Chapter 1

FIRST-ORDER MEREOTOPOLOGY

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1. Introduction

One of the many achievements of coordinate geometry has been to provide a conceptually elegant and unifying account of the nature of geometrical entities. According to this account, the one primitive spatial entity is the point, and the one primitive geometrical property of points is coordinate position. All other geometrical entities—lines, curves, surfaces and bodies—are nothing but collections of points; and all properties and relations involving these entities may be defined in terms of the relative positions of the points which make them up. The success and power of this reduction is so great that the identification of spatial regions with the sets of points they contain has come to seem virtually axiomatic.

Over the years, however, various authors have expressed disquiet with this conceptual régime. The primary source of the disquiet is the conviction that our theory of space should use only those resources absolutely necessary to systematize the data of spatial experience. For *points* are such remote abstractions from the objects with which we daily interact, and *co-ordinate position* such a distant relative of the spatial properties and relations which we directly perceive, that the question arises as to whether alternative mathematical models of space are not possible—in particular, models in which the primitive spatial entities are not points, but regions, and in which the primitive spatial properties and relations are qualitative rather than quantitative.

An example will help to make these worries more concrete. Consider any stable, medium-sized physical object, for example, a coffee cup. We all agree that this cup has a particular shape, which we may take to correspond to the region of space which it occupies at some instant. On the familiar point-based model of space, this region is a set of points. But suppose we now ask: is this set topologically open, semi-open or closed? That is: does it include none, some, or all of its boundary points? It is hard to see how we could answer this question. Not by microscopic analysis, since physical objects lose their definition on very small scales. And not by mathematical argument, since a world in which—say—cups are closed and saucers open is surely as logically possible as one where these topological characteristics are reversed. But if space really is made up of points as (modern) textbooks tell us, any assignment of a region of space to the coffee cup must answer the question. Perhaps then this model postulates too much.

This chapter addresses the question: what region-based accounts of the topological structure of space are possible? What can we say about them? How do they relate to each other and to the point-based models with which we are so familiar?

2. Mereotopologies

The purpose of section is to outline the conceptual framework for region-based theories of space adopted in this chapter. Specifically, we introduce the concept of a *mereotopology* over a topological space, we discuss the role of mereotopologies as interpretations of signatures of topological primitives, and we list some key mathematical questions concerning them.

We assume familiarity with fundamental concepts and standard facts of point-set topology and Boolean algebra: for details, see, e.g. Kelley, 1955 and Koppelberg, 1989, Ch. 1, respectively. In the context of point-set topology, if u is any subset of a topological space X, we denote the interior of u by u^0 and the closure of u by u^-. (The more usual notations of $\bar{u}$ and $[u]$ for the closure of u are reserved for other purposes.) We write $\mathcal{F}(u)$ to denote the *frontier of* u, namely $u^- \setminus u^0$.

2.1 Regular open sets

How might we go about building a region-based model of the space we inhabit? The example of the coffee cup suggests that any such model should resolve the issue of frontier points. The following technical details are well-suited to this purpose.

DEFINITION 1.1 *Let u be a subset of some topological space X. We say that u is* regular open (in X) *if u is equal to the interior of its closure. We denote the set of regular open subsets of X by* $\mathrm{RO}(X)$*.*

To fix our intuitions, consider the space $X = \mathbb{R}^2$. The elements of $\mathrm{RO}(\mathbb{R}^2)$ are the open subsets of $\mathbb{R}^2$ having no 'cracks' or 'pin-holes' (Fig. 1.1). Corresponding remarks apply to the case $X = \mathbb{R}^3$. Taking

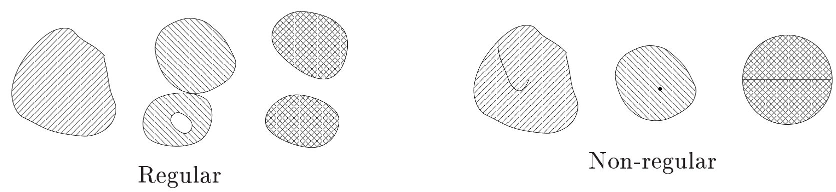


Figure 1.1. Some regular and non-regular open sets of the Euclidean plane.

regions of space to be regular open subsets of $\mathbb{R}^3$ finesses the issues encountered above concerning frontier points: regions are open by fiat. At the same time, however, it provides us with satisfying formal reconstructions of the intuitive notions of intersecting, merging and complementing regions, by means of the following standard theorem (see, for example, Koppelberg, 1989, pp. 25–27).

PROPOSITION 1.2 *Let X be a topological space. Then $\mathrm{RO}(X)$ is a Boolean algebra under the order $\subseteq$. In this Boolean algebra, top and bottom are defined by $1 = X$ and $0 = \emptyset$, and Boolean operations are defined by $x \cdot y = x \cap y$, $x + y = (x \cup y)^{-0}$ and $-x = X \setminus x^{-}$.*

Again, we can fix our intuitions regarding Proposition 1.2 by considering the case $X = \mathbb{R}^2$. The product, $x.y$, of two regular open sets x and y is simply their intersection, which is guaranteed to be a regular open set. The sum, $x + y$, of two regular open sets x and y is a little more complicated; *very* roughly, it is the union of x and y with any internal boundaries removed (Fig. 1.2). Finally, the complement, $-x$, of a regular open set x in $\mathrm{RO}(\mathbb{R}^2)$ is simply that part of the plane not occupied by x or its frontier. Corresponding remarks apply to the case $X = \mathbb{R}^3$.

It sometimes helps to reformulate the definition of regular open sets as follows. If $u \subseteq X$, then $\bigcup\{o \subseteq X | o \text{ open}, o \cap u = \emptyset\}$ is the largest open subset of X disjoint from u. We call this set the *pseudo-complement of* u, denoted u^*. From the above definitions, $u^* = X \setminus u^-$ and $u^{**} = (u^-)^0$. Hence, u is regular if and only if $u = u^{**}$; and, if u is regular open, u^* is simply $-u$. The following lemma shows that every subset of X is 'close' to a regular open subset.

LEMMA 1.3 *Let X be a topological space. For every $u \subseteq X$, the set $r = (u^-)^0$ is an element of* $\mathrm{RO}(X)$ *such that $u^0 \subseteq r \subseteq u^-$. If u is open, then r is unique.*

Proof Obviously $u^0 \subseteq r \subseteq u^-$. To show that $(u^-)^0 \in \mathrm{RO}(X)$, it suffices to show that $u^{****} = u^{**}$. If v is any set at all, then $v^{**} \cap v^* = \emptyset$, whence $v^* \subseteq v^{***}$. Moreover, if o is any open set, then o^{***} is an open set disjoint from o^{**} and hence disjoint from every open set disjoint from o^* and hence disjoint from o itself, whence $o^{***} \subseteq o^*$. Thus, for any open set o, $o^{***} = o^*$. Since u^* is open, we have $u^{****} = u^{**}$. For the final statement, if $s \in \mathrm{RO}(X)$ also satisfies $u \subseteq s \subseteq u^-$, then the (regular) open sets $s \cdot -r$ and $r \cdot -s$ are both in $u^- \setminus u$ and so are empty. QED

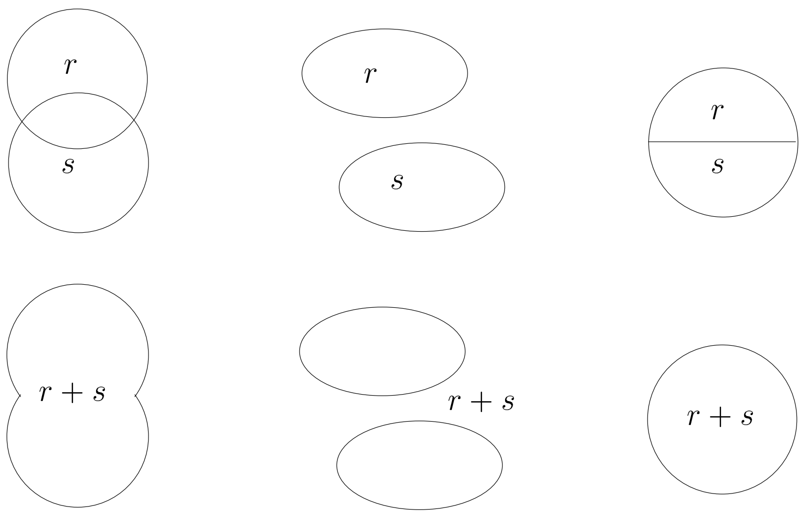


Figure 1.2. Three pairs of regions and their sums in $\mathrm{RO}(\mathbb{R}^2)$.

For the above reasons, it has become common practice in discussions of mereotopology to model regions of space as regular open subsets of $\mathbb{R}^3$; and that is the approach we shall take here. In the sequel, we shall always use the letters r, s, t to range over regular open sets; when we are concerned only with regular open sets, we write $r \leq s$ in preference to $r \subseteq s$, 0 in preference to $\emptyset$ and $r \cdot s$ in preference to $r \cap s$. Resorting to regular open sets is of course not the only way of dealing with boundary disputes. One obvious alternative is to use regular closed sets (sets equal to the closures of their interiors), since the regular closed sets of any topological space also form a Boolean algebra, which is in fact isomorphic to the Boolean algebra of regular open sets. Thus, in modelling regions as regular open sets of $\mathbb{R}^3$, it is understood that it is the resulting structure that is important, not the precise constitution of its elements. Understanding what this idea means in detail forms a central theme of this chapter.

We conclude our discussion of regular open sets by proving some technical results which will be useful below. Recall in this context that, if u, v are connected subsets of a topological space, with $u \cap v \neq \emptyset$, then $u \cup v$ is connected. Moreover, if u is connected and $u \subseteq v \subseteq u^-$, then v is connected.

LEMMA 1.4 *Let X be a topological space, let $u, v \subseteq X$ and let $r, s \in \mathrm{RO}(X)$. We have:*

(i) $(u \cup v)^{-0} = u^{-0} + v^{-0}$;

(ii) $r \cup s \subseteq r + s \subseteq r \cup s \cup (r^- \cap s^-) \subseteq (r \cup s)^-$;

(iii) $(r + s)^- = r^- \cup s^- = (r \cup s)^-$;

(iv) if r and s are connected with $r \cdot s > 0$, then $r + s$ is connected.

Proof (i) By Lemma 1.3, $(u \cup v)^{-0}$ is a regular open set which evidently contains the regular open sets u^{-0} and v^{-0}. Certainly, then $u^{-0} + v^{-0} \subseteq (u \cup v)^{-0}$. For the reverse inclusion, $(X \setminus u)^0 \cap (X \setminus v)^0 \cap (u \cup v)^{-0} = \emptyset$, whence $(X \setminus u)^{0-0} \cap (X \setminus v)^0 \cap (u \cup v)^{-0} = \emptyset$, whence $(X \setminus u)^{0-0} \cap (X \setminus v)^{0-0} \cap (u \cup v)^{-0} = \emptyset$, whence $((X \setminus u)^{0-} \cap (X \setminus v)^{0-})^0 \cap (u \cup v)^{-0} = \emptyset$, whence $((X \setminus u)^{0-} \cap (X \setminus v)^{0-})^{0-} \cap (u \cup v)^{-0} = \emptyset$. That is: $(u \cup v)^{-0} \subseteq (u^{-0} \cup v^{-0})^{-0}$. But by Proposition 1.2, $(u^{-0} \cup v^{-0})^{-0} = u^{-0} + v^{-0}$.

(ii) The only non-trivial inclusion is $r + s \subseteq r \cup s \cup (r^- \cap s^-)$. So suppose $p \notin s$ and $p \notin r^-$. That is, $p \in (-s)^-$ and $p \in -r$. But then, for all open o with $p \in o$, $o \cap -r$ is also open with $p \in o \cap -r$, whence $(o \cap -r) \cap -s \neq \emptyset$–that is, $o \cap (-r \cdot -s) \neq \emptyset$. Hence $p \in (-r \cdot -s)^-$ so $p \notin -(-r \cdot -s) = r + s$. A similar argument applies if $p \notin r$ and $p \notin s^-$.

(iii) $(r+s)^- = X \setminus -(r+s) = X \setminus (-r \cdot -s) = (X \setminus -r) \cup (X \setminus -s) = r^- \cup s^-$.

(iv) Certainly, $r \cup s$ is connected, and by (ii), $r \cup s \subseteq r + s \subseteq (r \cup s)^-$, whence $r + s$ is connected. QED

We note in passing that determining the validity of statements such as those of Lemma 1.4 is actually a decidable problem. See, e.g. Cantone and Cutello, 1994, Nutt, 1999, Pratt-Hartmann, 2002 and, for a fuller discussion, Ch. **??**.

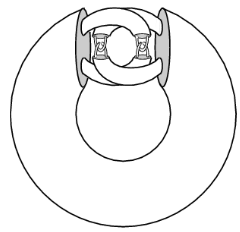


Figure 1.3. Alexander's horned sphere

2.2 Mereotopologies

We have argued, provisionally, that, for a subset of $\mathbb{R}^3$ to count as a region, it should be regular open. However, it would be hasty to assume that *all* regular open subsets of $\mathbb{R}^3$ should count as regions, at least if spatial regions are supposed to be parts of space occupied (or left unoccupied) by physical objects. Consider, for example, the bizarre region commonly known as Alexander's horned sphere and depicted in Fig. 1.3. (The reader is referred to Alexander, 1924a for details of the construction.) The interior of Alexander's horned sphere is certainly regular open, yet it is a poor candidate to represent the space occupied by a physical object. In fact, this region is a "ball" is so twisted in space that its complement in $\mathrm{RO}(\mathbb{R}^3)$ is not simply connected! Nor are such pathological objects to be found only in three-dimensional space, as we shall see below. And such examples suggest that we should at least be open to the possibility of region-based models of space in which only *some* regular open subsets of $\mathbb{R}^3$ qualify as regions. This immediately presents us with the question: if not all subsets of $\mathbb{R}^3$ qualify as *bona fide* regions, which do? As we shall see, the answers available and the issues which hinge on them require detailed analysis.

In view of these uncertainties, we adopt a very general notion of a region-based model of space—just sufficiently constrained that we can sensibly confine attention to the structure of regions in question without worrying about the points of which they are composed. In the context of point-set topology, a topological space is commonly said to be *semi-regular* if it has a basis of regular open sets, and *locally connected* if it has a basis of connected sets. It easy to see that, in a locally connected space, every component of an open set is open. Recall also, in the context of Boolean algebras, that, if B is a Boolean algebra and B' a Boolean subalgebra of B, then B' is said to be *dense* (*in* B) if, for every $b \in B$ with $0 < b$, there exists $b' \in B'$ with $0 < b' \leq b$.

DEFINITION 1.5 *Let X be a topological space. A* mereotopology over X *is a Boolean sub-algebra M of* $\mathrm{RO}(X)$ *such that, if o is an open subset of X and $p \in o$, there exists $r \in M$ such that $p \in r \subseteq o$. We refer to the elements of M as* regions. *If M is a mereotopology such that any component of a region in M is also a region in M, then we say that M* respects components.

Note that a mereotopology over X is always a dense subalgebra of $\mathrm{RO}(X)$. Our first task is to check that $\mathrm{RO}(X)$ *is* a mereotopology, for a suitable class of topological spaces.

LEMMA 1.6 *Let X be a semi-regular space. Then* $\mathrm{RO}(X)$ *is a mereotopology over $\mathbb{R}^n$; if X is also locally connected, then* $\mathrm{RO}(X)$ *respects components.*

Proof The first part of the lemma is instant from the relevant definitions. For the second part, let $r \in \mathrm{RO}(X)$, and let s be a component of r. Since X is locally connected, s is open, whence, by Lemma 1.3, $(s^-)^0$ is regular open with $s \subseteq (s^-)^0 \subseteq s^-$. Then, s^{-0} is a connected subset of r including s, whence $s^{-0} = s$ by the maximality of s. QED

Some etymological explanation is in order here. The term *mereology* was first introduced by Leśniewski,, and denotes the logic of the part-whole relationship. (For a survey, see, e.g. Simons, 1987.) The term *mereotopology* is a much more recent coinage, and standardly denotes the study of topological relationships in which regions, rather than points, are the primitive objects. (It is unclear where the word first appeared in print.) The employment of the word as a count-noun in Definition 1.5, to denote a certain class of mathematical structures, is new here, and prompted by analogy with the parallel usage of the word *topology*.

The foregoing discussion suggests that our search for a region-based model of space should begin with an examination of mereotopologies over $\mathbb{R}^3$. This approach may at first seem dissatisfying, because it depends for its formulation on the very point-based model of space we are trying to escape. As we shall see, however, it is the *structure* of the resulting collection of regions that will interest us—and the characterization of that structure in purely intrinsic terms form one of the main themes of this chapter. But before we can seek such intrinsic characterizations, we must first clarify what it is we want to characterize.

2.3 Geometric mereotopologies

The question before us is to identify the regular open subsets of $\mathbb{R}^3$ which we are prepared to count as 'sensible' regions of space. Here is a standard answer from the mathematical literature. Let L' be the first-order language with the arithmetic signature $(<, +, \cdot, 0, 1)$, interpreted over $\mathbb{R}$ in the usual way. (This interpretation is of course completely separate from our use of the same symbols to denote Boolean operations on regular open sets!) For the purposes of this chapter, we may say that a set $u \subseteq \mathbb{R}^n$ is *semi-algebraic* if there exists an L'-formula $\phi(\bar{x}, \bar{y})$ in $n + m$ variables $\bar{x}, \bar{y}$ and an m-tuple of real numbers $\bar{b}$ such that

$$u = \{\bar{a} \in \mathbb{R}^n \,|\, \text{the } (n+m)\text{-tuple } \bar{a}, \bar{b} \text{ satisfies the formula } \phi(\bar{x}, \bar{y})\}.$$

For a detailed discussion of semi-algebraic sets, see, e.g. van den Dries, 1998, Bochnak et al., 1998 and also Ch. **??**. (The more standard definition of semi-algebraic sets is equivalent to ours, and makes the name less puzzling.) For mereotopological purposes, we are exclusively interested in those semi-algebraic subsets of $\mathbb{R}^n$ which are *regular open.*

DEFINITION 1.7 *For $n > 0$, we denote the set of regular open, semi-algebraic sets in $\mathbb{R}^n$ by* $\mathrm{ROS}(\mathbb{R}^n)$*.*

LEMMA 1.8 *For $n > 0$,* $\mathrm{ROS}(\mathbb{R}^n)$ *is a mereotopology over $\mathbb{R}^n$.*

Proof We first show that $\mathrm{ROS}(\mathbb{R}^n)$ is a Boolean subalgebra of $\mathrm{RO}(\mathbb{R}^n)$. Evidently, $0, 1 \in \mathrm{ROS}(\mathbb{R}^n)$. Moreover, if a set u is definable by a first-order formula in the language of arithmetic, then so are its closure u^- and its interior u^0. Hence, if $r, s \in \mathrm{ROS}(\mathbb{R}^n)$, then so are $r \cdot s = r \cap s$, $r + s = (r \cup s)^{-0}$ and $-r = \mathbb{R}^n \setminus r^-$. We must establish that, for $p \in o$ with $o \subseteq \mathbb{R}^n$ open, there exists $r \in \mathrm{ROS}(\mathbb{R}^n)$ such that $p \in r \subseteq o$. But this is obvious since any open ball is an element of $\mathrm{ROS}(\mathbb{R}^n)$. QED

The structure of regular open semi-algebraic subsets of $\mathbb{R}^3$ might have a better claim to count as a region-based model of space than the whole of $\mathrm{RO}(\mathbb{R}^3)$, because it does a good job of ruling out pathological regular open sets. For example, the horned sphere of Fig. 1.3 is not semi-algebraic.

More generally, semi-algebraic sets count as well-behaved. One of their fundamental properties is that they admit of 'cell decompositions'. If $d > 0$, *d-cell* in $\mathbb{R}^n$ is any semi-algebraic subset of $\mathbb{R}^n$ homeomorphic to the open d-dimensional ball; a 0-*cell* in $\mathbb{R}^n$ is a singleton; and a *cell* is a d-cell for some d $(0 \leq d \leq n)$. The following result is standard (van den Dries, 1998, Ch. 3, Theorem 2.11).

PROPOSITION 1.9 (CELL DECOMPOSITION THEOREM) *If u is a semi-algebraic subset of $\mathbb{R}^n$, then u is the union of a finite collection of pairwise disjoint, semi-algebraic cells.*

For regular open semi-algebraic sets, this yields:

LEMMA 1.10 *Every* $r \in \mathrm{ROS}(\mathbb{R}^n)$ *is the sum of finitely many pairwise disjoint* n*-cells in* $\mathrm{ROS}(\mathbb{R}^n)$.

Proof By Proposition 1.9, let $r = u_1 \cup \ldots \cup u_m$ where the u_i are pairwise disjoint, semi-algebraic cells. Since r is regular, $r = r^{-0} = (u_1 \cup \ldots \cup u_m)^{-0} = u_1^{-0} + \ldots + u_m^{-0}$, by Lemma 1.4 (i). If u_i is a d-cell for $d < n$, then $u_i^{-0} = 0$; if u_i is an n-cell, $u_i^{-0} = u_i$. QED

The following notion will play an important part in the ensuing discussion.

DEFINITION 1.11 *A mereotopology M is* finitely decomposable *if every region in M is the sum of finitely many connected regions in M.*

LEMMA 1.12 $\mathrm{ROS}(\mathbb{R}^n)$ *is finitely decomposable.*

Proof By Lemma 1.10, since cells are connected. QED

LEMMA 1.13 *Every finitely decomposable mereotopology M over a locally connected space X respects components; moreover, every region in M is the sum of its components.*

Proof Suppose $r \in M$, and s is a component of r. By Lemma 1.6, $s \in \mathrm{RO}(X)$. Let $r_1, \ldots, r_n$ be connected elements of M such that $r = r_1 + \ldots + r_n$. By the maximality of s and Lemma 1.4 (iv), either $r_i \leq s$ or $r_i \cdot s = 0$ for all i $(1 \leq i \leq n)$. Thus, s is the sum of those r_i such that $r_i \leq s$. QED

Of course, the converse of Lemma 1.13 is false: although $\mathrm{RO}(X)$ respects components for any locally connected space X, it is easy to see that, for example, $\mathrm{RO}(\mathbb{R}^n)$ is not finitely decomposable for any $n > 0$.

The mereotopology $\mathrm{ROS}(\mathbb{R}^n)$ is thus at least a plausible region-based model of the space we inhabit. But it is not the only candidate for this job. Observe that any $(n-1)$-dimensional hyperplane of $\mathbb{R}^n$ cuts $\mathbb{R}^n$ into two residual domains, which we shall call *half-spaces*. It is easy to see that these half-spaces are regular open, with each being the pseudo-complement of the other. Hence, we can speak about the sums, products and complements of half-spaces in $\mathrm{RO}(\mathbb{R}^n)$.

DEFINITION 1.14 *A* basic polytope *in* $\mathbb{R}^n$ *is the product, in* $\mathrm{RO}(\mathbb{R}^n)$, *of finitely many half-spaces. A* polytope *in* $\mathbb{R}^n$ *is the sum, in* $\mathrm{RO}(\mathbb{R}^n)$, *of any finite set of basic polytopes. We denote the set of polytopes in* $\mathbb{R}^n$

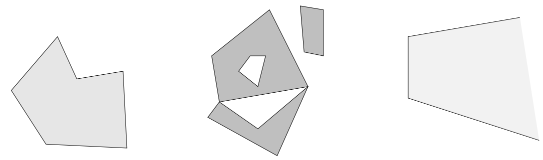


Figure 1.4. Three (differently shaded) regions in the mereotopology $\mathrm{ROP}(\mathbb{R}^2)$.

by $\mathrm{ROP}(\mathbb{R}^n)$*; we call the polytopes in* $\mathrm{ROP}(\mathbb{R}^2)$ polygons *and those in* $\mathrm{ROP}(\mathbb{R}^3)$ polyhedra.

Thus, polytopes (in our sense) may be unbounded, disconnected, and may have disconnected complements. Fig. 1.4 shows a selection of polygons. (In alternative parlance, the elements of $\mathrm{ROP}(\mathbb{R}^n)$ are the regular open *semi-linear* sets.) Evidently, the polyhedra constitute a more parsimonious region-based model of space than does $\mathrm{ROS}(\mathbb{R}^3)$.

Indeed, the following construction gives us a more parsimonious spatial ontology still. If an $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$ is defined by an equation $a_0 + a_1x_1 + \cdots a_nx_n = 0$, where the a_i $(0 \leq i \leq n)$, are rational numbers, we call it a *rational hyperplane*; and if a half-space is bounded by a rational hyperplane, we call it a *rational half-space.* Now we define:

Definition 1.15 *A* basic rational polytope *in* $\mathbb{R}^n$ *is the product, in* $\mathrm{RO}(\mathbb{R}^n)$*, of finitely many rational half-spaces. A* rational polytope *in* $\mathbb{R}^n$ *is the sum, in* $\mathrm{RO}(\mathbb{R}^n)$*, of any finite set of basic rational polytopes. We denote the set of rational polytopes in* $\mathbb{R}^n$ *by* $\mathrm{ROQ}(\mathbb{R}^n)$*; we call the elements of* $\mathrm{ROQ}(\mathbb{R}^2)$ rational polygons *and those of* $\mathrm{ROQ}(\mathbb{R}^3)$ rational polyhedra.

Evidently, $\mathrm{ROQ}(\mathbb{R}^n) \subsetneq \mathrm{ROP}(\mathbb{R}^n) \subsetneq \mathrm{ROS}(\mathbb{R}^n) \subsetneq \mathrm{RO}(\mathbb{R}^n)$. Note that $\mathrm{ROQ}(\mathbb{R}^n)$ is countable.

Lemma 1.16 *The collections* $\mathrm{ROP}(\mathbb{R}^n)$ *and* $\mathrm{ROQ}(\mathbb{R}^n)$ *are finitely decomposable mereotopologies over* $\mathbb{R}^n$*.*

Proof Basic polytopes are convex, and hence connected. QED

As models of the space we inhabit, $\mathrm{ROP}(\mathbb{R}^3)$ and $\mathrm{ROQ}(\mathbb{R}^3)$ may seem overly austere—for they contain no regions with curved boundaries. However, their study turns out to be instructive, as we shall see below.

2.4 Interpretations

So far, we have discussed various ways of selecting a collection of 'regions' from among the subsets of $\mathbb{R}^n$. But this selection process only really becomes interesting when we consider formal languages whose variables range over these collections, and whose non-logical constants belong to a limited repertoire of spatial primitives.

We assume familiarity with basic first-order logic: for details, see Hodges, 1993, Ch 1. In this context, we employ the following standard notation and terminology. Let Σ be a signature consisting of (zero or more) predicates, function-symbols and individual constants; we denote the first-order language with signature Σ by L_Σ. An L_Σ-formula with no free variables is called an L_Σ-*sentence*. Let $\mathfrak{A}$ be a structure interpreting the symbols in Σ over some domain A (assumed non-empty). For any L_Σ-formula $\phi(\bar{x})$, with $n > 0$ free-variables $\bar{x}$ and any n-tuple $\bar{a}$ from A, we write $\mathfrak{A} \models \phi[\bar{a}]$ if $\bar{a}$ satisfies $\phi(\bar{x})$ in $\mathfrak{A}$; similarly, for any L_Σ-sentence ϕ, we write $\mathfrak{A} \models \phi$ if ϕ is true in $\mathfrak{A}$. We call $\{\psi \mid \psi \text{ an } L_\Sigma\text{-sentence and } \mathfrak{A} \models \psi\}$ the L_Σ-*theory of* $\mathfrak{A}$, denoted $\mathrm{Th}_\Sigma(\mathfrak{A})$. Two structures $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent* (*for* Σ), written $\mathfrak{A} \equiv_\Sigma \mathfrak{B}$, if $\mathrm{Th}_\Sigma(\mathfrak{A}) = \mathrm{Th}_\Sigma(\mathfrak{B})$. We write $f : \mathfrak{A} \simeq_\Sigma \mathfrak{B}$ if f is a Σ-structure isomorphism from $\mathfrak{A}$ onto $\mathfrak{B}$ (and $\mathfrak{A} \simeq_\Sigma \mathfrak{B}$ if such an f exists). It is a simple result that if $f : \mathfrak{A} \simeq_\Sigma \mathfrak{B}$ and $\phi(\bar{x})$ is an L_Σ-sentence, then $\mathfrak{A} \models \phi[\bar{a}]$ implies $\mathfrak{B} \models \phi[f(\bar{a})]$ for every tuple $\bar{a}$ from A; in particular, $\mathfrak{A} \simeq_\Sigma \mathfrak{B}$ implies $\mathfrak{A} \equiv_\Sigma \mathfrak{B}$. We write $\mathfrak{A} \subseteq_\Sigma \mathfrak{B}$, if $\mathfrak{A}$ is a submodel of $\mathfrak{B}$ (i.e. $A \subseteq B$ and $\mathfrak{A}$ is the restriction of $\mathfrak{B}$ to A), and $\mathfrak{A} \preceq_\Sigma \mathfrak{B}$ if $\mathfrak{A}$ is an *elementary submodel* of $\mathfrak{B}$ (i.e. $A \subseteq B$ and and every tuple $\bar{a}$ of A satisfies the same L_Σ-formulas in both $\mathfrak{A}$ and $\mathfrak{B}$). We say that $\mathfrak{A}$ is *elementarily embeddable in* $\mathfrak{B}$ if $\mathfrak{A}$ is isomorphic to an elementary submodel of $\mathfrak{B}$. Trivially, $\mathfrak{A} \preceq_\Sigma \mathfrak{B}$ implies $\mathfrak{A} \equiv_\Sigma \mathfrak{B}$. Reference to the signature Σ, and the associated subscripts, is suppressed when clear from context.

Let M be a mereotopology over some topological space X. If Σ is a signature whose symbols conventionally denote familiar mereological or topological concepts, then M can always be regarded as a Σ-structure by interpreting the symbols of Σ in the familiar way. In particular, we take the symbols 0, 1, $+$, $\cdot$, $-$ and $\leq$ to have the obvious (Boolean algebra) interpretations over M; similarly, we take the unary predicate c to denote the property of being connected, and the binary predicate C to denote the relation which holds between two regions if and only if their topological closures intersect. Table 1.1 gives a formal summary. Under these interpretations, we may regard any mereotopology M as an interpretation for the signature $\Sigma = (0, 1, +, \cdot, -, \leq, c, C)$, or any subset thereof. That is: any L_Σ-sentence has a truth-value in M, and any L_Σ-formula $\phi(\bar{x})$ with $n > 0$ free-variables defines an n-ary relation over M, namely, the set of n-tuples from M satisfying $\phi(\bar{x})$. We remark that our interpretation of C is intended as a rational reconstruction of

Symbol	Type	Interpretation
0	individual constant	$0^M = \emptyset$
1	individual constant	$1^M = X$
$+$	binary function	$+^M(r,s) = ((r \cup s)^-)^0$
$\cdot$	binary function	$\cdot^M(r,s) = r \cap s$
$-$	unary function	$-^M(r) = X \setminus r^-$
$\leq$	binary predicate	$\leq^M = \{\langle r,s\rangle \in M^2 \mid r \subseteq s\}$
c	unary predicate	$c^M = \{r \in M \mid r \text{ connected}\}$
C	binary predicate	$C^M = \{\langle r,s\rangle \in M^2 \mid r^- \cap s^- \neq \emptyset\}$

Table 1.1. Interpretations of common mereotopological primitives, where M is a mereotopology over a topological space X.

the relation which Whitehead, 1929 called "extensive connection", and which has historically played a prominent role in region-based theories of space. Since Whitehead's term risks confusion with the standard topological notion of *connectedness*, we follow more recent usage and read $C(x,y)$ as "x contacts y".

Some examples will help to clarify the issues that arise concerning first-order languages interpreted over mereotopologies.

EXAMPLE 1.17 *Let $\Sigma = (C, c, \leq)$, and let ψ_{inf} be the L_Σ-sentence*

$$\forall x \forall y (C(x,y) \rightarrow \exists z (c(z) \wedge z \leq y \wedge C(x,z))).$$

This sentence 'says' that, if a region contacts another region, then it contacts some connected part of it. Let M be any finitely decomposable mereotopology; then $M \models \psi_{\text{inf}}$. For suppose $M \models C[r,s]$, and let $s_1, \ldots, s_m$, be connected regions of M summing to s. By Lemma 1.4(iii), $s^- = s_1^- \cup \cdots \cup s_m^-$, whence $M \models C[r, s_i]$ for some i. On the other hand, it is not difficult to see that $\mathrm{RO}(\mathbb{R}^2) \not\models \psi_{\text{inf}}$. Fig. 1.5 shows two regular open regions r, s in the plane, where r has infinitely many components, and s touches the closure of r but is separated from each of its components.

Example 1.5 shows, in particular, that the differences between the region-based models of space $\mathrm{RO}(\mathbb{R}^3)$ and $\mathrm{ROS}(\mathbb{R}^3)$ are 'visible' to certain first-order languages with signatures of topological primitives. In fact, the existence of regions with infinitely many components is not the only difference between these mereotopologies, as the next example shows.

EXAMPLE 1.18 *Let $\Sigma = (c, +)$, and let ψ_{sum} be the L_Σ-sentence*

$$\forall x_1 \forall x_2 \forall x_3 (c(x_1) \wedge c(x_2) \wedge c(x_3) \wedge c(x_1 + x_2 + x_3) \rightarrow (c(x_1 + x_2) \vee c(x_1 + x_3))).$$

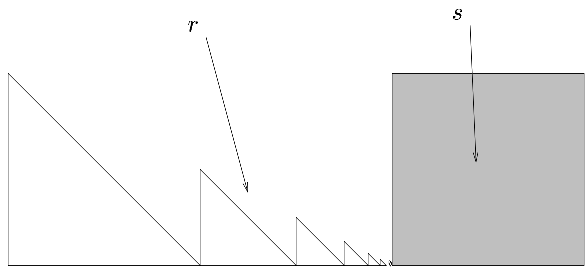


Figure 1.5. Two elements in $\mathrm{RO}(\mathbb{R}^2)$, one with infinitely many components.

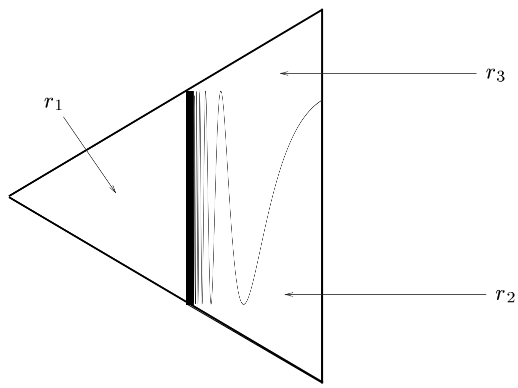


Figure 1.6. Three elements in $\mathrm{RO}(\mathbb{R}^2)$.

This sentence 'says' that if three connected regions have a connected sum, then the first must form a connected sum with one of the other two. We show in Lemma 1.56 below that, if M is any of $\mathrm{ROS}(\mathbb{R}^2)$, $\mathrm{ROP}(\mathbb{R}^2)$ *or* $\mathrm{ROQ}(\mathbb{R}^2)$*, then* $M \models \psi_{\mathrm{sum}}$*. However, it turns out that* $\mathrm{RO}(\mathbb{R}^2) \not\models \psi_{\mathrm{sum}}$*. For let*

$$\begin{aligned} r_1 &= \{(x,y)| -1 < x < 0 \; ; \; -1 - x < y < 1 + x\} \\ r_2 &= \{(x,y)|0 < x < 1 \; ; \; -1 - x < y < \sin(1/x)\} \\ r_3 &= \{(x,y)|0 < x < 1 \; ; \; \sin(1/x) < y < 1 + x\} \;, \end{aligned}$$

as depicted in Fig. 1.6. It is easy to check that $r_1 + r_2 + r_3$ *is the large triangle, and so is certainly connected, but that neither* $r_1 + r_2$ *nor* $r_1 + r_3$ *is connected.*

We shall see in Section 5 that, in some sense, Examples 1.17 and 1.18 represent the *only* differences between $\mathrm{RO}(\mathbb{R}^2)$ and $\mathrm{ROS}(\mathbb{R}^2)$.

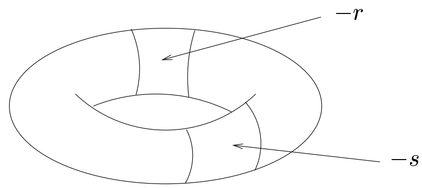


Figure 1.7. The (complements of) two connected elements r and s in the regular open algebra of a torus: $r \cdot s$ is not connected, and $(-r)^- \cap (-s)^- = \emptyset$.

Our final example illustrates a rather different set of issues concerning first-order mereotopological theories. We require the following fact about the topology of Euclidean spaces (Newman, 1964,p. 137).

PROPOSITION 1.19 *If d_1 and d_2 are non-intersecting closed sets in $\mathbb{R}^n$, and points p and q are connected in $\mathbb{R}^n \setminus d_1$ and also in $\mathbb{R}^n \setminus d_2$, then p and q are connected in $\mathbb{R}^n \setminus (d_1 \cup d_2)$.*

EXAMPLE 1.20 *Let $\Sigma = (C, c, \cdot, -)$, and let ψ_{sep} be the L_Σ-sentence*

$$\forall x \forall y (c(x) \wedge c(y) \rightarrow (c(x \cdot y) \vee C(-x, -y))).$$

This sentence 'says' that the closures of the complements of any two connected regions whose product is not connected intersect. Suppose that $r, s \in \mathrm{RO}(\mathbb{R}^n)$ are connected, with $r \cdot s$ not connected. Putting $d_1 = \mathbb{R}^n \setminus r$ and $d_2 = \mathbb{R}^n \setminus s$, we have $d_1 \cup d_2 = \mathbb{R}^n \setminus (r \cdot s)$, whence, by Proposition 1.19, $(-r)^- \cap (-s)^- \neq \emptyset$. Thus, if M is a mereotopology over any of the spaces $\mathbb{R}^n$, $M \models \psi_{\text{sep}}$. However, ψ_{sep} is not true for all mereotopologies. For example, let X be the surface of a torus, let M be $\mathrm{RO}(X)$, and let $r, s \in M$ be such that $-r$ and $-s$ are as illustrated in Fig. 1.7. By inspection, r and s are connected, $r \cdot s$ is not connected, and $-r$ does not contact $-s$. Hence, $M \models \neg\psi_{\text{sep}}$.

Thus the regular open algebra of the torus and the Euclidean plane have different first-order mereotopological theories over the signature $\{C, c, \cdot, -\}$.

There is nothing privileged about the above collection of primitives: in principle, we could employ any signature whose symbols can be given fixed interpretations over the structures we choose to confine our attention to. Since this chapter deals with *topological* notions, we consider only signatures with fixed *topological* interpretations—that is, signatures whose interpretations are preserved by homeomorphisms of the underlying topological space. For brevity, we speak of a 'signature of topological primitives'. For investigations of region-based theories with non-topological signatures, see, e.g. Davis et al., 1999,Pratt, 1999.

Given a mereotopology M and a signature Σ of topological primitives, three salient issues present themselves. The first concerns the *expressive power* of a first-order topological language L_Σ over a mereotopology M. Any L_Σ-formula $\phi(\bar{x})$ with free variables $\bar{x} = x_1, \ldots x_n$ defines an n-ary relation over M—namely, the set of n-tuples $\bar{r}$ satisfying $\phi(\bar{x})$ in M. And it is therefore natural to ask *which* relations can be so defined, and in particular, which primitives can be defined in terms of which others. Of particular interest in this regard is the property of being topologically indistinguishable from a specific object or tuple of objects. That is, given a tuple $\bar{r}$ from M, we would particularly like to know whether L_Σ is expressive enough to give a topologically complete characterization of $\bar{r}$. The answers to these questions depends heavily on the mereotopology M: Sections 3 and 4 analyse the expressive power of various first-order topological languages for well-behaved mereotopologies over the Euclidean plane. Section 6 analyses the much more difficult case of well-behaved mereotopologies over $\mathbb{R}^3$.

The second salient issue concerns the L_Σ-*theory* of M. Examples 1.17 and 1.18 show that restricting regions to be *semi-algebraic* (regular open) sets does affect the resulting first-order theory over some signatures of topological primitives. And the question therefore arises as to what other restrictions might be sensible, and what effect, if any, these restrictions have on the resulting first-order mereotopological theories. Most ambitiously, perhaps, we might ask whether the set of first-order sentences in various mereotopologies can be axiomatically characterized. Section 5 provides an example of such an axiomatic characterization. As a by-product of this analysis, we show that a wide range of plane mereotopologies share the same L_Σ-theory for (most) topological signatures Σ, and we venture to take that theory as the *standard* first-order L_Σ-theory of plane mereotopology. In this sense, the choice of what, exactly, counts as a region is much less critical than we might at first have supposed.

The third salient issue concerns the ontological commitments entailed by first-order mereotopological theories. To understand this issue, recall that a mereotopology M is a collection of subsets of some topological space, which we have chosen to regard as a Σ-structure, for some signature Σ of topological primitives. Any such mereotopology M thus defines an L_Σ-theory $\mathrm{Th}_\Sigma(M)$. But of course, *any* Σ structure $\mathfrak{A}$ with $\mathrm{Th}_\Sigma(\mathfrak{A}) = \mathrm{Th}_\Sigma(M)$ can be thought of as a (region-based) model of space which, from the point of view of L_Σ, makes exactly the same predictions as M. It is therefore natural to ask *which* structures these are, and what, if anything, we can say about their relationship to M. Notice that the elements of such Σ-structures need not be regions of topological spaces at all; as such they are genuinely region-based theories of space. In particular, we may ask whether mereotopologies in general admit of intrinsic characterizations making no reference to the topological spaces whose regions they make up. And we may further ask—particularly in the light of Example 1.20—what information those intrinsic characterizations yield about the topological spaces in question. Section 7 answers these, and related, questions.

The above three issues constitute the primary agenda of mereotopology, as conceived here.

3. Defining topological relations

Our task in this section is to compare the relative expressiveness of first-order languages having different signatures of topological primitives. Our main result is that L_C is at least as expressive as $L_{c,\leq}$ over all sensible mereotopologies. We also show that over some mereotopologies of interest, $L_{c,\leq}$ is also at least as expressive as L_C.

We assume familiarity with the standard (T_i-) separation properties of topological spaces. Terminology varies here: we adopt the convention according to which T_i-separation for $i > 2$ does not by definition imply T_1-separation; and we say that a space X is *Hausdorff* if it satisfies T_2-separation, *regular* if it satisfies both T_3- and T_1-separation, and *normal* if it satisfies both T_4- and T_1-separation.) In addition, we occasionally employ the following less familiar separation property (Düntsch and Winter, 2005).

Definition 1.21 *A topological space is* weakly regular *if it is semi-regular and, for any non-empty open set u, there exists a non-empty open set v with $v^- \subseteq u$.*

We have

$$X \text{ is normal} \Rightarrow X \text{ is regular} \Rightarrow X \text{ is weakly regular} \Rightarrow X \text{ is semi-regular.}$$

The reverse implications all fail (see Düntsch and Winter, 2005regarding weak regularity, and Steen and Seebach, 1995 for the other cases).

3.1 Contact

We begin by defining the part-of relation in L_C.

Lemma 1.22 *Let M be a mereotopology over a weakly regular space X, and let $r_1, r_2 \in M$. Then $r_1 \leq r_2$ if and only if $M \models \phi_{\leq}[r_1, r_2]$, where $\phi_{\leq}(x_1, x_2)$ is the L_C-formula $\forall z(C(x_1, z) \to C(x_2, z))$.*

Proof If $r_1 \leq r_2$ then $r_1^- \subseteq r_2^-$, so $s^- \cap r_1^- \neq \emptyset$ implies $s^- \cap r_2^- \neq \emptyset$ for any s. Conversely, if $r_1 \not\leq r_2$, by weak regularity, let u be a non-empty, open set such that $u^- \subseteq r_1 \cdot (-r_2)$. Since M is a mereotopology, let $s \in M$ be such that $0 \neq s \subseteq u$. Then $s^- \cap r_1^- \neq \emptyset$, but $s^- \cap r_2^- = \emptyset$.

QED

In dealing with mereotopologies over weakly regular spaces, we may therefore write the expression $u \leq v$ in L_C-formulas, as a shorthand for $\phi_{\leq}(u, v)$. It follows that the Boolean constants and functions 0, 1, $+$, $\cdot$ and $-$ are also L_C-definable for mereotopologies over weakly regular spaces, and we again freely employ these symbols in L_C-formulas as a shorthand for their definitions.

We now turn to defining the property of connectedness in L_C. We need some technical lemmas.

LEMMA 1.23 *Let M be a mereotopology over a regular topological space X. If $d \subseteq X$ is closed and $p \notin d$, there exists $r \in M$ such that $p \in r$ and $d \subseteq -r$. In fact, there exist $r, s \in M$ such that $p \in r$, $d \subseteq s$ and $r^- \cap s^- = \emptyset$.*

Proof For the first statement, by T_3-separation, let u, v be disjoint open subsets of X such that $p \in u$ and $d \subseteq v$. Since M is a mereotopology, there exists $r \in M$ such that $p \in r \subseteq u$, whence $d \subseteq v \subseteq X \setminus r^- = -r$. The second statement follows by two applications of the first: choose $s \in M$ such that $p \in -s$ and $d \subseteq s$; now choose $r \in M$ such that $p \in r$ and $s^- \subseteq -r$. QED

LEMMA 1.24 *Let $r, s \in \mathrm{RO}(X)$ for some topological space X. If $p \in r^-$ and $p \in s$, then $p \in (r \cdot s)^-$.*

Proof Let u be any open set containing p. Then $u \cap s$ is also an open set containing p, whence $(u \cap s) \cap r \neq \emptyset$, since $p \in r^-$. That is, $u \cap (s \cdot r) \neq \emptyset$.

QED

LEMMA 1.25 *Let M be a mereotopology over a regular topological space. For all $r_1, r_2 \in M$, $r_1^- \cap r_2^- \cap (r_1 + r_2) \neq \emptyset$ if and only if there exist $r_1', r_2' \in M$ such that $r_1' \leq r_1$, $r_2' \leq r_2$, ${r_1'}^- \cap {r_2'}^- \neq \emptyset$ and $(r_1' + r_2')^- \cap (-(r_1 + r_2))^- = \emptyset$.*

Proof The if-direction is immediate. For the only-if-direction, suppose $p \in r_1^- \cap r_2^- \cap (r_1 + r_2)$. By Lemma 1.23, let $s \in M$ be such that $p \in s$ and $(-(r_1 + r_2))^- \subseteq -s$; and let $r_1' = r_1 \cdot s$ and $r_2' = r_2 \cdot s$. By Lemma 1.24, $p \in {r_1'}^- \cap {r_2'}^-$, whence r_1' and r_2' have the required properties. QED

Lemma 1.26 *Let M be a mereotopology which respects components. Then $r \in M$ is connected if and only if $r_1^- \cap r_2^- \cap r \neq \emptyset$ for all nonempty, disjoint $r_1, r_2 \in M$ such that $r_1 + r_2 = r$.*

Proof Suppose r_1 and r_2 are non-empty, disjoint elements of M such that $r_1 + r_2 = r$ and $r_1^- \cap r_2^- \cap r = \emptyset$. By Lemma 1.4 (ii), $r = r_1 \cup r_2$, so that r is not connected. Conversely, suppose r is not connected. Let r_1 be a component of r and let $r_2 = r \setminus r_1$. Since M respects components, $r_1 \in M$. Since $r_1 \subseteq r_1 \cup (r_1^- \cap r_2) \subseteq r_1^-$, $r_1 \cup (r_1^- \cap r_2)$ is connected, whence $r_1^- \cap r_2 = \emptyset$ by maximality of components. Thus, $r_2 = r \setminus r_1^- = r \cdot (-r_1)$. Moreover, since r_1 is open and $r_1 \cap r_2 = \emptyset$, we have $r_1 \cap r_2^- = \emptyset$. Therefore $\emptyset = r_1^- \cap r_2^- \cap (r_1 \cup r_2) = r_1^- \cap r_2^- \cap r$ as required. QED

Lemma 1.27 *Let M be a mereotopology over a regular topological space X such that M respects components, and let $r \in M$. Then r is connected if and only if $M \models \phi_c[r]$, where $\phi_c(x)$ is the L_C-formula*

$$\begin{aligned} &\forall x_1 \forall x_2 (x_1 > 0 \wedge x_2 > 0 \wedge x_1 \cdot x_2 = 0 \wedge x_1 + x_2 = x \rightarrow \\ &\qquad \exists x_1' \exists x_2' (x_1' \leq x_1 \wedge x_2' \leq x_2 \wedge C(x_1', x_2') \wedge \neg C(x_1' + x_2', -x))). \end{aligned}$$

Proof Lemmas 1.25 and 1.26. QED

Together, Lemmas 1.22 and 1.27 guarantee that, for all mereotopologies over regular topological spaces which respect components, the language L_C is at least as expressive as $L_{c,\leq}$. We take it that all mereotopologies of interest fulfil these conditions: that is, the above reconstructions of the part-whole relation and the property of connectedness in L_C are very robust.

We present a further—and more surprising—demonstration of the expressive power of L_C in mereotopologies defined over $\mathbb{R}^2$. We require the following fact about the topology of Euclidean spaces (Newman, 1964,p. 112, c.f. Proposition 1.19).

Proposition 1.28 *Let d_1 and d_2 be closed sets in $\mathbb{R}^2$, at least one of which is bounded. If $\mathbb{R}^2 \setminus d_1$, $\mathbb{R}^2 \setminus d_2$ and $d_1 \cap d_2$ are all connected, then so is $\mathbb{R}^2 \setminus (d_1 \cup d_2)$.*

Lemma 1.29 *Let $s_1, s_2, t \in \mathrm{RO}(\mathbb{R}^2)$ such that: (i) either s_1 is bounded or s_2 is bounded; (ii) $-(s_1 + t)$, $-(s_2 + t)$ and t are all connected; and (iii) $s_1^- \cap s_2^- = \emptyset$. Then $-(s_1 + s_2 + t)$ is also connected.*

Proof Set $d_i = (s_i + t)^-$ (for $i = 1, 2$). Thus, the complement of d_i is $-(s_i + t)$ (for $i = 1, 2$), and the complement of $d_1 \cup d_2$ is $-(s_1 + s_2 + t)$.

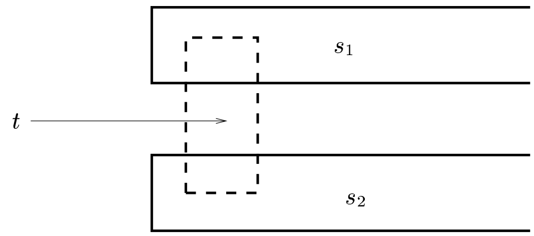


Figure 1.8. Expressing boundedness in L_C: s_1 and s_2 are unbounded to the right

Moreover, since t is connected, so is t^-, whence $d_1 \cap d_2 = (s_1 + t)^- \cap (s_2 + t)^- = (s_1^- \cup t^-) \cap (s_2^- \cup t^-) = (s_1^- \cap s_2^-) \cup t^- = t^-$ is connected. The result follows by Proposition 1.28. QED

Let ϕ_c be as defined in Lemma 1.27, and let $\phi_{ub}(y_1, y_2)$ be the L_C-formula

$$\exists z(\phi_c(-(y_1 + z)) \wedge \phi_c(-(y_2 + z)) \wedge \phi_c(z) \wedge \neg\phi_c(-(y_1 + y_2 + z))).$$

LEMMA 1.30 *Let M be a mereotopology over $\mathbb{R}^2$ such that M respects components and every unbounded element in M includes regions s_1, s_2 and t situated as in Fig. 1.8. Then for all $r \in M$, r is bounded if and only if $M \models \phi_{b^2}[r]$, where $\phi_{b^2}(x)$ is the L_C-formula:*

$$\forall y_1 \forall y_2 (y_1 \leq x \wedge y_2 \leq x \wedge \phi_{ub}(y_1, y_2) \rightarrow C(y_1, y_2)).$$

(The superscript 2 in ϕ_{b^2} refers to the fact that this formula works for mereotopologies over $\mathbb{R}^2$, and not, for example $\mathbb{R}^3$.)

Proof If r does not satisfy $\phi_{b^2}(x)$ then, by Lemma 1.29, r contains two unbounded regions, so is certainly itself unbounded. Conversely, if r is unbounded, let $s_1, s_2, t \in M$ be subsets of r situated as in Fig. 1.8. Thus, $s_1 \leq r$, $s_2 \leq r$ and $s_1^- \cap s_2^- = \emptyset$, but at the same time, s_1, s_2 satisfies $\phi_{ub}(y_1, y_2)$, with t a witness for the existentially quantified z. Hence r does not satisfy $\phi_{b^2}(x)$. QED

It is simple to verify that the mereotopologies $\mathrm{RO}(\mathbb{R}^2)$, $\mathrm{ROS}(\mathbb{R}^2)$, $\mathrm{ROP}(\mathbb{R}^2)$ and $\mathrm{ROQ}(\mathbb{R}^2)$ satisfy the conditions of Lemma 1.30. Hence, the property of boundedness is L_C-definable in all these mereotopologies. Nevertheless, Lemma 1.30, unlike Lemmas 1.22 and 1.27, has a *fragile* character, in that it depends on a very specific feature of the topological space $\mathbb{R}^2$; in particular, it fails to define boundedness for the corresponding mereotopologies over $\mathbb{R}^3$. We will see in Section 6 that boundedness is also L_C-definable in well-behaved mereotopologies over $\mathbb{R}^3$, but we have to go to much more trouble.

3.2 Reconstruction of points

In mereotopologies, the primitive objects—that is, the entities over which variables range—are regions, rather than points; but it is often simple to 'construct' points from regions, and 'simulate' statements about points using statements about regions. One way to construct the point p is as a pair of regions whose closures intersect in the singleton $\{p\}$, as we now proceed to show. (There are also more sophisticated ways, described in Section 7.1.)

LEMMA 1.31 *Let M be a mereotopology over a regular topological space, and let $r, s \in M$. Then $r^- \cap s^-$ is a singleton if and only if $M \models \phi_{\bowtie}[r, s]$, where $\phi_{\bowtie}(x_1, x_2)$ is the formula*

$$C(x_1,x_2) \wedge$$
$$\forall y_1 \forall y_2 (y_1 \leq x_1 \wedge y_2 \leq x_2 \wedge C(y_1, x_2) \wedge C(y_2, x_1) \rightarrow C(y_1, y_2)).$$

Furthermore, if $r^- \cap s^- = \{p\}$ and $t \in M$, then $p \in t$ if and only if $M \models \phi_{\in}[r, s, t]$, where $\phi_{\in}(x_1, x_2, x_3)$ is the formula

$$\exists y_1 (y_1 \leq x_1 \wedge C(y_1, x_2) \wedge \neg C(y_1, -x_3));$$

likewise, $p \in t^-$ if and only if $M \models \phi_{\bar{\in}}[r, s, t]$, where $\phi_{\bar{\in}}(x_1, x_2, x_3)$ is the formula

$$\forall y_1 (y_1 \leq x_1 \wedge C(y_1, x_2) \rightarrow C(y_1, x_3)).$$

Proof Routine by Lemmas 1.23 and 1.24. QED

If M is a mereotopology over a topological space X, let us say that M is *complete* if every point in X is the singleton intersection of some pair regions in M. For example, the mereotopologies $\mathrm{ROP}(\mathbb{R}^n)$, $\mathrm{ROS}(\mathbb{R}^n)$ evidently possess this property; by contrast, $\mathrm{ROQ}(\mathbb{R}^n)$ does not. We might say that, in a complete mereotopology, points can be 'simulated' by pairs of regions satisfying the formula $\phi_{\bowtie}$. If M is a complete mereotopology over a regular space, Lemma 1.31 gives us the right to include expressions such as, for example, $x_1 \cap x_2^- \neq \emptyset$ or $\mathcal{F}(x_1) \cap \mathcal{F}(x_2) \subseteq \mathcal{F}(x_3) \cap \mathcal{F}(x_4)$ etc. in L_C-formulas with the obvious interpretation, since such expressions can evidently be replaced by *bona fide* L_C-formulas with the appropriate extension over M.

The following lemma illustrates how easily we can express various topological relations in L_C:

LEMMA 1.32 *Let $r, s \in \mathrm{ROP}(\mathbb{R}^n)$. Then $r^- \cap s^-$ is connected if and only if $\mathrm{ROP}(\mathbb{R}^n) \models \phi_{\mathrm{ci}}[r, s]$, where $\phi_{\mathrm{ci}}(x, y)$ is the formula*

$$\forall z \neg (x^- \cap y^- \cap z \neq \emptyset \wedge x^- \cap y^- \cap -z \neq \emptyset \wedge x^- \cap y^- \subseteq z \cup -z).$$

Proof The only-if direction is immediate. So suppose $r^- \cap s^-$ is not connected; we must find a witness for z to show that $\mathrm{ROP}(\mathbb{R}^n) \models \neg\phi_{\mathrm{ci}}[r, s]$. But, by construction of $\mathrm{ROP}(\mathbb{R}^n)$, both r^- and s^- are expressible as finite unions of closed, convex sets; and so, therefore, is $r^- \cap s^-$. Since this latter set is not connected, it can be written as $d \cup e$, such that $d \cap e = \emptyset$ and d and e are both finite unions of non-empty, closed, convex sets—say, $d = d_1 \cup \cdots, \cup d_l$, $e = e_1 \cup \cdots, \cup e_m$. Given that any pair of disjoint, closed, convex sets in $\mathbb{R}^n$ can be separated by a hyperplane, we have half-spaces $h_{i,j}$ such that $d_i \subseteq h_{i,j}$ and $e_j \subseteq -h_{i,j}$ for all i, j ($1 \leq i \leq l$, $1 \leq j \leq m$). Then the required witness is

$$t = \sum_{1 \leq i \leq l} \prod_{1 \leq j \leq m} h_{i,j}.$$

QED

3.3 Compactifications

Before discussing the expressive power of $L_{c,\leq}$, we introduce some additional technical material that will be useful throughout this chapter. Recall that a topological space is said to be *locally compact* if every point has a compact neighbourhood. This property 'transfers', for Hausdorff spaces, to mereotopologies defined over them:

Lemma 1.33 *Let M be a mereotopology over a locally compact, Hausdorff space X, and let $p \in X$. Then p is contained within some $r \in M$ such that r^- is compact.*

Proof Let $p \in X$. Assuming X is locally compact, let $d \subseteq X$ be compact and $o \subseteq d$ be open such that $p \in o$. Now let $r \in M$ such that $p \in r \subseteq o \subseteq d$. But a closed subset of a compact set is always compact, and, in a Hausdorff space, every compact set is closed. Therefore $r^- \subseteq d^- = d$ is compact, as required. QED

Let X be a topological space, and let τ denote the collection of open sets of X. Now set $\dot{X} = X \cup \{\infty\}$, where ∞ is some object not in X. For $o \in \tau$, denote by $\dot{o}$ the set

$$\dot{o} = \begin{cases} o \cup \{\infty\} \text{ if } X \setminus o \text{ is compact;} \\ o \text{ otherwise,} \end{cases}$$

and denote by $\dot{\tau}$ the set $\tau \cup \{\dot{o} \mid o \in \tau\}$. Then we can take $\dot{X}$ to be a topological space whose collection of open sets is $\dot{\tau}$. Under this topology (which we always assume), we call $\dot{X}$ the *one-point* (or *Alexandroff*) *compactification* of X. The object ∞ is called the *point at infinity.* The space $\dot{X}$ is always compact. If X is locally compact and Hausdorff, then $\dot{X}$ is also Hausdorff, and hence normal.

NOTATION 1.34 *In this chapter, we denote spheres, open balls and closed balls in Euclidean spaces as follows*

$$
\begin{aligned}
\mathbf{S}^n &= \{(a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1} \mid a_1^2 + \cdots + a_{n+1}^2 = 1\} \\
\mathbf{B}^n &= \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid a_1^2 + \cdots + a_n^2 < 1\} \\
\mathbf{D}^n &= \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid a_1^2 + \cdots + a_n^2 \leq 1\};
\end{aligned}
$$

and we assume the usual topology on these sets.

(Recall that, by a *d-cell*, we mean any set homeomorphic to the open d-dimensional ball $\mathbf{B}^d$.) In the special cases $X = \mathbb{R}^n$, it is well-known that $\dot{X}$ is homeomorphic to $\mathbf{S}^n$ via the mapping:

$$
\begin{aligned}
\infty &\mapsto (0, \ldots, 0, 1) \\
(a_1, \ldots a_n) &\mapsto (a'_1, \ldots a'_{n+1}),
\end{aligned}
$$

where

$$
\begin{aligned}
a'_i &= 4a_i/(a_1^2 + \cdots + a_n^2 + 4) \qquad \text{for } 1 \leq i \leq n \\
a'_{n+1} &= (a_1^2 + \cdots + a_n^2 - 4)/(a_1^2 + \cdots + a_n^2 + 4).
\end{aligned}
$$

This mapping may be regarded as a stereographic projection by embedding $\mathbb{R}^n$ in the hyperplane of $\mathbb{R}^{n+1}$ defined in Cartesian geometry by the equation $x_{n+1} = -1$. This projection is depicted for the case $n = 2$ in Fig. 1.9. By way of allusion to this homeomorphism:

NOTATION 1.35 *Let $\mathbb{S}^n$ denote the 1-point compactification of $\mathbb{R}^n$.*

LEMMA 1.36 *Let X be a non-compact topological space. Then the mapping $r \mapsto \dot{r}$ is a Boolean algebra isomorphism from* $\mathrm{RO}(X)$ *to* $\mathrm{RO}(\dot{X})$.

Proof The function $o \mapsto \dot{o}$ is monotone, because a closed subset of a compact set is compact. Let o_1 and o_2 be open subsets of X, with $o = o_1 \cap o_2$. Since $X \setminus o = (X \setminus o_1) \cup (X \setminus o_2)$ is compact if and only if both $(X \setminus o_1)$ and $(X \setminus o_2)$ are compact, we have $\infty \in \dot{o}$ if and only if $\infty \in \dot{o_1} \cap \dot{o_2}$, whence $\dot{o} = \dot{o_1} \cap \dot{o_2}$.

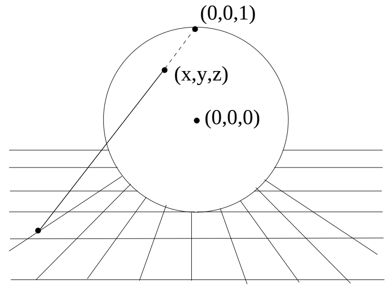


Figure 1.9. Stereographic projection of S^2 onto the 1-point compactification of $\mathbb{R}^2$.

If u is open in X, let u^* denote the pseudo-complement of u in X, and let $(\dot{u})^\star$ denote the pseudo-complement of $\dot{u}$ in $\dot{X}$. We claim that, for any open set u of X, with $v = u^*$, $(\dot{u})^\star = \dot{v}$. By definition, $\dot{v}$ is open in $\dot{X}$, and we have just shown that $\dot{u} \cap \dot{v} = \dot{\emptyset} = \emptyset$. Moreover, if w is any open set in $\dot{X}$ disjoint from $\dot{u}$, then for some open subset w' of X, we have either $w = w'$ or $w = \dot{w}'$. Either way $u \cap w' = \emptyset$, whence $w' \subseteq v$ and $\dot{w}' \subseteq \dot{v}$ by monotonicity. Hence $\dot{v}$ is the largest open subset of $\dot{X}$ disjoint from $\dot{u}$, i.e. $(\dot{u})^\star = \dot{v}$.

Note that if $r \in \mathrm{RO}(X)$, we have $r = r^{**}$ and $-r = r^*$. Hence $\dot{r}^{\star\star} = \dot{r}$, so that $\dot{r} \in \mathrm{RO}(\dot{X})$. Conversely, if $u' \in \mathrm{RO}(\dot{X})$, then $u' = \dot{u}$, for some open $u \subseteq X$. But then, we have $\dot{u} = \dot{u}^{**} = \dot{x}$, where $x = u^{**}$. Since the function $o \mapsto \dot{o}$ is injective, $u = u^{**}$. That is, $u \in \mathrm{RO}(X)$. QED

LEMMA 1.37 *Let X be a topological space and $o \subseteq X$ open. If o is connected in X, then $\dot{o}$ is connected in $\dot{X}$. Conversely, suppose X is non-compact, and for any closed subsets d_1 and d_2 of X with $X = d_1 \cup d_2$ and $d_1 \cap d_2$ compact, either d_1 is compact or d_2 is compact. If $\dot{o}$ is connected in $\dot{X}$, then o is connected in X.*

Proof Suppose o is open in X. If $\dot{o}$ is not connected in $\dot{X}$, let o_1, o_2 be non-empty open subsets of X such that $\dot{o} = \dot{o}_1 \cup \dot{o}_2$ and $\dot{o}_1 \cap \dot{o}_2 = \emptyset$. Then $o = o_1 \cup o_2$ and $o_1 \cap o_2 = \emptyset$, so o is not connected in X. Conversely, suppose o is not connected in X, so let o_1, o_2 be nonempty open subsets of X such that $o = o_1 \cup o_2$ and $o_1 \cap o_2 = \emptyset$. If $X \setminus o$ is not compact, then neither $X \setminus o_1$ nor $X \setminus o_2$ is compact, so that $\dot{o} = o = o_1 \cup o_2 = \dot{o}_1 \cup \dot{o}_2$ and $\dot{o}_1 \cap \dot{o}_2 = \emptyset$, whence $\dot{o}$ is not connected. If, on the other hand, $X \setminus o$ is compact, by the condition of the lemma, either $X \setminus o_1$ or $X \setminus o_2$ is compact, whence $\dot{o} = o \cup \{\infty\} = o_1 \cup o_2 \cup \{\infty\} = \dot{o}_1 \cup \dot{o}_2$. Moreover, by repeating the first paragraph of the proof of Lemma 1.36, we have $\dot{o}_1 \cap \dot{o}_2 = \dot{\emptyset} = \emptyset$. It follows that $\dot{o}$ is not connected. QED

The well-known Heine-Borel theorem states that, in $\mathbb{R}^n$, a set is compact if and only if it is closed and bounded. It is therefore easy to see that $\mathbb{R}^n$ satisfies the condition of Lemma 1.37.

LEMMA 1.38 *Let $n > 0$ and let M be any mereotopology over $\mathbb{R}^n$. Then the mapping $r \mapsto \dot{r}$ defines a structure isomorphism from M to $\dot{M}$ for the signature $(c, \leq)$: that is, $M \simeq_{c,\leq} \dot{M}$.*

Proof Lemmas 1.36 and 1.37. QED

LEMMA 1.39 *Let X be a locally compact, non-compact topological space and M a mereotopology over X. Define $\dot{M} = \{\dot{r} \mid r \in M\}$. Then $\dot{M}$ is a mereotopology over $\dot{X}$. We call $\dot{M}$ the 1-point compactification of M. If M is finitely decomposable, then so is $\dot{M}$.*

Proof Suppose that $\infty \in \dot{o}$ with o open in X; we show that there exists some $r \in \dot{M}$ such that $\infty \in r \subseteq \dot{o}$. Since M is a mereotopology over X and X is locally compact, Lemma 1.33 gives us a cover of $X \setminus o$ by elements of M whose closures are compact. Since $\infty \in \dot{o}$, $X \setminus o$ is compact, so that this cover has a finite sub-cover, say $r_1, \ldots, r_n$. Let $r = -(r_1 + \cdots + r_n)$. Thus, $X \setminus r = r_1^- \cup \cdots \cup r_n^-$ is compact and includes o, whence r has the required properties. The rest of the Lemma follows from Lemma 1.37. QED

Suppose now that $X = \mathbb{R}^n$ for some $n > 0$, and let M be a mereotopology over $\mathbb{R}^n$ respecting components. Then X satisfies the condition of Lemma 1.37, so by Lemma 1.39, $\dot{M}$ is a mereotopology over $\mathbb{R}^n$ respecting components. Since $\mathbb{S}^n$ denotes the 1-point-compactification of $\mathbb{R}^n$, the 1-point compactification of $\mathrm{RO}(\mathbb{R}^n)$ is thus $\mathrm{RO}(\mathbb{S}^n)$.

NOTATION 1.40 *Let* $\mathrm{ROS}(\mathbb{S}^n)$ *denote the 1-point compactification of* $\mathrm{ROS}(\mathbb{R}^n)$*, and similarly for* $\mathrm{ROP}(\mathbb{S}^n)$, $\mathrm{ROQ}(\mathbb{S}^n)$.

It is often more convenient to work with $\mathbb{S}^2$ and $\mathbb{S}^3$ rather than $\mathbb{R}^2$ and $\mathbb{R}^3$. When we need to make the distinction explicit, we refer to elements of $\mathrm{ROP}(\mathbb{R}^n)$ as polytopes (polyhedra, polygons) *in open space* and those of $\mathrm{ROP}(\mathbb{S}^n)$ as polytopes (polyhedra, polygons) *in closed space*. Note that, by Lemma 1.38, the mereotopologies $\mathrm{RO}(\mathbb{R}^n)$, $\mathrm{ROP}(\mathbb{R}^n)$, $\mathrm{ROQ}(\mathbb{R}^n)$ and $\mathrm{ROS}(\mathbb{R}^n)$ certainly all have the same $L_{c,\leq}$-theories as their respective 1-point compactifications.

3.4 Connectedness: the closed plane

We have seen that, over most mereotopologies of interest, the language L_C is as expressive as the language $L_{c,\leq}$. The question therefore arises as to whether a converse reduction is possible. In this section, we show that, for well-behaved mereotopologies over $\mathbb{S}^2$, the answer is positive.

We assume familiarity with basic geometric topology in the plane: for details, see Newman, 1964.Recall in this context that a *Jordan arc* in a topological space X is a homeomorphism from the unit interval $[0,1]$ into X, and a *Jordan curve* in X, a homeomorphism from the unit circle $\mathbf{S}^1$ into X. The Jordan curve, Theorem states that the locus of a Jordan curve in $\mathbb{R}^2$ separates $\mathbb{R}^2$ into two residual domains, exactly one of which is bounded. If we regard $\mathbf{S}^1$ as the intersection of the plane $x_1 = 0$ with $\mathbf{S}^2$, the Schönflies Theorem states that a Jordan curve $\gamma : \mathbf{S}^1 \to \mathbb{S}^2$ may be extended to a homeomorphism $\mathbf{S}^2 \leftrightarrow \mathbb{S}^2$. Thus, if γ is a Jordan curve in $\mathbb{S}^2$, the residual domains of $|\gamma|$ are 2-cells in $\mathbb{S}^2$; and if γ is a Jordan curve in $\mathbb{R}^2$, the bounded residual domain of γ is a 2-cell in $\mathbb{R}^2$.

The following concepts are important in understanding the good behaviour of the mereotopologies $\mathrm{ROS}(\mathbb{R}^2)$, $\mathrm{ROP}(\mathbb{R}^2)$ and $\mathrm{ROQ}(\mathbb{R}^2)$.

DEFINITION 1.41 *Let X be a topological space, $u \subseteq X$ and $p, q \in \mathcal{F}(u)$. An* end-cut *to p* in *u is a Jordan arc in X such that $f(1) = p$ and $f([0,1[) \subseteq u$. Likewise, a* cross-cut *from p to q* in *u is a Jordan arc in X such that $f(0) = p$, $f(1) = q$ and $f(]0,1[) \subseteq u$. Let M be a mereotopology over X. We say that M* has curve-selection *if, for all $r \in M$ and all $p \in \mathcal{F}(r)$, there exists an end-cut in r to p.*

The existence of end-cuts is by no means a universal property of regular open sets in $\mathbb{R}^n$ (for $n > 1$). However, the regions in $\mathrm{ROS}(\mathbb{R}^2)$, $\mathrm{ROP}(\mathbb{R}^2)$ and $\mathrm{ROQ}(\mathbb{R}^2)$ are well-behaved in this regard, as the following results show.

LEMMA 1.42 *Let $r \in \mathrm{ROP}(\mathbb{R}^n)$ and $p \in r^-$. Then there exists a linear function $f : [0,1] \to \mathbb{R}^n$ such that $f(1) = p$ and $f([0,1[) \subseteq r$. If p has rational coordinates, we may choose f so that it has parameters from $\mathbb{Q}$.*

Proof The proposition holds for basic polytopes because their closures are convex. It holds for all polytopes because if $r = r_1 + \cdots r_n$, $r^- = r_1^- \cup \cdots \cup r_n^-$ by Lemma 1.4 (iii). QED

The semi-algebraic case is much more involved. However, we have the following Theorem (van den Dries, 1998, Ch. 6, Corollary 1.5; Bochnak et al., 1998 Theorem 2.5.5).

PROPOSITION 1.43 (CURVE-SELECTION LEMMA) *Let S be a semi-algebraic subset of $\mathbb{R}^n$ and $p \in S^-$. Then there exists a continuous semi-algebraic function $f : [0,1] \to \mathbb{R}^n$ such that $f(1) = p$ and $f([0,1[) \subseteq S$.*

Thus, the mereotopologies $\mathrm{ROS}(\mathbb{R}^2)$, $\mathrm{ROP}(\mathbb{R}^2)$ and $\mathrm{ROQ}(\mathbb{R}^2)$ all certainly have curve-selection. Moreover, by making only minor modifications to the relevant arguments, it can be shown that $\mathrm{ROS}(\mathbb{S}^2)$, $\mathrm{ROP}(\mathbb{S}^2)$ and $\mathrm{ROQ}(\mathbb{S}^2)$ all have curve-selection too.

With these preliminaries behind us, we can turn to the expressive power of $L_{c,\leq}$. We note in passing that, since $\leq$ is a primitive of $L_{c,\leq}$, we may write the Boolean operators and constants $+$, $\cdot$, $-$, 0 and 1 in $L_{c,\leq}$-formulas, assuming them to be replaced by their usual definitions. In mereotopologies over the closed plane having curve-selection, we can express the property of being a 2-cell using an $L_{c,\leq}$-formula. To see this, we recall that the Jordan Curve Theorem has the following converse (see Newman, 1964Chapter VI, Theorem 16.1).

Proposition 1.44 (Converse of Jordan's theorem) *Let d be a closed subset of $\mathbb{S}^2$ such that $\mathbb{S}^2 \setminus d$ has two components, and suppose that, for each $p \in d$, and each component o of $\mathbb{S}^2 \setminus d$, there is an end-cut to p in o. Then d is the locus of a Jordan curve.*

Then we have:

Lemma 1.45 *Let M be any mereotopology over $\mathbb{S}^2$ having curve-selection. Then, for all $r \in M$, r is a 2-cell if and only if r is non-zero and connected with non-zero connected complement—that is, if and only if $M \models \psi_{\mathrm{J}}[r]$, where $\psi_{\mathrm{J}}(x)$ is the $L_{c,\leq}$-formula*

$$c(x) \wedge x > 0 \wedge c(-x) \wedge -x > 0.$$

Proof If $M \models \psi_{\mathrm{J}}[r]$, then $d = \mathcal{F}(r)$ satisfies the conditions of Proposition 1.44, since M has curve-selection. The other direction is immediate.

QED

Furthermore:

Lemma 1.46 *Let M be a mereotopology over $\mathbb{R}^2$ having curve-selection and also satisfying the conditions of Lemma 1.30. Then $r \in M$ is a 2-cell if and only if r satisfies the L_C-formula*

$$\phi_{\mathrm{c}}(x) \wedge x > 0 \wedge \phi_{\mathrm{c}}(-x) \wedge -x > 0 \wedge \phi_{\mathrm{b}^2}(x),$$

where $\phi_{\mathrm{c}}(x)$ and $\phi_{\mathrm{b}^2}(x)$ are as defined in Lemmas 1.27 and 1.30, respectively.

Proof If r satisfies the formula, then the bounded set $\mathcal{F}(r)$ is the locus of a Jordan curve in $\mathbb{S}^2$ and hence in $\mathbb{R}^2$ by the same reasoning as for

Lemma 1.45, and since r is the bounded residual domain of this set, it is a 2-cell. The other direction is again immediate. QED

We now proceed to a direct comparison between $L_{c,\leq}$ and L_C. Proposition 1.28 has a closed-plane variant, in which the condition that one of d_1 and d_2 is bounded may be dropped.

PROPOSITION 1.47 *Let d_1 and d_2 be closed sets in $\mathbb{S}^2$. If $\mathbb{S}^2 \setminus d_1$, $\mathbb{S}^2 \setminus d_2$ and $d_1 \cap d_2$ are all connected, then so is $\mathbb{S}^2 \setminus (d_1 \cup d_2)$.*

This leads to a closed-plane variant of Lemma 1.29:

LEMMA 1.48 *Let $s_1, s_2, t \in \mathrm{RO}(\mathbb{S}^2)$ such that: (i) $-(s_1 + t)$, $-(s_2 + t)$ and t are all connected; and (ii) $s_1^- \cap s_2^- = \emptyset$. Then $-(s_1 + s_2 + t)$ is also connected.*

Proof As for Lemma 1.29, using Proposition 1.47 in place of Proposition 1.28. QED

LEMMA 1.49 *Let M be any finitely decomposable mereotopology over $\mathbb{S}^2$ having curve-selection, let $\psi_{\mathrm{ub}}(y_1, y_2)$ be the $L_{c,\leq}$-formula*

$$\exists z(c(-(y_1 + z)) \wedge c(-(y_2 + z)) \wedge c(z) \wedge \neg c(-(y_1 + y_2 + z))),$$

and let $\psi_C(x_1, x_2)$ be the $L_{c,\leq}$-formula

$$\exists y_1 \exists y_2 (y_1 \leq x_1 \wedge y_2 \leq x_2 \wedge \psi_{\mathrm{ub}}(y_1, y_2)).$$

Then, for all $r_1, r_2 \in M$, $r_1^- \cap r_2^- \neq \emptyset$ if and only if $M \models \psi_C[r_1, r_2]$.

Proof The if-direction follows from Lemma 1.48. The only-if direction is left as a (fiddly) exercise. QED

Putting together Lemmas 1.22, 1.27 and 1.49, we see that $L_{c,\leq}$ is exactly as expressive as L_C in well-behaved mereotopologies over the closed plane $\mathbb{S}^2$.

As a final example of the expressiveness of the language L_C, we observe that it can distinguish between $\mathbb{R}^2$ and its 1-point compactification.

THEOREM 1.50 *Let M be any of $\mathrm{ROS}(\mathbb{R}^2)$, $\mathrm{ROP}(\mathbb{R}^2)$ or $\mathrm{ROQ}(\mathbb{R}^2)$. Then $M \not\equiv_C \dot{M}$.*

Proof Recall the L_C-formula $\phi_{\mathrm{b}^2}(x)$ defined in Lemma 1.30, and expressing the property of being bounded over M. Evidently, $M \not\models \forall x \phi_{\mathrm{b}^2}(x)$. But it is an easy consequence of Lemma 1.48 that $\dot{M} \models \forall x \phi_{\mathrm{b}^2}(x)$. QED

Theorem 1.50 stands in sharp contrast to the situation with the signature $\{c, \leq\}$ reported in Lemma 1.38.

4. Expressiveness of first-order languages in plane mereotopologies

In the previous section, we examined the relative expressive power of the languages L_C and $L_{c,\leq}$ for various mereotopologies, in particular those defined over $\mathbb{R}^2$ and $\mathbb{S}^2$. This section characterizes that expressive power in a more 'absolute' way. We employ the following terminology:

Definition 1.51 *Let X be a topological space and let $\bar{u} = u_1, \ldots, u_n$, $\bar{v} = v_1, \ldots, v_n$ be n-tuples of subsets of X. We say that $\bar{u}$ and $\bar{v}$ are* similarly situated (in X), *and write $\bar{u} \sim_X \bar{v}$, if there is a homeomorphism of X onto itself mapping $\bar{u}$ to $\bar{v}$. If X is clear from context, we omit reference to it, and simply write $\bar{u} \sim \bar{v}$. Now let M be a mereotopology over X and Σ a signature of topological primitives. For any L_Σ-formula ϕ with free-variables $\bar{x}$, we say that ϕ is* topologically complete (in M over X) *if any pair of tuples of the appropriate arity satisfying $\phi(\bar{x})$ in M are similarly situated in X.*

Readers familiar with basic geometric topology will recognize that the mereotopologies $\mathrm{ROS}(\mathbb{S}^2)$, $\mathrm{ROP}(\mathbb{S}^2)$ and $\mathrm{ROQ}(\mathbb{S}^2)$ are all (finitely) 'triangulable' (in the sense of van den Dries, 1998).Moreover, the observations of Section 3.2 strongly suggest that triangulations in these mereotopologies can be combinatorially described using first-order formulas with C as their only primitive. And since combinatorially isomorphic triangulations are similarly situated, it should be entirely unsurprising that every tuple in these mereotopologies satisfies a topologically complete L_C-formula (and hence also a topologically complete $L_{c,\leq}$-formula). That is: every tuple of regions in any of the mereotopologies $\mathrm{ROS}(\mathbb{S}^2)$, $\mathrm{ROP}(\mathbb{S}^2)$ and $\mathrm{ROQ}(\mathbb{S}^2)$ can be completely topologically described by an L_C-formula (or by an $L_{c,\leq}$-formula). Results of this general kind were proved, independently, by Kuijpers et al., 1995,Papadimitriou et al., 1999and Pratt and Schoop, 2000,by a variety of methods. Our objective here is a systematic and general investigation of this topic, using an approach which will prove useful in Sections 5 and 7.

4.1 Connected partitions

We have seen that, given a collection Σ of topological primitives, any mereotopology can be regarded as a Σ-structure by interpreting the symbols in Σ in the standard way. And the question then naturally arises as to whether we can obtain a converse to this observation. That is: under what conditions is a given Σ-structure isomorphic to some mereotopology—or perhaps, to some mereotopology belonging to a certain class? Since this question will preoccupy us in the sequel, some of the results below will be presented at a higher level of generality than their immediate applications warrant.

Accordingly, throughout Sections 4.1 and 4.2, $\mathfrak{A}$ shall denote an arbitrary structure interpreting the signature $\{0, 1, +, \cdot, -, c\}$, such that the reduct of $\mathfrak{A}$ to the signature $\{0, 1, +, \cdot, -\}$, is a Boolean algebra. To avoid notational clutter, if $a, b \in A$, we write $0, -a, a + b$ etc., rather than the more correct $0^{\mathfrak{A}}$, $-^{\mathfrak{A}}(\mathrm{a})$, $+^{\mathfrak{A}}(a, b)$ etc. In addition, abusing terminology slightly, we call an element $a \in A$ *connected* if $\mathfrak{A} \models c[a]$; and we say that $\mathfrak{A}$ is *finitely decomposable* if, for every $a \in A$, there exist connected elements $a_1, \ldots, a_n$ of $\mathfrak{A}$ such that $a = a_1 + \ldots + a_n$. Of course, in case $\mathfrak{A}$ is a mereotopology M, this usage is consistent with that adopted above. As usual in the context of Boolean algebras, we take a *partition* in $\mathfrak{A}$ to be a tuple of non-zero, pairwise disjoint elements summing to 1. If $\bar{a}$ is any tuple from $\mathfrak{A}$ (not necessarily a partition), and $\bar{b}$ a partition in $\mathfrak{A}$, we say that *$\bar{a}$ can be refined to $\bar{b}$* if every element of $\bar{a}$ can be written as the sum of (zero or more) elements of $\bar{b}$.

Definition 1.52 *A partition $\bar{a} = a_1, \ldots, a_n$ in $\mathfrak{A}$ such that a_i is connected for all i $(1 \leq i \leq n)$ is called a* connected partition.

Let ψ_{con} denote the $L_{c,\leq}$-sentence

$$\forall x \forall y (c(x) \wedge c(y) \wedge x \cdot y \neq 0 \rightarrow c(x + y)).$$

Thus, ψ_{con} 'says' that the sum of two overlapping connected regions is connected.

Lemma 1.53 *Let M any mereotopology. Then $M \models \psi_{\text{con}}$.*

Proof A restatement of Lemma 1.4 (iv). QED

Claim 1.54 *Suppose $\mathfrak{A}$ is finitely decomposable, and $\mathfrak{A} \models \psi_{\text{con}}$. Then every tuple in $\mathfrak{A}$ can be refined to a connected partition.*

Proof Given elements $a_1, \ldots, a_n$, collect all the non-zero products $b_1, \ldots, b_N$ of the form: $\pm a_1 \cdot \cdots \cdot \pm a_n$. For each j $(1 \leq j \leq N)$, let $b_{j,1}, \ldots, b_{j,N_j}$ be connected elements of $\mathfrak{A}$ summing to b_j. If, for any two of these elements, say $b_{j,k}$ and $b_{j,l}$, we have $b_{j,k} \cdot b_{j,l} > 0$, then we can replace them by their sum $b_{j,k} + b_{j,l}$ (which is connected, because $M \models \psi_{\text{con}}$). Proceeding in this way, we obtain the desired refinement. QED

Note that, in particular, every tuple in any finitely decomposable mereotopology can be refined to a connected partition.

Let us restrict attention now to finitely decomposable mereotopologies over $\mathbb{S}^2$ having curve-selection.

LEMMA 1.55 *Let M be a mereotopology over $\mathbb{R}^2$ or $\mathbb{S}^2$ having curve-selection. If r_1, r_2 and r_3 are pairwise disjoint, connected elements of M, then there exist at most two points lying on the frontiers of all three regions.*

Proof We suppose that p_1, p_2 and p_3 are distinct points all lying on the frontiers of r_1, r_2 and r_3 and derive a contradiction. Choose points q_1, q_2, q_3 such that $q_i \in r_i$ $(i = 1, 2, 3)$. By curve-selection, draw three end-cuts in r_i, say $\gamma_{i,1}$, $\gamma_{i,2}$ and $\gamma_{i,3}$ from q_i to p_1, p_2 and p_3, respectively. It is easy to see that, within each r_j $(1 \leq j \leq 3)$, the $\gamma_{i,j}$ can be chosen so that they intersect only at q_i. But since the r_j are disjoint, each $\gamma_{i,j}$ intersects any other $\gamma_{i',j'}$ only in p_i or q_i. And it is well known that this is impossible (see the right-hand graph in Fig. 1.11). QED

For $n > 2$, let ψ^n_{sum} denote the $L_{c,\leq}$-formula

$$\forall x_1 \ldots \forall x_n \Big(c(x_1 + \cdots + x_n) \wedge \bigwedge_{1 \leq i \leq n} c(x_i) \rightarrow \bigvee_{2 \leq i \leq n} c(x_1 + x_i) \Big).$$

(The formula ψ_{sum} of Example 1.18 is just ψ^3_{sum}.) Thus, ψ^n_{sum} 'says' that, if n connected regions have a connected sum, the first must form a connected sum with at least one of the others.

LEMMA 1.56 *Let M be a finitely decomposable mereotopology over $\mathbb{S}^2$ having curve-selection. Then $M \models \psi^n_{\text{sum}}$ for all $n > 1$.*

Proof Let $r_1, \ldots, r_n$ be connected with $r_1 + \cdots + r_n$ also connected. Assume first that the r_i are pairwise disjoint. Let $p \in r_1$ and $q \in r_2 + \cdots + r_n$. By the connectedness of $r_1 + \cdots + r_n$, draw a Jordan arc γ from p to q lying within $r_1 + \cdots + r_n$. By Lemma 1.55, only finitely many points can lie on the frontiers of more than two of the r_i, and we may certainly ensure that γ avoids all such points. By renumbering if necessary, we may assume that γ visits a point $p \in r_1^- \cap r_2^- \cap (r_1 + \ldots + r_n)$. But by the construction of γ, $p \notin r_i^-$ for all $i > 2$, whence $p \in -r_i$ for all such i. Therefore, $p \in r_1^- \cap r_2^- \cap (r_1 + r_2)$, whence $r_1 + r_2$ is connected. Finally, we relax the assumption that the r_i are pairwise disjoint. Since M is finitely decomposable, we have that each element of $\bar{r}$ is the sum of zero or more members of a tuple $\bar{s}$ of pairwise disjoint connected elements with the same sum. The result then follows easily by repeated applications of Lemma 1.53. QED

In the sequel, we abbreviate the formula

$$x_1 + x_2 = x \wedge x_1 > 0 \wedge x_2 > 0 \wedge x_1 \cdot x_2 = 0 \wedge c(x_1) \wedge c(x_2)$$

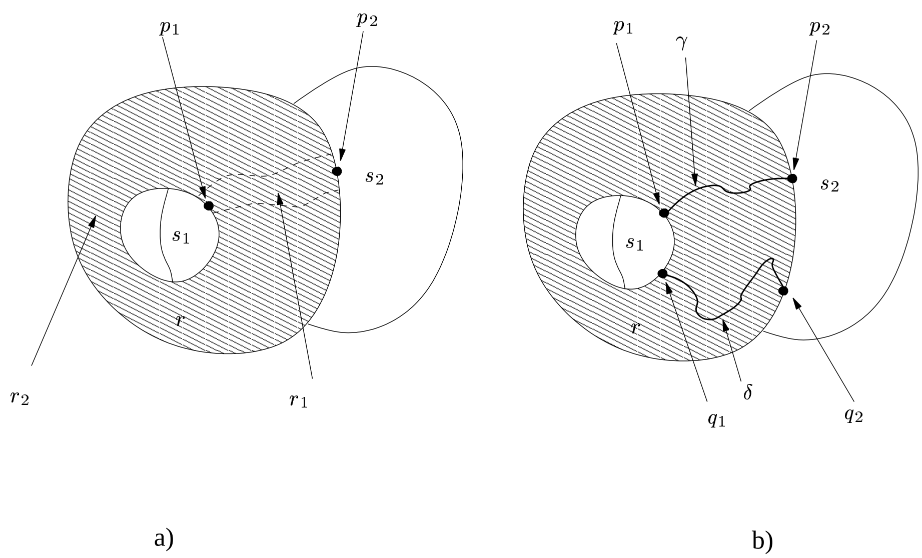


Figure 1.10. The configuration of ψ_{break}:

by $x_1 \oplus x_2 = x$; thus, $x_1 \oplus x_2 = x$ 'says' that x can be partitioned into non-empty, disjoint connected regions x_1 and x_2. Now let ψ_{break} denote the $L_{c,\leq}$-formula

$$\forall x \forall y_1 \forall y_2 \Big(\big(c(x) \wedge c(y_1) \wedge c(y_2) \wedge c(x + y_1) \wedge \\ c(x + y_2) \wedge x \cdot y_1 = 0 \wedge x \cdot y_2 = 0 \wedge x \neq 0 \big) \rightarrow \\ \exists x_1 \exists x_2 \big(x_1 \oplus x_2 = x \wedge c(x_1 + y_1) \wedge c(x_1 + y_2) \wedge c(x_2 + y_1) \wedge c(x_2 + y_2) \big) \Big).$$

Thus, ψ_{break} 'says' that, if r, s_1, s_2 are connected regions such that r is non-zero, disjoint from s_1 and s_2, and forms a connected sum with both s_1 and s_2, then r can be partitioned into connected, non-zero regions r_1, r_2 such that each of r_1 and r_2 forms a connected sum with each of s_1 and s_2. Fig. 1.10a illustrates this configuration; note that $-r$ need not be connected.

LEMMA 1.57 *Let M be a finitely decomposable mereotopology over $\mathbb{S}^2$ having curve-selection. Then $M \models \psi_{\text{break}}$.*

Proof Let r, s_1, s_2 be as above. We may assume that s_1 and s_2 are nonzero, since otherwise, similar or easier arguments apply. Refer to Fig. 1.10b. For $i = 1, 2$, since $r + s_i$ is connected, by Lemma 1.4 (ii), $r^- \cap s_i^- \cap (r + s_i) \neq \emptyset$. In fact, since the removal of finitely many points from the open set $r+s_i$ does not disconnect it, we can choose four distinct points p_i, q_i $(i = 1, 2)$ such that $p_i, q_i \in r^- \cap s_i^- \cap (r + s_i)$. Since M has curve-selection and r is connected it is easy to see that, by exchanging q_1 and q_2 if necessary, we can draw cross-cuts γ from p_1 to p_2 and δ from q_1 to q_2 such that $|\gamma|$ and $|\delta|$ are disjoint. Since $\mathbb{S}^2$ is normal and M a mereotopology, we can cover $|\delta|$ with elements of M whose closures are disjoint from $|\gamma|$. By compactness of $|\delta|$, this cover has a finite subcover, $t_1, \ldots, t_N$, say. Let $t = r \cdot (t_1 + \cdots + t_N)$; evidently, q_1 and q_2 lie on the frontier of the same component t' of t. Likewise, p_1 and p_2 lie on the frontier of the same component of $r \cdot -t'$: call this component $r_1 \in M$, and let $r_2 = r \cdot -r_1$. It is easy to check that r_1 and r_2 have the required properties. QED

4.2 Neighbourhood graphs

As before, $\mathfrak{A}$ shall denote an arbitrary structure interpreting the signature $\{0, 1, +, \cdot, -, c\}$, such that the reduct of $\mathfrak{A}$ to $\{0, 1, +, \cdot, -\}$, is a Boolean algebra. Recall the notion of connected partition introduced in Definition 1.52.

Definition 1.58 *Let $\bar{a} = a_1, \ldots, a_n$ be a connected partition in $\mathfrak{A}$. A connected partition $\bar{a}$ is called a* c^h-partition *if, for every $I \subseteq \{1, \ldots, n\}$ such that $|I| < h$, the element $(-\sum_{i \in I} a_i)$ is connected.*

If $\mathfrak{A} \models c(1)$, then a c^1-partition in $\mathfrak{A}$ is the same thing as a connected partition. Furthermore, if $\mathfrak{A}$ is in fact a mereotopology over $\mathbb{S}^2$ having curve-selection, then, by Lemma 1.45, a c^2-partition in $\mathfrak{A}$ is the same thing as a partition consisting entirely of 2-cells. It is c^3-partitions, however, that will mainly preoccupy us in the sequel.

We assume familiarity with basic graph theory: for details, see Diestel, 1991 Chapter 1. Recall in this context that a *graph* is a pair $G = (V, E)$ where V is a set (called *vertices*) and E is a set of 2-element subsets of V (called *edges*). We denote V by $V(G)$ and E by $E(G)$. Note that, on this definition, graphs have no 'loops' or 'multiple edges'. If G is a graph and U is a proper subset of $V(G)$, we denote by $G \setminus U$ the result of *deleting* all the nodes in U from G; and if $e = (v, v') \in E(G)$, we denote by the G/e the result of *contracting* G by merging v and v' into a single (new) node v'', such that (v'', w) is an edge of G/e just in case either (v, w) or (v', w) is an edge of G. If a graph H can be obtained from G by a sequence of deletions and contractions, then H is said to be a *minor* of G. Finally we take the terms *path*, *cycle*, *connected*, *component* to be defined in the standard way. In particular, recall that, for $h > 0$, G is said to be h-connected if $G \setminus U$ is connected for every $U \subseteq G$ such that $|U| < h$.

Definition 1.59 *Let* $\bar{a} = a_1, \ldots, a_n$ *be a tuple from* $\mathfrak{A}$. *If* $a_i + a_j$ *is connected for* $1 \leq i < j \leq n$, *we say that* a_i *and* a_j *are* neighbours. *The* neighbourhood, graph *of* $\bar{a}$, *denoted* $N_{\bar{a}}$, *is the graph with nodes* $\{a_1, \ldots, a_n\}$ *and edges* $\{(a_i, a_j) \mid a_i$ *and* a_j *are neighbours*$\}$.

Claim 1.60 *Suppose* $\mathfrak{A} \models \psi_{\text{con}}$ *and* $\mathfrak{A} \models \psi^n_{\text{sum}}$ *for all* $n > 2$. *Let* $\bar{a} = a_1, \ldots, a_n$ *be a tuple of connected elements of* $\mathfrak{A}$, *such that* $a_{n-1} + a_n$ *is connected. Let* $\bar{a}' = a_1, \ldots, a_{n-2}, (a_{n-1} + a_n)$. *Then* $N_{\bar{a}'} = N_{\bar{a}}/(n-1, n)$.

Proof For $1 \leq j < n-1$, $a_j + (a_{n-1} + a_n)$ is connected if and only if $a_j + a_{n-1}$ is connected or $a_j + a_n$ is connected. QED

Claim 1.61 *Suppose* $\mathfrak{A} \models \psi_{\text{con}}$ *and* $\mathfrak{A} \models \psi^n_{\text{sum}}$ *for all* $n > 2$. *Let* $\bar{a} = a_1, \ldots, a_n$ *be a tuple of connected elements of* $\mathfrak{A}$, *with* $a = a_1 + \ldots + a_n$. *Then* a *is connected if and only if* $N_{\bar{a}}$ *is a connected graph.*

Proof The if-direction follows easily from the fact that $\mathfrak{A} \models \psi_{\text{con}}$. For the only-if direction, note that the claim is trivial if $n = 1$, so assume $n > 1$, and that the claim holds for tuples of fewer than n elements. Since $\mathfrak{A} \models \psi^n_{\text{sum}}$ there exists i $(1 \leq i < n)$ such that a_i and a_n are neighbours. By renumbering if necessary, assume $i = n-1$, and let $\bar{a}'$ be as in Claim 1.60, so that $N_{\bar{a}'} = N_{\bar{a}}/(a_{n-1}, a_n)$. But $N_{\bar{a}'}$ is connected by inductive hypothesis, whence $N_{\bar{a}}$ is connected too. QED

Claim 1.62 *Suppose* $\mathfrak{A} \models \psi_{\text{con}}$ *and* $\mathfrak{A} \models \psi^n_{\text{sum}}$ *for all* $n > 2$. *Let* $\bar{a}$ *be a connected partition in* $\mathfrak{A}$, *and let* $h \geq 1$. *Then* $\bar{a}$ *is a* c^h*-partition if and only if* $N_{\bar{a}}$ *is an* h*-connected graph.*

Proof Immediate by Claim 1.61. QED

Claim 1.63 *Suppose* $\mathfrak{A} \models c(1)$, $\mathfrak{A} \models \psi_{\text{con}}$, $\mathfrak{A} \models \psi^n_{\text{sum}}$ *for all* $n > 2$, *and* $\mathfrak{A} \models \psi_{\text{break}}$. *Then every connected partition in* $\mathfrak{A}$ *can be refined to a* c^3*-partition.*

Proof We make free use of Claim 1.61. Let $\bar{a}$ be a connected partition. We show first that $\bar{a}$ can be refined to a c^2-partition. Choose an element a of $\bar{a}$ such that the number k of components of the graph $N_{\bar{a}} \setminus \{a\}$ is maximal. And let there be $m > 0$ elements a for which this maximum value is attained. If $\bar{a}$ is not already a c^2-partition, then $k > 1$. Let H_1, H_2 be distinct components of $N_{\bar{a}} \setminus \{a\}$. Since $N_{\bar{a}}$ is connected, there exist $b_1 \in H_1$, $b_2 \in H_2$ such that $a + b_1$ and $a + b_2$ are connected.

Since $\mathfrak{A} \models \psi_{\text{break}}$, let a_1, a_2 be non-empty, connected, disjoint elements summing to a with $a_1 + b_1$, $a_1 + b_2$, $a_2 + b_1$ and $a_2 + b_2$ all connected; and let $\bar{b}$ be the connected partition which results from replacing a by a_1 and a_2. Evidently, for $i = 1, 2$, $N_{\bar{b}} \setminus \{a_i\}$ has strictly fewer than k components. That is, the number of elements b in $\bar{b}$ such that $N_{\bar{b}} \setminus \{b\}$ has k components is strictly less than m. Proceeding in this way, we eventually obtain a c^2-partition.

Now let $\bar{a}$ be a c^2-partition. We show that $\bar{a}$ can be refined to a c^3-partition. If $\bar{a}$ is not a c^3-partition, choose a pair of distinct elements a and a' such that the number k of components of the graph $N_{\bar{a}} \setminus \{a, a'\}$ is maximal; and let there be $m > 0$ unordered pairs (a, a') for which this maximum value is attained. Let H_1, H_2 be distinct components of $N_{\bar{a}} \setminus \{a, a'\}$. Since $\bar{a}$ is a c^2-partition, there exist $b_1 \in H_1$, $b_2 \in H_2$ such that $a + b_1$ and $a + b_2$ are connected. And since $\mathfrak{A} \models \psi_{\text{break}}$, let a_1, a_2 be non-empty, connected, disjoint elements summing to a with $a_1 + b_1$, $a_1 + b_2$, $a_2 + b_1$ and $a_2 + b_2$ all connected; and let $\bar{b}$ be the connected partition which results from replacing a by a_1 and a_2. Evidently, for $i = 1, 2$, $N_{\bar{b}} \setminus \{a_i, a'\}$ has strictly fewer than k components. Moreover, suppose a'' is any other element of $\bar{a}$ (distinct from a and a') such that $N_{\bar{a}} \setminus \{a, a''\}$ also has k components. We claim that $N_{\bar{b}} \setminus \{a_1, a''\}$ and $N_{\bar{b}} \setminus \{a_2, a''\}$ cannot both have k components. Working for the moment on this assumption, we see that the number of pairs b, b' in $\bar{b}$ such that $N_{\bar{b}} \setminus \{b, b'\}$ has k components is strictly less than m. Proceeding in this way, we eventually obtain a c^3-partition.

It remains only to verify that the graphs $N_{\bar{b}} \setminus \{a_1, a''\}$ and $N_{\bar{b}} \setminus \{a_2, a''\}$ encountered above do not both have k components. If $a \in A$ and $B \subseteq A$, let us say that a is a neighbour of B if a is a neighbour of some element of B. Let the components of $N_{\bar{b}} \setminus \{a, a''\}$ be $H_1, \ldots H_k$. Since $\bar{b}$ is a c^2-partition, we have that, for all i $(1 \leq i \leq k)$, a is a neighbour of H_i, and therefore either a_1 or a_2 is a neighbour of H_i. Hence, we can re-order the H_i if necessary so that, for some p, q with $0 \leq p < q \leq k+1$, a_1 is a neighbour of H_i if and only if $i < q$ and a_2 is a neighbour of H_i if and only if $p < i$. Thus, the components of $N_{\bar{b}} \setminus \{a_1, a''\}$ are $H_1, \ldots, H_p, (\{a_2\} \cup H_{p+1} \cdots \cup H_k)$, and the components of $N_{\bar{b}} \setminus \{a_2, a''\}$ are $(\{a_1\} \cup H_1 \cdots \cup H_{q-1}), H_q, \ldots, H_k$. If these number k in each case, we have $p = k - 1$ and $q = 2$. But a' lies in one of the H_i, and a_1 and a_2 were chosen so that they are both neighbours of this a'. Hence a_1 and a_2 are both neighbours of H_i, whence $p < q - 1$. This yields $k \leq 1$, contradicting our assumption that $\bar{a}$ is not a c^3-partition. QED

We finish with a technical result which will be required later.

Definition 1.64 *If $\bar{a} = a_1, \ldots, a_N$ is a connected partition in $\mathfrak{A}$ such that, for any neighbour a_j of a_i, $-(a_i + a_j)$ is connected, we say that $\bar{a}$ is* radial about a_i.

Note incidentally that a c^3-partition is radial about each of its members.

Claim 1.65 *Suppose $\mathfrak{A} \models c(1)$, $\mathfrak{A} \models \psi_{\text{con}}$, $\mathfrak{A} \models \psi^n_{\text{sum}}$ for all $n > 2$, and $\mathfrak{A} \models \psi_{\text{break}}$. Let $n > 1$ and let $\bar{a} = a_1, \ldots, a_n$ be a connected partition in $\mathfrak{A}$ with $-a_1$ connected. Then $\bar{a}$ can be refined to a c^2-partition $a_1, b_2, \ldots, b_N$, radial about a_1, in which a_1 has at least three neighbours.*

Proof Similar to the above. QED

We conclude with a further corollary of Claim 1.61. We employ the following fact from graph theory, whose proof we leave to the reader.

Proposition 1.66 *If G is a finite 2-connected graph of order at least 2, and $v \in V(G)$, then there exists a $w \in V(G)$ such that $\{v, w\} \in E(G)$, and the removal of both v and w from G leaves a connected graph.*

Corollary 1.67 *Let M be a finitely decomposable mereotopology over $\mathbb{S}^2$ having curve-selection, and let $\bar{r} = r_1, \ldots, r_n$ be a partition in M consisting entirely of 2-cells. Then, by re-numbering if necessary, we have, for all k $(1 \leq k < n)$, $r_1 + \cdots + r_k$ is a 2-cell.*

That is: partitions of the closed plane into 2-balls are always 'shellable'. The analogous result for three-dimensional space fails (Rudin, 1958).

4.3 Partition graphs

We now prove that, if $\bar{r}$ is a c^3-partition in a finitely decomposable mereotopology over $\mathbb{S}^2$ having curve-selection, then the neighbourhood graph of $\bar{r}$ fixes its topological properties completely.

We assume familiarity with the basic theory of plane graphs: for details, see Diestel, 1991 Chapter 4. In this context, suppose that $e \subseteq \mathbb{S}^2$ is the locus of a Jordan arc. Then e has two *endpoints*; all other points are called *interior points*, and we denote the set of these interior points by (e). (Of course, e is not the *topological* interior of the set e in $\mathbb{S}^2$; but no confusion should arise in this regard.) A *plane graph* is a pair $G = (V, E)$, where V is a finite subset of $\mathbb{S}^2$ and E is a collection of sets $e \subseteq \mathbb{S}^2$ such that e is the locus of a Jordan arc, satisfying the following conditions for all $v \in V$ and all $e, e' \in E$:

1 if $e \in E$ and p is an endpoint of e, then $p \in V$;

2 $v \neq (e)$, and if $e \neq e'$ then $(e) \cap (e') = \emptyset$;

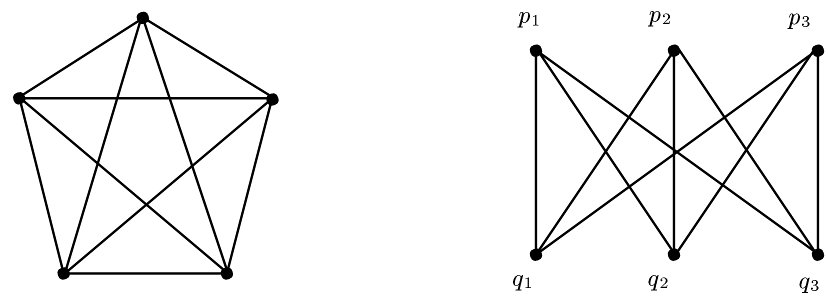


Figure 1.11. The non-planar graphs K^5 and $K_{3,3}$.

3 if $e \neq e'$, then e and e' do not join the same pair of endpoints.

The elements of V are called *vertices* of G, and the elements of E, the *edges* of G; an edge $e \in E$ is said to *join* the vertices at its endpoints. We denote V by $V(G)$, E by $E(G)$ and $V \cup \bigcup E$ by $|G|$. The components of $\mathbb{S}^2 \setminus |G|$ are called the faces of G, and we denote the set of these faces by $F(G)$. A plane graph is *semi-algebraic* if its edges are the loci of semi-algebraic Jordan arcs; similarly for the terms *piecewise linear* and *rational piecewise linear*. Notice that, on our definition, plane graphs have no 'loops' or 'multiple edges'. (Some authors prefer the term *simple graph*.) A plane graph will be regarded as an abstract graph in the obvious way, and we carry over notation and terminology accordingly. Conversely, if $G = (V, E)$ is an abstract graph, a *drawing* of G is a plane graph $G' = (V', E')$ for which there exists a function ϵ mapping V 1–1 onto V' and E 1–1 onto E' such that for all $(v, v') \in E$, $\epsilon((v, v'))$ joins $\epsilon(v)$ and $\epsilon(v')$. We call ϵ an *embedding*. If G has a drawing, G is *planar*. Not all abstract graphs are planar, of course: the graphs K^5 and $K_{3,3}$ illustrated in Fig. 1.11 are familiar non-planar graphs. Indeed, this fact has a converse:

Proposition 1.68 (Kuratowski, Wagner) *A graph is planar if and only if it has no minor isomorphic to either K^5 or $K_{3,3}$.*

We further assume familiarity with the notion of *duality* for plane graphs. Let G and G' be plane graphs. We say that G' is a *geometrical dual* of G if there are bijections $f_F : F(G) \to V(G')$ and $f_E : E(G) \to E(G')$ such that, for all $f \in F(G)$ and $e \in E(G)$:

1 $f_F(f) \in f$;

2 $f_E(e) \cap e$ is a single point interior to both $f_E(e)$ and e, and $f_E(e) \cap e' = \emptyset$ for all $e' \neq e$.

In our terminology, not every plane graph has a dual, because we do not allow graphs to contain loops or multiple edges. However, we rely below on the following sufficient condition (Wilson, 1979, p. 76).

PROPOSITION 1.69 *Every 3-connected plane graph has a dual.*

The following fact is also well-known.

LEMMA 1.70 *Let G and G' be connected plane graphs such that G' is a geometrical dual of G. Then there is a bijection $f_V : V(G) \to F(G')$ such that, for all $v \in V(G)$, $v \in f_V(v)$. Hence, G is a dual of G'.*

Proof Every face of G' contains at least one vertex of G by construction; it contains at most one by Euler's formula $|F(G)| - |E(G)| + |V(G)| = 2$ applied to G and G'. QED

Finally, duals are unique, in the following sense (Diestel, 1991, p. 88).

PROPOSITION 1.71 *Let G be a plane graph and let G' and G'' be plane graphs which are both geometric duals of G. Then there is a homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ mapping G' to G''. In fact, h can be chosen such that, for all $v \in G$, if f' and f'' are the faces of G' and G'', respectively, containing v, then h maps f' to f''.*

Now let us apply these ideas to the graphs whose faces are c^3-partitions in well-behaved, closed-plane mereotopologies.

LEMMA 1.72 *Let X be a topological space, and let r, s be disjoint elements of* $\mathrm{RO}(X)$ *with $p \in \mathcal{F}(r) \setminus \mathcal{F}(s)$. Then $p \in \mathcal{F}(-(r+s))$.*

Proof By Lemma 1.4 (ii), $p \notin r + s$. QED

LEMMA 1.73 *Let M be a mereotopology over $\mathbb{S}^2$ having curve-selection, and let $\bar{r} = r_1, \ldots, r_n$ be a c^3-partition in M. For all i, j ($1 \leq i < j \leq n$), $\mathcal{F}(r_i) \cap \mathcal{F}(r_j)$ is connected.*

Proof We may assume that $n \geq 3$. Since $\bar{r}$ is certainly a c^2-partition, every $\mathcal{F}(r_i)$ ($1 \leq i \leq n$) is a Jordan curve by Lemma 1.45. Suppose, for contradiction, that $\mathcal{F}(r_i) \cap \mathcal{F}(r_j)$ is not connected, and let $p, q \in \mathcal{F}(r_i) \cap \mathcal{F}(r_j)$ be separated in $\mathcal{F}(r_i)$ by $\{p', q'\} \subseteq \mathcal{F}(r_i) \setminus \mathcal{F}(r_j)$. By Lemma 1.72, $p', q' \in \mathcal{F}(-(r_i + r_j))$, so that, by the connectedness of $-(r_i + r_j)$, we can draw a cross-cut γ' (Definition 1.41) from p' to q' in $-(r_i + r_j) \subseteq -r_i$. By the connectedness of r_j, we can likewise draw a cross-cut γ from p to q in $r_j \subseteq -r_i$. But $-r_i$ is a 2-cell, whence γ and γ' are easily seen to intersect at an interior point, which is impossible, since $r_j \cap -(r_i + r_j)$ is empty. QED

Lemma 1.74 *Let M be a mereotopology over $\mathbb{S}^2$ having curve-selection, and let $\bar{r} = r_1, \ldots r_n$ $(n \geq 4)$ be a c^3-partition in M. Then there exists a unique plane graph G drawn in $\mathbb{S}^2$ such that the collection of sets $\{r_1, \ldots, r_n\}$ are exactly $F(G)$ and the collection of sets $\{\mathcal{F}(r_i) \cap \mathcal{F}(r_j) \mid 1 \leq i < j \leq n, r_i + r_j$ is connected$\}$ are exactly $E(G)$.*

Proof Let i, j, k be distinct integers in the range $[1, n]$. Since $\bar{r}$ is a c^3-partition, $r_j^- \cup r_k^- = \mathbb{S}^2 \setminus -(r_j + r_k)$ does not separate the nonempty sets r_i and $-(r_i + r_j + r_k)$, whence $\mathcal{F}(r_i) \cap (\mathcal{F}(r_j) \cup \mathcal{F}(r_k))$ is not the whole of the Jordan curve $\mathcal{F}(r_i)$. And since, by Lemma 1.73, $\mathcal{F}(r_i) \cap \mathcal{F}(r_j)$ is a connected subset of $\mathcal{F}(r_i)$, $\mathcal{F}(r_i) \cap \mathcal{F}(r_j)$ is either a point or the locus of a Jordan arc. Indeed, $\mathcal{F}(r_i)$ must include at least three Jordan arcs of the form $\mathcal{F}(r_i) \cap \mathcal{F}(r_j)$ for various j distinct from i. Let the vertices of G be the endpoints of all Jordan arcs of the form $\mathcal{F}(r_i) \cap \mathcal{F}(r_j)$, and let the edges of G be the segments of the various $\mathcal{F}(r_i)$ connecting them. To show that G is a plane graph, we must establish that if $\mathcal{F}(r_i) \cap \mathcal{F}(r_j)$ is a Jordan arc γ, then for all k $(1 \leq k \leq n)$ with $k \neq i, j$, $\mathcal{F}(r_k)$ contains no interior points of γ. For otherwise, let $p' \in \mathcal{F}(r_k)$ be an interior point of γ, and pick any $q' \in \mathcal{F}(r_i) \setminus \mathcal{F}(r_j)$. Then $p' \in \mathcal{F}(-(r_i + r_j))$ and also, by Lemma 1.72, $q' \in \mathcal{F}(-(r_i + r_j))$. If we now choose p and q in $\mathcal{F}(r_i) \cap \mathcal{F}(r_j)$ separating p' and q' on the Jordan curve $\mathcal{F}(r_i)$, the derivation of a contradiction proceeds as in Lemma 1.73. Hence no point of $\mathcal{F}(r_k)$ is an interior point of γ, as required. Moreover, no two Jordan arcs in E can have the same end-points, since $\bar{r}$ is a c^3-partition. It follows that G is a plane graph as required. Evidently, $F(G) = \{r_1, \ldots, r_n\}$ and $E(G')$ is the collection of sets $\mathcal{F}(r_i) \cap \mathcal{F}(r_j)$ for $1 \leq i < j \leq n$ which are Jordan arcs.

It therefore remains only to show that $\mathcal{F}(r_i) \cap \mathcal{F}(r_j)$ is a Jordan arc if and only if $r_i + r_j$ is connected. Note that $r_1 \cup r_2$ is trivially not connected. By Lemma 1.4 (ii), $r_i \cup r_j \subseteq r_i + r_j \subseteq r_i \cup r_j \cup (\mathcal{F}(r_i) \cap \mathcal{F}(r_j))$, and the removal of a single point from a connected, open set does not render it disconnected. Hence, if $r_i + r_j$ is connected, $\mathcal{F}(r_i) \cap \mathcal{F}(r_j)$ is neither empty nor a singleton, and hence is a Jordan arc. Conversely, suppose $\mathcal{F}(r_i) \cap \mathcal{F}(r_j)$ is a Jordan arc. We have already shown that, if p is an interior point of this arc, $p \notin \cup_{k \neq i,j} r_k^- = (\sum_{k \neq i,j} r_k)^-$. That is, $p \in -\sum_{k \neq i,j} r_k = r_i + r_j$. Hence $\mathcal{F}(r_i) \cap \mathcal{F}(r_j) \cap (r_i + r_j)$ is non-empty, whence $r_i + r_j$ is connected. QED

Definition 1.75 *Let M be a mereotopology over $\mathbb{S}^2$ having curve-selection, and let $\bar{r} = r_1, \ldots r_n$ $(n \geq 4)$ be a c^3-partition in M. We call the unique plane graph G satisfying the conditions of Lemma 1.74 the* the partition graph of $\bar{r}$.

Warning: the neighbourhood graph and the partition graph of a c^3-partition are not the same sort of thing. The former is an *abstract* graph whose nodes are regions and whose edges are pairs of regions; the latter is a *plane* graph, whose nodes are points and whose edges are the loci of Jordan arcs.

LEMMA 1.76 *Let M be a mereotopology over $\mathbb{S}^2$ having curve-selection, let $\bar{r} = r_1, \ldots r_n$ $(n \geq 4)$ be a c^3-partition in M, and let G be its partition graph. Then there is a plane embedding ϵ of $N_{\bar{r}}$ such that $\epsilon(N_{\bar{r}})$ is a geometrical dual of G and, for all i, $(1 \leq i \leq n)$, $\epsilon(r_i) \in r_i$.*

Proof Almost immediate from the definition of partition graph. QED

From Claim 1.62, c^3-partitions have 3-connected neighbourhood graphs. But 3-connected graphs have the crucial property that all their drawings are topologically the same.

PROPOSITION 1.77 (WHITNEY) *Let G and G' be 3-connected plane graphs and $f : G \to G'$ a graph isomorphism. Then f can be extended to a homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$.*

Let M be a finitely decomposable mereotopology over $\mathbb{S}^2$, and let $\bar{r} = r_1, \ldots, r_n$ and $\bar{s} = s_1, \ldots, s_n$, be n-tuples from M. We are interested in the case where the mapping $r_i \mapsto s_i$ is a graph isomorphism from $N_{\bar{r}}$ to $N_{\bar{s}}$—that is, where, for all i, j, $(1 \leq i < j \leq n)$, $r_i + r_j$ is connected if and only if $s_i + s_j$ is connected. We say in this case that $\bar{r}$ and $\bar{s}$ *have the same neighbourhood structure.*

THEOREM 1.78 *Let M be a finitely decomposable mereotopology over $\mathbb{S}^2$ having curve-selection. Then any two c^3-partitions in M having the same neighbourhood structure are similarly situated in $\mathbb{S}^2$.*

Proof It is straightforward to verify that, if $n \leq 3$, all n-element c^3-partitions in M are similarly situated in $\mathbb{S}^2$. Thus, we may assume that $n \geq 4$. Let $\bar{r} = r_1, \ldots, r_n$ and $\bar{s} = s_1, \ldots, s_n$ be c^3-partitions with the same neighbourhood structure, and let G and H be their respective partition graphs. By Lemma 1.76, let G^* and H^* be embeddings of $N_{\bar{r}}$ and $N_{\bar{s}}$, geometrically dual to G and H, respectively, let p_i be the vertex of G^* contained in r_i and let q_i be the vertex of H^* contained in s_i for all i $(1 \leq n)$. Hence, there is a graph isomorphism $f : G^* \to H^*$ mapping p_i to q_i. Since G^* and H^* are 3-connected, Proposition 1.77 guarantees that f can be extended to a homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$. Then $h(G)$ and H are both geometrical duals of the plane graph $h(G^*) = H^*$, such that, for all i $(1 \leq i \leq n)$ the faces $h(r_i)$ and s_i contain the vertex $h(p_i) = q_i$. By Proposition 1.71, let h' be a homeomorphism mapping

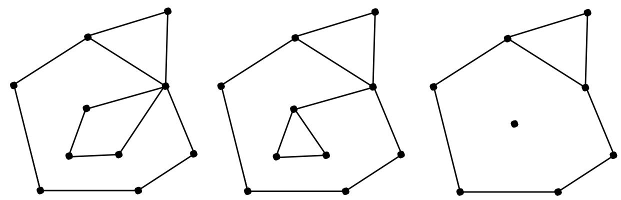


Figure 1.12. Only the left-hand graph defines a connected partition in $\mathrm{RO}(\mathbb{S}^2)$.

$h(G)$ to H such that $h'(h(r_i)) = s_i$. Thus, $\bar{r}$ and $\bar{s}$ are similarly situated.

QED

We finish this discussion of partition graphs with some 'obvious' lemmas concerning connected partitions in $\mathrm{ROP}(\mathbb{S}^2)$ and related mereotopologies. Readers irritated by proofs of such evident truths may skip to Theorem 1.82.

LEMMA 1.79 *Let G be a plane graph such that G has no isolated vertices, and every edge of G lies on the boundary of (at least) 2 faces of G. Then the members of $F(G)$ are regular open, and form a connected partition in* $\mathrm{RO}(\mathbb{S}^2)$. *Moreover, if G' is another such plane graph, with $|G| \subseteq |G'|$, then, for every $f \in F(G)$, $f = \sum\{f' \in F(G') \mid f' \subseteq f\}$.*

Proof Let $G = (V, E)$, and suppose $f \in F(G)$ and $p \in \mathcal{F}(f)$. Since G has no isolated vertices, there exists $e \in E$ such that $p \in e$ and hence some $f' \in F(G)$, distinct from f, such that $e \subseteq \mathcal{F}(f')$. Since f' is disjoint from f^-, $p \in (\mathbb{S}^2 \setminus f^-)^- = \mathbb{S}^2 \setminus (f^-)^0$, i.e. $p \notin (f^-)^0$. Thus, the open set f satisfies $(f^-)^0 \subseteq f$, and so is regular open. By Lemma 1.4 (ii), $\bigcup F(G) \subseteq \sum F(G) \subseteq (\bigcup F(G))^- = \bigcup\{f^- \mid f \in F(G)\} = \mathbb{S}^2$. But by Lemma 1.3, $\sum F(G)$ is the unique regular open set lying between $\bigcup F(G)$ and its closure; i.e. $\sum F(G) = 1$. Hence, the elements of $F(G)$ form a connected partition in $\mathrm{RO}(\mathbb{S}^2)$. The last part of the lemma then follows from Lemma 1.3, since, if $f \in F(G)$, then both f and $\sum\{f' \in F(G') \mid f' \subseteq f\}$ are regular open sets sandwiched between $\bigcup\{f' \in F(G') \mid f' \subseteq f\}$ and its closure.

QED

Of course, the converse of Lemma 1.79 is false: the configuration of Example 1.18 shows that not every connected partition in $\mathrm{RO}(\mathbb{S}^2)$ is the set of faces of some plane graph.

LEMMA 1.80 *If G is a piecewise linear plane graph such that G has no isolated vertices and every edge of G lies on the boundary of exactly 2 faces, then the faces of G form a connected partition in* $\mathrm{ROP}(\mathbb{S}^2)$.

Proof Let $L_1, \ldots, L_m$ be straight lines extending (in both directions) each of the line segments making up G. Let G' be the graph whose nodes are the points of intersection of the L_i (including ∞) and whose edges are the segments of the L_i joining them; and let P be the set of non-zero products $\pm s_1 \cdot \cdots \cdot \pm s_m$, where s_i is one of the residual half-planes of L_i for $1 \leq i \leq m$. By simple set-algebra, $\bigcup P = \bigcup F(G')$; and since every $r \in P$ is connected, and every $f \in F(G')$ is a maximal connected subset of $\mathbb{S}^2 \setminus |G'|$, $r \cap f \neq \emptyset$ implies $r \subseteq f$. Hence every $f \in F(G')$ is a union of elements of P. But since these elements are non-empty open and disjoint and f is connected, f simply is some element of P, and hence is an element of $\mathrm{ROP}(\mathbb{S}^2)$. Since $|G| \subseteq |G'|$, the result follows by the last part of Lemma 1.79. QED

Lemma 1.80 does have a converse:

LEMMA 1.81 *If $\bar{r}$ is a connected partition in* $\mathrm{ROP}(\mathbb{S}^2)$*, then $\bar{r}$ is the set of faces of some piecewise linear plane graph G; moreover, for any such plane graph G, G has no isolated vertices, and every edge of G lies on the boundary of exactly* 2 *faces.*

Proof By Claim 1.63, refine $\bar{r} = r_1, \ldots, r_n$ to a c^3-partition $\bar{t} = \{t_1, \ldots, t_N\}$, and let G_0 be the partition graph of $\bar{t}$. Suppose, by renumbering if necessary, that $r_1 = t_1 + \cdots + t_m$. Note that, if $e \in E(G_0)$, we have, for all j $(1 \leq j \leq N)$, $(e) \subseteq t_j^-$ or $(e) \cap t_j^- = \emptyset$. Hence if $(e) \not\subseteq r_1$, then $(e) \cap \bigcup_{m<j \leq N} t_j^- \neq \emptyset$, whence $e \subseteq t_j^-$ for some j $(m < j \leq N)$.

Let G_1 be the graph obtained from G_0 by first removing any edge e such that $(e) \subseteq r_1$, and then removing any vertex v such that $v \in r_1$. Since r_1 is open, the endpoints of every remaining arc are among the remaining vertices, so G_1 really is a plane graph. Moreover, if $m < j \leq N$, then $t_j^- \cap r_1 = \emptyset$, so that none of the vertices and edges removed from G_0 intersects t_j^-; hence t_j is a face of G_1. Therefore, the set of points

$$
\begin{aligned}
S = \{t_j \in F(G_0) \mid 1 \leq j \leq m\} \cup& \\
&\{e \in E(G_0) \mid (e) \subseteq r_1\} \cup \{v \in V(G_0) \mid v \in r_1\}
\end{aligned}
$$

must be the union of some faces of G_1. Trivially, $S \subseteq r_1$. We claim that $r_1 \subseteq S$. For if $p \in \mathbb{S}^2$, exactly one of the following three cases holds: (i) $p \in t_j$ for some j; (ii) $p \in V(G_0)$; or (iii) $p \in e$ for some $e \in E(G_0)$. In case (i), either $p \in S$ or $p \notin r_1$, according as $j \leq m$. In case (ii), trivially, either $p \in S$ or $p \notin r_1$. In case (iii), if $p \notin S$, then $p \in (e) \not\subseteq r_1$, whence $p \in e \subseteq t_j^-$ for some j $(m < j \leq N)$, whence $p \in (\sum_{m<j<N} t_j)^- = \mathbb{S}^2 \setminus r_1$. This proves that $r_1 \subseteq S$. Thus, $r_1 = S$ is

the union of a number of faces of G_1. But r_1 is by assumption connected, so r_1 is a face of G_1. Proceeding in the same way for $r_2, \ldots, r_n$, we obtain the desired graph $G = G_n$. QED

Lemmas 1.80 and 1.81 concern the mereotopology $\mathrm{ROP}(\mathbb{S}^2)$, but almost exactly similar arguments can be given for $\mathrm{ROS}(\mathbb{S}^2)$ and $\mathrm{ROQ}(\mathbb{S}^2)$. We omit the details, which are routine. Summarizing, we have:

THEOREM 1.82 *A tuple $\bar{u}$ of subsets of $\mathbb{S}^2$ is a connected partition in* $\mathrm{ROS}(\mathbb{S}^2)$ (*alternatively:* $\mathrm{ROP}(\mathbb{S}^2)$, $\mathrm{ROQ}(\mathbb{S}^2)$) *if and only if it is the set of faces of a semi-algebraic* (*respectively: piecewise linear, rational piecewise linear*) *graph with no isolated vertices and every edge lying on the boundary of two faces.*

4.4 Expressive power of first-order languages in plane mereotopologies

We are now in a position to give an absolute characterization of the expressive power of the languages $L_{c,\leq}$ and L_C over certain mereotopologies of interest. Recall the concept of topologically complete formula given in Definition 1.51. The following notation will be useful in constructing topologically complete formulas.

NOTATION 1.83 *Given a fixed Boolean algebra, a* Boolean matrix *is a rectangular matrix whose entries are the elements* 1 *and* 0. *If $\bar{r}$ is an n-tuple, $\bar{s}$ an N-tuple, and A a Boolean matrix with N rows and n columns, we write $\bar{r} = \bar{s}A$ to indicate that each element of $\bar{r}$ is the sum of certain elements of $\bar{s}$ as indicated by the elements of A via normal matrix multiplication. Similarly, we write $\bar{x} = \bar{z}A$ in first-order formulas to abbreviate the obvious conjunction of Boolean algebra equations.*

THEOREM 1.84 *Let M be any finitely decomposable mereotopology over $\mathbb{S}^2$ having curve-selection, and let Σ be the signature $(c, \leq, +, \cdot, -)$. Every tuple from M satisfies some* (*purely existential*) *L_Σ-formula which is topologically complete in M over $\mathbb{S}^2$.*

Proof Writing $\bar{z}$ for $z_1, \ldots, z_N$, let $\psi^N_{c^3}(\bar{z})$ be the formula:

$$\bigwedge_{1 \leq i \leq N} (c(z_i) \wedge z_i > 0) \wedge \bigwedge_{1 \leq i \leq j \leq N} (c(-(z_i + z_j)) \wedge z_i \cdot z_j = 0) \wedge \sum_{1 \leq i \leq N} z_i = 1.$$

Thus, $M \models \psi^N_{c^3}[\bar{s}]$ if and only if $\bar{s}$ is an N-element c^3-partition. If $\bar{s} = s_1, \ldots, s_N$ is a c^3-partition in M, let $\psi^{\bar{s}}_{+}(\bar{z})$ be the formula:

$$\bigwedge\{c(z_i + z_j) \mid 1 \leq i < j \leq N \text{ and } s_i + s_j \text{ is connected}\} \wedge$$
$$\bigwedge\{\neg c(z_i + z_j) \mid 1 \leq i < j \leq N \text{ and } s_i + s_j \text{ is not connected}\},$$

where $\bar{z}$ is the tuple of variables $z_1, \ldots, z_n$. Thus, $\psi^{\bar{s}}_{+}(\bar{z})$ encodes the neighbourhood structure of $\bar{s}$. Now let $\bar{r} = r_1, \ldots, r_n$ be any tuple of elements of M. By Claim 1.63, there exists a c^3-partition $\bar{s} = s_1, \ldots, s_N$ in M and a Boolean matrix A such that $\bar{r} = \bar{s}A$. Writing $\bar{x}$ for $x_1, \ldots, x_n$, let $\psi_{\bar{r}}(\bar{x})$ be the formula

$$\exists \bar{z}(\psi^N_{c^3}(\bar{z}) \wedge \psi^{\bar{s}}_{+}(\bar{z}) \wedge \bar{x} = \bar{z}A).$$

Certainly, $M \models \psi_{\bar{r}}[\bar{r}]$. And if $\bar{r}'$ is a tuple from M such that $M \models \psi_{\bar{r}}[\bar{r}']$, let $\bar{s}' = s'_1, \ldots, s'_N$ be corresponding witnesses for the existentially quantified variables $\bar{z}$. Then $s_1, \ldots, s_N$ and $s'_1, \ldots, s'_N$ are c^3-partitions in $\mathbb{S}^2$ which have the same neighbourhood structure, and hence which are similarly situated in $\mathbb{S}^2$, by Theorem 1.78. It follows that $\bar{r}$ and $\bar{r}'$ are similarly situated in $\mathbb{S}^2$ too. Thus, $\psi_{\bar{r}}(\bar{x})$ is topologically complete in M over $\mathbb{S}^2$. QED

Corollary 1.85 *Let M be any finitely decomposable mereotopology over $\mathbb{S}^2$ having curve-selection. Every tuple from M satisfies some L_C-formula which is topologically complete in M over $\mathbb{S}^2$.*

Proof Theorem 1.84 and Lemmas 1.22, 1.27 and 1.49. QED

Thus, for certain well-behaved mereotopologies over $\mathbb{S}^2$, both L_C and $L_{c,\leq}$ are, as we might put it, 'topologically fully descriptive'.

We now turn to the question of expressive power in mereotopologies over $\mathbb{R}^2$. We need some auxiliary lemmas.

Lemma 1.86 *Let $\bar{r} = r_1, \ldots, r_n$ be a c^3-partition in any mereotopology M over $\mathbb{S}^2$ having curve-selection. Let $p, p' \in \mathbb{S}^2$ such that, for all i $(1 \leq i \leq n)$, $p \in r_i^-$ if and only if $p' \in r_i^-$. Then there is a homeomorphism $h : \mathbb{S}^2 \rightarrow \mathbb{S}^2$ mapping p to p' and fixing each r_i.*

Proof Obvious, by viewing $\bar{r}$ as $F(G)$ for some plane graph G. QED

Lemma 1.87 *Let $\bar{r} = r_1, \ldots, r_n$ and $\bar{r}' = r'_1, \ldots, r'_n$ be similarly situated c^3-partitions in any mereotopology M over $\mathbb{S}^2$ having curve-selection. Let $p \in \mathbb{S}^2$ such that, for all i $(1 \leq i \leq n)$, $p \in r_i^-$ if and only if $p \in r_i'^-$.*

Then there is a homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ fixing p and mapping $\bar{r}$ to $\bar{r}'$.

Proof Let $h' : \mathbb{S}^2 \to \mathbb{S}^2$ be some homeomorphism mapping $\bar{r}$ to $\bar{r}'$. Then, for all i $(1 \leq i \leq N)$, $h'(p) \in {r'_i}^-$ if and only if $p \in {r_i}^-$. By Lemma 1.86, let $h'' : \mathbb{S}^2 \to \mathbb{S}^2$ be a homeomorphism fixing each r'_i, and mapping $h'(p)$ to p. Then $h := h'' \circ h'$ has the required properties. QED

THEOREM 1.88 *Let M be any finitely decomposable mereotopology over $\mathbb{R}^2$ such that $\dot{M}$ has curve-selection. Every tuple from M satisfies some L_C-formula which is topologically complete in M over $\mathbb{R}^2$.*

Proof Given any tuple $s_1, \ldots, s_N$ from M, let $\phi^{\bar{s}}_{\infty}(\bar{z})$ be the L_C-formula

$$\bigwedge\{\phi_{\mathrm{b}^2}(z_i) \mid 1 \leq i \leq N \text{ and } s_i \text{ is bounded}\} \wedge \\ \bigwedge\{\neg\phi_{\mathrm{b}^2}(z_i) \mid 1 \leq i \leq N \text{ and } s_i \text{ is not bounded}\},$$

where $\bar{z}$ is the tuple of variables $z_1, \ldots, z_N$, and ϕ_{b^2} is as in Lemma 1.30. Thus, $\phi^{\bar{s}}_{\infty}(\bar{z})$ encodes the pattern of boundedness in the tuple $\bar{s}$. Now, given a tuple $\bar{r}$, let $\bar{s}$ be an N-element c^3-partition in M refining $\bar{r}$, and let A be a Boolean matrix satisfying $\bar{r} = \bar{s}A$. Using the translation from $L_{c,\leq}$ to L_C established by Lemmas 1.22 and 1.27, let $\phi^N_{\mathrm{c}^3}(\bar{z})$ and $\phi^{\bar{s}}_{+}(\bar{z})$ be the L_C-formulas corresponding to $\psi^N_{\mathrm{c}^3}(\bar{z})$ and $\psi^{\bar{s}}_{+}(\bar{z})$ in the proof of Theorem 1.84. Writing $\bar{x}$ for $x_1, \ldots, x_n$, let $\phi_{\bar{r}}(\bar{x})$ be the formula

$$\exists\bar{z}(\phi^N_{\mathrm{c}^3}(\bar{z}) \wedge \phi^{\bar{s}}_{+}(\bar{z}) \wedge \phi^{\bar{s}}_{\infty}(\bar{z}) \wedge \bar{x} = \bar{z}A).$$

Certainly, $M \models \psi_{\bar{r}}[\bar{r}]$; and if $\bar{r}'$ is a tuple from M such that $M \models \psi_{\bar{r}}[\bar{r}']$, let $\bar{s}' = s'_1, \ldots, s'_N$ again be a corresponding witnesses for the existentially quantified variables $\bar{z}$. Then $\dot{s}_1, \ldots, \dot{s}_N$ and $\dot{s}'_1, \ldots, \dot{s}'_N$ are c^3-partitions in $\mathbb{S}^2$ which have the same neighbourhood structure, so that by Theorem 1.78 and Lemma 1.87, there is a homeomorphism $h : \mathbb{S}^2 \to \mathbb{S}^2$ fixing ∞ and mapping each $\dot{s}_i$ to $\dot{s}'_i$. Hence $\bar{s}$ and $\bar{s}'$ are similarly situated in $\mathbb{R}^2$, whence $\bar{r}$ and $\bar{r}'$ are similarly situated in $\mathbb{R}^2$ too. QED

Thus, for well-behaved mereotopologies over $\mathbb{R}^2$, L_C is, as we might put it, topologically fully descriptive.

4.5 Homogeneous mereotopologies

Up to this point, we have been concerned only to show that certain relations *can* be defined by first-order formulas with signatures of topological primitives. We turn now briefly to the question of which relations *cannot* be so defined.

At first glance, one might assume that languages with purely topological primitives can express only topological concepts in mereotopologies over which they are interpreted. However, this assumption is correct only if the mereotopologies in question have a certain property. Recall that, for a fixed topological space X, we write $\bar{u} \sim \bar{v}$ to mean that the tuples of subsets $\bar{u}$ and $\bar{v}$ are similarly situated in X (Definition 1.51).

Definition 1.89 *Let M be a mereotopology over X. We say M is* homogeneous (over X) *if, given any tuples $\bar{r}, \bar{s}$ from M with $\bar{r} \sim \bar{s}$ and any element $r \in M$, there exists an element $s \in M$ with $\bar{r}, r \sim \bar{s}, s$. Let M' also be a mereotopology over X, with $M' \subseteq M$. We say M' is* homogeneously embedded in M (*over X*) *if, given any tuple $\bar{r}$ from M', and any $r \in M$, there exists $s \in M'$ with $\bar{r}, r \sim \bar{r}, s$.*

Lemma 1.90 *Let X be either $\mathbb{R}^2$ or $\mathbb{S}^2$, and let M be any of* ROS(X), ROP(X) *or* ROQ(X). *Then M is homogeneous.*

Proof Assume $M = \mathrm{ROS}(\mathbb{S}^2)$; the other cases are identical. Let $\bar{r}$, $\bar{s}$ be tuples from M, and let $r \in M$. Let $\bar{t}$ be a connected partition refining $\bar{r}, r$ and so by Theorem 1.82 is the set of faces of some semi-algebraic plane graph G. If $h : \mathbb{S}^2 \to \mathbb{S}^2$ is a homeomorphism mapping $\bar{r}$ to $\bar{s}$, then h maps G to a plane graph H. But then it is not difficult to show that the edges of H can be deformed into a semi-algebraic plane graph H', and moreover, that this may be done in such a way that existing semi-algebraic edges are unaffected. By Theorem 1.82, the faces of the resulting graph are elements of M; hence we have a homeomorphism mapping $\bar{r}$ to $\bar{s}$ and taking r to some element s of M. QED

Homogeneity and homogeneous embedding are important because of the following facts.

Lemma 1.91 *Let M be a homogeneous mereotopology over a topological space X, and fix a signature Σ of topological primitives. If $\bar{r}$ and $\bar{s}$ are tuples of M which are similarly situated in X, then $\bar{r}$ and $\bar{s}$ satisfy the same L_Σ-formulas in M.*

Proof We show by induction on the complexity of $\phi(\bar{x}) \in L_\Sigma$ that, if $\bar{r}$ and $\bar{s}$ are tuples of the appropriate arity which are similarly situated in X, then $M \models \phi[\bar{r}]$ implies $M \models \phi[\bar{s}]$. The base case follows from the fact that the primitives in Σ have topological interpretations. The only non-trivial recursive case is where $\phi[\bar{x}] = \exists y \psi(\bar{x}, y)$. If $M \models \phi[\bar{r}]$, there exists $r \in M$ such that $M \models \psi[\bar{r}, r]$, and by homogeneity, if $\bar{r} \sim \bar{s}$, there exists $s \in M$ such that $\bar{r}, r \sim \bar{s}, s$, whence $M \models \psi[\bar{s}, s]$ by inductive hypothesis, so that $M \models \phi[\bar{s}]$ as required. QED

Lemma 1.91 gives an upper bound on the expressive power of first-order languages with signatures of topological primitives interpreted over homogeneous mereotopologies: such languages cannot distinguish between similarly situated tuples. It thus provides a partial converse to Theorems 1.84 and 1.88. It also yields an easy proof that, over well-behaved open-plane mereotopologies, $L_{c,\leq}$ cannot express the property of being bounded:

THEOREM 1.92 *Let M be a mereotopology over $\mathbb{R}^2$ such that $\dot{M}$ is homogeneous, and suppose M has curve-selection and contains a region r similarly situated in $\mathbb{R}^2$ to the open unit disc $\mathbf{B}^2$. Then there exists no formula $\psi(x)$ of $L_{c,\leq}$ such that, for all $r \in M$, r is bounded if and only if $M \models \psi[r]$.*

Proof Suppose such a formula $\psi(x)$ exists. Then $M \models \psi[r]$, and by Lemma 1.38, $\dot{M} \models \psi[\dot{r}]$. Since M has curve-selection, by Proposition 1.44 both $\dot{r}$ and its complement $-(\dot{r})$ in $\dot{M}$ are 2-cells in $\mathbb{S}^2$, and hence are similarly situated. By Lemma 1.91, $\dot{M} \models \psi[-(\dot{r})]$, and so by Lemma 1.38, $M \models \psi[-r]$. This contradicts the fact that $-r$ is unbounded. QED

Finally, we return to the relationship between ROS(X), ROP(X) and ROQ(X).

LEMMA 1.93 *Let X be either $\mathbb{R}^2$ or $\mathbb{S}^2$. Then* ROQ(X) *is homogeneously embedded in* ROP(X)*, which is in turn homogeneously embedded in* ROS(X).

Proof Virtually identical to the proof of Lemma 1.90. QED

The following result is well-known (see, for example, Hodges, 1993 p. 55).

PROPOSITION 1.94 (TARSKI-VAUGHT) *Let $\mathfrak{A}$, $\mathfrak{B}$ be structures with $\mathfrak{A} \subseteq \mathfrak{B}$, and suppose that, for any n-tuple $\bar{a}$ from A and any formula $\phi(\bar{x})$ of the form $\exists y \psi(\bar{x}, y)$ such that $\mathfrak{B} \models \phi[\bar{a}]$, there exists $a \in A$ such that $\mathfrak{B} \models \psi[\bar{a}, a]$. Then $\mathfrak{A} \preceq \mathfrak{B}$.*

LEMMA 1.95 *Let M, M' be mereotopologies over a topological space X, with M homogeneous and M' homogeneously embedded in M. Fix a signature of topological primitives. Then $M' \preceq M$.*

Proof By assumption, $M' \subseteq M$. Let $\bar{r}$ be an n-tuple of elements of M', and let $\phi(\bar{x})$ be any formula of L_Σ of the form $\exists y \psi(\bar{x}, y)$ such that $M \models \phi[\bar{r}]$. Then there exists $r \in M$ such that $M \models \psi[\bar{r}, r]$. Since M' is homogeneously embedded in M, there exists $s \in M'$ such that

$\bar{r}, r \sim \bar{r}, s$. Since M is homogeneous, $M \models \psi[\bar{r}, s]$ by Lemma 1.91. The result then follows by Proposition 1.94. QED

Hence, for X either $\mathbb{R}^2$ or $\mathbb{S}^2$, and over any signature Σ of topological primitives, we have $\mathrm{ROQ}(X) \preceq \mathrm{ROP}(X) \preceq \mathrm{ROS}(X)$. In particular, these three structures have identical L_Σ-theories. We show in the sequel that this is no accident: almost any 'reasonable' mereotopology over $\mathbb{S}^2$ has the same L_Σ-theory. Anticipating these results, we employ the following notation and terminology.

DEFINITION 1.96 *Let Σ be a signature of topological primitives. We call the theory* $\mathrm{Th}_\Sigma(\mathrm{ROS}(\mathbb{S}^2))$ *the* standard L_Σ-theory (of closed plane mereotopology)*, and denote it* $\mathbf{T}_\Sigma$.

5. Axiomatization

In this section, we provide an axiomatic characterization of $\mathbf{T}_{c,\leq}$, the standard $L_{c,\leq}$-theory of closed plane mereotopology. The material is essentially that of Pratt and Schoop, 1998.The axiom system in question will help us to identify mereotopologies over $\mathbb{S}^2$ having the standard $L_{c,\leq}$-theory.

As before, we write $\psi^n_{\mathrm{c}^3}(\bar{x})$ for the $L_{c,\leq}$-formula stating that $\bar{x}$ forms n-element c^3-partition, and $x = u \oplus v$ for the $L_{c,\leq}$-formula stating that u and v are disjoint, non-zero, connected regions summing to x. Let M be a mereotopology over $\mathbb{S}^2$ having curve-selection. Consider a triple r, s, t from M satisfying the formula $\psi^3_{\mathrm{c}^3}(x, y, z)$. By Lemma 1.45, each of these regions is a 2-cell, and it is easy to see that the closures of any two of these intersect in a Jordan arc. (Formally, this follows by Lemma 1.73.) Now let ψ_{split} denote the $L_{c,\leq}$-formula

$$\forall x \forall y \forall z \big(\psi^3_{\mathrm{c}^3}(x, y, z) \rightarrow \\ \exists u \exists v \big(u \oplus v = x \wedge c(u + y) \wedge \neg c(u + z) \wedge c(v + z) \wedge \neg c(v + y)\big)\big).$$

Informally, ψ_{split} 'says' that, given two 2-cells r and s whose frontiers intersect in a Jordan arc, r can be partitioned into two connected regions using a cross-cut whose end-points are the end-points of that Jordan arc (Fig. 1.13a).

DEFINITION 1.97 *A mereotopology M is* splittable *if $M \models \psi_{\mathrm{split}}$.*

The following lemma is unsurprising.

LEMMA 1.98 *The mereotopologies* $\mathrm{ROS}(\mathbb{S}^2)$*,* $\mathrm{ROP}(\mathbb{S}^2)$ *and* $\mathrm{ROQ}(\mathbb{S}^2)$ *are splittable.*

Proof Almost immediate from Theorem 1.84, Lemma 1.42 and Proposition 1.43. QED

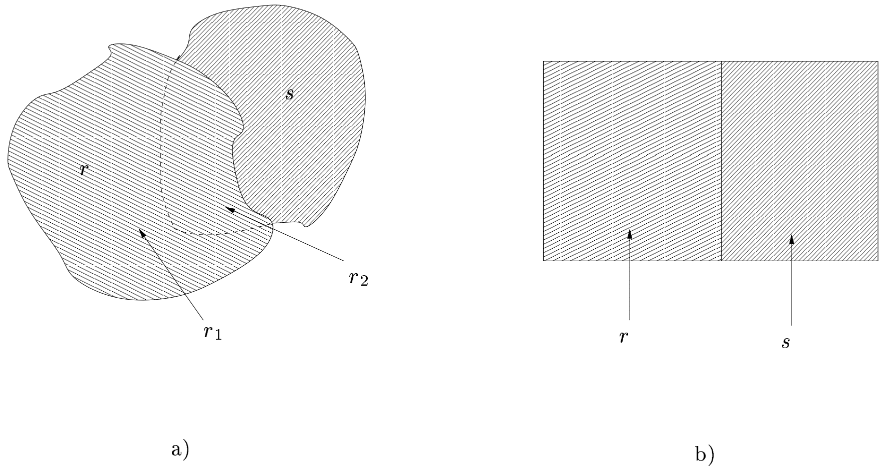


Figure 1.13. a) The configuration of ψ_{split}: r and s are disjoint 2-cells with $r^- \cap s^-$ a Jordan arc; r is broken into r_1 and r_2. b) A pair of regions in $\text{ROX}(\mathbb{R}^2)$ violating ψ_{split}.

However, not all finitely decomposable mereotopologies over $\mathbb{S}^2$ having curve-selection are splittable. If an $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$ is defined by an equation $x_i = 0$, where $0 \leq i \leq n$, we call it an *axis-oriented hyperplane*; and if a half-space is bounded by an axis-oriented hyperplane, we call it an *axis-oriented half-space.*

EXAMPLE 1.99 *Define* $\text{ROX}(\mathbb{S}^n)$ *to be the Boolean sub-algebra of* $\text{RO}(\mathbb{S}^n)$ *generated by the axis-oriented half-spaces. It is easy to see that* $\text{ROX}(\mathbb{S}^n)$ *is a finitely decomposable mereotopology over* $\mathbb{S}^n$ *having curve-selection. However,* $\text{ROX}(\mathbb{S}^n) \not\models \psi_{\text{split}}$*, as is clear in the case* $n = 2$ *by inspection of Fig. 1.13b).*

Thus, whereas $\text{RO}(\mathbb{S}^2)$ has, as it were, too many regions for the standard theory, $\text{ROX}(\mathbb{S}^2)$ has too few. As we have observed, $\text{RO}(\mathbb{S}^2)$ is not finitely decomposable, and lacks curve-selection, while $\text{ROX}(\mathbb{S}^2)$ is not splittable. It transpires that these represent the *only* ways of failing to exhibit the standard theory of closed plane mereotopology. Specifically, we show in this section that all splittable, finitely decomposable mereotopologies over $\mathbb{S}^2$ having curve-selection have the same $L_{c,\leq}$-theory. Our strategy is to pick one splittable, finitely decomposable mereotopologies over $\mathbb{S}^2$ having curve-selection—$\text{ROP}(\mathbb{S}^2)$ will do—and characterize its theory axiomatically. We then merely need to check that our axiom system is correct for all such mereotopologies.

5.1 The axioms

Our axiom system comprises three parts: a *general inference system*, a set of *proper axioms* and an ω*-rule*. (i) The general inference system is simply any complete Hilbert, system for first-order logic, restricted to the signature $\{+, \cdot, -, \leq, c\}$. (ii) The proper axioms are as follows:

1 the usual axioms of Boolean algebra, and the axiom $0 \neq 1$;

2 the axiom ψ_{con} (Lemma 1.53);

3 where $n > 2$, the axioms ψ^n_{sum} (Lemma 1.56);

4 the axiom

$$\neg\exists x_1 \ldots \exists x_5 \Big(\bigwedge_{1 \leq i \leq 5} (c(x_i) \wedge x_i \neq 0) \wedge \bigwedge_{1 \leq i < j \leq 5} (c(x_i + x_j) \wedge x_i \cdot x_j = 0) \Big);$$

5 the axiom

$$\neg\exists x_1 \ldots \exists x_6 \Big(\bigwedge_{1 \leq i \leq 6} (c(x_i) \wedge x_i \neq 0) \wedge \bigwedge_{1 \leq i < j \leq 6} x_i \cdot x_j = 0 \wedge \bigwedge_{\substack{1 \leq i \leq 3 \\ 4 \leq j \leq 6}} c(x_i + x_j) \Big);$$

6 the axioms $c(0)$ and $c(1)$;

7 the axiom ϕ_{break} (Lemma 1.57);

8 the axiom ϕ_{split} (Definition 1.97).

(iii) The final component of our axiom system is the ω-rule. If $n \geq 1$, we let $\psi^n_{\text{c}}(x)$ stand for the formula

$$\exists z_1 \ldots \exists z_n \Big(\bigwedge_{1 \leq i \leq n} c(z_i) \wedge (x = z_1 + \cdots + z_n) \Big).$$

Thus, $\psi^n_{\text{c}}(x)$ 'says' that x can be formed by summing n connected regions. The ω-rule is then the (infinitary) rule of inference:

$$\frac{\{\forall x(\psi^n_{\text{c}}(x) \rightarrow \phi(x)) | n \geq 1\}}{\forall x \phi(x)}.$$

Let Φ be a set of $L_{c,\leq}$-sentences. A *proof* with *premises* Φ in the above system is a sequence of $L_{c,\leq}$-formulas $\{\phi_\alpha\}_{\alpha<\beta}$, for some ordinal β (not necessarily finite) such that every ϕ_α is either (i) an element of Φ or (ii) an axiom or (iii) the result of applying a rule of inference to some formulas ϕ_γ with $\gamma < \alpha$. If ψ is the last line of some such proof, we write $\Phi \vdash \psi$. If $\Phi = \{\phi\}$ we write $\phi \vdash \psi$, and if $\Phi = \emptyset$ we write $\vdash \psi$ and call ψ a *theorem*. Let us denote the set of theorems by T_{Ax}. The main result of Section 5 is:

THEOREM 1.100 *T_{Ax} is the complete $L_{c,\leq}$-theory of any finitely decomposable, splittable mereotopology over $\mathbb{S}^2$ having curve-selection.*

Proof Lemmas 1.102 and 1.104, below. QED

Of course, this entails that all such mereotopologies, considered as $\{c, \leq\}$-structures are elementarily equivalent.

The ω-rule is less unfamiliar than one might at first think. Essentially, it says that if a property holds of every region which is the sum of finitely many connected regions, then it simply holds of every region. This conditional is obviously true in a finitely decomposable mereotopology. Thus, a proof involving the ω-rule is analogous to an argument of the kind encountered in elementary algebra textbooks in which one proves a property of all polynomials by showing that it holds of all polynomials of some arbitrary degree n. Nevertheless, the inclusion of an infinitary proof rule does mean that we ought to check the deduction theorem.

LEMMA 1.101 *Let ϕ be an $L_{c,\leq}$-sentence and ψ an $L_{c,\leq}$-formula such that $\phi \vdash \psi$. Then $\vdash \phi \rightarrow \psi$.*

Proof By assumption, there is a proof $\{\phi_\alpha\}_{\alpha<\beta+1}$ with premises $\{\phi\}$ and last line $\phi_\beta = \psi$. Without loss of generality, we may assume that the first (actually, zeroth) line of the proof ψ_0 is ϕ. We proceed by induction on β. The case $\beta = 0$ is trivial, since $\vdash \phi \rightarrow \phi$. If $\beta > 0$, then either ϕ_β is an axiom or is derived from applying a rule of inference to earlier lines of the proof. The only interesting case is where $\phi = \forall x \pi$ is derived by the ω-rule from the formulas $\forall x(\psi^n_{\mathrm{c}}(x) \rightarrow \pi)$ occurring earlier in the proof. But the inductive hypothesis then yields $\vdash \phi \rightarrow \forall x(\psi^n_{\mathrm{c}}(x) \rightarrow \pi)$, for each n, whence $\vdash \forall x(\psi^n_{\mathrm{c}}(x) \rightarrow (\phi \rightarrow \pi))$. The ω-rule then yields $\vdash \forall x(\phi \rightarrow \pi)$, whence $\vdash \phi \rightarrow \forall x \pi$ (note that ϕ is a sentence), as required. QED

We remark in passing that the axiom $c(0)$ is actually redundant: it can be derived from the other axioms and proof rules.

5.2 Correctness

In this section, we establish the easy half of Theorem 1.100.

LEMMA 1.102 *If M is a splittable, finitely decomposable mereotopology over $\mathbb{S}^2$ having curve-selection, then $M \models T_{\mathrm{Ax}}$.*

Proof We follow the enumeration in Section 5.1, showing that the proper axioms are all true in M and that the ω-rule is truth-preserving.

1 M is a mereotopology.

2 Lemma 1.53.

3 Lemma 1.56.

4 Suppose $r_1, \ldots, r_5$ are connected, non-empty and pairwise disjoint, and that any pair of them have a connected sum. By Lemma 1.26, choose points $p_i \in r_i$ and $q_{i,j} \in \mathcal{F}(r_i) \cap \mathcal{F}(r_j) \cap (r_i + r_j)$ $(1 \leq i < j \leq 5)$. For each i $(1 \leq i \leq 5)$, draw end-cuts in r_i from p_i to all the points $q_{i,j}$ and $q_{j,i}$; it is easy to see that these can be chosen so that any pair of these end-cuts intersect only in the point p_i. Ignoring the points $q_{i,j}$, we have a plane drawing of the graph K^5, which is known to be non-planar (Fig. 1.11).

5 As for axiom 4, but with $K_{3,3}$ instead of K^5.

6 Trivial.

7 Lemma 1.57.

8 M is splittable.

The ω-rule is obviously truth-preserving, because M is finitely decomposable. QED

5.3 Completeness

In this section, we establish the difficult half of Theorem 1.100. We make use of the *omitting types theorem*: for details, see, e.g. Hodges, 1993, pp 333. Let $\mathfrak{A}$ be a structure, $\Phi(x)$ a set of formulas with free variable x, and T a set of sentences. We say that $\mathfrak{A}$ *omits* $\Phi(x)$ if, for all $a \in A$, $\mathfrak{A} \not\models \Phi[a]$. We say that T *locally omits* $\Phi(x)$ if, for every formula $\theta(x)$ with free variable x such that θ is consistent with T, there exists $\phi(x) \in \Phi(x)$ such that $T \not\models \forall x(\theta(x) \rightarrow \phi(x))$. The following theorem is a well-known strengthening of the completeness theorem for first-order logic.

PROPOSITION 1.103 (OMITTING TYPES THEOREM) *If a consistent theory T locally omits a set of formulas $\Phi(x)$, then T has a countable model omitting $\Phi(x)$.*

With these preliminaries behind us, we can proceed with our completeness proof.

Lemma 1.104 *If ϕ is an $L_{c,\leq}$-sentence, and* $\mathrm{ROP}(\mathbb{S}^2) \models \phi$*, then* $\phi \in T_{\mathrm{Ax}}$.

Proof Suppose that $\phi \notin T_{\mathrm{Ax}}$. We are required to prove that $\mathrm{ROP}(\mathbb{S}^2) \models \neg\phi$. Let T be the set of all and only those $L_{c,\leq}$-sentences ψ such that $\neg\phi \vdash \psi$. By Lemma 1.101, T is a consistent set of sentences, and from the ω-rule, T locally omits the type $\{\neg\psi^n_{\mathrm{c}}(x) \mid n > 0\}$. By Proposition 1.103, there exists a countable model $\mathfrak{A} \models T$ omitting that type. Fix the structure $\mathfrak{A}$ for the remainder of this proof.

We now proceed in three stages. Stage 1 establishes some basic facts about $\mathfrak{A}$; Stage 2 shows that $\mathfrak{A}$ can be embedded in the $\{c, \leq\}$-structure $\mathrm{ROP}(\mathbb{S}^2)$; Stage 3 shows that the embedding we have chosen is in fact elementary.

Stage 1: Axioms 1 ensure that the reduct of $\mathfrak{A}$ to the signature $\{+, \cdot, -, \leq\}$ is a Boolean algebra. Such structures were discussed in Section 4.2, where various terminology and notational conventions were introduced. We carry these over to the present proof. Using that terminology, another way of saying that $\mathfrak{A}$ omits the type $\{\neg\psi^n_{\mathrm{c}}(x) \mid n > 0\}$ is to say that $\mathfrak{A}$ is finitely decomposable.

By Axioms 2, 3, 6 and 7, all the claims in Section 4.2 hold of $\mathfrak{A}$. In particular, every tuple can be refined to a connected partition, and thence to a c^2- and a c^3-partition. Furthermore, we have

Claim 1.105 *Let $\bar{b} = b_1, \ldots, b_n$ be a connected partition in $\mathfrak{A}$. Then the neighbourhood graph of $\bar{b}$ is planar.*

Proof By Proposition 1.68, if the neighbourhood graph G of $\bar{b}$ is not planar, it contains either K^5 or $K_{3,3}$ as a minor. But then there is a sequence of contractions of G resulting in a graph H which has either K^5 or $K_{3,3}$ as a sub-graph. By repeated applications of Claim 1.60 (renumbering the b_i as necessary), there is a connected partition $\bar{s}$ in $\mathfrak{A}$ whose neighbourhood graph contains K^5 or $K_{3,3}$ as a sub-graph. But this is impossible by Axioms 4 and 5. QED

Stage 2: Since $\mathfrak{A}$ is countable, let $A = \{a_1, a_2, \ldots\}$. Let $N_0 = 1$ and let $\bar{c}^0$ be the 1-tuple whose element is the unit of the Boolean algebra $\mathfrak{A}$. Trivially, $\bar{c}^0$ is a c^3-partition. For $n \geq 0$, suppose that the c^3-partition $\bar{c}^{(n)} = c^{(n)}_1, \ldots, c^{(n)}_{N_n}$ in $\mathfrak{A}$ has been defined; then, by Claim 1.63, let $\bar{c}^{(n+1)} = c^{(n+1)}_1, \ldots, c^{(n+1)}_{N_{n+1}}$ be a c^3-partition in $\mathfrak{A}$ refining the tuple

$c_1^{(n)}, \ldots, c_{N_n}^{(n)}, a_{n+1}$. It is then obvious that, for each $n > 0$, $\bar{c}^{(n)}$ refines the tuple $a_1, \ldots, a_n$ and also every tuple $\bar{c}^{(m)}$ for all m $(0 < m \leq n)$. We fix the enumerations $a_0, a_1, \ldots$ and $\bar{c}^{(0)}, \bar{c}^{(1)}, \ldots$ for the remainder of Stage 2.

For brevity, denote $\mathrm{ROP}(\mathbb{S}^2)$ by S. We now map each initial segment $a_1, \ldots, a_n$ of A into S. Let $w^{(n)}$ be the set of functions $g^{(n)}$: $\{c_1^{(n)}, \ldots, c_{N_n}^{(n)}\} \to S$ satisfying the conditions:

G1: the regions $g^{(n)}(c_1^{(n)}), \ldots, g^{(n)}(c_{N_n}^{(n)})$ form a connected partition;

G2: for all i, j $(1 \leq i < j \leq N_n)$, $g^{(n)}(c_i^{(n)}) + g^{(n)}(c_j^{(n)})$ is connected if and only if $c_i^{(n)} + c_j^{(n)}$ is connected.

We remark that, in G2, we have $g^{(n)}(c_i^{(n)}), g^{(n)}(c_j^{(n)}) \in S$ and $c_i^{(n)}, c_j^{(n)} \in A$. Hence, different senses of "+" and "connected" apply in the two cases.

Claim 1.106 *For all $n \in \mathbb{N}$, $w^{(n)} \neq \emptyset$.*

Proof For the proof of this claim, we shall drop the n-sub- and superscripts and write N for N_n and c_i for $c_i^{(n)}$. Let G be the neighbourhood graph on $c_1, \ldots, c_N$. By Claim 1.105, G is planar. By Axioms 6 and Claim 1.61, G is connected. Let H be a drawing of G in $\mathbb{S}^2$ (under some mapping $\epsilon : V(G) \to V(H)$); we may assume that H is piecewise linear. By Proposition 1.69, let H^* be a geometric dual of H, which we may likewise assume to be piecewise linear. By Lemma 1.70, every vertex of H lies in exactly one face of H^*. It follows that every edge of H^* is on the boundary of two faces; moreover, H^* by construction contains no isolated nodes. By Theorem 1.82, the faces of H^* form a connected partition in S. So define $g(c_i)$ to be the face of H^* containing the H-vertex $\epsilon(c_i)$. Properties G1 and G2 are then almost immediate. QED

Claim 1.107 *Let $I \subseteq \{1, \ldots, N_n\}$, and let $g^{(n)} \in w^{(n)}$. Then $\sum_{i \in I} c_i$ is connected if and only if $\sum_{i \in I} g^{(n)}(c_i)$ is connected.*

Proof Claim 1.61 and property G2. QED

Suppose $n > m \geq 0$, so that $\bar{c}^{(n)}$ refines $\bar{c}^{(m)}$. For all i $(1 \leq i \leq N_n)$, let $c_{i,1}, \ldots, c_{i,M_i}$ be the collection of elements of $\bar{c}^{(n)}$ which sum to $c_i^{(m)}$. If $g^{(n)} \in w^{(n)}$, then, we may define the *restriction* of $g^{(n)}$ to $\bar{c}^{(m)}$, written $g^{(n)}|_m$, as follows:

$$g^{(n)}|_m(c_i^{(m)}) = g^{(n)}(c_{i,1}^{(n)}) + \ldots + g^{(n)}(c_{i,M_i}^{(n)})$$

Claim 1.108 *Let $g^{(n)} \in w^{(n)}$ with $0 \leq m < n$. Then $g^{(n)}|_m \in w^{(m)}$.*

Proof We must prove that G1 and G2 hold of $g^{(n)}|_m$. G1 is trivial. For G2, we note that, by construction,

$$g^{(n)}|_m(c_i^{(m)}) + g^{(n)}|_m(c_j^{(m)}) = g^{(n)}(c_{i,1}^{(n)}) + \ldots + g^{(n)}(c_{i,M_i}^{(n)}) + g^{(n)}(c_{j,1}^{(n)}) + \ldots + g^{(n)}(c_{j,M_j}^{(n)}).$$

By Claim 1.107, this element of S is connected if and only if the element of A

$$c_{i,1}^{(n)} + \ldots + c_{i,M_i}^{(n)} + c_{j,1}^{(n)} + \ldots + c_{j,M_j}^{(n)} = c_i^{(m)} + c_j^{(m)}$$

is connected. Hence G2 holds as required. QED

Claim 1.109 *Let $g \in w^{(n)}$. Then there exists a $g' \in w^{(n+1)}$ such that $g'|_n = g$.*

Proof Choose any $g'' \in w^{(n+1)}$. By Claim 1.108, $g''|_n \in w^{(n)}$. Letting $\bar{r} = g(c_1), \ldots, g(c_{N_n})$ and $\bar{s} = g''|_n(c_1), \ldots, g''|_n(c_{N_n})$, we see that $\bar{r}$ and $\bar{s}$ are c^3-partitions in S with the same neighbourhood graphs—namely, the neighbourhood graph of $c_1, \ldots, c_{N_n}$. By Theorem 1.78, let $h : \mathbb{S}^2 \to \mathbb{S}^2$ be a homeomorphism taking $\bar{s}$ to $\bar{r}$. Thus, $h \circ g''$ maps $\bar{c}^{(n+1)}$ to the faces of a plane graph G in $\mathbb{S}^2$ whose edges include the frontiers of the elements $\bar{r}$. Now let $h' : \mathbb{S}^2 \to \mathbb{S}^2$ be a deformation making all the curved edges of G piecewise linear, while leaving any already piecewise linear edges unaffected. By Theorem 1.82, $g' = h' \circ h \circ g'' \in w^{(n+1)}$ maps $\bar{c}^{(n+1)}$ to an N_{n+1}-tuple in S and it is easy to see that g' satisfies the conditions of the claim. QED

By Claim 1.109, there exists a sequence of embeddings:

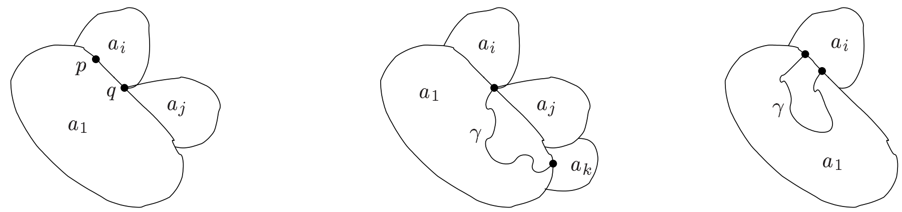
$$\emptyset = g^{(0)}, g^{(1)}, g^{(2)}, \ldots$$

such that, for all n $(0 < n)$, $g^{(n)}$ maps $\bar{c}^{(n)}$ to S, and, for all m, n $(0 \leq m < n)$, $g^{(n)}|_m = g^{(m)}$.

Now let $a \in A$ be such that $a = c_{i_1}^{(n)} + \ldots + c_{i_k}^{(n)}$. Then we define

$$g(a) = g^{(n)}(c_{i_1}^{(n)}) + \ldots + g^{(n)}(c_{i_k}^{(n)}) \ .$$

The fact that $g^{(n)}|_m = g^{(m)}$ whenever $0 \leq m < n$ means that this mapping is well defined. It is easy to see that $g : A \to S$ is a Boolean algebra isomorphism; moreover, by Claim 1.107, $g(a)$ is connected if and only if a is connected. That is, we have proved:



$\chi(p) = \{i\}$ $\chi(q) = \{i, j\}$ $\chi(\gamma) = \{\{i, j\}, \{k\}\}$ $\chi(\gamma) = \{\{i\}\}$

a) b) c)

Figure 1.14. The 'hub' a_1 of a radial partition

Claim 1.110 *$\mathfrak{A}$ can be isomorphically embedded in* $\mathrm{ROP}(\mathbb{S}^2)$, *regarded as a* $\{c, \leq\}$*-structure.*

In view of Claim 1.110, and in order to simplify notation, we might as well take $\mathfrak{A}$ to *be* a substructure of $\mathrm{ROP}(\mathbb{S}^2)$. Note that the previously distinct uses of the Boolean functions and the term "connected" become unambiguous, as do "connected partition", "c^h-partition", "neighbour", and so on. Moreover, since $A \subseteq S$, we may meaningfully talk about the *frontier* $\mathcal{F}(a)$ of any $a \in A$, and apply all the results established previously about elements of $\mathrm{ROP}(\mathbb{S}^2)$. For example, by Lemma 1.73, if $r_1, \ldots, r_n$ is a c^2-partition in A radial about r_1 such that r_1 has at least 2 neighbours, then, for any neighbour r_i of r_1, $\mathcal{F}(r_1) \cap \mathcal{F}(r_i)$ is a Jordan arc. Recall that, for tuples $\bar{r}$ and $\bar{s}$ from $\mathrm{ROP}(\mathbb{S}^2)$, we write $\bar{r} \sim \bar{s}$ if $\bar{r}$ and $\bar{s}$ are similarly situated (in $\mathbb{S}^2$).

Stage 3: In the previous stage, we established that $\mathfrak{A}$ can be chosen to be a substructure of $\mathrm{ROP}(\mathbb{S}^2)$. In this stage, we show that, in that case, $\mathfrak{A}$ is in fact an elementary substructure of $\mathrm{ROP}(\mathbb{S}^2)$.

Claim 1.111 *Let $a_1, \ldots, a_n \in A$ be a c^2-partition radial about a_1 such that a_1 has at least 3 neighbours. Let $r_1, r_2 \in S$ be disjoint 2-cells with $a_1 = r_1 + r_2$. Then there exist $c_1, c_2 \in A$ such that $a_1, \ldots, a_n, c_1, c_2 \sim a_1, \ldots, a_n, r_1, r_2$.*

Proof Since a_1, r_1, r_2 are 2-cells with a_1 equal to the disjoint sum of r_1 and r_2, r_1 and r_2 must be separated by a cross-cut γ in a_1. For any neighbour a_i of a_1, $\mathcal{F}(a_1) \cap \mathcal{F}(a_i)$ is a Jordan arc. Let $p \in \mathcal{F}(a_1)$. By inspection, p lies on either one or two Jordan arcs of the form $\mathcal{F}(a_1) \cap \mathcal{F}(a_i)$ where a_i is a neighbour of a_1. We define the *character of* p, written $\chi(p)$ to be the set of those i $(2 \leq i \leq n)$ such that a_i is a neighbour of a_1

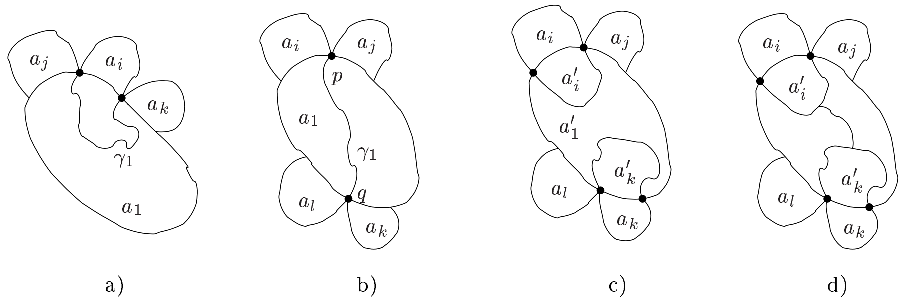


Figure 1.15. The construction of a cross-cut with a given character

and $p \in \mathcal{F}(a_i)$ (Fig. 1.14a). Note that $\chi(p)$ has either 1 or 2 elements. If $\chi(p)$ has one element, then p lies on some Jordan arc $\mathcal{F}(a_1) \cap \mathcal{F}(a_i)$, but not at its endpoints. If $\chi(p)$ has two elements, then since a_1 has at least three neighbours, $\chi(p)$ determines p. Now let γ be a cross-cut in a_1. We define the *character of* γ, written $\chi(\gamma)$ to be the set of characters of its endpoints. (See Fig. 1.14b and Fig. 1.14c for examples.) It is routine to show that, if γ_1 and γ_2 are two such cross-cuts and $\chi(\gamma_1) = \chi(\gamma_2)$, there is a homeomorphism of the closed plane onto itself taking a_i to itself for all i $(1 \leq i \leq n)$ and taking γ_1 to γ_2. So, to prove the lemma, it suffices to establish that, if γ_1 is any cross-cut in a_1, there exist disjoint 2-cells $c_1, c_2 \in A$ with $a_1 = c_1 + c_2$ such that the cross-cut γ_2 separating c_1 and c_2 in a_1 satisfies $\chi(\gamma_1) = \chi(\gamma_2)$.

Let the endpoints of γ_1 be p and q. We prove the result for the special case where $\chi(\gamma)$, $\chi(p)$ and $\chi(q)$ all contain two elements; the other cases are dealt with similarly. Fig. 1.15a shows the sub-case where $\chi(p)$ and $\chi(q)$ are non-disjoint; Fig. 1.15b shows the sub-case where $\chi(p)$ and $\chi(q)$ are disjoint.

The sub-case of Fig. 1.15a is trivial: Axiom 8 with a_1 substituted for x and a_i for y immediately guarantees the existence of $c_1, c_2 \in A$ partitioning a_1, and hence separated by a cross-cut γ_2; moreover the connectivity conditions on c_1 and c_2 mean that γ_1 and γ_2 have the same endpoints, so that $\chi(\gamma_1) = \chi(\gamma_2)$.

The sub-case of Fig. 1.15b requires a little more work. However, two applications of Axiom 8 guarantee the existence in A of the regions a'_i, a'_k as in Fig. 1.15c. Axiom 7 then guarantees that the region labelled a'_1 in Fig. 1.15c can be split into two regions as shown in Fig. 1.15d.

Summing together appropriate subdivisions of a_1 produces $c_1, c_2 \in A$ separated by an arc γ_2 satisfying $\chi(\gamma_1) = \chi(\gamma_2)$. QED

The rest of this section is devoted to showing that we can relax the conditions of Claim 1.111.

CLAIM 1.112 *Let $n > 1$ and let $a_1, \ldots, a_n \in A$ be a partition such that a_1 is a 2-cell. Let $r_1, r_2 \in S$ be disjoint 2-cells with $a_1 = r_1 + r_2$. Then there exist $c_1, c_2 \in A$ such that $a_1, \ldots, a_n, c_1, c_2 \sim a_1, \ldots, a_n, r_1, r_2$.*

Proof Immediate given claims 1.65 and 1.111. QED

CLAIM 1.113 *Let $n > 1$ and let $a_1, \ldots, a_n \in A$ be a partition such that a_1 is a 2-cell. Let $r \in S$ be such that $r \leq a_1$. Then there exists $c \in A$ such that $a_1, \ldots, a_n, c \sim a_1, \ldots, a_n, r$.*

Proof By the construction of $S = \mathrm{ROP}(\mathbb{S}^2)$, we can partition a_1 into 2-cells $r_1, \ldots, r_m$ such that r can be expressed as the sum of various r_j. It suffices to show that there are $c_1, \ldots, c_m \in A$ such that

$$a_1, \ldots, a_n, r_1, \ldots, r_m \sim a_1, \ldots, a_n, c_1, \ldots, c_m.$$

We proceed by induction on m. If $m = 1$, then $r_1 = a_1$ and we are done. If $m > 1$, by Corollary 1.67, we can renumber the r_i if necessary so that r_1 and $r_2' = r_2 + \ldots + r_m$ are 2-cells. By Claim 1.112, there exist $c_1, c_2' \in A$ such that $a_1, \ldots, a_n, r_1, r_2' \sim a_1, \ldots, a_n, c_1, c_2'$. Let h be a homeomorphism of the closed plane onto itself mapping a_i to itself, r_1 to c_1 and r_2' to c_2'. By exactly the same argument as for Lemma 1.90, h can be chosen so that $h(r_i) \in S$ for all i $(2 \leq i \leq m)$. But then the $h(r_i)$ partition the 2-cell c_2' into 2-cells. So consider the partition $c_2', c_1, a_2, \ldots, a_n$. By inductive hypothesis, there exist $c_2, \ldots, c_m \in A$ such that

$$c_2', c_1, a_2, \ldots, a_n, h(r_2), \ldots, h(r_m) \sim c_2', c_1, a_2, \ldots, a_n, c_2, \ldots, c_m \ .$$

The result then follows. QED

CLAIM 1.114 *Let $n > 1$ and let $a_1, \ldots, a_n \in A$ be a c^2-partition. Let $r \in S$. Then there exists $c \in A$ such that $a_1, \ldots, a_n, c \sim a_1, \ldots, a_n, r$.*

Proof Write r as the sum $r \cdot a_1 + \ldots + r \cdot a_n$ By considering these terms separately, we use Claim 1.113 and an induction similar to that used in the proof of Claim 1.113. The details are routine. QED

Claim 1.115 *Let* $n \geq 0$ *and let* $a_1, \ldots, a_n \in A$. *Let* $r \in S$. *Then there exists* $c \in A$ *such that* $a_1, \ldots, a_n, r \sim a_1, \ldots, a_n, c$.

Proof Immediate given Claims 1.54, 1.63 and 1.114.

Claim 1.116 $\mathfrak{A} \preceq \mathrm{ROP}(\mathbb{S}^2)$.

Proof We certainly have $\mathfrak{A} \subseteq \mathrm{ROP}(\mathbb{S}^2)$. Let $n \geq 0$ and let $\phi(x_1, \ldots, x_n)$ be any formula of the form $\exists y \psi(x_1, \ldots, x_n, y)$. Let $a_1, \ldots, a_n \in A$ such that $\mathrm{ROP}(\mathbb{S}^2) \models \phi[a_1, \ldots, a_n]$. Then there exists $r \in S$ such that $\mathrm{ROP}(\mathbb{S}^2) \models \psi[a_1, \ldots, a_n, r]$. By Claim 1.115, there exists $c \in A$ such that $a_1, \ldots, a_n, r \sim a_1, \ldots, a_n, c$. By Lemmas 1.90 and 1.91, $\mathrm{ROP}(\mathbb{S}^2) \models \psi[a_1, \ldots, a_n, c]$. The claim then follows by Proposition 1.94. QED

By Claim 1.116, $\mathfrak{A}$ and $\mathrm{ROP}(\mathbb{S}^2)$ have the same theory. But by construction, $\mathfrak{A} \models \neg\phi$, whence $\mathrm{ROP}(\mathbb{S}^2) \models \neg\phi$, which completes the proof of Lemma 1.104. QED

Corollary 1.117 *All splittable, finitely decomposable mereotopologies over* $\mathbb{S}^2$ *with curve-selection have the same* $L_{c,\leq}$*-theory, and hence also the same* L_C*-theory.*

Thus, while Examples 1.17, 1.18 and 1.99 show that there certainly *are* elementarily inequivalent mereotopologies over $\mathbb{R}^2$ and $\mathbb{S}^2$, Corollary 1.117 indicates that there is nothing like the free-for-all one might initially expect. At least for the signatures $\{c, \leq\}$ and $\{C\}$, the reference to $\mathbf{T}_\Sigma$ as the *standard* first-order mereotopological theory of the closed plane is justified. Corollary 1.173 generalizes this result to apply to any signature of topological primitives.

For reasons of simplicity (which we trust the reader will appreciate) we have provided an axiomatization of well-behaved plane mereotopologies only for the language $L_{c,\leq}$. It should be clear from the foregoing discussion, however, that an analogous result could be obtained for the language L_C, which as we noted, is more expressive over $\mathrm{ROP}(\mathbb{R}^2)$. Such an axiomatization was developed in Schoop, 1999.

Of course, it is one thing to have an axiomatic characterization of the $L_{c,\leq}$-theory of $\mathrm{ROP}(\mathbb{S}^2)$—quite another to determine whether a given $L_{c,\leq}$-sentence is a member of it. The question therefore arises as to the computational characteristics of this problem. Dornheim, 1998showed (in effect) that this theory is undecidable and hence (since it is a complete theory), not r.e. It follows that the ω-rule (or some equivalent mechanism) is indispensable in this axiomatization. In fact, Schaefer and Štefankovič, 2004showed (in effect) that the decision problem for $\mathrm{Th}_{c,\leq}\mathrm{ROP}(\mathbb{S}^2)$ is at least as hard as that of second-order arithmetic.

Specifically, Schaefer and Štefankovič effectively encode second-order arithmetic in a first-order language with variables ranging over 2-cells in $\mathbb{R}^2$ and primitive predicates expressing the so-called RCC-relations (see Randell et al., 1992,Egenhofer, 1991;but it is easy to see that that theory can in turn be effectively encoded in $\mathrm{Th}_{c,\leq}\mathrm{ROP}(\mathbb{S}^2)$. Schaefer and Štefankovič also consider the complexity of the quantifier-free fragment of their logic, a problem closely related to the well-known problem of recognizing so-called string-graphs (see e.g. Erlich et al., 1976,Kratochvíl, 1988,and show that it is in NEXPTIME. In Schaefer et al., 2003, this bound is improved to NP—a very surprising result.

6. Spatial mereotopology

In this section, we extend the main results of Section 4 to the spatial mereotopology $\mathrm{ROP}(\mathbb{R}^3)$. This material is a tidied up version of Pratt and Schoop, 2002.

6.1 Facts about $\mathrm{ROP}(\mathbb{R}^3)$ and $\mathrm{ROP}(\mathbb{S}^3)$

Recall that a 2-*manifold* is a Hausdorff space locally homeomorphic at every point to the open disc $\mathbf{B}^2$, and that a *surface* is a connected 2-manifold.

LEMMA 1.118 *Let X be either $\mathbb{R}^n$ or $\mathbb{S}^n$, and let M be a mereotopology over X having curve-selection. If $r \in M$ with r and $-r$ both connected, then $\mathcal{F}(r)$ is connected.*

Proof Consider the case $X = \mathbb{R}^n$. Let $r \in M$ be connected and non-empty with connected, non-empty complement, and suppose the closed set $\mathcal{F}(r)$ is not connected. Let d_1 and d_2 be closed sets partitioning $\mathcal{F}(r)$, and let $p \in r$, $q \in -r$. Since r is connected with connected complement, it is easy to see that the conditions of Proposition 1.19 are fulfilled, so that p and q are connected in $\mathbb{R}^n \setminus (d_1 \cup d_2)$. But this is absurd given that $d_1 \cup d_2 = \mathcal{F}(r)$. The case $X = \mathbb{S}^n$ follows easily. QED

LEMMA 1.119 *Let $r \in \mathrm{ROP}(\mathbb{S}^3)$ such that r and $-r$ are non-empty and connected, and $\mathcal{F}(r)$ is not a surface. Then the graph K^5 can be drawn in $\mathcal{F}(r)$.*

Proof It is easy to see that $\mathcal{F}(r)$ can be finitely triangulated. Call any point where $\mathcal{F}(r)$ is not locally homeomorphic to $\mathbf{B}^2$ a *bad point*; and call any edge of the triangulation all of whose points are bad a *bad edge*. By the properties of triangulations, any bad point either occurs on a bad edge or else is an isolated bad point at a vertex of the triangulation.

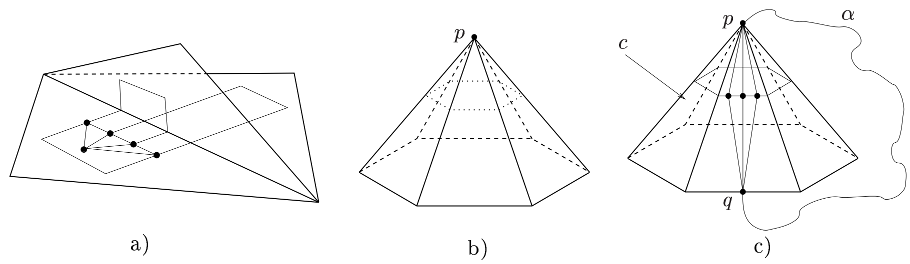


Figure 1.16. Embedding K^5 in non-surfaces (Proof of Lemma 1.124).

If there is a bad edge, then more than two triangles must share this edge, and the embedding of K^5 in $\mathcal{F}(r)$ proceeds as shown in Fig. 1.16a. Assume, then, that there are no bad edges, but that some vertex p of the triangulation is an isolated bad point. Call two triangles with p as a vertex *neighbours* if they share an edge having p as a vertex. Since all edges are good, these triangles can clearly be arranged into disjoint cycles such that each triangle belongs to the same cycle as its two neighbours. Choose one such cycle. By applying a homeomorphism if necessary, we may assume that this triangle-cycle forms a cone with vertex p as shown in Fig. 1.16b. Since there are only finitely many triangles in the triangulation, we can ensure that we choose a triangle-cycle such that the points inside the tip of the cone either all belong to r or all belong to $-r$. Let s be either r or $-r$ depending on which of these possibilities is realized. Note that, since r is non-empty and connected with non-empty, connected complement, so is s.

Let $t \in \mathrm{ROP}(\mathbb{S}^3)$ be a small element representing the tip of the cone, indicated by the light dotted lines in Fig. 1.16b. Removing t from s visibly does not disconnect s, so that $s \cdot -t$ is connected; moreover, t shares some face with $-s$, so that $t + -s = -(s \cdot -t)$ is also connected. Thus, $s \cdot -t$ is non-empty and connected with nonempty-connected complement, whence, by Lemma 1.118, $\mathcal{F}(s \cdot -t)$ is connected. Moreover, since p is bad, there must be at least two triangle-cycles with p as vertex; whence $p \in \mathcal{F}(s \cdot -t)$. Thus we may choose a point q on the base rim of t and connect it to p by a Jordan arc α in $\mathcal{F}(s)$ such that the locus of α is disjoint from $\mathcal{F}(t)$ except for its endpoints, as shown in Fig. 1.16c. The embedding of K^5 in $\mathcal{F}(s) = \mathcal{F}(r)$ then proceeds as depicted. QED

One notable difference between $\mathbb{S}^2$ and $\mathbb{S}^3$ is that the Schönflies Theorem, which holds in the former, fails in the latter. In fact, the pathological 'region' known as Alexander's horned sphere, and depicted in Fig. 1.3 is the best-known counterexample: the frontier of this region is homeomorphic to $\mathbb{S}^2$, but its exterior is not simply connected, and is certainly therefore not homeomorphic to $\mathbf{B}^3$. Nevertheless, Alexander, 1924balso proved a Schönflies-type result for polyhedra, which, in our notation, can be written as follows. (See also Moise, 1977, Ch. 17.)

Proposition 1.120 *Let $r \in \mathrm{ROP}(\mathbb{S}^3)$ be such that $\mathcal{F}(r)$ is homeomorphic to $\mathbb{S}^2$. Then both r^- and $(-r)^-$ are homeomorphic to $\mathbf{D}^3$.*

To avoid cumbersome locutions in the sequel, we define:

Definition 1.121 *Let X be either $\mathbb{R}^n$ or $\mathbb{S}^n$. A* ball *in X is a subset of X similarly situated in X to the unit ball $\mathbf{B}^3$. A* polyhedral ball *in X is a ball which is an element of* $\mathrm{ROP}(X)$.

Thus, if $r \in \mathrm{ROP}(\mathbb{S}^3)$ with $\mathcal{F}(r)$ homeomorphic to $\mathbb{S}^2$, then r and $-r$ are both balls in $\mathbb{S}^3$. Furthermore, if $r \in \mathrm{ROP}(\mathbb{R}^3)$ with $\mathcal{F}(r)$ homeomorphic to $\mathbb{S}^2$ (and hence bounded), then then *exactly one of* r and $-r$ is a ball in $\mathbb{R}^3$. We note in passing:

Lemma 1.122 *If $r \in \mathrm{RO}(\mathbb{S}^3)$ is a (polyhedral) ball in $\mathbb{S}^3$, then so is $-r$.*

Proof By definition, r is similarly situated in $\mathbb{S}^3$ to $u = \mathbf{B}^3(0,1)$. By considering a spherical inversion, u is similarly situated in $\mathbb{S}^3$ to $-u$.

QED

The following well-known theorem will also prove useful in the sequel (see, e.g. Massey, 1967, p. 10).

Proposition 1.123 (Classification Theorem for surfaces) *Every compact surface is homeomorphic to either (i) $\mathbb{S}^2$ or (ii) the sum of finitely many connected tori or (iii) the sum of finitely many projective planes.*

6.2 Expressing familiar spatial concepts in L_C

Our next task is to show that certain familiar concepts defined on the mereotopology $\mathrm{ROP}(\mathbb{R}^3)$ can be expressed using L_C-formulas. As a preliminary, recall the discussion of Section 3.2, which showed that: (i) expressions such as $x^- \cap y^- \cap z \neq \emptyset$ etc. can be regarded as L_C-formulas; and (ii) there is an L_C-formula $\phi_{\mathrm{ci}}(x, y)$ which we may read as "$x^- \cap y^-$ is connected".

Now suppose r and s are elements of $\mathrm{ROP}(\mathbb{R}^3)$, and consider, for example, the set $\mathcal{F}(r) \setminus \mathcal{F}(s)$. Evidently, this set is connected if and only if it is piecewise-linear arc-connected, and therefore if and only if any two points in it are contained within some connected set of the form $r^- \cap t^- \subseteq \mathcal{F}(r) \setminus \mathcal{F}(s)$ with $t \in \mathrm{ROP}(\mathbb{R}^3)$. It follows from the discussion of Section 3.2 that there is an L_C-formula satisfied by a pair of regions r, s if and only if $\mathcal{F}(r) \setminus \mathcal{F}(s)$ is connected. In the sequel, then, we write, without further commentary, expressions such as $c(\mathcal{F}(x) \setminus \mathcal{F}(y))$ etc. as L_C-formulas having the obvious interpretations.

LEMMA 1.124 *There exists an L_C-formula $\phi_{K^5}(x)$ such that, for all $r \in \mathrm{ROP}(\mathbb{R}^3)$, $\mathrm{ROP}(\mathbb{R}^3) \models \phi_{K^5}[r]$ if and only if K^5 is embeddable in $\mathcal{F}(r)$.*

Proof The graph K^5 is evidently embeddable in $\mathcal{F}(r)$ if and only if there exist polyhedra v_i $(1 \leq i \leq 5)$ and $e_{i,j}$ $(1 \leq i < j \leq 5)$, all disjoint from r and from each other, satisfying the following conditions:

1 For all i $(1 \leq i \leq 5)$, $v_i^- \cap r^-$ is a singleton

2 For all i, j $(1 \leq i < j \leq 5)$, $e_{i,j}^- \cap r^-$ is connected

3 For all i, j, i', j' $(1 \leq i < j \leq 5,\ 1 \leq i' < j' \leq 5)$, $\{i,j\} \cap \{i',j'\} = \emptyset$ implies $e_{i,j}^- \cap e_{i',j'}^- \cap r^- = \emptyset$, and $\{i,j\} \cap \{i',j'\} = \{k\}$ implies $e_{i,j}^- \cap e_{i',j'}^- \cap r^- = v_k^- \cap r^-$.

(Note incidentally that the polyhedra $e_{i,j}$ are not themselves required to be connected—only the sets $e_{i,j}^- \cap r^- = \mathcal{F}(e_{i,j}) \cap \mathcal{F}(r)$.) But the above conditions are expressible in L_C over $\mathrm{ROP}(\mathbb{R}^3)$. QED

LEMMA 1.125 *There exists an L_C-formula $\phi_{\mathrm{b}^*}(x)$ such that, for all $r \in \mathrm{ROP}(\mathbb{R}^3)$:*

1 if $\mathcal{F}(r)$ is connected and unbounded, then $\mathrm{ROP}(\mathbb{R}^3) \models \phi_{\mathrm{b}^*}[r]$;

2 if $\mathcal{F}(r)$ is homeomorphic to $\mathbb{S}^2$, then $\mathrm{ROP}(\mathbb{R}^3) \not\models \phi_{\mathrm{b}^*}[r]$.

Proof Let $\phi_{\mathrm{b}^*}(x)$ be

$$\exists y_1 \exists y_2 (y_1 \cdot x = 0 \wedge y_2 \cdot x = 0 \wedge c(\mathcal{F}(x) \cap \mathcal{F}(y_1) \cap \mathcal{F}(y_2)) \wedge \\ c(\mathcal{F}(x) \setminus \mathcal{F}(y_1)) \wedge c(\mathcal{F}(x) \setminus \mathcal{F}(y_2)) \wedge \neg c(\mathcal{F}(x) \setminus (\mathcal{F}(y_1) \cup \mathcal{F}(y_2)))).$$

Thus, $\phi_{\mathrm{b}^*}(x)$ 'says' that there exist polyhedra y_1 and y_2, disjoint from x, such that the sets $\mathcal{F}(x) \cap \mathcal{F}(y_1) \cap \mathcal{F}(y_2)$, $\mathcal{F}(x) \setminus \mathcal{F}(y_1)$ and $\mathcal{F}(x) \setminus \mathcal{F}(y_2)$ are all connected, but the set $\mathcal{F}(x) \setminus (\mathcal{F}(y_1) \cup \mathcal{F}(y_2))$ is not.

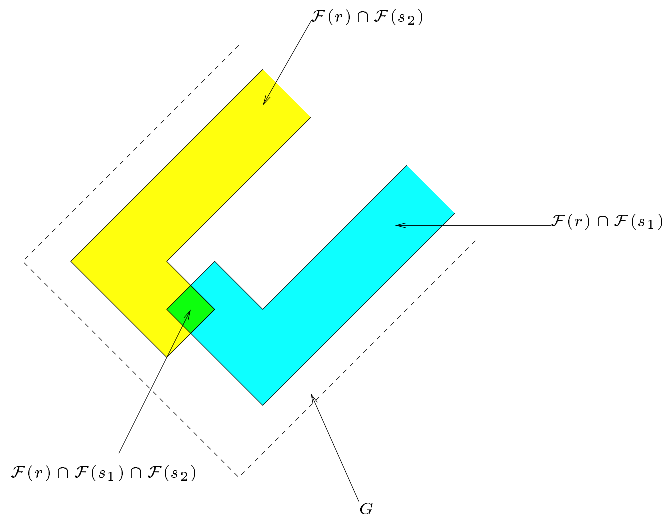


Figure 1.17. Arrangement of $\mathcal{F}(r) \cap \mathcal{F}(s_1)$ and $\mathcal{F}(r) \cap \mathcal{F}(s_2)$ on G (Proof of Lemma 1.125).

Suppose $\mathcal{F}(r)$ is connected and unbounded. Let r be a Boolean combination of finitely many half-spaces, corresponding to a finite set of planes, say, $P_1, \ldots, P_m$; it is then easy to see that $\mathcal{F}(r) \subseteq P_1 \cup \cdots \cup P_m$. Since $\mathcal{F}(r)$ is unbounded, we can draw in $\mathcal{F}(r)$ a rectangular figure G, unbounded on one side (dotted lines in Fig. 1.17), such that G intersects only one of the P_i. Let $s_1, s_2 \in \mathrm{ROP}(\mathbb{R}^3)$ be laminas, infinitely extended in one direction, and placed on G (on the outside of r) so that $\mathcal{F}(r) \cap \mathcal{F}(s_1)$ and $\mathcal{F}(r) \cap \mathcal{F}(s_2)$ are arranged as shown. Since $\mathcal{F}(r)$ is connected, $\mathcal{F}(r) \setminus \mathcal{F}(s_1)$ and $\mathcal{F}(r) \setminus \mathcal{F}(s_2)$ are also connected; and since G lies on just one of the P_i, $\mathcal{F}(r) \setminus (\mathcal{F}(s_1) \cup \mathcal{F}(s_2))$ is not connected. Thus $\mathrm{ROP}(\mathbb{R}^3) \models \gamma[r]$. The second part of the Lemma follows by Proposition 1.47. QED

Let $\phi_{\mathrm{c}}(x)$ be the L_C-formula defined in Lemma 1.27 and satisfied by $r \in \mathrm{ROP}(\mathbb{R}^3)$ if and only if r is connected, and let $\phi_{\mathrm{J}}(x)$ abbreviate the formula $x \neq 0 \wedge x \neq 1 \wedge \phi_{\mathrm{c}}(x) \wedge \phi_{\mathrm{c}}(-x)$.

LEMMA 1.126 *For all $r \in \mathrm{ROP}(\mathbb{R}^3)$, r satisfies $\phi_{\mathrm{J}}(x) \wedge \neg\phi_{K^5}(x) \wedge \neg\phi_{\mathrm{b}^*}(x)$ if and only if $\mathcal{F}(r)$ is homeomorphic to $\mathbb{S}^2$.*

Proof Suppose $\mathcal{F}(r)$ is homeomorphic to $\mathbb{S}^2$. Certainly, by Proposition 1.120, $\mathrm{ROP}(\mathbb{R}^3) \models \phi_{\mathrm{J}}[r]$; by Lemma 1.124, $\mathrm{ROP}(\mathbb{R}^3) \models \neg\phi_{K^5}[r]$; by Lemma 1.125, $\mathrm{ROP}(\mathbb{R}^3) \models \neg\phi_{\mathrm{b}^*}[r]$. Conversely, suppose that r satisfies $\phi_{\mathrm{J}}(x) \wedge \neg\phi_{K^5}(x) \wedge \neg\phi_{\mathrm{b}^*}(x)$. By Lemma 1.118, $\mathcal{F}(r)$ is connected, and by the first part of Lemma 1.125, $\mathcal{F}(r)$ is bounded. Moreover, K^5 cannot be embedded in $\mathcal{F}(r)$, by Lemma 1.124. Hence $\mathcal{F}(r) = \mathcal{F}(\dot{r})$ is a compact surface, by Lemma 1.119. The result then follows from Proposition 1.123. QED

LEMMA 1.127 *Let* $r \in \mathrm{ROP}(\mathbb{R}^3)$ *satisfy* $\phi_{\mathrm{J}}(x) \wedge \neg\phi_{K^5}(x) \wedge \phi_{\mathrm{b}^*}(x)$*. Then* r *is unbounded.*

Proof Suppose for contradiction that r is bounded, so that we also have $r \in \mathrm{ROP}(\mathbb{S}^3)$. By Lemma 1.119, $\mathcal{F}(r)$ is a surface. Moreover, since r is bounded, $\mathcal{F}(r)$ is compact, and since K^5 cannot be drawn in $\mathcal{F}(s)$, $\mathcal{F}(r)$ is homeomorphic to $\mathbb{S}^2$ by Proposition 1.123. But since $\mathrm{ROP}(\mathbb{R}^3) \models \phi_{\mathrm{b}^*}[r]$, this contradicts the second part of Lemma 1.125. Hence, r is unbounded. QED

LEMMA 1.128 *There exists an* L_C*-formula* $\phi_{\mathrm{b}^3}(x)$ *such that, for all* $r \in \mathrm{ROP}(\mathbb{R}^3)$, $\mathrm{ROP}(\mathbb{R}^3) \models \phi_{\mathrm{b}^3}[r]$ *if and only if* r *is bounded.*

Proof Let $\phi_{\mathrm{b}^3}(x)$ be the formula

$$\exists y \exists z(x \leq y \wedge y \cdot z = 0 \wedge \\ \phi_{\mathrm{J}}(y) \wedge \neg\phi_{K^5}(y) \wedge \neg\phi_{\mathrm{b}^*}(y) \wedge \phi_{\mathrm{J}}(z) \wedge \neg\phi_{K^5}(z) \wedge \phi_{\mathrm{b}^*}(z)).$$

If r is bounded, let $s \in \mathrm{ROP}(\mathbb{R}^3)$ be a ball in $\mathbb{R}^3$ such that $r \leq s$; and let $t \in \mathrm{ROP}(\mathbb{R}^3)$ be a half-space disjoint from s. By Lemma 1.125, $\mathrm{ROP}(\mathbb{R}^3) \models \neg\phi_{\mathrm{b}^*}[s]$ and $\mathrm{ROP}(\mathbb{R}^3) \models \phi_{\mathrm{b}^*}[t]$. Thus, s and t are suitable witnesses for y and z in $\phi_{\mathrm{b}^3}(x)$, so that $\mathrm{ROP}(\mathbb{R}^3) \models \phi_{\mathrm{b}^3}[r]$.

Conversely, suppose that $\mathrm{ROP}(\mathbb{R}^3) \models \phi_{\mathrm{b}^3}[r]$. Let s and t be witnesses for y and z. By Lemma 1.126, $\mathcal{F}(s)$ is homeomorphic to $\mathbb{S}^2$, whence, by Proposition 1.120, exactly one of s and $-s$ is a ball in $\mathbb{R}^3$. By Lemma 1.127, t is unbounded, and so intersects the complement of every ball in $\mathbb{R}^3$. Therefore $-s$ is not a ball in $\mathbb{R}^3$, so s is. Hence, r is bounded. QED

THEOREM 1.129 *There exists a formula* $\phi_{\mathrm{B}}(x)$ *such that, for all* $r \in \mathrm{ROP}(\mathbb{R}^3)$, $\mathrm{ROP}(\mathbb{R}^3) \models \phi_{\mathrm{B}}[r]$ *if and only if* r *is a polyhedral ball in* $\mathbb{R}^3$*.*

Proof Let $\phi_{\mathrm{B}}(x)$ be

$$\phi_{\mathrm{J}}(x) \wedge \neg\phi_{K^5}(x) \wedge \neg\phi_{\mathrm{b}^*}(x) \wedge \phi_{\mathrm{b}^3}(x),$$

and apply Lemmas 1.126 and 1.128. QED

Thus, with a little effort, we can define certain familiar topological notions over $\mathrm{ROP}(\mathbb{R}^3)$ using L_C-formulas. The following technical material, which is devoted to defining some decidedly *unfamiliar* topological notions over $\mathrm{ROP}(\mathbb{R}^3)$, will be used in the sequel. We recall the discussion of compactifications in Section 3.3, and consider the mapping $r \mapsto \dot{r}$ from $\mathrm{ROP}(\mathbb{R}^3)$ to its 1-point compactification $\mathrm{ROP}(\mathbb{S}^3)$. By Lemmas 1.36 and 1.37, this mapping is a Boolean algebra isomorphism and preserves the properties of connectedness and non-connectedness. For technical reasons, we will occasionally need to consider properties of elements in $\mathrm{ROP}(\mathbb{R}^3)$ whose defining conditions make reference to their counterparts in $\mathrm{ROP}(\mathbb{S}^3)$.

For all $r \in \mathrm{ROP}(\mathbb{R}^3)$, $\infty \in \dot{r}$ if and only if $-r$ is bounded, and $\infty \in \dot{r}^-$ if and only if r is unbounded (where the closure operator $^-$ refers to the topology on $\mathbb{S}^3$). By Lemma 1.128 then, it is harmless to employ the expression $\infty \in \dot{x}$ in L_C-formulas, since we can take it as a mnemonic for $\phi_{\mathrm{b}^3}(-x)$; and similarly for expressions such as $\infty \in \dot{x}^-$, $\infty \in \mathcal{F}(\dot{x})$, etc.

LEMMA 1.130 *There exists a formula $\phi_{\dot{K}^5}(x)$ satisfied by $r \in \mathrm{ROP}(\mathbb{R}^3)$ if and only if K^5 is embeddable in $\mathcal{F}(\dot{r})$.*

Proof As for Lemma 1.124, making the obvious adjustments to accommodate the point at infinity. QED

LEMMA 1.131 *There exists a formula $\phi_{\dot{\mathrm{B}}}(x)$ such that, for all $r \in \mathrm{ROP}(\mathbb{R}^3)$, $\mathrm{ROP}(\mathbb{R}^3) \models \phi_{\dot{\mathrm{B}}}[r]$ if and only if $\dot{r}$ is a ball in $\mathbb{S}^3$.*

Proof Let $\phi_{\dot{\mathrm{B}}}(x)$ be $\phi_{\mathrm{J}}(x) \wedge \neg\phi_{\dot{K}^5}(x)$. If $\dot{r}$ is a ball in $\mathbb{S}^3$, it is evident that $\mathrm{ROP}(\mathbb{R}^3) \models \phi_{\dot{\mathrm{B}}}[r]$. Conversely, suppose $\mathrm{ROP}(\mathbb{R}^3) \models \phi_{\dot{\mathrm{B}}}[r]$. By Lemmas 1.119 and 1.130, $\mathcal{F}(\dot{r})$ is a surface in $\mathbb{S}^3$. Furthermore, by Proposition 1.123, $\mathcal{F}(\dot{r})$ is homeomorphic to $\mathbb{S}^2$. The result then follows by Proposition 1.120. QED

6.3 Characterizing triangulations in L_C

In Section 4, we showed that every tuple in $\mathrm{ROP}(\mathbb{R}^2)$ satisfies a topologically complete L_C-formula—that is, an L_C-formula with the property that all tuples satisfying it are similarly situated. Our proof exploited Whitney's theorem on 3-connected graphs in the plane to show that any c^3-partition in $\mathrm{ROP}(\mathbb{S}^2)$ is determined up to similar situation by its neighbourhood graph. However, Whitney's theorem is not available for $\mathbb{S}^3$, and so we must adopt an alternative approach to analysing the expressive power of L_C over $\mathrm{ROP}(\mathbb{R}^3)$. This approach has the advantage of

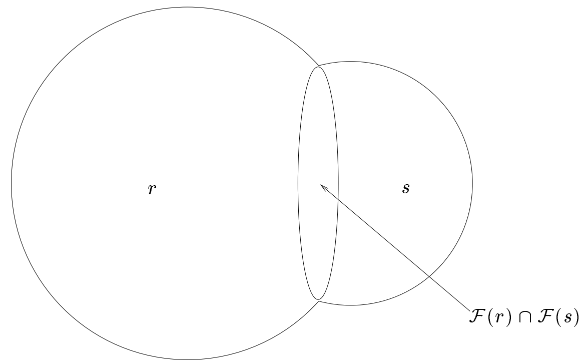


Figure 1.18. The configuration of Proposition 1.132

being, in some ways, more straightforward than that of Section 4, though the topologically complete formulas it constructs are more complicated.

We assume familiarity with the basic theory of triangulations and PL-complexes: for details, see, e.g., Moise, 1977, Ch. 7. We also require the following 'obvious' result about balls in $\mathbb{S}^3$ (Pratt and Schoop, 2002,Theorem 3.14).

PROPOSITION 1.132 *Let $r, s \in \mathrm{RO}(\mathbb{S}^3)$ be disjoint balls in $\mathbb{S}^3$ such that $r + s$ is also a ball in $\mathbb{S}^3$. Then $\mathcal{F}(r) \cap \mathcal{F}(s) \cap \mathcal{F}(r + s)$ is the locus of a Jordan curve, and $\mathcal{F}(r) \cap \mathcal{F}(s)$ is homeomorphic to the closed disc $\mathbf{D}^2$.*

The situation is illustrated in Fig. 1.18.

DEFINITION 1.133 *A quadruple $q = \langle r_1, r_2, r_3, r_4 \rangle$ of pairwise disjoint elements of* $\mathrm{ROP}(\mathbb{S}^3)$ *is a* q-cell in $\mathbb{S}^3$ *if, for all non-empty $J \subseteq \{1, 2, 3, 4\}$, the polyhedron $\sum_{j \in J} r_j$ is a ball in $\mathbb{S}^3$.*

The reference to the containing space $\mathbb{S}^3$ is significant: in the sequel, we introduce q-cells in $\mathbb{R}^3$. However, we sometimes speak simply of *q-cells* if it is clear from context which space we are talking about (or if it makes no difference).

EXAMPLE 1.134 *Consider the regular open tetrahedron t_0 with vertices $v_1 = (0, 0, 0)$, $v_2 = (1, 0, 0)$, $v_3 = (0, 1, 0)$, $v_4 = (0, 0, 1)$. Let t_1, t_2, t_3, t_4 be the four regular open tetrahedra (taken in some fixed order) each having three vertices from $\{v_1, \ldots, v_4\}$ and the point $(1/4, 1/4, 1/4)$ as the fourth vertex (Fig. 1.19). Evidently, the quadruple $q_0 = \langle t_1, t_2, t_3, t_4 \rangle$ is a q-cell.*

THEOREM 1.135 *All q-cells in $\mathbb{S}^3$ are similarly situated in $\mathbb{S}^3$.*

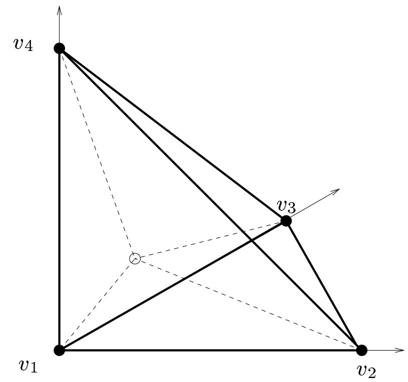


Figure 1.19. The q-cell q_0

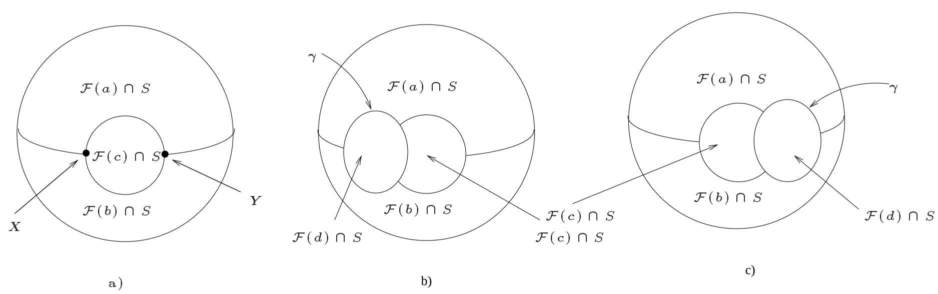


Figure 1.20. Possible arrangements of $\mathcal{F}(a) \cap S$, $\mathcal{F}(b) \cap S$, $\mathcal{F}(c) \cap S$ and $\mathcal{F}(d) \cap S$, where $S = \mathcal{F}(a + b + c)$ (Proof of Theorem 1.135).

Proof Let $\langle a, b, c, d \rangle$ be a q-cell. Since a, b, c, $a + b$, $b + c$, $a + c$ and $a + b + c$ are balls, by Proposition 1.132, the sets $\mathcal{F}(a) \cap \mathcal{F}(b)$, $\mathcal{F}(a) \cap \mathcal{F}(c)$, $\mathcal{F}(b) \cap \mathcal{F}(c)$ and $\mathcal{F}(a+b) \cap \mathcal{F}(c)$ are all closed discs. Letting $S = \mathcal{F}(a+b+c)$, it is then easy to show that the sets $\mathcal{F}(a) \cap S$, $\mathcal{F}(b) \cap S$ and $\mathcal{F}(c) \cap S$ must be arranged on S as shown in Fig. 1.20a, up to similar situation. Let $e = -(a+b+c+d)$; then, by Lemma 1.122, $\sum(B \cup \{e\})$ is a ball for any proper subset $B \subset \{a, b, c, d\}$. Thus, all of the sets $a+b+c$, d, e, $a+b+c+d$ and $a+b+c+e$ are balls. By Proposition 1.132 again, $\mathcal{F}(d) \cap S$ and $\mathcal{F}(e) \cap S$ are both closed discs, whose common frontier in the space S is the locus of some Jordan curve γ, say.

Consider how γ might be drawn on S. Since $a + d$ and $a + e$ are balls, by Proposition 1.132, $\mathcal{F}(a) \cap \mathcal{F}(d)$ and $\mathcal{F}(a) \cap \mathcal{F}(e)$ are closed discs. Similarly, $\mathcal{F}(b) \cap \mathcal{F}(d)$, $\mathcal{F}(b) \cap \mathcal{F}(e)$, $\mathcal{F}(c) \cap \mathcal{F}(d)$ and $\mathcal{F}(c) \cap \mathcal{F}(e)$ are closed discs. Hence γ divides each of the three sets $\mathcal{F}(a) \cap S$, $\mathcal{F}(b) \cap S$ and

$\mathcal{F}(c) \cap S$ into two residual domains. Moreover, γ cannot pass through either of the points X or Y in Fig. 1.20a; for otherwise, one of the sets $\mathcal{F}(a) \cap \mathcal{F}(d)$, $\mathcal{F}(b) \cap \mathcal{F}(d)$, $\mathcal{F}(c) \cap \mathcal{F}(d)$, $\mathcal{F}(a) \cap \mathcal{F}(e)$, $\mathcal{F}(b) \cap \mathcal{F}(e)$ or $\mathcal{F}(c) \cap \mathcal{F}(e)$ would fail to be a closed disc. It is then easy to see that γ and the region $\mathcal{F}(d) \cap S$ it encloses must lie in S as shown in Fig. 1.20b or Fig. 1.20c, up to similar situation. But these two arrangements of a, b, c, d are mirror images. QED

NOTATION 1.136 *If $q = \langle t_1, \ldots, t_4 \rangle$ is a q-cell, denote the component polyhedron t_i by $q[i]$ for all i $(1 \leq i \leq 4)$. Denote the polyhedron $t_1 + \cdots + t_4$ by $\hat{q}$.*

In Example 1.134, $\hat{q}_0$ is the interior of the convex hull of the points $V = \{v_1, \ldots, v_4\}$. We employ familiar terms from discussions of simplicial complexes: a *face* of q_0 is the convex closure of any non-empty subset of V; a face of q_0 is proper if it is not the whole of $(\hat{q}_0)^-$; a *vertex* of q_0 is an element of V.

DEFINITION 1.137 *Let q be any q-cell, and h a homeomorphism of $\mathbb{S}^3$ onto itself taking q_0 to q. A* (proper) face *of q is a set of points $h(F)$, where F is a (proper) face of q_0. A* vertex *of q is a point $h(v)$, where v is a vertex of q_0.*

We remark that, in Definition 1.137, a suitable homeomorphism h can always be found, by Theorem 1.135; moreover, since the faces of q_0 are expressible as set-algebraic combinations of the polyhedra $t_1, \ldots, t_4$ and their topological closures, the precise choice of h does not matter. Thus, q-cells are simply homeomorphic images of the q-cell q_0 of Example 1.134, with the notions of *face* and *vertex* transferred in the obvious way.

DEFINITION 1.138 *A* q-cell partition *(in* ROP$(\mathbb{S}^3)$*) is a sequence $\bar{q} = q_1, \ldots, q_n$ of q-cells in $\mathbb{S}^3$ such that (i) $\hat{q}_1, \ldots, \hat{q}_n$ is a partition in* ROP$(\mathbb{S}^3)$*; and (ii) for all i, j $(1 \leq i < j \leq n)$, if F is a face of q_i and G a face of q_j, then $F \cap G$ is either empty or a face of both q_i and q_j. A* vertex *of a q-cell partition is a vertex of one of its elements.*

Thus, q-cell partitions define (finite) PL-complexes in the obvious way: each q-cell in the partition corresponds to a PL 3-simplex, and its proper faces to PL d-simplices for $d < 3$.

DEFINITION 1.139 *Let $\bar{q} = q_1, \ldots, q_N$ and $\bar{q}' = q'_1, \ldots, q'_N$ be q-cell partitions in* ROP$(\mathbb{S}^3)$*. We say that $\bar{q}$ and $\bar{q}'$ are* isomorphic *if there is a bijection between the vertices of $\bar{q}$ and the vertices of $\bar{q}'$ such that, for all*

i, j $(1 \leq i \leq N,\ 1 \leq j \leq 4)$, the vertices of q_i lying on the frontier of $q_i[j]$ are mapped to the vertices of q'_i lying on the frontier of $q'_i[j]$.

Lemma 1.140 *Isomorphic q-cell partitions in* $\mathrm{ROP}(\mathbb{S}^3)$ *are similarly situated in* $\mathbb{S}^3$.

Proof Isomorphic q-cell partitions define isomorphic PL-complexes. QED

We conclude this sub-section by extending the notions of q-cell and q-cell partition to the open space $\mathbb{R}^3$.

Definition 1.141 *A quadruple* $q = \langle r_1, r_2, r_3, r_4 \rangle$ *of elements of* $\mathrm{ROP}(\mathbb{R}^3)$ *is a* q-cell in $\mathbb{R}^3$ *if* $\dot{q} = \langle \dot{r}_1, \dot{r}_2, \dot{r}_3, \dot{r}_4 \rangle$ *is a q-cell in* $\mathbb{S}^3$. *A sequence* $\bar{q} = q_1, \ldots, q_n$ *of q-cells in* $\mathbb{R}^3$ *is a* q-cell-partition in $\mathrm{ROP}(\mathbb{R}^3)$ *if* $\dot{q}_1, \ldots, \dot{q}_n$ *is a q-cell partition in* $\mathrm{ROP}(\mathbb{S}^3)$.

Definition 1.142 *Let* $\bar{q} = q_1, \ldots, q_n$ *and* $\bar{q}' = q'_1, \ldots, q'_n$ *be q-cell partitions in* $\mathrm{ROP}(\mathbb{R}^3)$. *We say that* $\bar{q}$ *and* $\bar{q}'$ *are* isomorphic *if: (i) the corresponding q-cell partitions* $\dot{q}_1, \ldots, \dot{q}_n$ *and* $\dot{q}'_1, \ldots, \dot{q}'_n$ *in* $\mathrm{ROP}(\mathbb{S}^3)$ *are isomorphic; and (ii) for all* i, j $(1 \leq i \leq n,\ 1 \leq j \leq 4)$, $q_i[j]$ *is bounded if and only if* $q'_i[j]$ *is bounded.*

Intuitively, knowing which $q_i[j]$ are bounded for a q-cell partition $q_1, \ldots, q_n$ in $\mathrm{ROP}(\mathbb{R}^3)$ amounts to knowing, up to homeomorphism, where the point at infinity is in the corresponding q-cell partition in $\mathrm{ROP}(\mathbb{S}^3)$. More precisely, we have:

Lemma 1.143 *Let* $\bar{q} = q_1, \ldots q_n$ *and* $\bar{q}' = q'_1, \ldots q'_n$ *be similarly situated q-cell partitions in* $\mathrm{ROP}(\mathbb{S}^3)$. *Let* $p \in \mathbb{S}^3$ *such that, for all* i, j $(1 \leq i \leq n,\ 1 \leq j \leq 4)$, $p \in (q_i[j])^-$ *if and only if* $p \in (q'_i[j])^-$. *Then there is a homeomorphism* $h : \mathbb{S}^3 \to \mathbb{S}^3$ *fixing* p *and mapping* $\bar{q}$ *to* $\bar{q}'$.

Proof Parallel to the proof of Lemma 1.87. QED

Theorem 1.144 *Isomorphic q-cell partitions in* $\mathrm{ROP}(\mathbb{R}^3)$ *are similarly situated in* $\mathbb{R}^3$.

Proof Let $q_1, \ldots q_n$ and $q'_1, \ldots q'_n$ be isomorphic q-cell partitions in $\mathrm{ROP}(\mathbb{R}^3)$. Then $\dot{q}_1, \ldots \dot{q}_n$ and $\dot{q'}_1, \ldots \dot{q'}_n$ are isomorphic q-cell partitions such that, for all i, j $(1 \leq i \leq n,\ 1 \leq j \leq 4)$, $\infty \in (\dot{q}_i[j])^-$ if and only if $\infty \in (\dot{q'}_i[j])^-$. By Lemmas 1.140 and 1.143, there exists a homeomorphism h of $\mathbb{S}^3$ onto itself mapping $\dot{q}_1, \ldots \dot{q}_n$ to $\dot{q'}_1, \ldots \dot{q'}_n$, and fixing ∞. Thus, $h' = h \setminus \{\langle \infty, \infty \rangle\}$ is a homeomorphism of $\mathbb{R}^3$ onto itself mapping $q_1, \ldots q_n$ to $q'_1, \ldots q'_n$. QED

6.4 Expressive power of L_C in ROP($\mathbb{R}^3$)

We are now ready to show that every tuple in ROP($\mathbb{R}^3$) satisfies a formula which is topologically complete in ROP($\mathbb{R}^3$) over $\mathbb{R}^3$.

LEMMA 1.145 *For all $N > 0$, there exists a formula $\phi_{\mathrm{q}}^N(\bar{z})$ such that, for any $4N$-tuple $\bar{t}$ from* ROP($\mathbb{R}^3$), ROP($\mathbb{R}^3$) $\models \phi_{\mathrm{q}}^N[\bar{t}]$ *if and only if $\bar{t}$ is a q-cell partition in* ROP($\mathbb{R}^3$).

Proof Let $\phi_{\dot{\mathrm{B}}}(x)$ be as defined in Lemma 1.131, and suppose $s_1, \ldots, s_4 \in$ ROP($\mathbb{R}^3$). Then the quadruple $\langle \dot{s}_1, \ldots, \dot{s}_4 \rangle$ is a q-cell in $\mathbb{S}^3$ if and only if ROP($\mathbb{R}^3$) $\models \phi_{\mathrm{q}}[s_1, \ldots, s_4]$, where $\phi_{\mathrm{q}}(y_1, \ldots, y_4)$ is the formula

$$\bigwedge \Big\{ \phi_{\dot{\mathrm{B}}}\Big(\sum_{j \in J} y_j \Big) \mid \emptyset \neq J \subseteq \{1,2,3,4\} \Big\}.$$

The result then follows easily. QED

LEMMA 1.146 *Let $\bar{t}$ be a $4N$-tuple forming an N-element q-cell partition in* ROP($\mathbb{R}^3$). *Then we can find a formula $\gamma(\bar{z})$ such that, for any $4N$-tuple $\bar{t}'$ of* ROP($\mathbb{R}^3$), ROP($\mathbb{R}^3$) $\models \gamma[\bar{t}']$ *if and only if $\bar{t}'$ is an N-element q-cell partition isomorphic to $\bar{t}$.*

Proof Almost immediate from Lemmas 1.128 and 1.145 and the discussion of Section 3.2. QED

LEMMA 1.147 *Every q-cell partition in* ROP($\mathbb{R}^3$) *satisfies a L_C-formula which is topologically complete in* ROP($\mathbb{R}^3$) *over $\mathbb{R}^3$.*

Proof Theorem 1.144 and Lemma 1.146. QED

LEMMA 1.148 *Any n-tuple $\bar{r}$ from* ROP($\mathbb{R}^3$) *can be refined to an N-element q-cell partition. That is: there exists a $4N$-tuple $\bar{t}$ from* ROP($\mathbb{R}^3$) *and a $(4N \times n)$ Boolean array A such that $\bar{t}$ forms a q-cell partition in* ROP($\mathbb{R}^3$) *and $\bar{r} = \bar{t}A$.*

Proof By the definition of ROP($\mathbb{R}^3$), we can certainly refine $\bar{r}$ to a partition of convex regions of $\mathbb{R}^3$, each of which is bounded by a finite number of planes, and thence, by triangulating these convex regions, into a partition of polyhedra $t_1, \ldots t_N$, such that each $\dot{t}_i$ is a ball in $\mathbb{S}^3$, and the boundary of each t_i ($1 \leq i \leq N$) is composed of 4 'triangles' (in the sense used earlier in this proof). By subdividing each t_i, we can construct a q-cell q_i whose faces are exactly the triangles bounding t_i, and such that $\hat{q}_i = t_i$. Then $q_1, \ldots q_N$ is the required q-cell partition.

QED

THEOREM 1.149 *Every tuple in* $\mathrm{ROP}(\mathbb{R}^3)$ *satisfies some* L_C*-formula which is topologically complete in* $\mathrm{ROP}(\mathbb{R}^3)$ *over* $\mathbb{R}^3$*.*

Proof Let $\bar{r}$ be a tuple from $\mathrm{ROP}(\mathbb{R}^3)$. Let $\bar{t}$ and A be as in Lemma 1.148, and by Lemma 1.147 let $\phi_{\bar{t}}(\bar{z})$ be a formula satisfied by $\bar{t}$ which is topologically complete in $\mathrm{ROP}(\mathbb{R}^3)$ over $\mathbb{R}^3$. Then the formula $\exists \bar{z}(\phi_{\bar{t}}(\bar{z}) \wedge \bar{x} = \bar{z}A)$, which is also topologically complete in $\mathrm{ROP}(\mathbb{R}^3)$ over $\mathbb{R}^3$, is satisfied by $\bar{r}$. QED

This concludes the main business of this section: the language L_C is sufficiently expressive that every tuple of polyhedra in $\mathbb{R}^3$ can be characterized up to the relation of similar situation in $\mathbb{R}^3$ by one of its formulas. Moreover, it is easy to see that an analogous result must obtain for polyhedra in $\mathbb{S}^3$. Of course, the characterizing formulas for tuples of polyhedra obtained in this section are much more complicated than the corresponding $L_{c,\leq}$-formulas for tuples of polygons obtained in Section 4.

In Section 5, we exploited the high expressive power of $L_{c,\leq}$ in $\mathrm{ROP}(\mathbb{S}^2)$ to obtain an axiomatization of $\mathrm{Th}_{c,\leq}(\mathrm{ROP}(\mathbb{S}^2))$, and thence, a formulation of the conditions under which an arbitrary mereotopology over $\mathbb{S}^2$ has the same $L_{c,\leq}$-theory as $\mathrm{ROP}(\mathbb{S}^2)$. The question therefore arises as to whether an analogous approach is possible for characterizing 'reasonable' spatial mereotopologies using the results of this section. The major disincentive to such an undertaking is the relative weakness of the requirement of finite decomposability in $\mathbb{S}^3$. For the plane case, the requirement of finite decomposability led very easily to the existence of c^3-partition refinements, which paved the way for an axiomatic characterization of $\mathrm{Th}_{c,\leq}(\mathrm{ROP}(\mathbb{S}^2))$. In the spatial case, by contrast, much stronger assumptions are needed to guarantee the existence of q-cell partitions, as examples such as the region depicted in Fig. 1.3 show. Thus, while the identification of a standard theory of spatial mereotopology is certainly conceivable, it is not obvious, at the time of writing, how best to approach this matter.

7. Model Theory

In Section 2, we defined a *mereotopology* over a topological space X to be a Boolean sub-algebra M of $RO(X)$ in which, for all $p \in o \subseteq X$, with o open, there exists $r \in M$ such that $p \in r \subseteq o$. However, we also promised a purely intrinsic characterization of such structures—one making no reference to points or topological spaces. In this section, we fulfil that promise, and (partially) realize the vision with which we started this chapter, of a reconstruction of topology where the fundamental objects are not points, but regions.

7.1 Abstract models of mereotopological theories

We begin by noting some simple facts about mereotopologies over topological spaces of various kinds.

LEMMA 1.150 *Let M be a mereotopology over a topological space X, considered as a structure interpreting the signature $\{C, +, \cdot, -, 0, 1, \leq\}$. (i) The sentences Φ_{CA} consisting of the usual axioms of Boolean algebra together with*

$$\begin{array}{l}\forall x \neg C(x,0)\\ \forall x(x>0 \to C(x,x))\\ \forall x\forall y(C(x,y) \to C(y,x))\\ \forall x\forall y(C(x,y) \wedge y \leq z \to C(x,z))\\ \forall x\forall y(C(x,y+z) \to C(x,y) \vee C(x,z))\end{array}$$

are all true in M. (ii) If X is weakly regular, then the sentence ϕ_{ext} given by

$$\forall x\forall y(\forall z(C(x,z) \to C(y,z)) \to x \leq y)$$

is true in M. (iii) If X is compact and Hausdorff, then the sentence ϕ_{int} given by

$$\forall x\forall y(\neg C(x,y) \to \exists z(\neg C(x,-z) \wedge \neg C(y,z)))$$

is true in M.

Proof (i) Straightforward. (ii) Lemma 1.22. (iii) Suppose $r, s \in M$ with $r^- \cap s^- = \emptyset$. Since X is regular, by Lemma 1.23, for each point in $p \in r^-$, there is $r_p \in M$ with $p \in r_p$ and $s^- \subset -r_p$. Since the r_p cover r^-, choose a finite subcover, and let the sum of this subcover be t. Then $r^- \subseteq t$ and $s^- \subseteq -t$. QED

The three claims in Lemma 1.150 all have converses. Specifically:

PROPOSITION 1.151 *Let $\mathfrak{A}$ be a structure interpreting the signature $\Sigma = \{C, +, \cdot, -, 0, 1\ \leq\}$. (i) If $\mathfrak{A} \models \Phi_{\mathrm{CA}}$, then $\mathfrak{A}$ is isomorphic (as a Σ-structure) to a mereotopology over a topological space X; in fact, X can always be chosen so as to be semi-regular and T_0. (ii) If $\mathfrak{A} \models \Phi_{\mathrm{CA}} \cup \{\phi_{\mathrm{ext}}\}$, then X can be chosen so as to be weakly regular and T_1.*

(iii) If $\mathfrak{A} \models \Phi_{\mathrm{CA}} \cup \{\phi_{\mathrm{ext}}, \phi_{\mathrm{int}}\}$*, then* X *can be chosen so as to be compact and Hausdorff.*

These results first appeared (in equivalent form) in Dimov and Vakarelov, 2006,Düntsch and Winter, 2005and Roeper, 1997,respectively. In the literature, structures satisfying Φ_{CA} are sometimes referred to as *contact algebras*, the sentence ϕ_{ext} as the *extensionality axiom*, and the sentence ϕ_{int} as the *interpolation axiom*. Together, Lemma 1.150 and Proposition 1.151 show that mereotopologies over certain classes of topological spaces can be characterized purely intrinsically, without reference to those spaces or the points that make them up. We note in passing that Proposition 1.151 speaks of *mereotopologies* over X (Definition 1.5), where the sources cited refer only to dense sub-algebras of $\mathrm{RO}(X)$. This slight strengthening is immediate from the relevant proofs, and improves the match between Lemma 1.150 and Proposition 1.151. For a fuller discussion, see Ch. **??**.

Furthermore, it turns out that the topological realizations in Proposition 1.151 (iii) are, in an important sense, unique. We motivate this result with a simple observation.

LEMMA 1.152 *Let* M_i *be a mereotopology over the topological space* X_i*, for* $i = 1, 2$*. Suppose there is a homeomorphism* $h : X_1 \to X_2$ *which maps* M_1 *onto* M_2*. Then, for any signature* Σ *of topological primitives,* h *induces a structure isomorphism* $h : M_1 \simeq_\Sigma M_2$*.*

Proof Immediate. QED

The uniqueness of the topological realizations in Proposition 1.151 (iii) takes the form of a partial converse of Lemma 1.152:

THEOREM 1.153 (ROEPER, 1997) *Let*M_i *be a mereotopology over a compact, Hausdorff space* X_i $(i = 1, 2)$*. Suppose there is a structure isomorphism* $f : M_1 \simeq_C M_2$*. Then there exists a homeomorphism* $h : X_1 \to X_2$ *which induces* f*—that is, one such that, for all* $r \in M_1$*,* $f(r) = h(r)$*.*

Thus, every model of $\Phi_{\mathrm{CA}} \cup \{\phi_{\mathrm{ext}}, \phi_{\mathrm{int}}\}$ is isomorphic to exactly one mereotopology over a compact Hausdorff space (up to homeomorphism). Since this fact is important for the development here, we present details of the proof.

We assume familiarity with the theory of ultrafilters: for details, see Koppelberg, 1989, Ch. 1, Sec. 2. In this context, recall that, for B a Boolean algebra, a *filter* on B is a set $F \subseteq B$ such that $a, b \in F$ implies $a \cdot b \in F$, and $a \in F$, $a \leq b \in B$ implies $b \in F$. A filter is *proper* if it is not the whole of B, or equivalently, if it does not contain 0. A proper filter U is an *ultrafilter* if it is maximal under set-inclusion, or equivalently, if $b_1 + b_2 \in U$ implies $b_1 \in U$ or $b_2 \in U$. The following result is standard (Koppelberg, 1989, Chapter 1, 2.16).

Proposition 1.154 (Prime Ideal Theorem) *Any proper filter on a Boolean algebra can be extended to an ultrafilter.*

In the following lemmas, let M be a mereotopology over a compact, Hausdorff space X. Since a compact Hausdorff space is normal (and hence regular), Lemma 1.23 applies.

Lemma 1.155 *Let U be an ultrafilter on M. Then the set $\bigcap\{r^-|r \in U\}$ is a singleton. We denote the member of this set by p_U and say that U* converges to *p_U.*

Proof We first show that $\bigcap\{u^-|u \in U\}$ contains at least one point. For otherwise, $\bigcup\{X \setminus u^-|u \in U\} = X$, whence $\{-u|u \in U\}$ covers X. By compactness of X, let $-u_1, \ldots, -u_n$ be a finite subcover. Then $-u_1 + \cdots + -u_n = 1$; i.e. $u_1 \cdot \cdots \cdot u_n = 0 \in U$, contradicting the fact that U is proper. Next we show that $\bigcap\{u^-|u \in U\}$ contains at most one point. For suppose p, q are distinct points of X. By Lemma 1.23, there exists $r \in M$ such that $p \in r$ and $q \in -r$. Hence $p \notin (-r)^-$ and $q \notin r^-$. Since U is an ultrafilter, either r or $-r$ is in U, so that either p or q is not in $\bigcap\{u^-|u \in U\}$. QED

Lemma 1.156 *Let U be an ultrafilter on M, and let $r \in M$. If $p_U \in r$, then there exists $s \in U$ such that $p_U \in s$ and $s^- \subseteq r$. Hence also, $r \in U$.*

Proof Suppose $p_U \in r \in M$. Then $p_U \notin (-r)^-$, and by Lemma 1.23, there exists $s \in M$ such that $p_U \in s$ and $s^- \subseteq r$. But since $p_U \notin (-s)^-$ we have $-s \notin U$, and thus $s \in U$. QED

Definition 1.157 *If U and V are ultrafilters on M, we say U and V are* contacting *if $r^- \cap s^- \neq \emptyset$ for all $r \in U$ and $s \in V$.*

Lemma 1.158 *If U and V are ultrafilters on M, then U and V are contacting if and only if $p_U = p_V$.*

Proof The if-direction is trivial. For the only-if direction, suppose that $p_U \neq p_V$. By Lemma 1.23, there exist $r, s \in M$ such that $p_U \in r$, $p_V \in s$ and $r^- \cap s^- = \emptyset$. By Lemma 1.156, $r \in U$, $s \in V$, so that U and V are not contacting. QED

LEMMA 1.159 *Let M_1 and M_2 be mereotopologies over weakly regular topological spaces, let $f : M_1 \simeq_C M_2$ be an isomorphism, and let U and V be contacting ultrafilters on M_1. Then $f(U)$ and $f(V)$ are contacting ultrafilters on M_2.*

Proof Almost immediate given the definability of $\leq$ in terms of C (Lemma 1.22). QED

LEMMA 1.160 *Let M_1 and M_2 be mereotopologies over weakly regular topological spaces, such that $f : M_1 \simeq_C M_2$. Let $r \in M$, and let U be an ultrafilter on M_1 with $p_U \in r$. Then $p_{f(U)} \in f(r)$.*

Proof By Lemma 1.156, there exists $s \in U$ such that $p_U \in s$ and $s^- \subseteq r$, so that $s^- \cap (-r)^- = \emptyset$. Since f is also a Boolean algebra isomorphism, $f(s)^- \cap (-f(r))^- = \emptyset$, i.e. $f(s)^- \subseteq f(r)$. Since $f(s) \in f(U)$, $p_{f(U)} \in f(s)^- \subseteq f(r)$. QED

Proof [Theorem 1.153] Suppose that $f : M_1 \simeq_C M_2$. Define the map h by $h(p_U) = p_{f(U)}$, for U a compact ultrafilter on M_1. We show: (i) h is well-defined and 1–1, (ii) the domain of h is the whole of X_1 and the range of h is the whole of X_2, (iii) for all $r \in M_1$, $f(r) = h(r)$, and for all $s \in M_2$, $f^{-1}(s) = h^{-1}(s)$, and (iv) h and h^{-1} are continuous. To prove (i), let U, V be compact ultrafilters on M_1, both converging to p. By Lemma 1.159, the isomorphism f maps contacting ultrafilters to contacting ultrafilters. Hence, h is well-defined. Applying the same reasoning to f^{-1}, h is 1–1. To prove (ii), let $p \in X_1$. Then $\{r \in M_1 | p \in r\}$ is a proper filter on M_1, and by Proposition 1.154, this filter can be extended to an ultrafilter U on M_1. By Lemma 1.155, U converges to some point p_U. Since X_1 is Hausdorff $p = p_U$. Thus, the domain of h is the whole of X_1. Similarly, if $q \in X_2$, we have an ultrafilter V on M_2 such that $q = p_V$. Thus $q = p_V = p_{f(f^{-1}(V))} = h(p_{f^{-1}(V)})$, so that the range of h is the whole of X_2. To prove (iii), let $p_V \in f(r)$ with V an ultrafilter on M_2. By Lemma 1.160, $p_{f^{-1}(V)} \in r$. Hence, $p_V = h(p_{f^{-1}(V)}) \in h(r)$. Conversely, let $p_V \in h(r)$. By the definition of h, $p_{f^{-1}(V)} \in r$, and by Lemma 1.160, $p_V \in f(r)$. Hence $f(r) = h(r)$. Now if $s \in M_2$, $f^{-1}(s) \in M_1$, so, applying the results just obtained to this set, we have $f^{-1}(s) = h^{-1}(h(f^{-1}(s))) = h^{-1}(f(f^{-1}(s))) = h^{-1}(s)$. (iv) Let $u \subseteq X_1$ be an open set. Since M_1 is a mereotopology, for each point $p \in u$, there exists $r_p \in M_1$ with $p \in r_p \subseteq u$. Thus the set $\mathcal{U} = \{r_p \in M | p \in u\}$ satisfies $\bigcup \mathcal{U} = u$. Then $h(u) = h(\bigcup \mathcal{U}) = \bigcup_{r \in \mathcal{U}} h(r) = \bigcup_{r \in \mathcal{U}} f(r)$ is a union of open sets in X_2 and hence is itself an open set in X_2. Therefore, h^{-1} is continuous. By substituting h^{-1} and for h and repeating the argument, h is continuous. QED

7.2 Abstract models of geometrical mereotopological theories

We have shown that mereotopologies over certain classes of topological spaces can be characterized in terms of certain first-order sentences which they make true. But what of specific mereotopologies of interest—for instance, those defined over the open or closed plane? This is the topic we now address, based on the results of Pratt and Lemon, 1997.

We employ standard results on prime models: for details, see Chang and Keisler, 1990, Ch. 2. A structure $\mathfrak{A}$ is said to be a *prime model* if it is elementarily embeddable in any elementarily equivalent submodel. Prime models are considered the 'simplest' or 'smallest' models of their theories, a view which is justified by the following proposition (Chang and Keisler, 1990, Theorem 2.3.3). In the sequel, all signatures are silently assumed to be countable.

PROPOSITION 1.161 *Elementarily equivalent prime models are isomorphic.*

The following notion is closely related to that of primeness. A formula ϕ is said to be *complete with respect to* a theory T if, for all formulas θ having the same free variables of ϕ, exactly one of $T \models \phi \to \theta$ or $T \models \phi \to \neg\theta$ hold. A structure $\mathfrak{A}$ is said to be *atomic* if any n-tuple $\bar{a}$ in A satisfies a formula $\phi(\bar{x})$ in $\mathfrak{A}$ such that ϕ is complete with respect to $\mathrm{Th}(\mathfrak{A})$. We have the following standard result (see, for example, Chang and Keisler, 1990, Theorem 2.3.4).

PROPOSITION 1.162 *A structure is countable atomic if and only if it is a prime model.*

Recall the concepts of topologically complete formula and homogeneous mereotopology given in Definitions 1.89 and 1.51, respectively.

LEMMA 1.163 *Let M be a homogeneous mereotopology over a topological space X, and let Σ be a signature of topological primitives. If $\phi \in L_\Sigma$ is topologically complete in X over M, then ϕ is complete with respect to $\mathrm{Th}_\Sigma(M)$.*

Proof Immediate from Lemma 1.91. QED

Theorem 1.176 below is a partial converse of this result.

For the next theorem, recall that $\mathrm{ROQ}(\mathbb{S}^2)$ is the rational polygonal mereotopology over the closed plane, and that its $L_{c,\leq}$-theory is $\mathbf{T}_{c,\leq}$, the standard $L_{c,\leq}$-theory of closed plane mereotopology, which we axiomatized in Section 5. Recall further that $\psi^N_{\mathrm{c}^3}(\bar{z})$ is the $L_{c,\leq}$-formula stating that $\bar{z}$ forms a c^3-partition, employed in the proof of Theorem 1.84

THEOREM 1.164 *The mereotopology* ROQ($\mathbb{S}^2$)*, considered as a* $\{c, \leq\}$*-structure, is a prime model of* $\mathbf{T}_{c,\leq}$*. In fact, for any* N*, there exist formulas* $\gamma_1(\bar{z}), \ldots, \gamma_K(\bar{z})$ *(with* K *depending on* N*), complete with respect to* $\mathbf{T}_{c,\leq}$*, such that*

$$\mathbf{T}_{c,\leq} \models \forall \bar{z}(\psi^N_{c^3}(\bar{z}) \to (\gamma_1(\bar{z}) \vee \cdots \vee \gamma_K(\bar{z}))).$$

Proof The first part of the theorem is immediate from Theorem 1.84 and Lemma 1.163. For the second part, observe that, for a given N, there are only finitely many neighbourhood structures on an N-element c^3-partition, each one giving rise to a topologically complete formula of the form

$$\exists \bar{z}(\psi^N_{c^3}(\bar{z}) \wedge \psi^{\bar{s}}_{+}(\bar{z}) \wedge \bar{x} = \bar{z}A),$$

as described in the proof of Theorem 1.84. QED

Note that, by Lemma 1.38, ROQ($\mathbb{S}^2$) and ROQ($\mathbb{R}^2$) are the same $\{c, \leq\}$-structure, so we could replace $\mathbb{S}^2$ in Theorem 1.164 by $\mathbb{R}^2$.

Similarly, we have

THEOREM 1.165 *The mereotopologies* ROQ($\mathbb{R}^2$) *and* ROQ($\mathbb{S}^2$)*, considered as* $\{C\}$*-structures, are prime models.*

Proof As for Theorem 1.164, but using Theorem 1.88 and Corollary 1.85, respectively. QED

Analogues of Theorem 1.165 hold in three dimensions, of course. For example, we have:

THEOREM 1.166 *The mereotopology* ROQ($\mathbb{R}^3$) *is a prime model of the* L_C*-theory of* ROP($\mathbb{R}^3$)*.*

The proof strategy is essentially identical to the plane case, using Theorem 1.149. Note, however, that *much* more care is required to show that the topologically complete formulas identified in Theorem 1.149 are complete with respect to the L_C-theory of ROP($\mathbb{R}^3$). We leave the details to the interested reader.

Returning to mereotopologies over $\mathbb{S}^2$, the question then arises as to whether ROQ($\mathbb{S}^2$) is *strictly* simplest among countable models of $\mathbf{T}_{c,\leq}$, in that there are countable models of that theory not isomorphic to ROQ($\mathbb{S}^2$). The answer is: yes and no. Recall that a theory is said to be *ω-categorical* if it has exactly one countable model up to isomorphism. Recall also that a *type* in variables $\bar{x} = x_1, \ldots, x_n$ is a maximal consistent set of formulas whose free variables are among the $x_1, \ldots, x_n$, and that a theory T is said to *have* a type $\Phi(\bar{x})$ if $\Phi(\bar{x})$ is consistent with T. The following result is standard (see, for example, Chang and Keisler, 1990, Theorem 2.3.13).

PROPOSITION 1.167 *Let T be a complete theory. Then T is ω-categorical if and only if, for each n, T has only finitely many types in $x_1, \ldots, x_n$.*

THEOREM 1.168 *$\mathbf{T}_{c,\leq}$ is not ω-categorical.*

Proof By Proposition 1.167, it suffices to prove that $\mathbf{T}_{c,\leq}$ has countably many types in the single variable x. It is easy to see that, for every positive integer m, the formula $\psi_m(x)$

$$\exists z_1 \ldots \exists z_m \left(\bigwedge_{1 \leq i \leq m} (c(z_i) \wedge z_i \neq 0) \wedge \bigwedge_{1 \leq i < j \leq m} \neg c(z_i + z_j) \wedge x = \sum_{1 \leq i \leq m} z_i \right)$$

is satisfied in $\mathrm{ROQ}(\mathbb{S}^2)$ by all and only those regions having exactly m components. Hence, the $\psi_m(x)$ are all satisfied in $\mathrm{ROQ}(\mathbb{S}^2)$; so each can be extended to a type $\Gamma_m(x)$ of $\mathrm{Th}_{c,\leq}(\mathrm{ROQ}(\mathbb{S}^2))$. But the $\psi_m(x)$ are also pairwise mutually exclusive in $\mathbf{T}_{c,\leq}$; so no two of them can be extended to the same type. Hence, $\mathbf{T}_{c,\leq}$ has infinitely many types in x. QED

One the other hand, it turns out that $\mathbf{T}_{c,\leq}$ is *almost* countably categorical, in the following sense. Note that, since any model of $\mathbf{T}_{c,\leq}$ is a Boolean algebra interpreting the predicate c, we may employ the terminology introduced at the start of Section 4.1.

THEOREM 1.169 *All countable finitely decomposable models of $\mathbf{T}_{c,\leq}$ are isomorphic.*

Proof Let $\mathfrak{A} \models \mathbf{T}_{c,\leq}$ be finitely decomposable. By Claims 1.54 and 1.63, every tuple from A can be refined to a c^3-partition. Theorem 1.164 then implies that $\mathfrak{A}$ is prime. The result follows by Proposition 1.161. QED

The above results show that, while specific mereotopologies such as $\mathrm{ROS}(\mathbb{S}^2)$ cannot be characterized in terms of the first-order sentences which they make true, they *almost* can. Specifically, we have the following abstract characterization of the mereotopology $\mathrm{ROQ}(\mathbb{S}^2)$.

COROLLARY 1.170 *If $\mathfrak{A}$ is a countable, finitely decomposable model of Axioms 1—8 in Section 5.1, then $\mathfrak{A}$ is isomorphic (as a $\{c, \leq\}$-structure) to the mereotopology $\mathrm{ROQ}(\mathbb{S}^2)$.*

Proof Theorem 1.169 and the fact that, by Theorem 1.100, any finitely decomposable model $\mathfrak{A}$ of Axioms 1—8 is elementarily equivalent to $\mathrm{ROQ}(\mathbb{S}^2)$. QED

7.3 Loose ends

We end this section with some matters touched on earlier in this chapter. We continue to assume all signatures to be countable. The following proposition is a special case of the Löwenheim-Skolem Theorem (see, for example, Hodges, 1993, p. 90).

PROPOSITION 1.171 *Let $\mathfrak{A}$ be a Σ-structure and Z a countable subset of A. Then $\mathfrak{A}$ has a countable elementary submodel whose domain includes Z.*

Recall that a topological space X is said to be *second countable* if its topology has a countable basis.

LEMMA 1.172 *Let M be a mereotopology over a compact, second-countable, Hausdorff space X, and let $P \subseteq M$ be countable. Then there is a countable mereotopology Q over X such that $P \subseteq Q$ and $Q \preceq M$.*

Proof We construct a countable subset $P' \subseteq M$ such that, for all $p \in o \subseteq X$ with o open, there exists $r \in P'$ such that $p \in r \subseteq o$. The lemma then follows from Proposition 1.171 by putting $\mathfrak{A} = M$ and $Z = P \cup P'$. Let B be a countable basis for the topology on X. For any $b, c \in B$ with $b^- \subseteq c$, take a cover of b^- by elements $s \in M$ such that $s \subseteq c$ (possible because M is a mereotopology), choose a finite subcover (possible because X is compact), and let $r_{b,c}$ be the sum, in M, of the elements of this finite subcover. Certainly, $b \subseteq r_{b,c} \subseteq c^-$. Let $P' = \{r_{b,c} \mid b, c \in B, b^- \subseteq c\}$. Since X is normal, for all $p \in o \subseteq X$ with o open, we can find $b, c \in B$ with $p \in b$, $b^- \subseteq c$ and $c^- \subseteq o$. But then $p \in r_{b,c} \subseteq o$ as required. QED

Note that Lemma 1.172 holds for all (countable) signatures.

We may now derive the promised strengthening of Corollary 1.117.

COROLLARY 1.173 *All splittable, finitely decomposable mereotopologies over $\mathbb{S}^2$ with curve-selection have the same L_Σ-theory for any topological signature Σ.*

Proof Let M_1, M_2 be two such mereotopologies. Extend the signature Σ if necessary so that it contains the predicates C, c and $\leq$, and expand M_1 and M_2 by interpreting these predicates in the normal way. By Lemma 1.172, let Q_i be a countable mereotopology over $\mathbb{S}^2$ such that $Q_i \preceq_\Sigma M_i$, for $i = 1, 2$. Thus, Q_1 and Q_2 are splittable, finitely decomposable mereotopologies over $\mathbb{S}^2$ having curve-selection. By Corollary 1.117, $Q_1 \equiv_{c,\leq} Q_2$. By Theorem 1.169, $Q_1 \simeq_{c,\leq} Q_2$. By Lemma 1.49, $Q_1 \simeq_C Q_2$. By Theorem 1.153, there is a homeomorphism mapping Q_1 onto Q_2. Finally, by Lemma 1.152, $Q_1 \simeq_\Sigma Q_2$, whence $M_1 \equiv_\Sigma M_2$. QED

Recall from Definition 1.96 that, if Σ is a signature of topological primitives, $\mathbf{T}_\Sigma$ denotes $\mathrm{Th}_\Sigma(\mathrm{ROS}(\mathbb{S}^2))$. By Corollary 1.173, $\mathbf{T}_\Sigma$ is the L_Σ-theory of any splittable, finitely decomposable mereotopology over $\mathbb{S}^2$ having curve-selection. This justifies our decision to call it the *standard* L_Σ-theory of closed plane mereotopology.

Theorem 1.169 now has the following corollary.

Corollary 1.174 *Let M be a countable, finitely decomposable mereotopology over a locally connected, compact Hausdorff space X, such that* $\mathrm{Th}_C(M) = \mathbf{T}_C$. *Then there is a homeomorphism $h : X \leftrightarrow \mathbb{S}^2$ taking M to* $\mathrm{ROQ}(\mathbb{S}^2)$.

Proof By Lemmas 1.22 and 1.27, $\mathrm{Th}_{C,c,\leq}(M) = \mathbf{T}_{C,c,\leq}$. By Theorem 1.169, $M \simeq_{c,\leq} \mathrm{ROQ}(\mathbb{S}^2)$. But $\mathbf{T}_{C,c,\leq}$ contains a formula defining C explicitly in terms of c and $\leq$. Hence $M \simeq_C \mathrm{ROQ}(\mathbb{S}^2)$. Now apply Theorem 1.153. QED

We remark that there is no prospect of removing the compactness condition from Corollary 1.174. For example, let p_π be, say, the point of $\mathbb{S}^2$ with coordinates $(0, \pi)$, and consider the topological space $X = \mathbb{S}^2 \setminus \{p_\pi\}$ and the mereotopology M over X given by $M = \{r \setminus \{p_\pi\} \mid r \in \mathrm{ROQ}(X)\}$. Then $\mathrm{ROQ}(\mathbb{S}^2) \simeq_{C,c,\leq} M$; but $\mathbb{S}^2$ and X are not homeomorphic.

A further consequence of Theorem 1.153 is the promised partial converse of Lemma 1.163. We require the following fact about prime models.

Lemma 1.175 *Let $\mathfrak{A}$ be a countable, atomic model and let $\bar{a}$,$\bar{b}$ be tuples from A which satisfy the same formulas in $\mathfrak{A}$. Then there is an automorphism of $\mathfrak{A}$ taking $\bar{a}$ to $\bar{b}$.*

Proof Almost immediate from Proposition 1.161, by adding a tuple of individual constants to stand alternatively for $\bar{a}$ and $\bar{b}$. QED

Theorem 1.176 *Let M be a mereotopology over a compact, second-countable Hausdorff space X, and let Σ be a signature of topological primitives such that C (contact) is first-order definable over M. If every tuple from M satisfies an L_Σ-formula which is complete with respect to* Th_Σ, *then that L_Σ-formula is topologically complete in M over X.*

Proof Let ϕ be complete with respect to $\mathrm{Th}_\Sigma(M)$, and suppose that $M \models \phi[\bar{r}]$, $M \models \phi[\bar{s}]$. We must show that $\bar{r} \sim \bar{s}$. By Lemma 1.172, let M' be any countable mereotopology over X containing the tuples $\bar{r}$

and $\bar{s}$, such that $M' \preceq M$. Thus, M' is countable and atomic, and ϕ is a complete formula with respect to $\mathrm{Th}_\Sigma(M')$ satisfied by both $\bar{r}$ and $\bar{s}$. By Lemma 1.175, there exists an automorphism $f : M' \simeq_\Sigma M'$ such that $f(\bar{r}) = \bar{s}$. Then, by Theorem 1.153, there is a homeomorphism $h : X \to X$ taking $\bar{r}$ to $\bar{s}$. QED

Lemma 1.163 and Theorem 1.176 establish the close connection between the notions of topological completeness with respect to a topological space and completeness with respect to a mereotopological theory.

8. Philosophical Considerations

The earliest modern work on region-based theories of space is that of Whitehead and de Laguna (Whitehead, 1919; Whitehead, 1920; Whitehead, 1929; de Laguna, 1922a; de Laguna, 1922b; de Laguna, 1922c). Both authors propose a system of postulates governing a small collection of primitive spatial relations, together with reconstructions of familiar spatial concepts in terms of those relations. The postulates serve implicitly to define the primitive relations they constrain (and perhaps the domain of entities over which they quantify), while the reconstructions of familiar spatial concepts connect the whole system to the data of spatial experience. To be sure, both Whitehead and de Laguna motivate their postulates by providing informal interpretations for their respective spatial primitives. Thus, for example, Whitehead illustrates his relation of *extensive connection* (as he calls it) using diagrams suggesting that two regions are extensively connected just in case their topological closures share a point in common (this is the interpretation given to the binary predicate C in this chapter). However, such explanations are intended only as a heuristic guide. Officially, spatial primitives acquire their content solely from the entire system postulates in which they participate. Primitives, by definition, are not explicitly definable.

The inspiration for such systems was presumably the axiomatic treatment of geometry found in Euclid (and latterly Hilbert); and the motivation for carrying out the procedure on a purely region-based footing seems, for both authors, to have been a certain disquiet about the empirical distance between the concept of a point as a primitive geometrical entity and the character of everyday spatial experience. The great difficulty of this approach, of course, is the problem of evaluating the system of postulates and conceptual reconstructions proposed. Whitehead's system has thirty-one postulates (or *assumptions*, in Whitehead's terminology) and a similar number of definitions. De Laguna's system, though far tidier, is also hardly self-evident. The only obvious sources of justification for such systems are their ability to chime with our pre-theoretic intuition and their eventual integration into a larger, empirically successful, physical theory. Neither source is very satisfactory. On the one hand, as we have seen in this chapter, almost any collection of spatial primitives enables us to write down propositions on which pre-theoretic intuition cannot be expected to return a reliable verdict. On the other hand, although empirical confirmation of a general physical theory must provide some support for the account of space it contains, the size of the undertaking and the difficulty of assigning credit when theories perform well (or blame when they perform badly) means that there is little practical prospect of any such justification for such systems of postulates and conceptual reconstructions.

An alternative approach to developing a region-based theory of space is illustrated by Tarski's *Geometry of Solids* (Tarski, 1956). Tarski too develops a geometry in which regions, not points, are the primitive objects; however, in contrast to Whitehead and Laguna, he does not build his theory by writing a collection of plausible-looking, but unprovable, axioms. Rather, beginning with the familiar model of space as $\mathbb{R}^3$, he considers a formal language whose variables range over the set of of spheres in $\mathbb{R}^3$ (defined in the standard way), and whose sole non-logical constant is the part-whole relation (again, defined in the standard way). Because the 'primitives' in Tarski's geometry of solids are well-defined mathematical objects and relations, the question of what postulates they satisfy is a well-defined mathematical problem, not a matter for intuition or experiment. And because many familiar spatial concepts have rational reconstructions in terms of the standard model, the question of how, if at all, these concepts can be expressed using formulas of Tarski's language is again a purely mathematical affair. Having thus specified the structure under consideration and the language used to describe it, Tarski then goes on to examine the kinds of logical issues that should by now be familiar to us. In fact, Tarski obtains a system of axioms (in higher-order logic) for which the standard Euclidean interpretation is, up to isomorphism, the only model.

This alternative approach is, in contrast to the 'postulationist' strategy of Whitehead and de Laguna, conservative and rationalist: conservative, because no attempt is made to build systems of axioms and definitions from the ground up; rationalist, because the appropriateness of the resulting region-based theories is secured by means of their logical relations to point-based models whose usefulness as representations of the space we inhabit—at least approximately and for mesoscopic objects—is anyway beyond doubt. It is this approach that we have taken in this chapter. Latterly, region-based theories of space have increased in popularity, following the seminal work of Clarke, 1981, Clarke, 1985, Biacino and Gerla, 1991,Randell et al., 1992,Gotts et al., 1996and Renz and Nebel, 1997.One reason for this resurgence of interest, particularly within the A.I. community, is the requirement to quantify over spatial regions without leaving the realm of first-order logic. The technology of theorem-proving for first-order logic is more highly developed than for higher-order logics; and, more generally, formalisms with limited expressive power enjoy a premium in A.I. if they give rise to entailment and satisfiability problems which have (theoretically or practically) efficient algorithmic solutions. Insofar as the study of region-based theories of space is motivated by computational considerations, the best approach to developing and analysing such theories is surely that of Tarski, not that of Whitehead.

These matters notwithstanding, the most striking outcome of the investigation undertaken here is just how much information it gives us about the possibilities for developing a truly region-based theory of space, along the lines apparently envisaged by Whitehead and de Laguna. Consider, for example, the issue of the 'correct' set of postulates. True, Examples 1.17 and 1.18 show that different mereotopologies defined over the spaces $\mathrm{RO}(\mathbb{R}^2)$ indeed have different first-order theories. Nevertheless, the discussion of Section 5 shows that the choices on offer are much more limited than these examples might initially lead one to suppose. In particular, all finitely decomposable, splittable mereotopologies over $\mathbb{S}^2$ having curve-selection have identical L_Σ-theories, for any signature of topological primitives. We proposed that this common L_Σ-theory should therefore be regarded as standard.

Or take again the issue of reconstructing familiar spatial concepts in terms of a chosen collection of primitives. We have seen that first-order topological languages interpreted over well-behaved mereotopologies have surprising—but not unlimited—expressive power. In particular, we provided formulas expressing a variety of familiar spatial relationships (as defined by their familiar point-based definitions, of course) over a wide range of mereotopologies. In addition, we showed that the first-order language $L_{c,\leq}$ is sufficiently expressive that every tuple of polygons in $\mathbb{S}^2$ can be characterized up to similar situation by one of its formulas, and that the first-order language L_C is sufficiently expressive that every tuple of polygons in $\mathbb{R}^2$ and every tuple of polyhedra in $\mathbb{R}^3$ can be characterized up to similar situation by one of its formulas.

Most striking of all, however, is what the foregoing analysis tells us about the view of space to which any first-order mereotopological theory commits us. While almost all interesting mereotopologies have first-order theories which are not categorical in any infinite cardinal, we nevertheless showed that the plane mereotopology $\mathrm{ROQ}(\mathbb{S}^2)$ and the spatial mereotopology ROQ($\mathbb{S}^3$) are prime models of their first-order theories over standard signatures of topological primitives. We further showed that ROQ($\mathbb{S}^2$) is, up to isomorphism, the only countable, finitely decomposable model of its $L_{c,\leq}$-theory; and we remarked that a corresponding observation—albeit with a more complex version of the finite decomposability condition—must apply in three dimensions as well. Finally, we showed that mereotopologies over compact, Hausdorff spaces, regarded as structures interpreting suitably rich topological signatures, determine their underlying spaces up to homeomorphism. In conclusion, the logical possibilities for region-based topological theories of space are more constrained than their earliest proponents might perhaps have thought.

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