbb{R}^3$) over $\mathbb{R}^3$.

**Proof** Theorem 1.144 and Lemma 1.146. \qed

**Lemma 1.148** Any $n$-tuple $\vec{r}$ from ROP($\mathbb{R}^3$) can be refined to an $N$-element $q$-cell partition. That is: there exists a $4N$-tuple $\vec{t}$ from ROP($\mathbb{R}^3$) and a $(4N \times n)$ Boolean array $A$ such that $\vec{t}$ forms a $q$-cell partition in ROP($\mathbb{R}^3$) and $\vec{r} = \vec{t}A$.

**Proof** By the definition of ROP($\mathbb{R}^3$), we can certainly refine $\vec{r}$ to a partition of convex regions of $\mathbb{R}^3$, each of which is bounded by a finite number of planes, and thence, by triangulating these convex regions, into a partition of polyhedra $t_1, \ldots, t_N$, such that each $t_i$ is a ball in $\mathbb{S}^3$, and the boundary of each $t_i$ ($1 \leq i \leq N$) is composed of 4 ‘triangles’ (in the sense used earlier in this proof). By subdividing each $t_i$, we can construct a $q$-cell $q_i$ whose faces are exactly the triangles bounding $t_i$, \ldots.
and such that $q_i = t_i$. Then $q_1, \ldots, q_N$ is the required $q$-cell partition.

QED

**Theorem 1.149** Every tuple in $\text{ROP}(\mathbb{R}^3)$ satisfies some $L_C$-formula which is topologically complete in $\text{ROP}(\mathbb{R}^3)$ over $\mathbb{R}^3$.

**Proof** Let $\vec{r}$ be a tuple from $\text{ROP}(\mathbb{R}^3)$. Let $\vec{t}$ and $A$ be as in Lemma 1.148, and by Lemma 1.147 let $\phi_T(\vec{x})$ be a formula satisfied by $\vec{t}$ which is topologically complete in $\text{ROP}(\mathbb{R}^3)$ over $\mathbb{R}^3$. Then the formula $\exists \vec{x}(\phi_T(\vec{x}) \land \vec{x} = \vec{z}A)$, which is also topologically complete in $\text{ROP}(\mathbb{R}^3)$ over $\mathbb{R}^3$, is satisfied by $\vec{r}$.

This concludes the main business of this section: the language $L_C$ is sufficiently expressive that every tuple of polyhedra in $\mathbb{R}^3$ can be characterized up to the relation of similar situation in $\mathbb{R}^3$ by one of its formulas. Moreover, it is easy to see that an analogous result must obtain for polyhedra in $S^3$. Of course, the characterizing formulas for tuples of polyhedra obtained in this section are much more complicated than the corresponding $L_{c, \leq}$-formulas for tuples of polygons obtained in Section 4.

In Section 5, we exploited the high expressive power of $L_{c, \leq}$ in $\text{ROP}(S^2)$ to obtain an axiomatization of $\text{Th}_{c, \leq}(\text{ROP}(S^2))$, and thence, a formulation of the conditions under which an arbitrary mereotopology over $S^2$ has the same $L_{c, \leq}$-theory as $\text{ROP}(S^2)$. The question therefore arises as to whether an analogous approach is possible for characterizing 'reasonable' spatial mereotopologies using the results of this section. The major disincentive to such an undertaking is the relative weakness of the requirement of finite decomposability in $S^3$. For the plane case, the requirement of finite decomposability led very easily to the existence of $c^3$-partition refinements, which paved the way for an axiomatic characterization of $\text{Th}_{c, \leq}(\text{ROP}(S^2))$. In the spatial case, by contrast, much stronger assumptions are needed to guarantee the existence of $\alpha$-cell partitions, as examples such as the region depicted in Fig. 1.3 show. Thus, while the identification of a standard theory of spatial mereotopology is certainly conceivable, it is not obvious, at the time of writing, how best to approach this matter.

### 7. Model Theory

In Section 2, we defined a **mereotopology over a topological space** $X$ to be a Boolean sub-algebra $M$ of $RO(X)$ in which, for all $p \in o \subseteq X$, with $o$ open, there exists $r \in M$ such that $p \subseteq r \subseteq o$. However, we also promised a purely intrinsic characterization of such structures—one making no
reference to points or topological spaces. In this section, we fulfill that promise, and (partially) realize the vision with which we started this chapter, of a reconstruction of topology where the fundamental objects are not points, but regions.

7.1 Abstract models of mereotopological theories

We begin by noting some simple facts about mereotopologies over topological spaces of various kinds.

**Lemma 1.150** Let $M$ be a mereotopology over a topological space $X$, considered as a structure interpreting the signature $\{C, +, \cdot, -, 0, 1, \leq\}$. (i) The sentences $\Phi_{CA}$ consisting of the usual axioms of Boolean algebra together with

$$
\forall x \neg C(x, 0) \\
\forall x (x > 0 \rightarrow C(x, x)) \\
\forall x \forall y (C(x, y) \rightarrow C(y, x)) \\
\forall x \forall y (C(x, y) \land y \leq z \rightarrow C(x, z)) \\
\forall x \forall y (C(x, y + z) \rightarrow C(x, y) \lor C(x, z))
$$

are all true in $M$. (ii) If $X$ is weakly regular, then the sentence $\phi_{\text{ext}}$ given by

$$
\forall x \forall y (\forall z (C(x, z) \rightarrow C(y, z)) \rightarrow x \leq y)
$$

is true in $M$. (iii) If $X$ is compact and Hausdorff, then the sentence $\phi_{\text{int}}$ given by

$$
\forall x \forall y (\neg C(x, y) \rightarrow \exists z (\neg C(x, -z) \land \neg C(y, z)))
$$

is true in $M$.

**Proof** (i) Straightforward. (ii) Lemma 1.22. (iii) Suppose $r, s \in M$ with $r^{-} \cap s^{-} = \emptyset$. Since $X$ is regular, by Lemma 1.23, for each point in $p \in r^{-}$, there is $r_{p} \in M$ with $p \in r_{p}$ and $s^{-} \subseteq -r_{p}$. Since the $r_{p}$ cover $r^{-}$, choose a finite subcover, and let the sum of this subcover be $t$. Then $r^{-} \subseteq t$ and $s^{-} \subseteq -t$.

The three claims in Lemma 1.150 all have converses. Specifically:

**Proposition 1.151** Let $\mathfrak{A}$ be a structure interpreting the signature $\Sigma = \{C, +, \cdot, -, 0, 1, \leq\}$. (i) If $\mathfrak{A} \models \Phi_{CA}$, then $\mathfrak{A}$ is isomorphic (as a $\Sigma$-structure) to a mereotopology over a topological space $X$; in fact, $X$ can always be chosen so as to be semi-regular and $T_{0}$. (ii) If $\mathfrak{A} \models \Phi_{CA} \cup \{\phi_{\text{ext}}\}$, then $X$ can be chosen so as to be weakly regular and $T_{1}$. 

QED
(iii) If $\mathfrak{A} \models \Phi_{\text{CA}} \cup \{\phi_{\text{ext}}, \phi_{\text{int}}\}$, then $X$ can be chosen so as to be compact and Hausdorff.

These results first appeared (in equivalent form) in Dimov and Vakarellov, 2006, Düntsch and Winter, 2005 and Roep, 1997, respectively. In the literature, structures satisfying $\Phi_{\text{CA}}$ are sometimes referred to as contact algebras, the sentence $\phi_{\text{ext}}$ as the extensionality axiom, and the sentence $\phi_{\text{int}}$ as the interpolation axiom. Together, Lemma 1.150 and Proposition 1.151 show that mereotopologies over certain classes of topological spaces can be characterized purely intrinsically, without reference to those spaces or the points that make them up. We note in passing that Proposition 1.151 speaks of mereotopologies over $X$ (Definition 1.5), where the sources cited refer only to dense sub-algebras of RO($X$). This slight strengthening is immediate from the relevant proofs, and improves the match between Lemma 1.150 and Proposition 1.151. For a fuller discussion, see Ch. ??.

Furthermore, it turns out that the topological realizations in Proposition 1.151 (iii) are, in an important sense, unique. We motivate this result with a simple observation.

**Lemma 1.152** Let $M_i$ be a mereotopology over the topological space $X_i$, for $i = 1, 2$. Suppose there is a homomorphism $h : X_1 \to X_2$ which maps $M_1$ onto $M_2$. Then, for any signature $\Sigma$ of topological primitives, $h$ induces a structure isomorphism $h : M_1 \cong_{\Sigma} M_2$.

**Proof** Immediate. \quad QED

The uniqueness of the topological realizations in Proposition 1.151 (iii) takes the form of a partial converse of Lemma 1.152:

**Theorem 1.153** (Roep, 1997) Let $M_i$ be a mereotopology over a compact, Hausdorff space $X_i$ ($i = 1, 2$). Suppose there is a structure isomorphism $f : M_1 \cong_{C} M_2$. Then there exists a homeomorphism $h : X_1 \to X_2$ which induces $f$—that is, one such that, for all $r \in M_1$, $f(r) = h(r)$.

Thus, every model of $\Phi_{\text{CA}} \cup \{\phi_{\text{ext}}, \phi_{\text{int}}\}$ is isomorphic to exactly one mereotopology over a compact Hausdorff space (up to homeomorphism). Since this fact is important for the development here, we present details of the proof.

We assume familiarity with the theory of ultrafilters: for details, see Koppelberg, 1989, Ch. 1, Sec. 2. In this context, recall that, for $B$ a Boolean algebra, a filter on $B$ is a set $F \subseteq B$ such that $a, b \in F$ implies $a \cdot b \in F$, and $a \in F$, $a \leq b \in B$ implies $b \in F$. A filter is *proper* if it is not the whole of $B$, or equivalently, if it does not contain 0. A proper filter $U$ is an *ultrafilter* if it is maximal under set-inclusion, or
equivalently, if \( b_1 + b_2 \in U \) implies \( b_1 \in U \) or \( b_2 \in U \). The following result is standard (Koppelberg, 1989, Chapter 1, 2.16).

**Proposition 1.154** (Prime Ideal Theorem) Any proper filter on a Boolean algebra can be extended to an ultrafilter.

In the following lemmas, let \( M \) be a mereotopology over a compact, Hausdorff space \( X \). Since a compact Hausdorff space is normal (and hence regular), Lemma 1.23 applies.

**Lemma 1.155** Let \( U \) be an ultrafilter on \( M \). Then the set \( \bigcap \{ r^- | r \in U \} \) is a singleton. We denote the member of this set by \( p_U \) and say that \( U \) converges to \( p_U \).

**Proof** We first show that \( \bigcap \{ u^- | u \in U \} \) contains at least one point. For otherwise, \( \bigcup \{ X \setminus u^- | u \in U \} = X \), whence \( \{-u | u \in U\} \) covers \( X \). By compactness of \( X \), let \( -u_1, \ldots, -u_n \) be a finite subcover. Then \( -u_1 + \cdots + -u_n = 1 \); i.e. \( u_1 \cdot \cdots \cdot u_n = 0 \in U \), contradicting the fact that \( U \) is proper. Next we show that \( \bigcap \{ u^- | u \in U \} \) contains at most one point. For suppose \( p, q \) are distinct points of \( X \). By Lemma 1.23, there exists \( r \in M \) such that \( p \in r \) and \( q \in -r \). Hence \( p \not\in (-r)^- \) and \( q \not\in r^- \). Since \( U \) is an ultrafilter, either \( r \) or \( -r \) is in \( U \), so that either \( p \) or \( q \) is not in \( \bigcap \{ u^- | u \in U \} \). QED

**Lemma 1.156** Let \( U \) be an ultrafilter on \( M \), and let \( r \in M \). If \( p_U \in r \), then there exists \( s \in U \) such that \( p_U \in s \) and \( s^- \subseteq r \). Hence also, \( r \in U \).

**Proof** Suppose \( p_U \in r \in M \). Then \( p_U \not\in (-r)^- \), and by Lemma 1.23, there exists \( s \in M \) such that \( p_U \in s \) and \( s^- \subseteq r \). But since \( p_U \not\in (-s)^- \), we have \( -s \not\in U \), and thus \( s \in U \). QED

**Definition 1.157** If \( U \) and \( V \) are ultrafilters on \( M \), we say \( U \) and \( V \) are contacting if \( r^- \cap s^- \neq \emptyset \) for all \( r \in U \) and \( s \in V \).

**Lemma 1.158** If \( U \) and \( V \) are ultrafilters on \( M \), then \( U \) and \( V \) are contacting if and only if \( p_U = p_V \).

**Proof** The if-direction is trivial. For the only-if direction, suppose that \( p_U \neq p_V \). By Lemma 1.23, there exist \( r, s \in M \) such that \( p_U \in r \), \( p_V \in s \) and \( r^- \cap s^- = \emptyset \). By Lemma 1.156, \( r \in U \), \( s \in V \), so that \( U \) and \( V \) are not contacting. QED
Lemma 1.159 Let $M_1$ and $M_2$ be meretopologies over weakly regular topological spaces, let $f : M_1 \simeq C M_2$ be an isomorphism, and let $U$ and $V$ be contacting ultrafilters on $M_1$. Then $f(U)$ and $f(V)$ are contacting ultrafilters on $M_2$.

Proof Almost immediate given the definability of $\leq$ in terms of $C$ (Lemma 1.22). \hfill \QED

Lemma 1.160 Let $M_1$ and $M_2$ be meretopologies over weakly regular topological spaces, such that $f : M_1 \simeq C M_2$. Let $r \in M$, and let $U$ be an ultrafilter on $M_1$ with $p_U \in r$. Then $p_{f(U)} \in f(r)$.

Proof By Lemma 1.156, there exists $s \in U$ such that $p_U \in s$ and $s^{-} \subseteq r$, so that $s^{-} \cap (-r)^{-} = \emptyset$. Since $f$ is also a Boolean algebra isomorphism, $f(s)^{-} \cap (-f(r))^{-} = \emptyset$, i.e. $f(s)^{-} \subseteq f(r)$. Since $f(s) \in f(U)$, $p_{f(U)} \in f(s)^{-} \subseteq f(r)$. \hfill \QED

[Theorem 1.153] Suppose that $f : M_1 \simeq C M_2$. Define the map $h$ by $h(p_U) = p_{f(U)}$, for $U$ a compact ultrafilter on $M_1$. We show: (i) $h$ is well-defined and 1–1, (ii) the domain of $h$ is the whole of $X_1$ and the range of $h$ is the whole of $X_2$, (iii) for all $r \in M_1$, $f(r) = h(r)$, and for all $s \in M_2$, $f^{-1}(s) = h^{-1}(s)$, and (iv) $h$ and $h^{-1}$ are continuous. To prove (i), let $U$, $V$ be compact ultrafilters on $M_1$, both converging to $p$. By Lemma 1.159, the isomorphism $f$ maps contacting ultrafilters to contacting ultrafilters. Hence, $h$ is well-defined. Applying the same reasoning to $f^{-1}$, $h$ is 1–1. To prove (ii), let $p \in X_1$. Then $\{r \in M_1 | p \in r\}$ is a proper filter on $M_1$, and by Proposition 1.154, this filter can be extended to an ultrafilter $U$ on $M_1$. By Lemma 1.155, $U$ converges to some point $p_U$. Since $X_1$ is Hausdorff $p = p_U$. Thus, the domain of $h$ is the whole of $X_1$. Similarly, if $q \in X_2$, we have an ultrafilter $V$ on $M_2$ such that $q = p_V$. Thus $q = p_V = p_{f(f^{-1}(V))} = h(p_{f^{-1}(V)})$, so that the range of $h$ is the whole of $X_2$. To prove (iii), let $p_U \in f(r)$ with $V$ an ultrafilter on $M_2$. By Lemma 1.160, $p_{f^{-1}(V)} \in r$. Hence, $p_U = h(p_{f^{-1}(V)}) \in h(r)$. Conversely, let $p_U \in h(r)$. By the definition of $h$, $p_{f^{-1}(V)} \in r$, and by Lemma 1.160, $p_U \in f(r)$. Hence $f(r) = h(r)$. Now if $s \in M_2$, $f^{-1}(s) \subseteq M_1$, so, applying the results just obtained to this set, we have $f^{-1}(s) = h^{-1}(h(f^{-1}(s))) = h^{-1}(f(f^{-1}(s))) = h^{-1}(s)$. (iv) Let $u \subseteq X_1$ be an open set. Since $M_1$ is a meretopology, for each point $p \in u$, there exists $r_p \in M_1$ with $p \in r_p \subseteq u$. Thus the set $U = \{r_p \in M | p \in u\}$ satisfies $\bigcup U = u$. Then $h(u) = h(\bigcup U) = \bigcup_{r \in U} h(r) = \bigcup_{r \in U} f(r)$ is a union of open sets in $X_2$ and hence is itself an open set in $X_2$. Therefore, $h^{-1}$ is continuous. By substituting $h^{-1}$ and for $h$ and repeating the argument, $h$ is continuous. \hfill \QED
7.2 Abstract models of geometrical mereotopological theories

We have shown that mereotopologies over certain classes of topological spaces can be characterized in terms of certain first-order sentences which they make true. But what of specific mereotopologies of interest—for instance, those defined over the open or closed plane? This is the topic we now address, based on the results of Pratt and Lemon, 1997.

We employ standard results on prime models: for details, see Chang and Keisler, 1990, Ch. 2. A structure $\mathfrak{A}$ is said to be a prime model if it is elementarily embeddable in any elementarily equivalent submodel. Prime models are considered the ‘simplest’ or ‘smallest’ models of their theories, a view which is justified by the following proposition (Chang and Keisler, 1990, Theorem 2.3.3). In the sequel, all signatures are silently assumed to be countable.

**Proposition 1.161** Elementarily equivalent prime models are isomorphic.

The following notion is closely related to that of primeness. A formula $\phi$ is said to be complete with respect to a theory $T$ if, for all formulas $\theta$ having the same free variables of $\phi$, exactly one of $T \models \phi \leftrightarrow \theta$ or $T \models \phi \rightarrow \neg \theta$ hold. A structure $\mathfrak{A}$ is said to be atomic if any $n$-tuple $\bar{a}$ in $A$ satisfies a formula $\phi(\bar{x})$ in $\mathfrak{A}$ such that $\phi$ is complete with respect to $\text{Th}(\mathfrak{A})$. We have the following standard result (see, for example, Chang and Keisler, 1990, Theorem 2.3.4).

**Proposition 1.162** A structure is countable atomic if and only if it is a prime model.

Recall the concepts of topologically complete formula and homogeneous mereotopology given in Definitions 1.89 and 1.51, respectively.

**Lemma 1.163** Let $M$ be a homogeneous mereotopology over a topological space $X$, and let $\Sigma$ be a signature of topological primitives. If $\phi \in L_\Sigma$ is topologically complete in $X$ over $M$, then $\phi$ is complete with respect to $\text{Th}_\Sigma(M)$.

**Proof** Immediate from Lemma 1.91. QED

Theorem 1.176 below is a partial converse of this result.

For the next theorem, recall that $\text{ROQ}(S^2)$ is the rational polygonal mereotopology over the closed plane, and that its $L_{c,\leq}$-theory is $T_{c,\leq}$, the standard $L_{c,\leq}$-theory of closed plane mereotopology, which we axiomatized in Section 5. Recall further that $\psi_3^N(\bar{z})$ is the $L_{c,\leq}$-formula stating that $\bar{z}$ forms a $c^3$-partition, employed in the proof of Theorem 1.84
Theorem 1.164 The meretopology $\text{ROQ}(\mathbb{S}^2)$, considered as a $\{c, \leq\}$-structure, is a prime model of $\text{T}_{c, \leq}$. In fact, for any $N$, there exist formulas $\gamma_1(x), \ldots, \gamma_K(x)$ (with $K$ depending on $N$), complete with respect to $\text{T}_{c, \leq}$, such that

$$\text{T}_{c, \leq} \models \forall x (\psi_{c, \leq}^N(x) \rightarrow (\gamma_1(x) \lor \cdots \lor \gamma_K(x))).$$

Proof The first part of the theorem is immediate from Theorem 1.84 and Lemma 1.163. For the second part, observe that, for a given $N$, there are only finitely many neighbourhood structures on an $N$-element $c^3$-partition, each one giving rise to a topologically complete formula of the form

$$\exists x (\psi_{c, \leq}^N(x) \land \psi_{+}^N(x) \land x = A),$$

as described in the proof of Theorem 1.84.\qed

Note that, by Lemma 1.38, $\text{ROQ}(\mathbb{S}^2)$ and $\text{ROQ}(\mathbb{R}^2)$ are the same $\{c, \leq\}$-structure, so we could replace $\mathbb{S}^2$ in Theorem 1.164 by $\mathbb{R}^2$.

Similarly, we have

Theorem 1.165 The meretopologies $\text{ROQ}(\mathbb{R}^2)$ and $\text{ROQ}(\mathbb{S}^2)$, considered as $\{C\}$-structures, are prime models.

Proof As for Theorem 1.164, but using Theorem 1.88 and Corollary 1.85, respectively.\qed

Analogues of Theorem 1.165 hold in three dimensions, of course. For example, we have:

Theorem 1.166 The meretopology $\text{ROQ}(\mathbb{R}^3)$ is a prime model of the $L_C$-theory of $\text{ROP}(\mathbb{R}^3)$.

The proof strategy is essentially identical to the plane case, using Theorem 1.149. Note, however, that much more care is required to show that the topologically complete formulas identified in Theorem 1.149 are complete with respect to the $L_C$-theory of $\text{ROP}(\mathbb{R}^3)$. We leave the details to the interested reader.

Returning to meretopologies over $\mathbb{S}^2$, the question then arises as to whether $\text{ROQ}(\mathbb{S}^2)$ is strictly simplest among countable models of $\text{T}_{c, \leq}$, in that there are countable models of that theory not isomorphic to $\text{ROQ}(\mathbb{S}^2)$. The answer is: yes and no. Recall that a theory is said to be $\omega$-categorical if it has exactly one countable model up to isomorphism. Recall also that a type in variables $x = x_1, \ldots, x_n$ is a maximal consistent set of formulas whose free variables are among the $x_1, \ldots, x_n$, and that
a theory $T$ is said to have a type $\Phi(\bar{x})$ if $\Phi(\bar{x})$ is consistent with $T$. The following result is standard (see, for example, Chang and Keisler, 1990, Theorem 2.3.13).

**Proposition 1.167** Let $T$ be a complete theory. Then $T$ is $\omega$-categorical if and only if, for each $n$, $T$ has only finitely many types in $x_1, \ldots, x_n$.

**Theorem 1.168** $T_{c,\leq}$ is not $\omega$-categorical.

**Proof** By Proposition 1.167, it suffices to prove that $T_{c,\leq}$ has countably many types in the single variable $x$. It is easy to see that, for every positive integer $m$, the formula $\psi_m(x)$

$$\exists z_1 \ldots \exists z_m \left( \bigwedge_{1 \leq i \leq m} (c(z_i) \land z_i \neq 0) \land \bigwedge_{1 \leq i < j \leq m} -c(z_i + z_j) \land x = \sum_{1 \leq i \leq m} z_i \right)$$

is satisfied in ROQ($S^2$) by all and only those regions having exactly $m$ components. Hence, the $\psi_m(x)$ are all satisfied in ROQ($S^2$); so each can be extended to a type $\Gamma_m(x)$ of $\text{Th}_{c,\leq}(\text{ROQ}(S^2))$. But the $\psi_m(x)$ are also pairwise mutually exclusive in $T_{c,\leq}$; so no two of them can be extended to the same type. Hence, $T_{c,\leq}$ has infinitely many types in $x$.

QED

One the other hand, it turns out that $T_{c,\leq}$ is almost countably categorical, in the following sense. Note that, since any model of $T_{c,\leq}$ is a Boolean algebra interpreting the predicate $c$, we may employ the terminology introduced at the start of Section 4.1.

**Theorem 1.169** All countable finitely decomposable models of $T_{c,\leq}$ are isomorphic.

**Proof** Let $\mathcal{A} \models T_{c,\leq}$ be finitely decomposable. By Claims 1.54 and 1.63, every tuple from $A$ can be refined to a $c^3$-partition. Theorem 1.164 then implies that $\mathcal{A}$ is prime. The result follows by Proposition 1.161. QED

The above results show that, while specific mereotopologies such as ROS($S^2$) cannot be characterized in terms of the first-order sentences which they make true, they almost can. Specifically, we have the following abstract characterization of the mereotopology ROQ($S^2$).

**Corollary 1.170** If $\mathcal{A}$ is a countable, finitely decomposable model of Axioms 1—8 in Section 5.1, then $\mathcal{A}$ is isomorphic (as a $\{c, \leq\}$-structure) to the mereotopology ROQ($S^2$).

**Proof** Theorem 1.169 and the fact that, by Theorem 1.100, any finitely decomposable model $\mathcal{A}$ of Axioms 1—8 is elementarily equivalent to ROQ($S^2$). QED
7.3 Loose ends

We end this section with some matters touched on earlier in this chapter. We continue to assume all signatures to be countable. The following proposition is a special case of the Löwenheim-Skolem Theorem (see, for example, Hodges, 1993, p. 90).

**Proposition 1.171** Let $\mathfrak{A}$ be a $\Sigma$-structure and $Z$ a countable subset of $A$. Then $\mathfrak{A}$ has a countable elementary submodel whose domain includes $Z$.

Recall that a topological space $X$ is said to be second countable if its topology has a countable basis.

**Lemma 1.172** Let $M$ be a mereotopology over a compact, second-countable, Hausdorff space $X$, and let $P \subseteq M$ be countable. Then there is a countable mereotopology $Q$ over $X$ such that $P \subseteq Q$ and $Q \subseteq M$.

**Proof** We construct a countable subset $P' \subseteq M$ such that, for all $p \in o \subseteq X$ with $o$ open, there exists $r \in P'$ such that $p \in r \subseteq o$. The lemma then follows from Proposition 1.171 by putting $\mathfrak{A} = M$ and $Z = P \cup P'$. Let $B$ be a countable basis for the topology on $X$. For any $b, c \in B$ with $b^- \subseteq c$, take a cover of $b^-$ by elements $s \in M$ such that $s \subseteq c$ (possible because $M$ is a mereotopology), choose a finite subcover (possible because $X$ is compact), and let $r_{b,c}$ be the sum, in $M$, of the elements of this finite subcover. Certainly, $b \subseteq r_{b,c} \subseteq c^-$. Let $P' = \{ r_{b,c} \mid b, c \in B, b^- \subseteq c \}$. Since $X$ is normal, for all $p \in o \subseteq X$ with $o$ open, we can find $b, c \in B$ with $p \in b, b^- \subseteq c$ and $c^- \subseteq o$. But then $p \in r_{b,c} \subseteq o$ as required.

QED

Note that Lemma 1.172 holds for all (countable) signatures.

We may now derive the promised strengthening of Corollary 1.117.

**Corollary 1.173** All splittable, finitely decomposable mereotopologies over $\mathbb{S}^2$ with curve-selection have the same $L_\Sigma$-theory for any topological signature $\Sigma$.

**Proof** Let $M_1, M_2$ be two such mereotopologies. Extend the signature $\Sigma$ if necessary so that it contains the predicates $C, c$ and $\leq$, and expand $M_1$ and $M_2$ by interpreting these predicates in the normal way. By Lemma 1.172, let $Q_i$ be a countable mereotopology over $\mathbb{S}^2$ such that $Q_i \sqsubseteq M_i$, for $i = 1, 2$. Thus, $Q_1$ and $Q_2$ are splittable, finitely decomposable mereotopologies over $\mathbb{S}^2$ having curve-selection. By Corollary 1.117, $Q_1 \preceq_{e, \leq} Q_2$. By Theorem 1.169, $Q_1 \simeq_{e, \leq} Q_2$. By Lemma 1.49, $Q_1 \simeq_c Q_2$. By Theorem 1.153, there is a homeomorphism...
mapping $Q_1$ onto $Q_2$. Finally, by Lemma 1.152, $Q_1 \simeq_\Sigma Q_2$, whence $M_1 \equiv_\Sigma M_2$. \hfill \QED

Recall from Definition 1.96 that, if $\Sigma$ is a signature of topological primitives, $T_\Sigma$ denotes $\text{Th}_\Sigma(\text{ROS}(\mathbb{S}^2))$. By Corollary 1.173, $T_\Sigma$ is the $L_\Sigma$-theory of any splittable, finitely decomposable mereotopology over $\mathbb{S}^2$ having curve-selection. This justifies our decision to call it the standard $L_\Sigma$-theory of closed plane mereotopology.

Theorem 1.169 now has the following corollary.

**Corollary 1.174** Let $M$ be a countable, finitely decomposable mereotopology over a locally connected, compact Hausdorff space $X$, such that $\text{Th}_C(M) = T_C$. Then there is a homeomorphism $h : X \leftrightarrow \mathbb{S}^2$ taking $M$ to $\text{ROQ}(\mathbb{S}^2)$.

**Proof** By Lemmas 1.22 and 1.27, $\text{Th}_{C,c,\leq}(M) = T_{C,c,\leq}$. By Theorem 1.169, $M \simeq_{C,\leq} \text{ROQ}(\mathbb{S}^2)$. But $T_{C,c,\leq}$ contains a formula defining $C$ explicitly in terms of $c$ and $\leq$. Hence $M \simeq_C \text{ROQ}(\mathbb{S}^2)$. Now apply Theorem 1.153. \hfill \QED

We remark that there is no prospect of removing the compactness condition from Corollary 1.174. For example, let $p_\pi$ be, say, the point of $\mathbb{S}^2$ with coordinates $(0,\pi)$, and consider the topological space $X = \mathbb{S}^2 \setminus \{p_\pi\}$ and the mereotopology $M$ over $X$ given by $M = \{r \setminus \{p_\pi\} \mid r \in \text{ROQ}(X)\}$. Then $\text{ROQ}(\mathbb{S}^2) \simeq_{C,c,\leq} M$; but $\mathbb{S}^2$ and $X$ are not homeomorphic.

A further consequence of Theorem 1.153 is the promised partial converse of Lemma 1.163. We require the following fact about prime models.

**Lemma 1.175** Let $\mathfrak{A}$ be a countable, atomic model and let $\bar{a}$, $\bar{b}$ be tuples from $A$ which satisfy the same formulas in $\mathfrak{A}$. Then there is an automorphism of $\mathfrak{A}$ taking $\bar{a}$ to $\bar{b}$.

**Proof** Almost immediate from Proposition 1.161, by adding a tuple of individual constants to stand alternatively for $\bar{a}$ and $\bar{b}$. \hfill \QED

**Theorem 1.176** Let $M$ be a mereotopology over a compact, second-countable Hausdorff space $X$, and let $\Sigma$ be a signature of topological primitives such that $C$ (contact) is first-order definable over $M$. If every tuple from $M$ satisfies an $L_\Sigma$-formula which is complete with respect to $\text{Th}_\Sigma$, then that $L_\Sigma$-formula is topologically complete in $M$ over $X$.

**Proof** Let $\phi$ be complete with respect to $\text{Th}_\Sigma(M)$, and suppose that $M \models \phi[\bar{r}]$, $M \models \phi[\bar{s}]$. We must show that $\bar{r} \sim \bar{s}$. By Lemma 1.172, let $M'$ be any countable mereotopology over $X$ containing the tuples $\bar{r}$
and $\bar{s}$, such that $M' \preceq M$. Thus, $M'$ is countable and atomic, and $\phi$ is a complete formula with respect to $\text{Th}_{\Sigma}(M')$ satisfied by both $\bar{r}$ and $\bar{s}$. By Lemma 1.175, there exists an automorphism $f : M' \cong \Sigma M'$ such that $f(\bar{r}) = \bar{s}$. Then, by Theorem 1.153, there is a homeomorphism $h : X \to X$ taking $\bar{r}$ to $\bar{s}$.

QED

Lemma 1.163 and Theorem 1.176 establish the close connection between the notions of topological completeness with respect to a topological space and completeness with respect to a mereotopological theory.

8. Philosophical Considerations

The earliest modern work on region-based theories of space is that of Whitehead and de Laguna (Whitehead, 1919; Whitehead, 1920; Whitehead, 1929; de Laguna, 1922a; de Laguna, 1922b; de Laguna, 1922c). Both authors propose a system of postulates governing a small collection of primitive spatial relations, together with reconstructions of familiar spatial concepts in terms of those relations. The postulates serve implicitly to define the primitive relations they constrain (and perhaps the domain of entities over which they quantify), while the reconstructions of familiar spatial concepts connect the whole system to the data of spatial experience. To be sure, both Whitehead and de Laguna motivate their postulates by providing informal interpretations for their respective spatial primitives. Thus, for example, Whitehead illustrates his relation of extensive connection (as he calls it) using diagrams suggesting that two regions are extensively connected just in case their topological closures share a point in common (this is the interpretation given to the binary predicate $C$ in this chapter). However, such explanations are intended only as a heuristic guide. Officially, spatial primitives acquire their content solely from the entire system postulates in which they participate. Primitives, by definition, are not explicitly definable.

The inspiration for such systems was presumably the axiomatic treatment of geometry found in Euclid (and latterly Hilbert); and the motivation for carrying out the procedure on a purely region-based footing seems, for both authors, to have been a certain disquiet about the empirical distance between the concept of a point as a primitive geometrical entity and the character of everyday spatial experience. The great difficulty of this approach, of course, is the problem of evaluating the system of postulates and conceptual reconstructions proposed. Whitehead’s system has thirty-one postulates (or assumptions, in Whitehead’s terminology) and a similar number of definitions. De Laguna’s system, though far tidier, is also hardly self-evident. The only obvious sources of justification for such systems are their ability to chime with our pre-
theoretic intuition and their eventual integration into a larger, empirically successful, physical theory. Neither source is very satisfactory. On the one hand, as we have seen in this chapter, almost any collection of spatial primitives enables us to write down propositions on which pre-theoretic intuition cannot be expected to return a reliable verdict. On the other hand, although empirical confirmation of a general physical theory must provide some support for the account of space it contains, the size of the undertaking and the difficulty of assigning credit when theories perform well (or blame when they perform badly) means that there is little practical prospect of any such justification for such systems of postulates and conceptual reconstructions.

An alternative approach to developing a region-based theory of space is illustrated by Tarski’s *Geometry of Solids* (Tarski, 1956). Tarski too develops a geometry in which regions, not points, are the primitive objects; however, in contrast to Whitehead and Laguna, he does not build his theory by writing a collection of plausible-looking, but unprovable, axioms. Rather, beginning with the familiar model of space as $\mathbb{R}^3$, he considers a formal language whose variables range over the set of of spheres in $\mathbb{R}^3$ (defined in the standard way), and whose sole non-logical constant is the part-whole relation (again, defined in the standard way). Because the ‘primitives’ in Tarski’s geometry of solids are well-defined mathematical objects and relations, the question of what postulates they satisfy is a well-defined mathematical problem, not a matter for intuition or experiment. And because many familiar spatial concepts have rational reconstructions in terms of the standard model, the question of how, if at all, these concepts can be expressed using formulas of Tarski’s language is again a purely mathematical affair. Having thus specified the structure under consideration and the language used to describe it, Tarski then goes on to examine the kinds of logical issues that should by now be familiar to us. In fact, Tarski obtains a system of axioms (in higher-order logic) for which the standard Euclidean interpretation is, up to isomorphism, the only model.

This alternative approach is, in contrast to the ‘postulationist’ strategy of Whitehead and de Laguna, conservative and rationalist: conservative, because no attempt is made to build systems of axioms and definitions from the ground up; rationalist, because the appropriateness of the resulting region-based theories is secured by means of their logical relations to point-based models whose usefulness as representations of the space we inhabit—at least approximately and for mesoscopic objects—is anyway beyond doubt. It is this approach that we have taken in this chapter. Latterly, region-based theories of space have increased in popularity, following the seminal work of Clarke, 1981, Clarke, 1985,
Biacino and Gerla, 1991; Randell et al., 1992; Gotts et al., 1996; and Renz and Nebel, 1997. One reason for this resurgence of interest, particularly within the A.I. community, is the requirement to quantify over spatial regions without leaving the realm of first-order logic. The technology of theorem-proving for first-order logic is more highly developed than for higher-order logics; and, more generally, formalisms with limited expressive power enjoy a premium in A.I. if they give rise to entailment and satisfiability problems which have (theoretically or practically) efficient algorithmic solutions. Insofar as the study of region-based theories of space is motivated by computational considerations, the best approach to developing and analysing such theories is surely that of Tarski, not that of Whitehead.

These matters notwithstanding, the most striking outcome of the investigation undertaken here is just how much information it gives us about the possibilities for developing a truly region-based theory of space, along the lines apparently envisaged by Whitehead and de Laguna. Consider, for example, the issue of the ‘correct’ set of postulates. True, Examples 1.17 and 1.18 show that different mereotopologies defined over the spaces RO($\mathbb{R}^2$) indeed have different first-order theories. Nevertheless, the discussion of Section 5 shows that the choices on offer are much more limited than these examples might initially lead one to suppose. In particular, all finitely decomposable, splittable mereotopologies over $\mathbb{S}^2$ having curve-selection have identical $L_\Sigma$-theories, for any signature of topological primitives. We proposed that this common $L_\Sigma$-theory should therefore be regarded as standard.

Or take again the issue of reconstructing familiar spatial concepts in terms of a chosen collection of primitives. We have seen that first-order topological languages interpreted over well-behaved mereotopologies have surprising—but not unlimited—expressive power. In particular, we provided formulas expressing a variety of familiar spatial relationships (as defined by their familiar point-based definitions, of course) over a wide range of mereotopologies. In addition, we showed that the first-order language $L_{\Sigma} \leq$ is sufficiently expressive that every tuple of polygons in $\mathbb{S}^2$ can be characterized up to similar situation by one of its formulas, and that the first-order language $L_{C}$ is sufficiently expressive that every tuple of polygons in $\mathbb{R}^2$ and every tuple of polyhedra in $\mathbb{R}^3$ can be characterized up to similar situation by one of its formulas.

Most striking of all, however, is what the foregoing analysis tells us about the view of space to which any first-order mereotopological theory commits us. While almost all interesting mereotopologies have first-order theories which are not categorical in any infinite cardinal, we nevertheless showed that the plane mereotopology RO($\mathbb{S}^2$) and the spatial
mereotopology ROQ(S^3) are prime models of their first-order theories over standard signatures of topological primitives. We further showed that ROQ(S^2) is, up to isomorphism, the only countable, finitely decomposable model of its L_e, S-theory; and we remarked that a corresponding observation—albeit with a more complex version of the finite decomposability condition—must apply in three dimensions as well. Finally, we showed that mereotopologies over compact, Hausdorff spaces, regarded as structures interpreting suitably rich topological signatures, determine their underlying spaces up to homeomorphism. In conclusion, the logical possibilities for region-based topological theories of space are more constrained than their earliest proponents might perhaps have thought.

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