A short introduction to
Ordinal Notations
Harold Simmons
University of Manchester

I will give a pseudo-historical account. The account is historical for I will describe some of the principal developments of the subject in the chronological order of occurrence (with one or two time warps). The account is pseudo for several reasons.

Some of the historical remarks may not be strictly accurate. I am not an historian, and I haven’t checked all the appropriate primary sources. However, I haven’t tried to deliberately falsify the story.

As with most mathematics, some things we can do better now than when they were first developed. Sometimes I take the more recent approach.

This is only a short account, so several, perhaps many, topics have been omitted. At this stage these topics can be seen as peripheral, but in a longer account they would have greater prominence.

This account is historical not because I want to write a history, but because I find that mode the easiest way to present this particular material. For some other subjects I would not try an historical account, but for this subject it seems to work.

It is hard to find a decent overview or true history of this subject. However, I do recommend [6] for a short history (as it then stood).

Preamble on ordinals

In a longer, well organized, course on the subject there wold be a quite early section on ordinals, as ordinals. Perhaps a knowledge of ordinals would be a prerequisite. Certainly I expect you to know something of ordinals, but not very much.

In the Appendix I hope I have gathered together all the relevant facts about ordinals. Here let me set down the first few facts, just to start us off.

Ordinals come in three kinds.

\[
\begin{array}{ccc}
\text{zero} & \alpha & \text{successor} \alpha + 1 & \text{limit} \lambda \\
0 & \alpha & \alpha + 1 & \lambda \\
\end{array}
\]

Mostly we write

\[
\alpha, \beta, \gamma, \delta, \ldots \text{ for arbitrary ordinals} \\
\lambda \text{ for limit ordinals}
\]

with some variations in places, such as \(\theta\) for a special kind of ordinal. Let

\[
\omega \quad \Omega
\]

be the least

\[
\text{infinite} \quad \text{uncountable}
\]

ordinal, respectively. Mostly we are concerned with

\[
\text{Ord} = [0, \Omega)
\]

the stretch of countable ordinals, but some uncountable ordinals do appear in Section 7.
1 Cantor

G. Cantor was working on fourier analysis when he came across a problem that led him to invent the ordinals. A variant of that problem now gives the Cantor-Bendixson process in point set topology.

As we will see shortly, we don’t need to know any topology to understand the problem, but it does no harm to have a brief look at the process.

A topological space is a set \( S \) furnished with two families of subsets, its family \( OS \) of open sets and its family \( CS \) of closed sets. These families have to satisfy certain conditions, but we don’t need to know those here.

Given a closed set \( X \in CS \), a point \( s \in X \) is isolated in \( X \) if

\[
X \cap U = \{s\}
\]

for some open set \( U \). For some topological purposes these isolated points can be seen as a flaw. In fact, we say a closed set \( X \) is perfect if it has no isolated points.

What if \( X \in CS \) is not perfect? The obvious thing to do is to throw away the isolated points to obtain the subset

\[
\lim(X) \subseteq X
\]

of limit points of \( X \). The set \( \lim(X) \) is also closed and we hope that it is perfect. Unfortunately, it need not be.

The set \( \lim(X) \) can have some isolated points. There may be points in \( \lim(X) \) which are not isolated in \( X \), but become isolated in \( \lim(X) \). We will see some examples shortly.

What should we now do? Obviously, we should repeat the process of throwing away isolated points.

Given a closed set \( X \in CS \) we let

\[
\lim^0(X) = X \quad \lim^{r+1}(X) = \lim(\lim^r(X))
\]

for each \( r \in \mathbb{N} \). This gives us a descending chain

\[
X = \lim^0(X) \supseteq \lim(X) \supseteq \lim^2(X) \supseteq \cdots \supseteq \lim^r(X) \supseteq \cdots \quad (r \in \mathbb{N})
\]
of closed sets, and it can happen that not one of them is perfect. However, the intersection
\[ \lim^{\omega}(X) = \bigcap \{ \lim^{r}(X) \mid r \in \mathbb{N} \} \]
is a closed set, and surely this is perfect.

Unfortunately not. To obtain the perfect part of a closed set we may have to iterate \( \lim \) for quite a long time. But how do we index such a long iteration? The ordinal were invented precisely to do that job.

We have just seen the standard name, \( \omega \), of the smallest infinite ordinal. Some other infinite ordinal have standard names, and we will meet some later. In fact, this subject is, in part, about ways of naming infinite ordinals.

We can now look at some examples of the CB-process where we don’t need to understand the topological significance.

Let \( S \) be an poset with \( \leq \) as its carried comparison. Let \( \mathcal{O}S = \Upsilon S \quad \mathcal{C}S = \mathcal{L}S \)
be the families of
upper \quad lower
sections of \( S \), respectively. These are the open subsets and the closed subsets of a topological space, but we don’t need to worry about that.

Given a lower section \( X \in \mathcal{L}S \), a node \( s \in S \) is isolated if
\[ X \cap U = \{ s \} \]
for some upper section \( U \in \Upsilon S \). For any such \( U \) we have
\[ \uparrow s \subseteq U \]
and then
\[ X \cap \uparrow s = \{ s \} \]
to show that \( s \) is a maximal member of \( X \). Thus, for this simplified case,
\[ \lim(X) \]
is the lower section of non-maximal members of \( X \).

1.1 EXAMPLES. (a) Consider \( \mathbb{Z} \) as a poset. The whole set \( X = \mathbb{Z} \) does not have a maximal member, and so is perfect. Similarly, the empty set is perfect. Every other lower section \( X \) has a unique maximal member, which is removed to form \( \lim(X) \). In turn, this new set has a maximal member which is removed to form \( \lim^{2}(X) \), and so on. Thus, for each \( X \in \mathcal{L}Z \) we have
\[ \lim^{\omega}(X) = \begin{cases} \mathbb{Z} & \text{if } X = \mathbb{Z} \\ \emptyset & \text{if } X \neq \mathbb{Z} \end{cases} \]
to show that the two extremes are the only perfect subsets.

(b) Consider \( \mathbb{R} \) as a poset. The whole set \( \mathbb{R} \) does not have a maximal member, and neither does \( \emptyset \). These two set are perfect. Each other lower section of \( \mathbb{R} \) has one of the two forms
\[ X = (\neg \infty, a) \quad Y = (\neg \infty, a] \]
for some $a \in \mathbb{R}$. For these we have

$$\lim(X) = X \quad \lim(Y) = X$$

so that perfect parts are achieved quite quickly.

To produce a couple of slightly more exotic examples of the CB-process, we use the notion of a tree. Since there are various kinds of trees, let’s put our trees in a more general context.

1.2 DEFINITION. Let $S$ be a poset with $\leq$ as its carried comparison. For each node $s \in S$ let

$$\downarrow s = \{ x \in S \mid x \leq s \}$$

be the principal lower section generated by $s$.

(a) A tree (of the most general kind) is a poset $S$ such that $\downarrow s$ is linear for each node $s \in S$.

(b) A well-founded tree is a poset $S$ such that $\downarrow s$ is linear and well-founded for each node $s \in S$.

(c) A common tree is a poset $S$ such that $\downarrow s$ is linear and finite for each node $s \in S$.

(d) For us a tree is a common tree $S$ which is also rooted, that is it has a unique bottom node $\bot$, its root.

This kind of tree grows in levels. The root $\bot$ is the only node on level 0. An arbitrary node $s$ is on level $l$ if the linearly ordered set $\downarrow s$ has $1 + l$ members.

Each node $s$ has a set $I(s)$ of immediate successors given by

$$t \in I(s) \iff s < t \text{ with nothing between the two}$$

for $t \in S$. Thus if $s$ lives on level $l$ then $I(s)$ is a collection of nodes on level $l + 1$. The set $I(s)$ could be empty (in which case $s$ is a maximal node of the tree), it could be finite, or it could be very large.

Two particular trees are worth noting.

In the Cantor tree the set $I(s)$ has just two members for each node $s$.

In the Baire tree the set $I(s)$ is a copy of $\mathbb{N}$ for each node $s$.

With this notion of tree it isn’t too hard to prove the following. You need to remember a version of the pigeon hole principle. If you put infinitely many birds into finitely many boxes, then some squawking takes place in at least one of the boxes.

1.3 LEMMA. Let $S$ be a finite splitting tree (such as the Cantor tree). Then for each lower section $X$ of $S$ the set $\lim^\omega(X)$ is perfect.

This shows that for ‘nice’ trees the Cantor-Bendixson properties are reasonably civilized. Wilder behaviour is quite easy to come across.

1.4 LEMMA. Let $S$ be a tree for which each node has infinitely many immediate successors (such as the Baire tree). Then for each node $s \in S$ and countable ordinal $\alpha$, we have

$$\lim^\alpha(X) = \downarrow s$$

for some lower section $X$.

In a nutshell, this is the problem that Cantor came across.
Although the ordinals are part of the harvest of Cantor’s work, a more dramatic windfall was the realization (and understanding by some) that there are different sizes of infinite sets. In particular, the set \( \mathbb{N}, \mathbb{Z}, \) and \( \mathbb{Q} \) are all the same countable size, but \( \mathbb{R} \) is bigger. How big is the continuum?

Hardy’s idea was to try to construct a set of size \( \Omega \) inside a version of the continuum. He took the set

\[
\mathbb{N}' = (\mathbb{N} \rightarrow \mathbb{N})
\]

of all number theoretic functions as (a copy of the) continuum. In fact, he used only strictly monotone functions. His idea was to attach to each ordinal \( \alpha < \Omega \)

\[
\alpha \mapsto h_\alpha
\]

a different function \( h_\alpha \). In other words, he tried to produce an injection

\[
\text{Ord} \rightarrow \mathbb{N}'
\]

by explicit construction. As we will see, this was a bit ambitious.

Hardy works almost entirely by example. I must say that the paper isn’t particularly well written. It reads like something he knocked off in the tea interval at Fenners.

At \( 0 \) he took

\[
h_0 = \text{id}
\]

(the identity function on \( \mathbb{N} \)) as the base function. For each step \( \alpha \mapsto \alpha + 1 \) he set

\[
h_{\alpha+1}(x) = h_\alpha(x + 1)
\]

for each \( x \in \mathbb{N} \). Thus we have

\[
h_r = x + r
\]

for each \( r < \omega \).

What did he do at limit ordinals? He took a diagonalization through the earlier list of functions. For instance, he took

\[
h_\omega(x) = h_x(x) = 2x
\]

at \( \omega \).

The problem that Hardy didn’t address is that of attaching a fundamental sequence to each limit ordinal. Perhaps he didn’t realize that it is a significant problem,

Each countable limit ordinal \( \lambda < \Omega \) is the supremum of a countable set of strictly smaller ordinals, its set of predecessors. Clearly, we don’t need all of \([0, \lambda)\), for any cofinal subset will do.

2.1 Definition. For a countable limit ordinal \( \lambda < \Omega \), a fundamental sequence for \( \lambda \) is (at least) a function

\[
\lambda[\cdot] : \mathbb{N} \rightarrow [0, \lambda)
\]

picking out an \( \omega \)-chain of predecessors.
I say ‘at least’ since we usually require other conditions on the function $\lambda[\cdot]$. For instance,

$$
\lambda[0] < \lambda[1] < \cdots < \lambda[r] < \cdots \quad (r < \omega)
$$

and

$$
\lambda = \bigvee \{\lambda[r] \mid r < \omega\}
$$

seem not unreasonable requests.

How can we describe fundamental sequences for a sufficiently long stretch of countable limit ordinals? By now there are standard methods that go at least as far as $\epsilon_0$, the least critical ordinal. Beyond that things are not so simple. In fact, the problem of producing fundamental sequences is almost the same as the problem of producing ordinal notations.

All this is jumping the gun. So let’s simple assume that we have a selection of fundamental sequences for enough limit ordinals, say up to some ordinal $\Lambda$. I won’t try to pin down $\Lambda$ here (it’s bigger than me).

With this selection the **Hardy hierarchy** is given by

$$
\begin{align*}
\text{for each ordinal } & \alpha \text{ and limit ordinal } \lambda \text{ (both below } \Lambda) \text{ and each } x \in \mathbb{N}, \\
\quad h_0(x) &= x \\
\quad h_{\alpha+1}(x) &= h_\alpha(x + 1) \\
\quad h_\lambda(x) &= h_{\lambda[r]}(x)
\end{align*}
$$

These hierarchies all seem to be defined in a similar fashion.

The names

[Accermann, Robinson, Péter Grzegorczyk, Fast, Slow]

are all used in connection with such hierarchies. (The last two used to be a comedy act.)

Let

$$
\begin{align*}
\mathbb{N}' &= (\mathbb{N} \rightarrow \mathbb{N}) \\
\mathbb{N}'' &= (\mathbb{N}' \rightarrow \mathbb{N}') = (\mathbb{N}' \rightarrow \mathbb{N} \rightarrow \mathbb{N}) \\
\mathbb{N}''' &= (\mathbb{N}'' \rightarrow \mathbb{N}'') = (\mathbb{N}'' \rightarrow \mathbb{N}' \rightarrow \mathbb{N} \rightarrow \mathbb{N})
\end{align*}
$$

be the set of first level, second level, and third level functions on $\mathbb{N}$, respectively. Often we refer to a second level function as an operator.

Each operator

$$
J : \mathbb{N}''
$$

required two inputs

$$
\begin{align*}
f : \mathbb{N}' \\
x : \mathbb{N}
\end{align*}
$$

to produce

$$
Jfx : \mathbb{N}
$$
its eventual output. We are interested in those operators \( J \) which ‘jump up’ the complexity of its first input \( f \) (provided \( f \) is of a suitable kind). We need not try to make that idea precise here.

Given such a jump operator \( J \) its ordinal iterates

\[
(J^\alpha \mid \alpha < \Lambda)
\]

are generated by

\[
\begin{align*}
J^0 &= \text{Id} \\
J^{\alpha+1} &= J \circ J^\alpha \\
J^\lambda f x &= J^{\lambda[x]} f x
\end{align*}
\]

for each ordinal \( \alpha \) and limit ordinal \( \lambda \) (both below \( \Lambda \)), each \( f : \mathbb{N}' \) and each \( x \in \mathbb{N} \). Here \( \text{Id} : \mathbb{N}'' \) is the identity operator.

2.2 EXAMPLE. (a) Let

\[
l : \mathbb{N}' \quad r : \mathbb{N}'
\]

be a fixed pair of functions. Then

\[
J f = l \circ f \circ r
\]

gives an operator \( J \).

With

\[
l = \text{id} \quad r = \text{Suc}
\]

we see that

\[
J^\alpha \text{id}
\]

is the Hardy hierarchy.

With

\[
l = \text{Suc} \quad r = \text{id}
\]

we obtain an operator that generates the Slow growing hierarchy.

(b) With

\[
\begin{align*}
\text{Ackermann} & \quad \text{Robinson, Péter} & \quad \text{Grzegorczyk} \\
J f x &= f^{x+1} x & J f x &= f^{x+1} & J f x &= f^x 2
\end{align*}
\]

(for \( f : \mathbb{N}', x : \mathbb{N} \)) we obtain operators that generate variant of the Fast growing hierarchy. These operators are derived from the work of the indicated people. Each is such that

\[
J^\omega \text{Suc}
\]

is an example of a recursive function that is not primitive recursive.

From this description we begin to see what the central problem is. The construction converts each notation for an ordinal \( \alpha < \Lambda \) into a third level operator

\[
\{\alpha\} : \mathbb{N}'''
\]
where

\[
\begin{align*}
\{0\}J &= Id \\
\{\alpha + 1\}J &= J \circ \{\alpha\}J \\
\{\lambda\}Jf_x &= \{\lambda[x]\}J
\end{align*}
\]

(for each ordinal \(\alpha\) and limit ordinal \(\lambda\), both below \(\Lambda\), and each \(J : \mathbb{N}'' \to \mathbb{N}'', f : \mathbb{N}' \to \mathbb{N}, x \in \mathbb{N}\)). It is the properties of these third level functions that should be investigated.

In [15], Chapters 7,8, and 9, I tried to organize some of the general features of number theoretic hierarchies. I'm not sure I did it very well. Another attempt can be found in [20], Section 19. That section can be read almost in isolation from the rest of [20].

### 3 The critical problem

How might we name all ordinals or a decent stretch of the countable ordinals? We use something similar to the representation of natural numbers by decimal expansion, except we use an expansion to base \(\omega\).

The function

\[\omega^*\]

is inflationary and monotone, that is

\[\alpha \leq \omega^\alpha \quad \alpha \leq \beta \implies \omega^\alpha \leq \omega^\beta\]

for all ordinals \(\alpha, \beta\). Using this we can produce a representation of ordinals.

Consider any ordinal \(\theta\). To avoid a couple of silly exceptions we can suppose \(\theta \neq 0\). We have

\[\theta < \theta + 1 \leq \omega^{\theta + 1}\]

and hence

\[\theta < \omega^{\theta + 1}\]

for at least one ordinal \(\alpha\). Consider the smallest such ordinal. Thus we have

\[\omega^\alpha \leq \theta < \omega^{\alpha + 1}\]

for a unique ordinal \(\alpha\).

With this \(\alpha\) consider the multiples

\[0 = \omega^\alpha \cdot 0 < \omega^\alpha \cdot 1 < \omega^\alpha \cdot 2 < \cdots < \omega^\alpha \cdot m < \cdots\]

for \(m \in \mathbb{N}\). We have

\[\omega^\alpha \cdot 1 = \omega^\alpha \leq \theta < \omega^{\alpha + 1} = \omega^\alpha \cdot \omega = \bigvee \{\omega^\alpha \cdot m \mid m \in \mathbb{N}\}\]

and hence

\[\omega^\alpha \cdot m \leq \theta < \omega^\alpha \cdot (m + 1)\]

for some unique \(m \in \mathbb{N}\).

With this \(\alpha\) and \(m\) consider the ordinals \(\beta\) such that

\[\theta < \omega^\alpha \cdot m + \beta\]
holds. We know there is at least one such ordinal, namely \( \omega^\alpha \). We look at the least such ordinal, and check that it is not a limit ordinal.

By way of contradiction suppose we have a limit ordinal \( \lambda \) with
\[
\theta < \omega^\alpha \cdot m + \lambda
\]
and with
\[
\omega^\alpha \cdot m + \gamma \leq \theta
\]
for all \( \gamma < \lambda \). With this we have
\[
\omega^\alpha \cdot m + \lambda = \bigvee \{\omega^\alpha \cdot m + \gamma \mid \gamma \leq \theta\} \leq \theta
\]
which is the contradiction.

This shows that the least ordinal \( \beta \), as above, is a successor ordinal. Thus we obtain
\[
\omega^\alpha \cdot m + \theta' \leq \theta < \omega^\alpha \cdot m + \theta' + 1
\]
for some unique ordinal \( \theta' \). In fact, since the upper bound is the successor of the lower bound, we have
\[
\theta = \omega^\alpha \cdot m + \theta'
\]
for this unique \( \theta' \). Note also that
\[
\theta' < \theta' + 1 \leq \omega^\alpha \leq \theta
\]
so we have a descending construction \( \theta \mapsto \theta' \) on ordinals.

We iterate this construction. Thus, starting from any (non-zero) ordinal \( \theta \) we set
\[
\begin{align*}
\theta(0) &= \theta \\
\theta(0) &= \omega^{\alpha(0)} \cdot m(0) + \theta(1) \quad \text{where } \theta(1) = \theta(0)' \\
\theta(1) &= \omega^{\alpha(1)} \cdot m(1) + \theta(2) \quad \text{where } \theta(2) = \theta(1)' \\
\theta(2) &= \omega^{\alpha(2)} \cdot m(2) + \theta(3) \quad \text{where } \theta(3) = \theta(2)'
\end{align*}
\]
where the generated ordinals and exponents strictly descend
\[
\begin{align*}
\theta(0) &> \theta(1) > \theta(2) > \theta(3) > \cdots \\
\alpha(0) &> \alpha(1) > \alpha(2) > \alpha(3) > \cdots
\end{align*}
\]
and the multipliers \( m(0), m(1), m(2), m(3), \ldots \) are taken from \( \mathbb{N} \).

This descent must eventually stop, and so we obtain a decomposition
\[
\theta = \omega^{\alpha(0)} \cdot m(0) + \cdots + \omega^{\alpha(s)} \cdots m(s)
\]
the **Cantor normal form** of \( \theta \) to base \( \omega \).

But perhaps you have spotted a problem.

Consider the first step. Starting from \( \theta \) we obtain
\[
\alpha \leq \omega^\alpha \leq \theta < \omega^{\alpha+1}
\]
for some ordinal \( \alpha \). How do we know that \( \alpha < \theta \)?

We don’t! There are ordinal \( \theta \) with \( \omega^\theta = \theta \). For such an ordinal the recursion simply cycles getting nowhere. There is a further discussion of this towards the end of the Appendix.

This shows that if we want to name ordinals in this way then we have to generate critical ordinals. That is the problem we concentrate on for the remainder of these notes.
How might we generate long chains of critical ordinals? This question was addressed by O. Veblen in [21], a paper that (I believe) introduced some of the fundamental ideas of the subject. Some of the notation and terminology has changed since then, but with some patience the paper can be read. At least the first part can be. What is going on in §3 and §4 is a mystery to me. I take it on trust that others have sorted out these ideas, and we will look at that material later.

Veblen’s paper introduced what is now called the Veblen hierarchy. Let’s get a rough idea of what this is before going into the details.

Let \( \text{Ord} \) be the set of countable ordinals, and let

\[
\text{Ord}' = (\text{Ord} \rightarrow \text{Ord})
\]

be the set of ordinal functions. We are interested in certain functions \( f: \text{Ord}' \) which have an abundance of fixed points. We require \( f \) to have arbitrarily large fixed points, each of which should be critical. Furthermore, we require the function that enumerates these fixed points to have similar properties. For the time being let us say that such a function is nice. We will look at a precise definition later.

We start from a nice function \( f \). Quite often this function is taken to be \( \omega \), but it is neater if we let \( f \) be a parameter of the construction.

Using \( f \) we generate a family

\[
(\phi_f \alpha \mid \alpha \in \text{Ord})
\]

of nice functions where the sets of fixed points become sparser and sparser. We set

\[
\begin{align*}
\phi_f 0 &= f \\
\phi_f (\alpha + 1) &= \text{enumeration of fixed points of } \phi_f \alpha \\
\phi_f \lambda &= \text{enumeration of common fixed points of } \phi_f \alpha \text{ for all } \alpha < \lambda
\end{align*}
\]

for ordinals \( \alpha \) and limit ordinals \( \lambda \). The fact that the base function \( f \) is nice ensures that each function \( \phi_f \alpha \) is nice.

4.1 EXAMPLE. Using \( f = \omega \) we have

\[
\begin{align*}
\phi_f 0 \beta &= \omega^\beta \\
\phi_f 1 \beta &= \epsilon_\beta \\
\phi_f 2 0 &= \epsilon_\epsilon_\epsilon_\cdots \\
\phi_f 2 \beta &= \text{the } \beta^{\text{th}} \text{ ordinal } \nu \text{ with } \epsilon_\nu = \nu \\
\phi_f 3 \beta &= \cdots
\end{align*}
\]

and so on. As you can see, we quickly run out of names for the ordinals generated. Part of the idea of the hierarchy is to provide a uniform system of names.

Let’s now look at some of the details.

Sometimes the hierarchy is generated using closed and unbounded sets of ordinals. This is the approach taken in [13] pages 48 and 49, and in [12] pages 349-355.
This approach involves flitting between the sets and their enumerating functions. In fact, it is easier if we deal only with the functions.

Almost always the base function is taken to be normal. It turns out that it is more convenient if we relax that restriction slightly. The following account is based on [17]. I will not give all the proof here, they can be found in [17].

Ordinal functions can be iterated through the ordinals. To show how that is done it is convenient to introduce a more general notion.

4.2 DEFINITION. (a) For each non-empty and countable set \( G \subseteq \text{Ord}' \) of ordinals functions the pointwise supremum

\[
\bigvee G : \text{Ord}'
\]

is the function given by

\[
(\bigvee G)\zeta = \bigvee \{g\zeta \mid g \in G\}
\]

for each input ordinal \( \zeta \).

(b) For each function \( g : \text{Ord}' \), the ordinal iterates

\[
(g^\alpha \mid \alpha \in \text{Ord})
\]

are generated by

\[
g^0 = id \quad g^{\alpha + 1} = g \circ g^\alpha \quad g^\lambda = \bigvee \{g^\alpha \mid \alpha < \lambda\}
\]

for each \( \alpha \in \text{Ord} \) and limit ordinal \( \lambda \in \text{Ord} \). Here \( id \) is the identity function on \( \text{Ord} \), we use composition at successor steps, and pointwise suprema at limit leaps.

(c) We say a class \( S \subseteq \text{Ord}' \) of ordinal functions is smooth if it is closed under composition and pointwise suprema of non-empty, countable subsets \( G \subseteq S \).

We can make the iterates more explicit by giving each component \( g^\alpha \) an input. Thus, for each \( \zeta \in \text{Ord} \) we have

\[
g^0\zeta = \zeta \quad g^{\alpha + 1}\zeta = g(g^\alpha \zeta) \quad g^\lambda\zeta = \bigvee \{g^\alpha \zeta \mid \alpha < \lambda\}
\]

for each \( \alpha \in \text{Ord} \) and limit ordinal \( \lambda \in \text{Ord} \).

Notice that the ordinal iterates of a function taken from a smooth class stay inside that class (where \( id \) might just be an exception).

Making use of the conditions listed in Table 4 of the Appendix, an ordinal function is normal if it is strictly monotone, continuous, and big. (Sometimes the big condition isn’t imposed, but it is technically convenient to do so here.)

It turns out that the class of normal functions is slightly too small, so we enlarge it a bit and give it a companion.

4.3 DEFINITION. An ordinal function \( f \in \text{Ord}' \) is fruitful if it is inflationary, monotone, continuous, and big. Let \( \text{Fruit} \) be the class of fruitful functions.

An ordinal function \( h \in \text{Ord}' \) is helpful if it is strictly inflationary, monotone, and strictly big. Let \( \text{Help} \) be the class of helpful functions.

Each normal function is fruitful. A fruitful function is like a normal function except it can remain dormant for long stretches.

It is not too hard to see that both \( \text{Fruit} \) and \( \text{Help} \) are smooth.
Fruitful function are so called because they have lots of fixed points, and these are critical. Helpful functions enable us to generate fruitful functions. To explain this we use an auxiliary construction.

Given any set $S$ we let

$$S' = (S \rightarrow S)$$

be the set of functions from $S$ to $S$. We have already used this notation with $\text{Ord}$. The idea can be iterated to produce

$$S', S'', S''', S'''', \ldots$$

generations of functions at higher and higher levels. Here we need only $\text{Ord}', \text{Ord}''$, $\text{Ord}'''$, so lets make sure we know what these are.

A general function of type

$$g : \text{Ord}' \quad \text{that is} \quad g : \text{Ord} \rightarrow \text{Ord} \quad \text{requires} \quad 1 \quad \text{inputs to return a value} \quad g\zeta$$

$$G : \text{Ord}'' \quad \text{that is} \quad G : \text{Ord}' \rightarrow \text{Ord}' \quad \text{requires} \quad 2 \quad \text{inputs to return a value} \quad Gg\zeta$$

$$\Gamma : \text{Ord}''' \quad \text{that is} \quad \Gamma : \text{Ord}'' \rightarrow \text{Ord}'' \quad \text{requires} \quad 3 \quad \text{inputs to return a value} \quad \Gamma Gg\zeta$$

which is an ordinal. Here $\zeta$ is an input ordinal. Notice that

$$\text{Ord}' \rightarrow \text{Ord}' \quad \text{expands as} \quad \text{Ord}' \rightarrow \text{Ord} \rightarrow \text{Ord}$$

and

$$\text{Ord}'' \rightarrow \text{Ord}'' \quad \text{expands as} \quad \text{Ord}'' \rightarrow \text{Ord}' \rightarrow \text{Ord}'$$

which is

$$\text{Ord}' \rightarrow \text{Ord} \rightarrow \text{Ord}$$

using the standard currying convention.

Here is a second level function we need.

4.4 DEFINITION. Let $\text{Fix} : \text{Ord}''$ be the second level function given by

$$\text{Fix} f\zeta = f^{\omega}(\zeta + 1)$$

for each function $f : \text{Ord}'$ and ordinal $\zeta$. ■

We have

$$\text{Fix} f\zeta = \sqrt{\{f^r(\zeta + 1) \mid r < \omega\}}$$

and this makes sense for any function $f : \text{Ord}'$. However, we use $\text{Fix}$ only on $f \in \text{Fruit}$. For such $f$ we see that $\text{Fix}$ is a fixed point extractor.

4.5 LEMMA. For each $f \in \text{Fruit}$ and $\zeta \in \text{Ord}$, the value $\text{Fix} f\zeta$ is the least ordinal $\nu$ such that $\zeta < \nu = f\nu$. Furthermore, this value $\nu$ is critical.

Much of the standard material on ordinal notations is about extracting fixed points, so we can see why $\text{Fix}$ might be useful. It seems strange that $\text{Fix}$ is rarely, if ever, named in the literature.

Fruitful and helpful functions work hand in hand.
4.6 LEMMA. (a) For each $f \in \text{Fruit}$ the function $\text{Fix} f$ is helpful.

(b) For each $h \in \text{Help}$ and ordinal $\zeta$, the ordinal function $\alpha \mapsto h^\alpha \zeta$ is normal.

By part (b) of this result, for each helpful function $h$ and ordinal $\zeta$ the function $\alpha \mapsto h^\alpha \zeta$ is fruitful (in fact, normal) and provides an enumeration of a set of critical ordinals. This will be useful if only we can find some helpful functions. That is where part (a) comes into play.

The smallest fruitful function (we are interested in) is $\omega^\bullet$, exponentiation to base $\omega$. If we hit this function with $\text{Fix}$ then we obtain a helpful function. Thus

$$\text{Next} = \text{Fix} \omega^\bullet$$

is helpful and we find that for each ordinal $\zeta$ the output $\text{Next} \zeta$ is the next critical ordinal strictly beyond $\zeta$. In particular, by iterating $\text{Next}$

$$\epsilon_\alpha = \text{Next}^\alpha \epsilon_0 = \text{Next}^{1+\alpha} \omega$$

we generate a long list of critical ordinals. This shows how the gadget allows us to use smaller ordinals to name larger critical ordinals.

Of course, this process will run out of steam at the least ordinal $\nu$ where

$$\nu = \text{Next}^\nu \omega = \epsilon_\nu$$

which is

$$\epsilon_\epsilon \epsilon \ldots$$

or $\phi_{\omega^\bullet 20}$ in Veblen’s system. To generate larger critical ordinals we need more powerful functions. Veblen’s methods organize the production of these in a systematic fashion.

Using helpful function we can give a more compact description of the Veblen hierarchy. To do this we use another second level function which, I believe, has rarely appeared explicitly in the literature.

4.7 DEFINITION. Let $\text{Veb} : \text{Ord}''$ be the second level function given by

$$\text{Veb} f \alpha = h^{1+\alpha} 0 \quad \text{where} \quad h = \text{Fix} f$$

for $f : \text{Ord}'$ and $\alpha \in \text{Ord}$. We call $\text{Veb}$ the Veblen derivative. $\blacksquare$

I won’t try to explain why this function is a derivative, but let’s see what it does.

Let $f : \text{Ord}'$ be an arbitrary fruitful function. By Lemma 4.6(a) the function

$$h = \text{Fix} f$$

is helpful. By Lemma 4.6(b) the function

$$\text{Veb} f$$

is fruitful (in fact, normal). Using Lemma 4.5 we see that $\text{Veb} f$ enumerates, in order, the fixed points of $f$, and each such fixed point is critical.

Now consider some level

$$\phi_f \alpha$$
of the Veblen hierarchy based on \( f \). We assume that \( \phi_f \) is fruitful. The next level is

\[
\phi_f(\alpha + 1) = \text{Veb}(\phi_f \alpha)
\]

and this is fruitful (in fact, normal). After that the next few levels are

\[
\begin{align*}
\phi_f(\alpha + 2) &= \text{Veb}(\phi_f(\alpha + 1)) = \text{Veb}^2(\phi_f \alpha) \\
\phi_f(\alpha + 3) &= \text{Veb}(\phi_f(\alpha + 2)) = \text{Veb}^3(\phi_f \alpha) \\
\phi_f(\alpha + 4) &= \text{Veb}(\phi_f(\alpha + 3)) = \text{Veb}^4(\phi_f \alpha)
\end{align*}
\]

and so on. This suggests that the Veblen hierarchy is generated by iterating the second level function \( \text{Veb} \). To make this precise we lift up a level the ideas of Definition 4.2

4.8 **DEFINITION.** (a) For each non-empty and countable set \( \mathcal{G} \subseteq \text{Ord}'' \) of second level ordinals functions the pointwise supremum

\[
\bigvee \mathcal{G} : \text{Ord}''
\]

is the function given by

\[(\bigvee \mathcal{G})g = \bigvee \{Gg \mid G \in \mathcal{G}\}\]

for each \( g \in \text{Ord}' \).

(b) For each second level function \( G : \text{Ord}'' \), the ordinal iterates

\[(G^\alpha \mid \alpha \in \text{Ord})\]

are generated by

\[
\begin{align*}
G^0 &= \text{Id} \\
G^{\alpha+1} &= G \circ G^\alpha \\
G^\lambda &= \bigvee \{G^\alpha \mid \alpha < \lambda\}
\end{align*}
\]

for each \( \alpha \in \text{Ord} \) and limit ordinal \( \lambda \in \text{Ord} \). Here \( \text{Id} \) is the identity function on \( \text{Ord} \), we use composition at successor steps, and pointwise suprema at limit leaps.

Let’s see what is going on here.

For \( \mathcal{G} \subseteq \text{Ord}'' \) the second level pointwise supremum \( \bigvee \mathcal{G} \) is computed using a first level pointwise supremum. In turn this is computed using an actual supremum in \( \text{Ord} \). Thus for each \( g : \text{Ord}' \) and \( \zeta \in \text{Ord} \) we have

\[(\bigvee \mathcal{G})g\zeta = \bigvee \{Gg \mid G \in \mathcal{G}\}\zeta = \bigvee \{Gg\zeta \mid G \in \mathcal{G}\}\]

which gives an eventual output ordinal.

Similarly, for \( G : \text{Ord}'' \) the ordinal iterates \( G^\bullet \) can be defined by

\[
\begin{align*}
G^0g &= g \\
G^{\alpha+1}g &= G(g^\alpha g) \\
G^\lambda g\zeta &= \bigvee \{G^\alpha g\zeta \mid \alpha < \lambda\}
\end{align*}
\]

for each ordinal \( \alpha \), limit ordinal \( \lambda \), function \( g : \text{Ord}' \), and input ordinal \( \zeta \).

Clearly this idea can be lifted to even higher levels.

With this notion we can form the ordinal iterates

\[
\text{Veb}^\bullet
\]

of the Veblen derivative. Each such second level function can be applied to a fruitful function, and we have more or less proved the following.
4.9 **Lemma.** For each \( f \in \text{Fruit} \) and ordinal \( \alpha \in \text{Ord} \), the function 
\[
Veb^\alpha f
\]
is fruitful.

**Proof.** We proceed by induction on \( \alpha \).
Since \( Veb^0 f = f \), the base case is trivial.
The step case follows since \( Veb \) converts a fruitful function into a fruitful (in fact, normal) function.
The limit leap holds since \( \text{Fruit} \) is smooth. ■

Notice that even if the base function \( f \) is normal, the limit levels \( Veb^\lambda f \) need not be normal, merely fruitful. This is one reason why fruitful functions are more amenable that normal functions. Why do we bother with these limit levels?

4.10 **Lemma.** For each \( f \in \text{Fruit} \) and limit ordinal \( \lambda \), the common fixed points of the family 
\[
\{ Veb^\alpha f \mid \alpha < \lambda \}
\]
are precisely the fixed points of \( Veb^\lambda f \).

**Proof.** Suppose \( \nu \) is a common fixed point of the family, that is 
\[
Veb^\alpha f \nu = \nu
\]
for each \( \alpha < \lambda \). But then 
\[
Veb^\lambda f \nu = \bigvee \{ Veb^\alpha f \nu \mid \alpha < \lambda \} = \nu
\]
to show that \( \nu \) is a fixed point of \( Veb^\lambda f \).

Conversely, suppose that 
\[
Veb^\lambda f \nu = \nu
\]
and consider any \( \alpha < \lambda \). Remembering that \( Veb^\alpha f \) is fruitful, and hence inflationary, we have 
\[
\nu \leq Veb^\alpha f \nu \leq Veb^\lambda f \nu = \nu
\]
to show that \( \nu \) is a common fixed point of the family. ■

With this it doesn’t take too long to prove the following.

4.11 **Theorem.** We have 
\[
\phi_f(1 + \alpha) = Veb^{\alpha+1} f
\]
for each \( f \in \text{Fruit} \) and \( \alpha \in \text{Ord} \).

Notice the flip 
\[
1 + \alpha \quad \alpha + 1
\]
between the indexing ordinals. This kind of thing is not uncommon in this subject. The Veblen hierarchy misses out limit levels and then re-indexes to hide this omission. Thus 
\[
\phi_f(\lambda) = Veb^{\lambda+1} f
\]
for each limit ordinal \( \lambda \). This hiccup passes all the way up to the next limit level.
A revisionist view

The hierarchy \( \phi_f \) generates many critical ordinals. It is not hard to check that

\[
\alpha, \beta \leq \phi_f \alpha \beta
\]

(for all \( \alpha, \beta \in \text{Ord} \)) so that previously generated ordinals can be used to index the next phase of generation. Of course, eventually this hierarchy runs out of steam. We have

\[
\nu = \phi_f \nu 0
\]

for many ordinal \( \nu \). For the case \( f = \omega^* \) these are the strongly critical ordinals. Just like the least critical ordinal \( \varepsilon_0 \), the least strongly critical ordinal \( \Gamma_0 \) is an important marker in several parts of proof theory.

What should we do when we meet such an obstruction? In \S 3 and \S 4 of [21] Veblen does try something but, at least to me, it is far from clear what is going on. The obvious thing to do is to look at the function \( f^+ : \text{Ord}' \) given by

\[
f^+ \alpha = \phi(1 + \alpha)0 = \text{Web}^{\alpha+1} 0
\]

(for each \( \alpha \in \text{Ord} \)). I believe that Veblen did do something like this, but I can’t be sure.

Let’s see if we can make this idea work.

We modify the fixed point extractor \( \text{Fix} : \text{Ord}'' \).

5.1 DEFINITION. Let \([0] : \text{Ord}'' \) be the second level function given by

\[
[0]h = \text{Fix} f \quad \text{where} \quad f \alpha = h^0 \alpha 0 \quad \text{for } \alpha \in \text{Ord}
\]

for each \( h : \text{Ord}' \).

The infix here indicates that \([0] \) is the zeroth of a whole family of functions

\([0], [1], [2], \ldots, [i], \ldots\)

at higher and higher levels. We will meet \([1] \) later, but the others are not needed here.

The use of ‘\( h \)’ indicates that we apply \([0] \) only to helpful functions. For \( h \in \text{Help} \) the auxiliary function \( f \) where

\[
f \alpha = h^\alpha
\]

(for \( \alpha \in \text{Ord} \)) is fruitful by Lemma 4.6(b), and hence

\[
[0]h = \text{Fix} f
\]

is helpful by Lemma 4.6(a). Thus \([0] \) converts helpful functions into helpful functions.

With a bit of fiddling about we can prove the following.

5.2 LEMMA. For each \( h \in \text{Help} \) we have

\[
[0]h \zeta = (\text{the least } \nu \text{ with } \zeta < \nu = h^\nu 0) = (\text{the least } \nu \text{ with } 0 < \nu = h^\nu \zeta)
\]

for each \( \zeta \in \text{Ord} \).
We can also check that \([0]h \in \text{Help}\) for \(h \in \text{Help}\). More generally, using the ordinal iterates of \([0]\) we have the following.

5.3 LEMMA. For each \(h \in \text{Help}\) we have

\([0]^\alpha h \in \text{Help}\)

for each \(\alpha \in \text{Ord}\), and for each \(\zeta \in \text{Ord}\) the function

\(\alpha \mapsto [0]^\alpha h \zeta\)

is normal.

I don’t believe it is worth trying to prove this result in isolation. There is a more general development, as given in [17], which sets the result in a broader context.

Why are we interested in this? Because it leads to an even neater description of the Veblen hierarchy.

5.4 LEMMA. We have

\((\text{Fix} \circ \text{Veb})f = ([0] \circ \text{Fix})f\)

for each \(f \in \text{Fruit}\).

**Proof.** For each \(h \in \text{Help}\) and \(\zeta \in \text{Ord}\) we have

\([0]h \zeta = \text{(least } \nu \text{ with } \zeta < \nu = h^\nu 0)\)

and we will apply this with \(h = \text{Fix} f\).

Also, with this \(h\), we have

\((\text{Fix} \circ \text{Veb})f \zeta = \text{(least } \nu \text{ with } \zeta < \nu = \text{Veb} f \nu) = \text{(least } \nu \text{ with } \zeta < \nu = h^{1+\nu} 0)\)

so that, remembering that \(\nu\) will be infinite, we have the required result.

This is a simple example of a shuffle technique which enables us to reorganize various hierarchies. With the result we can almost prove the following.

5.5 LEMMA. We have

\((\text{Fix} \circ \text{Veb}^\alpha)f = ([0]^\alpha \circ \text{Fix})f\)

for each \(f \in \text{Fruit}\) and \(\alpha \in \text{Ord}\).

**Proof.** We proceed by induction on \(\alpha\). The base case is trivial, and Lemma 5.4 gives the induction step. For the induction leap to a limit ordinal we use an observation about how \(\text{Fix}\) passes across directed pointwise suprema. (This, of course, should have been proved earlier.)

We got into this by worrying about the modification

\(f \mapsto f^+\)
of a fruitful function $f$, as given by
\[ f^\alpha = \text{Veblen}^{\alpha+1} f0 \]
for each $\alpha \in \text{Ord}$. We can now give a neater description of this.

Observe that
\[ \text{Veblen} f0 = \text{Fix} f0 \]
for $f \in \text{Fruit}$. With this and with $h = \text{Fix} f$ we have
\[
\begin{align*}
  f^+ \alpha &= \text{Veblen}^{\alpha+1} f0 \\
  &= \text{Veblen}(\text{Veblen}^\alpha f)0 \\
  &= \text{Fix}(\text{Veblen}^\alpha f)0 \\
  &= [0]^\alpha(\text{Fix} f)0 = [0]^\alpha h0
\end{align*}
\]
for each $\alpha \in \text{Ord}$. Using Lemma 5.3 this shows that $f^+$ is normal.

A similar calculation gives what is, perhaps, a more enlightening description of the Veblen hierarchy. We have
\[
\begin{align*}
  \phi_f(1 + \alpha) \beta &= \text{Veblen}^{\alpha+1} f \beta \\
  &= \text{Veblen}(\text{Veblen}^\alpha f) \beta \\
  &= (\text{Fix}(\text{Veblen}^\alpha f))^{1+\beta}0 \\
  &= ([0]^\alpha(\text{Fix} f))^{1+\beta}0 = ([0]^\alpha h)^{1+\beta}0
\end{align*}
\]
which brings out the different roles played by the two inputs.

To conclude this section let’s have a look at a third level analogue of the second level function $[0]$.

5.6 DEFINITION. Let $[1] : \text{Ord}''$ be the third level function given by
\[
[1]Hh = \text{Fix} f \quad \text{where } f\alpha = H^\alpha h0 \quad \text{(for } \alpha \in \text{Ord})
\]
for each $H : \text{Ord}''$, $h : \text{Ord}'$.

In the grand scheme of things we should show that $[1]$ has certain properties. We don’t need to do that here, provided we take certain things on trust.

Consider the Veblen hierarchy
\[
\alpha, \beta \mapsto ([0]^\alpha h)^{1+\beta}0
\]
based on $f \in \text{Fruit}$ with $h = \text{Fix} f$. We know that this grinds to a halt at the least ordinal $\nu$ with
\[
\nu = [0]^{\nu} h0
\]
which, for the case $f = \omega^*$, is the least strongly critical ordinal. This ordinal is just
\[
[1][0]h0
\]
which suggests how we might continue.

A much longer hierarchy is given by
\[
\alpha, \beta, \gamma \mapsto ((([1][0])^\beta h)^{1+\gamma}0
\]

which eventually grinds to a halt at the least ordinal $\nu$ with

$$\nu = [1]^{\nu}[0]h0$$

which, for the case $f = \omega^\bullet$, is sometimes called the Ackermann ordinal. (Don’t ask me why.)

You can see a theme developing here, but that’s not part of the history.

6 Bachmann and Schütte

In the literature on this subject H. Bachmann is often mentioned, and sometimes his works [2] and [3] are cited. But rarely does anyone say what he did. I used to think that he was one of the originators of the method to be described in Section 7. However, I have been quite firmly told: ‘This is oversimplifying (if not falsifying) the history. Bachmann’s approach was quite different . . . ’.

There is a brief account of his ideas on page 64 of [6]. A mode detailed account is given around page 342 of [9]. There are some discrepancies between the two accounts, but in general the idea is the same.

What is that idea?

Using the base function $f = \omega^\bullet$ the Veblen hierarchy produces a family of functions

$$\phi\alpha : \text{Ord} \rightarrow \text{Ord}$$

indexed by $\alpha < \Omega$. This array has many fixed points

$$\beta = \phi\alpha\beta \quad \alpha = \phi\alpha\beta$$

so there are many obstructions which prevent the use of $\phi$ to generate notations for all ordinals $\alpha < \Omega$. As we have seen, using this $\phi$ we can not get beyond the least ordinal $\nu$ with

$$\nu = \phi\nu0$$

which is $\Gamma_0$.

Bachmann’s idea was to allow the first input of $\phi$ (the $\alpha$ above) to move into the uncountable ordinals $\xi$, and use the new values

$$\phi\xi\beta$$

to enumerate some of the earlier barriers.

For instance, we may set

$$\phi\Omega\beta = \phi\beta0$$

for $\beta < \Omega$, and then continue as before. In other words, using the function

$$f = \phi \cdot 0$$

we have

$$\phi(\Omega + \alpha) = \phi f\alpha$$

in the notation of Section 4.
Of course, to produce $\phi \xi$ for many uncountable ordinals $\xi$ requires a systematic organization of these ordinals. We need not describe that here.

Eventually Bachmann’s method transmogrified into the method described in Section 7.

Around the same time, but without a knowledge of Bachmann’s work, K. Schütte was reorganizing and extending Veblen’s methods. There had been some slightly earlier work by W. Ackermann, but I don’t know that material.

I believe that, in part, Schütte was trying to sort out what is going on in §3 and §4 of [21]. This work appeared in [14] and introduced the ‘Klammersymbole’. There is nothing untranslatable about this word, it simply means bracket.¹

Let me explain the general idea.

A Schütte bracket is an array of ordinals

\[
\begin{pmatrix}
 \zeta & \alpha(1) & \cdots & \alpha(s) \\
 r & i(1) & \cdots & i(s) \\
\end{pmatrix}
\]

where each of $i(1), \ldots, i(s)$ is non-zero. It is usual to assume that $r < i(1) < \cdots < i(s)$ but this ascending condition is not necessary.

Let $f : \text{Ord}'$ be any normal function. In fact, as we will see, we can use any fruitful function. We are asked to view the bracket as an input to $f$ to return

\[
f \left( \begin{pmatrix}
 \zeta & \alpha(1) & \cdots & \alpha(s) \\
 r & i(1) & \cdots & i(s) \\
\end{pmatrix} \right)
\]

an ordinal. I know, but be patient, things will get sorted out.

The evaluation of this output is given be a rather intricate recursion in which the size of the bracket can increase or decrease and, of course, the components can change. To describe this recursion it is convenient to have some terminology (which is my own).

We call

- $\zeta$ the input
- $r$ the rank
- $\alpha(1), \ldots, \alpha(s)$ the exponents
- $i(1), \ldots, i(s)$ the indexes

of the bracket. We obtain the motor from the bracket by removing the rank and the input. In particular, a motor may be empty. We decompose a bracket in this way because, as we will see, the motor, the rank, and the input interact with the function in different ways.

The evaluation rules are given in Table 1. As we can see, for most of the rules the step is determined by the left-most column of the motor. We thus write

\[
\begin{pmatrix}
 \cdot & \alpha & - \\
 0 & i & - \\
\end{pmatrix}
\]

for such a bracket and its associated motor. The two dashes indicate the rest of the motor, and this could be empty.

¹I am grateful to Andrea Schalk for some technical advise in this section.
(A) If any exponent is zero, then that column (exponent and associated index) should be omitted. The effect of this is that we may assume that each exponent is non-zero.

(B) For rank 0 with an empty motor we set $f\left(\frac{\zeta}{0}\right) = f\zeta$ for each $f$ and $\zeta$.

(C) For rank 0 with a non-empty motor the value $f\left(\frac{\zeta}{0} \frac{1 + \alpha}{1 + i} \frac{-}{-}\right)$ is the $\zeta$th common fixed point of the family

$$G = \left\{ f\left(\frac{\cdot}{r} \frac{1 + \xi}{1 + i} \frac{-}{-}\right) \mid r < 1 + i, \xi < \alpha \right\}$$

of functions, some of which use brackets with non-zero rank.

(D) For non-zero rank $1 + r$ we set

$$f\left(\frac{1 + \eta}{1 + r} \frac{-}{-}\right) = f\left(\frac{0}{0} \frac{1 + \eta}{1 + r} \frac{-}{-}\right)$$

for each ordinal $\eta$. (The value $f\left(\frac{0}{1 + r} \frac{-}{-}\right)$ can be taken to be 1.)

Table 1: The condensed evaluation rules for brackets

Let’s look at the rules in more detail.

Each index in a motor must be non-zero, and so has the form $1 + i$ for some ordinal $i$. Rule (A) ensures that we can assume that each exponent is non-zero, and so has the form $1 + \alpha$ for some ordinal $\alpha$. We will make use of these restrictions. Rule (D) is not often stated but is used in practice. This rule increases the size of the motor, and so makes the recursion slightly tricky.

The crucial rule is (C) which asks for the common fixed point of certain ‘earlier’ functions. In other words, (C) is a use of $\text{Veb}$. It becomes clearer if we expand (C) into nine clauses by considering whether or not each of the two ordinal $i$ and $\alpha$ is 0, a successor, or a limit. This leads to 11 clauses (E0, …, E10) as in Table 2. In this table $i$, $\alpha$, and $r$ are arbitrary ordinals, and $\theta, \lambda$ are limit ordinals. In rules (E4, E5, E6) the ordinal $i$ is rigid, but in rules (E7, E8, E9) it varies over $i < \theta$. In rules (E2, E5, E8) the ordinal $\alpha$ is rigid, but in rules (E3, E6, E9) it varies over $\alpha < \lambda$.

Rule (B) converts into clause (E0) and rule (D) converts into clause (E10). Rule (C) converts into clauses (E1, …, E9) depending on the nature of the two ordinals $i$ and $\alpha$. In general, for $0 < k < 10$, clause (Ek) gives

$$f\left(\frac{\cdot}{0} \frac{1 + \beta}{1 + j} \frac{-}{-}\right) = \text{Veb}(\bigvee \mathcal{G})$$

where $\mathcal{G}$ is a certain family of functions $g, g_\alpha$ or $g_i$ as given in the right hand column. The required variation of the $i$ and $\alpha$ are indicated above. Rule (C) asks for the common fixed points of the family $\mathcal{G}$ of (fruitful) functions. It turns out that this family is directed, so the rule can be written as shown. In some instances this family $\mathcal{G}$ has a maximum
Table 2: The expanded evaluation rules for brackets

member $g$, and this has been given in the table.

When written out in this way it becomes clearer how the recursion works, and that it is well founded.

It also becomes clear that we should not view a bracket as an input to a function. (But then, who would?) I won’t go into details here. A full analysis is given in [18], and some of the proofs are a bit finicky. Here I will simply tell you what the results are.

Each motor

\[
\begin{bmatrix}
1 + \alpha(1) & \cdots & 1 + \alpha(s) \\
1 + i(1) & \cdots & 1 + i(s)
\end{bmatrix}
\]

should be viewed as a certain operator

\[
\nabla \begin{bmatrix}
\alpha(1) + 1 & \cdots & \alpha(s) + 1 \\
i(1) + 1 & \cdots & i(s) + 1
\end{bmatrix}
\]
which converts a helpful function $h : \text{Ord}'$ into a helpful function. Notice the flips

$$1 + \bullet \mapsto \bullet + 1$$

here. This is not a mistake. The flips arise in a similar way to those in the original Veblen hierarchy.

After quite a bit of work we find that for each $f \in \text{Fruit}$ we have

$$'f \left(\begin{array}{} \zeta \\ 0 \\ 0 \end{array}\right)' = \left(\nabla \left[\begin{array}{} \zeta \\ 0 \\ 0 \end{array}\right] h\right)^{1+\zeta} 0 \quad \text{where} \quad h = \text{Fix} f$$

for each input $\zeta$, and motor, as indicated. Furthermore, the operator

$$\nabla \left[\begin{array}{} \zeta \end{array}\right]$$

can be described explicitly without any need for recursion.

Recall the operators

$$[0] : \text{Ord}'' \quad [1] : \text{Ord}'''$$

we met in Section 5. Using ordinal iterations we set

$$\nabla \left[\begin{array}{} \alpha + 1 \\ i + 1 \end{array}\right] = ([1][0])^{1+\alpha}$$

for each pair $\alpha, i$ of ordinals. Using these we set

$$\nabla \left[\begin{array}{} \alpha(1) + 1 \\ i(1) + 1 \\ \cdots \\ \alpha(s) + 1 \\ i(s) + 1 \end{array}\right] = \nabla \left[\begin{array}{} \alpha(1) + 1 \\ i(1) + 1 \\ \cdots \\ \alpha(s) + 1 \\ i(s) + 1 \end{array}\right] \circ [0] \circ \cdots \circ \nabla \left[\begin{array}{} \alpha(s) + 1 \\ i(s) + 1 \end{array}\right] \circ [0]$$

for each motor, as indicated.

In many cases the odd [0] scattered around can be omitted. Actually, the definition of

$$\nabla \left[\begin{array}{} \zeta \end{array}\right]$$

is not quite right when $i = 0$ and $\alpha$ is finite, but it is near enough to give the right impression.

There is a lesson to be learned here. All that we are doing is combining [0] and [1] in various ways to produce operators

$$\text{Help} \quad \cdots \quad \text{Help}$$

which convert helpful function into ‘faster’ helpful functions. The Schütte brackets are merely selecting some of these operators and using a rather esoteric recursion to do it. With the base function

$$h = \text{Next} = \text{Fix} \omega^*$$

these ordinals produced never get beyond the Ackermann ordinal mentioned at the end of Section 5. As explained in [16] and [17], this idea can be lifted to higher levels to obtain the ordinals usually generated by the method described in the next section.
7 The prescribed development

So far we have been generating notations from below. By that I mean we first use the natural numbers to index some method of generating a stretch of ordinals. We then use these generated ordinals to index the generation of a longer stretch of ordinals, which we then use to index an even longer stretch, and so on. Of course, every now and then the method runs out of steam, we come up against an obstructing fixed point ordinal. We then have to look for a new trick to continue.

Thus 'from below' means we start at the bottom and climb higher and higher using whatever techniques are available.

The current standard method is quite different in that it generates ordinals from above. By that I mean it uses uncountable ordinals to index the generation of countable ordinals. I know that sounds a bit strange, but we will see how it works in due course.

Although there are precursors, it seems that this method was introduced by W. Buchholz in [4]. A description of idea can be found in [11, 12, 13], and a more general method is describe in [5]. However, these accounts are not easy to read.

Here I will describe on of the simplest variants of the method.

We are going to deal with countable and some uncountable ordinals, so we need to set up some notation.

Let $\Omega$ be the least uncountable ordinal. This is just $\omega_1$, but $\Omega$ is a neater notation for our purposes. We are interested in generating a long stretch of critical ordinals in $[0, \Omega)$ and for that we use some ordinals beyond $\Omega$.

Since $\omega^\Omega = \Omega$
the ordinal $\Omega$ is critical. It is $\epsilon_\Omega$. Let $\Omega^+$ be the next critical beyond $\Omega$. Thus $\Omega^+$ is the limit of $\Omega, \Omega^\Omega, \Omega^{\Omega^\Omega}, \ldots$
formed by repeated use of $\Omega^\ast$. We use the stretch $[0, \Omega^+)$ to index the generation of countable ordinals.

We don’t want to confuse the two kinds or ordinals, so we use

$\alpha, \beta, \gamma, \delta < \Omega \quad \xi, \eta, \zeta < \Omega^+$

with indicated range of variation. We also reserve $\theta$ for a critical ordinal below $\Omega$. Once or twice we also use $\phi$ for such a critical ordinal.

In the first instance we produce two functions

$\psi : [0, \Omega^+) \longrightarrow [0, \Omega)$
$\Psi : [0, \Omega^+) \longrightarrow P[0, \Omega)$

where $\psi$ enumerates a stretch of critical ordinals (below $\Omega$) in ascending order but with many dormant segments, and $\Psi$ helps in the process.

There are several variants of the function $\psi$. Here we look at the simplest version.
7.1 DEFINITION. The outputs of the enumerating function

\[ \psi : [0, \Omega^+) \longrightarrow [0, \Omega) \]

are generated by recursion on the inputs.

Consider \( \xi < \Omega^+ \) and suppose \( \psi \eta \) is known for all \( \eta < \xi \). Consider those sets

\[ \Xi \subseteq [0, \Omega^+) \]

where

\[
\begin{align*}
(E1) \quad 0, \Omega & \in \Xi \\
(E2) \quad \Xi \text{ is closed under } +, \omega^* \\
(E3) \quad \eta < \xi \quad \Longrightarrow \quad \psi \eta \in \Xi
\end{align*}
\]

(for \( \eta < \Omega^+ \)). Let \( \Psi \xi \) be the least such set \( \Xi \). Then \( \psi \xi \) is the least non-member of \( \Psi \xi \).

This is not exactly the clearest definition you have ever seen, is it? We are going to do some work on it to sort out what is going on. Let's begin by looking at what the definition actually says.

Suppose \( \psi \eta \) is known for all \( \eta < \xi \), and we wish to calculate \( \psi \xi \). As set up this evaluation is a search for the set \( \Psi \xi \). We look at those sets \( \Xi \) satisfying \((E1, E2, E3)\). The whole stretch \([0, \Omega^+)\) is one such set. The intersection of any family of such sets is itself such a set. Thus there is a smallest such set, and this is \( \Psi \xi \). We require

\[ [0, \Omega) \not\subseteq \Psi \xi \]

so we may take \( \psi \xi \) to be the smallest ordinal not captured by \( \Psi \xi \).

At this stage it is not obvious that \( \psi \) is total, that it is defined for all inputs \( \xi < \Omega^+ \). We will verify this in due course, but for the time being let us say an ordinal \( \xi < \Omega^+ \) is acceptable if \( \psi \eta \) exists for all \( 0 \leq \eta \leq \xi \).

Notice that 0 is acceptable. For the case \( \xi = 0 \) the condition \((E3)\) is vacuous. So all we need is the smallest set satisfying \((E1, E2)\). A few moment's thought gives \( \psi 0 = \epsilon_0 \).

After that we find that \( \psi 1 = \epsilon_1, \psi 2 = \epsilon_2, \ldots \), but this behaviour does not continue for ever.

We will eventually show that each \( \xi < \Omega^+ \) is acceptable. We also develop a better understanding of the set \( \Psi \xi \) so that we obtain a neater description of this enumerating function which doesn't even mention \( \Psi \).

There is one obvious requirement which we can get out of the way immediately.

7.2 LEMMA. For each acceptable \( \xi < \Omega^+ \) the value \( \psi \xi \) is critical.

Proof. Using the closure property \((E2)\) we have

\[ \alpha, \beta \in \Psi \xi \Longrightarrow \omega^\alpha \cdot m + \beta \in \Psi \xi \]

for all \( \alpha, \beta < \Omega \) and \( m \in \mathbb{N} \).

Let \( \theta = \psi \xi \), so that \([0, \theta) \subseteq \Psi \xi \) with \( \theta \notin \Psi \xi \). In the usual way we have

\[ \theta = \omega^\alpha \cdot m + \beta \]
for some
\[ \alpha \leq \omega^\alpha \leq \theta < \omega^{\alpha+1} \quad \beta < \omega^\alpha \leq \theta \]
and some \( m \in \mathbb{N} \). If \( \alpha < \theta \) then
\[ \theta = \omega^\alpha \cdot m + \beta \in \Psi \]
which is not so. Thus \( \alpha = \theta \) to give
\[ \theta \leq \omega^\theta \leq \theta \]
and hence \( \theta \) is critical. \( \blacksquare \)

This shows that \( \psi \) does indeed produce a collection of criticals. But which ones and how does it do it? The problem with Definition 7.1 is the use of the auxiliary set \( \Psi \). Let's try to get a better understanding of this set.

7.3 DEFINITION. Let \( \theta < \Omega \) be a critical ordinal. A set
\[ \Xi \subseteq [0, \Omega^+] \]
is \( \theta \)-striated if
\[ (S1) \quad 0, \Omega \in \Xi \]
\[ (S2) \quad \Xi \text{ is closed under } +, \omega^\cdot \]
\[ (S3) \quad \eta \in \Xi \left\{ \begin{array}{l} \eta < \omega^\cdot \\ \alpha < \theta \end{array} \right\} \implies \eta + \alpha \in \Xi \]
for all \( \eta < \omega^\cdot \) and \( \alpha < \Omega \).

Let
\[ \Xi(\theta) \]
be the least \( \theta \)-striated set. \( \blacksquare \)

Trivially, the whole stretch \( [0, \Omega^+] \) is \( \theta \)-striated. Also, the intersection of any family of \( \theta \)-striated sets is itself \( \theta \)-striated. Thus \( \Xi(\theta) \), the least \( \theta \)-striated set, does exist.

What does \( \Xi(\theta) \) look like? By \( (S1) \) we have \( 0 \in \Xi(\theta) \), and hence
\[ [0, \theta) \subseteq \Xi(\theta) \]
by \( (S3) \). A few moment’s thought shows that
\[ [0, \theta) \cup [\Omega, \Omega^+] \]
is \( \theta \)-striated, so this set is an upper bound for \( \Xi(\theta) \). In particular, we have
\[ \alpha \in \Xi(\theta) \iff \alpha < \theta \]
for \( \alpha < \Omega \). Later we will get a better picture of the uncountable part of \( \Xi(\theta) \).

Here is why we are interested in \( \theta \)-striation.

7.4 LEMMA. Suppose \( \xi < \Omega^+ \) is acceptable. Then
\[ \Psi \xi = \Xi(\psi \xi) \]
and the value \( \psi \xi \) is the least critical \( \theta < \Omega \) such that
\[ \langle \theta \rangle \left\{ \begin{array}{l} \eta \in \Xi(\theta) \\ \alpha < \theta \end{array} \right\} \implies \psi \eta < \theta \]
for \( \eta < \Omega^+ \).
Proof. We show first that $\Psi_\xi$ is $\theta$-striated. To do that only the closure property $(S3)$ is a problem.

Since $[0, \theta] \subseteq \Psi_\xi$ we have

$$\eta \in \Psi_\xi \land \alpha < \theta \implies \begin{cases} \eta \in \Psi_\xi \\ \alpha \in \Psi_\xi \end{cases} \implies \eta + \alpha \in \Psi_\xi$$

to verify $(S3)$. The second implication holds by $(E2)$.

By the minimality of $\Xi(\theta)$, this gives

$$\Xi(\theta) \subseteq \Psi_\xi$$

so we now need the converse inclusion. To obtain that we verify that the set $\Xi(\theta)$ has $(E1, E2, E3)$. As with the first inclusion, only $(E3)$ is a problem.

A use of $(S3)$ with $\eta = 0$ gives $[0, \theta] \subseteq \Xi(\theta)$. Thus for arbitrary $\eta < \Omega^+$ the first inclusion gives

$$\eta \in \Xi(\theta) \land \eta < \xi \implies \begin{cases} \eta \in \Psi_\xi \\ \eta < \xi \end{cases} \implies \psi \eta \in \Psi_\xi \implies \psi \eta < \theta \implies \psi \eta \in \Xi(\theta)$$

for the required result.

This also shows that $\theta = \psi_\xi$ satisfies the implication $\langle \theta \rangle$.

Finally, consider any critical $\theta$ that satisfies $\langle \theta \rangle$. Then $\Xi(\theta)$ satisfies $(E1, E2, E3)$, so

$$\Psi_\xi \subseteq \Xi(\theta)$$

by the minimality of $\Psi_\xi$. If $\theta < \psi_\xi$ then $\theta \in \Psi_\xi$, so that $\theta \in \Xi(\theta)$, which is not so. Thus $\psi_\xi \leq \theta$.

That’s a bit better, isn’t it. The evaluation of $\psi_\xi$ is a search for a critical $\theta$, the least one such that $\langle \theta \rangle$ holds. The problem with this is that we are not quite sure what $\Xi(\theta)$ looks like. And why is the uncountable part of $\Xi(\theta)$ needed? Also, we still don’t know that $\psi$ is total.

The hypothesis of Lemma 7.4 requires that $\xi$ is acceptable. In fact, another look at the proof gives the following.

7.5 SCHOLIUM. Suppose $\xi < \Omega^+$ and suppose $\psi \eta$ is defined for all $\eta < \xi$. Suppose also there is a critical ordinal $\theta < \Omega$ such that $\langle \theta \rangle$ holds. Then $\Psi_\xi$ is defined (and $\psi_\xi \leq \theta$).

Proof. The condition $\langle \theta \rangle$ ensures that $\Xi(\theta)$ is $\theta$-striated. ■

This could be the step clause in the proof of the totality of $\psi$. However, if you think about it the leap to a limit ordinal is not entirely straightforward. We return to this in due course.

Before that we simplify this characterization a bit more.

Each non-zero ordinal $\eta < \Omega^+$ has a decomposition to base $\Omega$. Thus

$$\eta = \Omega^{\eta(0)} \cdot \alpha(0) + \cdots + \Omega^{\eta(s)} \cdot \alpha(s)$$

for certain exponents

$$\eta > \eta(0) > \cdots > \eta(s)$$
and multipliers $\alpha(i) < \Omega$. The strict comparison $\eta > \eta(0)$ holds since $\Omega^+$ is the next critical beyond $\Omega$. Note that if $\eta < \Omega$ then $s = 0$ with $\eta(0) = 0$ and $\alpha(0) = \eta$, so that $\eta$ is its own $\Omega$-decomposition. This also applies to $\eta = 0$.

For nonzero $\eta < \Omega^+$ each exponent in its $\Omega$-decomposition is strictly smaller. Each such non-zero exponent has its own $\Omega$-decomposition to produce a family of second level exponents and multipliers. We may then decompose each second level exponent, and so on.

In this way we generate
\[
\nabla(\eta)
\]
the full $\Omega$-decomposition of $\eta < \Omega^+$. This is a finite tree decorated by the multipliers that arise in the full decomposition process. The shape of $\nabla(\eta)$ enables us to reconstruct $\eta$ from the decorated tree (for we just do the exponentiations, multiplications, and additions). In particular, the assignment
\[
\eta \mapsto \nabla(\eta)
\]
is injective. Of course, not all decorated trees arise in this way, but that doesn’t matter here.

The tree $\nabla(\eta)$ is finite, and so contains a largest multiplier. Let $|\eta|$ be this countable ordinal. Thus
\[
|0| = 0
\]
with
\[
|\eta| = \max\{|\eta(0)|, \ldots, |(\eta(s)|, \alpha(0), \ldots, \alpha(s)\}
\]
for the 1-step decomposition above.

We use this idea to get a better understanding of $\Xi(\theta)$.

7.6 LEMMA. For each critical $\theta < \Omega$ we have
\[
\eta \in \Xi(\theta) \implies |\eta| < \theta
\]
for all $\eta < \Omega^+$.

Proof. For each critical $\theta < \Omega$ let
\[
\Xi[\theta] \subseteq [0, \Omega^+)
\]
begiven by
\[
\eta \in \Xi[\theta] \iff |\eta| < \theta
\]
for $\eta < \Omega^+$. It suffices to show that $\Xi[\theta]$ is $\theta$-striated, and then use the minimality of $\Xi(\theta)$.

Look at the required conditions $(S1, S2, S3)$. We see that the only problem is to show that $\Xi[\theta]$ is closed under $\omega^\ast$.

Consider any $\eta \in \Xi[\theta]$.

If $\eta < \omega$ then $\eta < \theta$, so that $\omega^\eta < \theta$ (since $\theta$ is critical), to give $\omega^\eta \in \Xi[\theta]$.

Suppose $\Omega \leq \eta$. Then
\[
\eta = \Omega^{n(0)} \cdot \alpha(0) + \cdots + \Omega^{n(s)} \cdot \alpha(s) + \alpha
\]
for non-zero exponents $\eta(0), \ldots, \eta(s) \in \Xi[\theta]$, and multipliers $\alpha(0), \ldots, \alpha(s), \alpha < \theta$. Let

$$\eta(i) = 1 + \zeta(i)$$

for each $0 \leq i \leq s$. We have $\eta(i) \neq \zeta(i)$ only when $\eta(i) < \omega$. In particular, each $\zeta(i)$ is in $\Xi[\theta]$, and hence

$$\zeta = \Omega^{\zeta(0)} \cdot \alpha(0) + \cdots + \Omega^{\zeta(s)} \cdot \alpha(s)$$

is also in $\Xi[\theta]$.

With this we have

$$\eta = \Omega \cdot (\Omega^{\zeta(0)} \cdot \alpha(0) + \cdots + \Omega^{\zeta(s)} \cdot \alpha(s)) + \alpha = \Omega \cdot \zeta + \alpha$$

and hence

$$\omega^\eta = \omega^{\Omega \cdot \zeta + \alpha} = \omega^{\Omega \cdot \zeta} \cdot \omega^\alpha = (\omega^\Omega)^\zeta \cdot \omega^\alpha = \Omega^\zeta \cdot \beta$$

where $\beta = \omega^\alpha < \theta$.

Since

$$|\omega^\eta| = \max\{|\zeta|, \beta\} < \theta$$

we have the required result.

You might think that using the two notations

$$\Xi(\theta) \quad \Xi[\theta]$$

is a bit confusing, and you might be right. But we will stick with it just for the next result.\footnote{Lemma 7.6 gives $\Xi(\theta) \subseteq \Xi[\theta]$ and in [19] I ‘show’ that these two sets are the same. However, part of that proof is nonsense. It would simplify things if these two sets are the same, but I do not know if this is the case.}

The following is the crucial observation.

7.7 LEMMA. For each critical ordinal $\theta < \Omega$ the set

$$\Xi[\theta] = \{\eta < \Omega^+ \mid |\eta| < \theta\}$$

is countable.

Proof. We have

$$\eta \in \Xi[\theta] \iff |\eta| < \theta \iff \nabla(\eta) \subseteq [0, \theta)$$

where $\nabla(\eta)$ is a certain finite tree decorated with countable ordinals. The right hand inclusion means that these decorating ordinals are all in $[0, \theta)$. The assignment

$$\eta \mapsto \nabla(\eta)$$

is injective.

There are only countably many ordinals $\alpha < \theta$. Thus each finite tree can be decorated with such ordinals in just countably many ways.

There are only countably many finite trees.
Thus there are only countably many tree decorated with ordinals $\alpha < \theta$. Each such tree puts at most one ordinal into $\Xi[\theta]$, and hence this set is countable.

This is the crucial result that ensure that $\psi$ is total. In fact, there is a more general result.

**7.8 Lemma.** Consider any $\xi < \Omega^+$ and suppose
\[ \varphi : [0, \xi) \rightarrow [0, \Omega) \]
is any function. Then there is at least one critical ordinal $\theta < \Omega$ such that
\[ |\eta| < \theta \quad \eta < \xi \}
\[ \varphi \eta < \theta \]
for all $\eta < \Omega^+$.

**Proof.** Since $|\xi| < \Omega$, we have $|\xi| < \phi$ for a final section of critical ordinals $\phi < \Omega$. We set up an inflationary function
\[ \phi \mapsto \phi' \]
on these critical ordinals.

Given such a critical $\phi$, Lemma 7.7 ensures that the set
\[ \Phi = \{ \phi \} \cup \{ \varphi \eta \mid |\eta| < \phi, \eta < \xi \} \]
is countable. Thus
\[ \bigvee \Phi < \Omega \]
and hence
\[ \phi < \bigvee \Phi < \phi' < \Omega \]
for some critical ordinal $\phi'$. Observe that
\[ |\eta| < \phi \quad \eta < \xi \}
\[ \varphi \eta \in \Phi \implies \varphi \eta < \phi' \]
for $\eta < \Omega^+$.

Starting with any critical $\phi$ with $|\xi| < \phi$, we iterate this construction
\[ \phi_0 = \phi \quad \phi_{r+1}' \]
to produce an ascending chain
\[ \phi = \phi_0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq \phi_r \leq \cdots \quad (r < \omega) \]
of criticals. The supremum
\[ \theta = \bigvee \{ \phi_r \mid r < \omega \} \]
is critical, and $\theta < \Omega$ on cardinality grounds.

For each $\eta < \Omega^+$ we have
\[ |\eta| < \theta \quad \eta < \xi \}
\[ (\exists r) \left( |\eta| \in \phi_r \right) \implies (\exists r) [\varphi \eta < \phi_{r+1}] \implies \varphi \eta < \theta \]
for the required result.

With this we have the result we have been waiting for.
7.9 **THEOREM.** The enumerating function \( \psi \) is total.

**Proof.** By way of contradiction suppose \( \psi \) is not total. There is at least one \( \xi < \Omega^+ \) where \( \psi \xi \) is not defined. We look at the smallest such \( \xi \).

With this \( \xi \) we show there is a critical \( \theta < \Omega \) with

\[
\langle \theta \rangle \quad \eta \in \Xi(\theta) \quad \alpha < \theta \quad \Rightarrow \quad \psi \eta < \theta
\]

for \( \eta < \Omega^+ \). A use of Scholium 7.5 then shows that \( \psi \xi \) is defined.

We certainly have a function

\[
\psi : [0, \xi) \rightarrow [0, \Omega)
\]

(by the minimality of \( \xi \)). Thus Lemma 7.8 gives at least one critical \( \theta \) with

\[
\left\{ \begin{array}{l}
|\eta| < \theta \\
\eta < \xi
\end{array} \right. \quad \Rightarrow \quad \psi \eta < \theta
\]

(for \( \eta < \Omega^+ \)). A use of Lemma 7.6 gives \( \langle \theta \rangle \), and hence we have the required result. ■

We can now quite quickly obtain a simpler, and neater, description of \( \psi \). To do that we first set up what could be a different function.

7.10 **DEFINITION.** The outputs of the enumerating function

\[
\varphi : [0, \Omega^+) \rightarrow [0, \Omega)
\]

are generated by recursion on the inputs.

Consider \( \xi < \Omega^+ \) and suppose \( \psi \eta \) is known for all \( \eta < \xi \). Then \( \varphi \xi \) is the least critical ordinal \( \theta < \Omega \) with

\[
\left\{ \begin{array}{l}
|\eta| < \theta \\
\eta < \xi
\end{array} \right. \quad \Rightarrow \quad \varphi \eta < \theta
\]

for all \( \eta < \Omega^+ \). ■

By Lemma 7.8 this function \( \varphi \) is total. More importantly, its the function we have been worrying about all along.

7.11 **THEOREM.** The two enumerating functions \( \psi \) and \( \varphi \) are the same.

**Proof.** We show first that

\[
\psi \xi \leq \varphi \xi
\]

by a progressive induction on \( \xi < \Omega^+ \). Thus, with \( \theta = \varphi \xi \), a use of Lemma 7.6 gives

\[
\left\{ \begin{array}{l}
\eta \in \Xi(\theta) \\
\eta < \xi
\end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l}
|\eta| < \theta \\
\eta < \xi
\end{array} \right. \quad \Rightarrow \quad \psi \xi \leq \varphi \eta < \theta
\]

and hence \( \psi \xi \leq \theta = \varphi \xi \) by Scholium 7.5.
Now, by way of contradiction, suppose $\varphi \xi < \psi \xi$ for some $\xi < \Omega^+$. Let $\xi$ be the least such $\xi$, and let $\theta = \varphi \xi$. As above we have

$$\eta \in \Xi(\theta) \quad \eta < \xi \implies \psi \xi \leq \varphi \eta < \theta$$

using the minimality of $\xi$. This gives

$$\psi \xi \leq \theta = \varphi \xi$$

which is the contradiction. ■

I think we can agree that Definition 7.10 is much easier to understand than Definition 7.1. Furthermore, the properties of $\varphi$ can be developed quite quickly. For instance, almost trivially $\varphi$ is monotone. Various properties of $\varphi$ are developed in [16], so we need not go into those here. However, there is one aspect that should be mentioned.

For $\xi < \Omega^+$ think of comparing the two countable ordinals $|\xi|$ and $\varphi \xi$. We say

$$\xi \text{ is tame if } |\xi| < \varphi \xi \quad \xi \text{ is wild if } \varphi \leq |\xi|$$

and we find that

$$\varphi \xi = \begin{cases} \text{Next}(\varphi \xi) & \text{if } \xi \text{ is tame} \\ \varphi \xi & \text{if } \xi \text{ is wild} \end{cases}$$

where, recall, $\text{Next}$ produces the next critical ordinal.

We find that as $\xi$ increases this ordinal is tame for a while, and then becomes wild at some limit ordinal. It can be wild for quite a long time, and the value $\varphi \xi$ remains constant until, at some later stage, the input again becomes tame. The nature of these wild stretches is not entirely clear. An earlier version of [16] contained some information about this, but the referee seemed not to understand what was going on. (But then again, neither do I.)

To conclude let me say a few words about the use of the uncountable ordinals up to $\Omega^+$.

This is a red herring (which, partly due to the EEC fishing polices, have almost disappeared).

What is being used is a hidden type structure. This is partly exposed by the full $\Omega$-decomposition tree. Some information about this is given in [16], but there is still much work to be done.

An appendix on ordinals

This subject, Ordinal notations, is concerned with ways of naming – describing – ordinals as iteration gadgets. As we see in Section 2, there is more to an iteration than just an ordinal. Roughly speaking, an iteration is an ordinal together with a selection of its internal structure, such as various fundamental sequences. (Even this description is a little short of the mark. As iteration gadgets $\omega$ and $1 + \omega$ are not the same.) However, for this short course we needn’t worry about that aspect.

We are concerned here with ways of generating countable ordinals. And in the grand scheme of things, only small countable ordinals. In Section 7 we also need some uncountable ordinals, but not very many of these.
I assume you know something about ordinals, so I won’t present a watertight development of these gadgets. I will simply gather together the relevant facts.

Each ordinal is one of three kinds.

(base) 0 the least ordinal
(step) \( \alpha + 1 \) the successor of \( \alpha \), the next ordinal after \( \alpha \)
(leap) \( \lambda \) a limit ordinal, the supremum of all \( \alpha < \lambda \)

I have slipped in some notation here. Mostly we write

\[ \alpha, \beta, \gamma, \delta, \ldots \] for arbitrary ordinals
\[ \lambda \] for limit ordinals

with some variations in places.

The ordinals are linearly ordered, in fact well-ordered. The whole family is full of gaps. There is nothing strictly between an ordinal \( \alpha \) and its successor.

\[ \alpha \leq \beta \leq \alpha + 1 \implies \alpha = \beta \text{ or } \beta = \alpha + 1 \]

Each set \( A \) of ordinals has a supremum \( \bigvee A \), the least ordinal \( \gamma \) with \( \alpha \leq \gamma \) for all \( \alpha \in A \). If \( A \) has a maximum member then this is \( \bigvee A \), otherwise \( \bigvee A \) is a limit ordinal. These two cases are not exclusive.

The first few ordinals are

\[ 0, 1, 2, 3, \ldots \]

a copy of \( \mathbb{N} \) as a linearly ordered set. Immediately after that we have

\[ \omega \]

the least infinite ordinal, and the least limit ordinal. We then have

\[ \omega, \omega + 1, \omega + 2, \ldots, \omega + \omega = \omega \cdot 2 \]
\[ \omega \cdot 2, \omega \cdot 2 + 1, \omega \cdot 2 + 2, \ldots, \omega \cdot 2 + \omega = \omega \cdot 3 \]
\[ \omega \cdot 3, \ldots \]

followed by

\[ \omega \cdot \omega = \omega^2, \ldots, \omega^3, \ldots, \omega^\omega, \ldots \]

and so on. Using this idea we can cobble together notations for the first few ordinals, but it doesn’t get very far. We will see just how far shortly.

Beyond all the countable ordinals we have

\[ \Omega \]

the least uncountable ordinal. This is just \( \omega_1 \), but the notation \( \Omega \) is more convenient here.

The well-ordering of the ordinals gives us a principle of proof by induction. The two most common variants are as follows.
**addition** \(\gamma + \bullet\) \hspace{1cm} **multiplication** \(\gamma \times \bullet\) \hspace{1cm} **exponentiation** \(\gamma^\bullet\)

(base) \(\gamma + 0 = \gamma\) \hspace{1cm} (step) \(\gamma + \alpha' = (\gamma + \alpha) + 1\) \hspace{1cm} (leap) \(\gamma + \lambda = \bigvee \{\gamma + \alpha | \alpha < \lambda\}\)

\(\gamma \times 0 = 0\) \hspace{1cm} \(\gamma \times \alpha' = (\gamma \times \alpha) + \gamma\) \hspace{1cm} \(\gamma \times \lambda = \bigvee \{\gamma \times \alpha | \alpha < \lambda\}\)

\(\gamma^0 = 1\) \hspace{1cm} \(\gamma^\alpha' = \gamma^\alpha \times \gamma\) \hspace{1cm} \(\gamma^\lambda = \bigvee \{\gamma^\alpha | \alpha < \lambda\}\)

Here we have written \(\alpha'\) for \(\alpha + 1\) to get the array on the page.

<table>
<thead>
<tr>
<th>Table 3: The arithmetic or ordinals</th>
</tr>
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**Step-wise induction.** Suppose \(P(\cdot)\) is a property of ordinals, and suppose

\[
\begin{align*}
\text{(base)} & \quad P(0) \\
\text{(step)} & \quad (\forall \alpha)[P(\alpha) \implies P(\alpha + 1)] \\
\text{(leap)} & \quad (\forall \lambda)[(\forall \alpha < \lambda)P(\alpha) \implies P(\lambda)]
\end{align*}
\]

are known. Then

\[(\forall \alpha)P(\alpha)\]

holds.

**Progressive induction.** Suppose \(P(\cdot)\) is a property of ordinals, and suppose

\[(\forall \alpha)[(\forall \beta < \alpha)P(\beta) \implies P(\alpha)]\]

is known. Then

\[(\forall \alpha)P(\alpha)\]

holds.

Both variants are used here. Sometimes we use the progressive version in the contrapositive form as a proof by contradiction.

Alongside these principles of proof by induction, there are principles of construction by recursion. We will see something of this shortly, and more in the body of the notes.

The arithmetic of ordinals is concerned with the

**addition** \(\gamma, \alpha \mapsto \gamma + \alpha\) \hspace{1cm} **multiplication** \(\gamma, \alpha \mapsto \gamma \times \alpha\) \hspace{1cm} **exponentiation** \(\gamma, \alpha \mapsto \gamma^\alpha\)

of ordinals. Each of these operations can be generated by a step-wise recursion on \(\alpha\) with \(\gamma\) fixed. The rules are given in Table 3.

In the table multiplication is written as on the left

\(\gamma \times \alpha\) \hspace{1cm} \(\gamma \cdot \alpha\)

but more often than not the right hand notation is used. We follow that custom here.

These operation can catch out the unwary.
A function $g : \text{Ord}'$ is, respectively

(i) inflationary if $\alpha \leq g\alpha$

(ii) strictly inflationary if $\alpha < g\alpha$

(m) monotone if $\beta \leq \alpha \Rightarrow g\beta \leq g\alpha$

(sm) strictly monotone if $\beta < \alpha \Rightarrow g\beta < g\alpha$

(c) continuous if $g(\bigvee A) = \bigvee g[A]$

(b) big if $\omega^\alpha \leq g\alpha$ (except possibly for $\alpha = 0$)

(sb) strictly big if $g\alpha$ is critical

for all ordinals $\alpha, \beta$, and each non-empty countable set $A$ of ordinals.

Table 4: Some properties of ordinal functions

<table>
<thead>
<tr>
<th>Property</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>inflationary if $\alpha \leq g\alpha$</td>
</tr>
<tr>
<td>(ii)</td>
<td>strictly inflationary if $\alpha &lt; g\alpha$</td>
</tr>
<tr>
<td>(m)</td>
<td>monotone if $\beta \leq \alpha \Rightarrow g\beta \leq g\alpha$</td>
</tr>
<tr>
<td>(sm)</td>
<td>strictly monotone if $\beta &lt; \alpha \Rightarrow g\beta &lt; g\alpha$</td>
</tr>
<tr>
<td>(c)</td>
<td>continuous if $g(\bigvee A) = \bigvee g[A]$</td>
</tr>
<tr>
<td>(b)</td>
<td>big if $\omega^\alpha \leq g\alpha$ (except possibly for $\alpha = 0$)</td>
</tr>
<tr>
<td>(sb)</td>
<td>strictly big if $g\alpha$ is critical</td>
</tr>
</tbody>
</table>

Both addition and multiplication are associative, but neither is commutative. There are also some strange equalities. For instance, we have

$$1 + \omega = \omega$$

as ordinals. (However, if we seriously consider how we might capture iteration, then $1 + \omega$ and $\omega$ are not the same. In fact, there are many different versions of $\omega$.)

We do have

$$\gamma \cdot (\beta + \alpha) = \gamma \cdot \beta + \gamma \cdot \alpha$$

but the other distributive law doesn’t hold. We do have

$$\gamma^\beta \cdot \gamma^\alpha = \gamma^{\beta + \alpha}$$

which should be a bit of a relief.

Let

$$\text{Ord} = [0, \Omega)$$

be the set of countable ordinals, and let

$$\text{Ord}' = (\text{Ord} \longrightarrow \text{Ord})$$

be the family of functions on $\text{Ord}$. We use various properties of such functions $g : \text{Ord}'$. These are listed in Table 4.

These are standard properties except for big and strictly big. It is technically convenient to impose these restriction at times. (The notion of a critical ordinal is discussed shortly.)

Notice that

$$\gamma + \cdot \gamma \times \cdot \gamma^*$$

are strictly inflationary, strictly monotone, and continuous (if we exclude $\gamma = 0$).

In the first instance the arithmetic of ordinals is a bit like a non-commutative version of the arithmetic of natural numbers. Except that we soon come up against a fundamental obstruction.

For each ordinal $\alpha$ we have

$$\alpha \leq \omega^\alpha$$
and, in fact,

\[ \alpha < \omega^\omega \]

for the ordinals we first get to know. This doesn’t last.

7.12 DEFINITION. An ordinal \( \theta \) is critical if

\[ \omega^\theta = \theta \]

holds. \[\square\]

For instance, we have

\[ \alpha \leq \omega^\alpha \leq \Omega \]

for all \( \alpha < \Omega \). Thus

\[ \omega^\Omega = \bigvee \{ \omega^\alpha \mid \alpha < \omega \} \leq \Omega \]

to show that

\[ \omega^\Omega = \Omega \]

and hence \( \Omega \) is critical. In itself this is no big problem, except there are much smaller critical ordinals.

Given an ordinal \( \delta \) we set

\[ \delta[0] = \delta + 1 \quad \delta[r+1] = \omega^{\delta[r]} \]

for each \( r \in \mathbb{N} \). This generates an ascending \( \omega \)-chain

\[ \delta < \delta[0] \leq \delta[1] \leq \cdots \leq \delta[r] \leq \cdots \ (r < \omega) \]

of ordinals. We set

\[ \text{Next} \delta = \bigvee \{ \delta[r] \mid r < \omega \} \]

to obtain the limit of this chain.

7.13 LEMMA. For each ordinal \( \delta \), the ordinal \( \text{Next} \delta \) is the next critical ordinal beyond \( \delta \), the least critical ordinal strictly larger than \( \delta \).

Proof. We certainly have

\[ \delta < \delta[0] \leq \text{Next} \delta \]

to show that \( \text{Next} \delta \) is strictly larger than \( \delta \). Also

\[ \omega^{\text{Next} \delta} = \bigvee \{ \omega^{\delta[r]} \mid r < \omega \} = \bigvee \{ \delta[r+1] \mid r < \omega \} = \text{Next} \delta \]

to show that \( \text{Next} \delta \) is critical.

Consider any critical ordinal \( \epsilon \) with \( \delta < \epsilon \). We show that \( \delta[r] \leq \epsilon \) for each \( r \in \mathbb{N} \), and hence \( \text{Next} \delta \leq \epsilon \). We proceed by induction on \( r \).

We have \( \delta < \epsilon \), so that

\[ \delta[0] = \delta + 1 \leq \epsilon \]

for the base case, \( r = 0 \). Assuming

\[ \delta[r] \leq \epsilon \]


we have

$$\delta[r+1] = \omega^{\delta[r]} \leq \omega^r = \epsilon$$

to give the induction step, $r \mapsto r + 1$. ■

For instance, the limit of the ascending chain

$$1, \omega, \omega^\omega, \omega^{\omega^\omega}, \ldots$$

is the least critical ordinal. This is

$$\epsilon_0$$

and ordinal that has a fundamental significance in various parts of proof theory. (Sometimes, without a hint of sarcasm, critical ordinals are called epsilon number, and sometimes $\varepsilon$ is used as the generic symbol.)

More generally, we set

$$\epsilon_r = \text{Next}^{1+r}0 = \text{Next}^{\epsilon_0} (r < \omega)$$

to generate

$$\epsilon_0, \epsilon_1, \epsilon_2, \ldots$$

the first few critical ordinals. What happens beyond that?

7.14 LEMMA. The supremum of any (non-empty) set of critical ordinals is itself critical.

Proof. Let $\Theta$ be any set of critical ordinals, and let

$$\phi = \bigvee \Theta$$

be the supremum. For each $\theta \in \Theta$ we have

$$\omega^\theta = \theta \leq \phi$$

so that

$$\omega^\phi = \omega^{\bigvee \Theta} = \bigvee \{\omega^\theta \mid \theta \in \Theta\} \leq \phi$$

to show that

$$\omega^\phi = \phi$$

and hence $\phi$ is critical. ■

The next critical beyond

$$(\epsilon_r \mid r < \omega)$$

is

$$\epsilon_\omega = \bigvee \{\epsilon_r \mid r < \omega\}$$

and this is followed by

$$\epsilon_\omega, \epsilon_{\omega+1}, \epsilon_{\omega^2}, \ldots, \epsilon_{\omega+1}, \epsilon_{\omega^2}, \ldots, \epsilon_{\omega^2}, \ldots$$

where we use previously ordinals to generate the next stretch of critical ordinals. Eventually we get to

$$\epsilon_{\epsilon_0}$$

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followed by
\[ \epsilon_{\epsilon_0}, \epsilon_{\epsilon_{\epsilon_0}}, \epsilon_{\epsilon_{\epsilon_{\epsilon_0}}}, \ldots \]
until we arrive at
\[ \epsilon_{\epsilon_{\cdot \cdot \cdot}} \]
a critical ordinal which is its own index. What do we do now?

This is what the subject of ordinal notations is about.

Each (non-zero) countable ordinal \( \alpha < \Omega \) can be decomposed as
\[ \alpha = \omega^{\alpha(0)} \cdot m(0) + \cdots + \omega^{\alpha(s)} \cdot m(s) \]
for a unique list
\[ \alpha \geq \alpha(0) > \cdots > \alpha(s) \]
of countable exponents, and an associated list
\[ m(0), \ldots, m(s) \]
of multipliers from \( \mathbb{N} \). This is the \( \omega \)-decomposition of \( \alpha \).

Notice that the exponents strictly descend, except possibly at the first step. If \( \alpha \) is critical then
\[ \alpha = \omega^\alpha \]
and the decomposition gets nowhere. This shows why criticals are important. The problem of naming ordinals is essentially the problem of generating criticals.

We have the following result.

7.15 THEOREM. Let \( \delta \) be any ordinal and suppose we can name all ordinals \( \beta \leq \delta \). hen, using the \( \omega \)-decomposition we can name all ordinals \( \alpha < \text{Next}\delta \)

In Section 7 we need a variant of this idea to handle certain uncountable ordinals. Let
\[ \Omega^+ = \text{Next}\Omega \]
the next critical beyond \( \Omega \). Thus \( \Omega^+ \) is the limit of
\[ \Omega, \Omega^\Omega, \Omega^{\Omega^\Omega}, \Omega^{\Omega^{\Omega^\Omega}}, \ldots \]
formed by repeated use of \( \Omega^* \). We use ordinals \( \eta < \Omega^+ \).

Each (non-zero) ordinal \( \eta < \Omega^+ \) has a decomposition to base \( \Omega \). Thus
\[ \eta = \Omega^{\eta(0)} \cdot \alpha(0) + \cdots + \Omega^{\eta(s)} \cdot \alpha(s) \]
for certain exponents
\[ \eta > \eta(0) > \cdots > \eta(s) \]
and multipliers \( \alpha(i) < \Omega \). Here we do have a strict comparison \( \eta > \eta(0) \) since \( \Omega^+ \) is the next critical beyond \( \Omega \). Notice that here we use \textit{countable} multipliers rather than finite multipliers. In particular, if \( \eta < \Omega \) then \( s = 0 \) with \( \eta(0) = 0 \) and \( \alpha(0) = \eta \), so that \( \eta \) is its own \( \Omega \)-decomposition.
An appendix on odds and ends

In this last part I gather together some stuff that, in a longer course, would appear in the main body.

Ordinal iterates

Given a function

\[ f : S \longrightarrow S \]

on a set \( S \) we easily generate its finite iterates by

\[ f^0 = id \quad f^{r+1} = f \circ f^r \quad (r < \omega) \]

where \( id \) is the identity function on \( S \). In some situations these can be extended to ordinal iterates

\[ f^0 = id \quad f^{\alpha+1} = f \circ f^\alpha \quad f^\lambda = ??? \]

for ordinals \( \alpha \) and limit ordinals \( \lambda \). As indicated, the problem is passing across a limit leap. We look at two ways this can be done in the appropriate circumstances.

Let

\[ S = \mathbb{N} \quad \text{or} \quad S = \text{Ord} \]

and consider functions at various levels.

1. \( f : S' = S \longrightarrow S \)
2. \( F : S'' = S' \longrightarrow S \longrightarrow S \)
3. \( \phi : S''' = S'' \longrightarrow S' \longrightarrow S \longrightarrow S \)
4. \( \Phi : S'''' = S''' \longrightarrow S'' \longrightarrow S' \longrightarrow S \longrightarrow S \)

For these two cases we can pass across the limit as follows.

\[
\begin{array}{llll}
\mathbb{N} & \text{Ord} \\
\text{Don’t try} & (1) & f^\lambda \zeta = \bigvee \{ f^\alpha \zeta \mid \alpha < \lambda \} \\
F^\lambda f x = F^\lambda \alpha f x & (2) & F^\lambda f \zeta = \bigvee \{ F^\alpha f \zeta \mid \alpha < \lambda \} \\
\phi^\lambda F f x = \phi^\lambda \alpha F f x & (3) & \phi^\lambda F f \zeta = \bigvee \{ \phi^\alpha F f \zeta \mid \alpha < \lambda \} \\
\Phi^\lambda \phi F f x = \Phi^\lambda \alpha \phi F f x & (4) & \Phi^\lambda \phi F f \zeta = \bigvee \{ \Phi^\alpha \phi F f \zeta \mid \alpha < \lambda \} \\
\end{array}
\]

for \( x \in \mathbb{N} \) for \( \zeta \in \text{Ord} \)

Clearly these two tricks can be lifted up to all finite levels. The method on the left is the basis of many number theoretic hierarchies, and that on the right is used to generate ordinal notations.
Number theoretic functions hierarchies

In [7] Exercise 6, TF introduces a 2-paced function

\[ A : \mathbb{N}^2 \rightarrow \mathbb{N} \]

by the 2-recursion

\[
A(0, y) = y + 1
\]
\[
A(x', 0) = A(x, 1)
\]
\[
A(x', y') = A(x, A(x'y))
\]

for \( x, y \in \mathbb{N} \). Here \((\cdot)'\) is the successor function. Here refers to this as Ackermann’s function. Admittedly this exercise is nicked from an obscure exam paper, but this is function devised, independently, by R.M. Robinson and R. Péter to simplify Ackermann’s example of a 2-recursive function that is not 1-recursive (primitive recursive). Ackermann used a 3-placed function.

This and similar functions can be placed in a more general context.

Consider the R-P jump operator

\[ R : \mathbb{N}'' \]

(as in Example 2.2(b)) given by

\[ Rf y = f^{1+y}1 \]

for each \( f : \mathbb{N}' \) and \( y \in \mathbb{N} \). By iterating \( R \) each suitable function \( f \) generates a hierarchy of functions of increasing rate of growth. Thus

\[
f_0 = f
\]
\[
f_{\alpha+1} = Rf_\alpha \quad \text{that is } f_\alpha = R^\alpha f
\]
\[
f_\lambda y = f_{\lambda[y]} y
\]

for each ordinal \( \alpha \), limit ordinal \( \lambda \), and \( y \in \mathbb{N} \).

As an exercise you should show that

\[ A(x, y) = R^x S y \]

where \( S \) is the successor function. As a second exercise you might calculate

\[ A(4, 2) \]

and show that \( A(4, 3) \) is bigger than any known measure of the universe.

Starting with the successor function \( S \) the hierarchy

\[ S_\alpha | \alpha < \epsilon_0 \]

stratifies the functions definable in peano arithmetic. As observed by Gentzen, there is a tight connection between the required quantifier depth and the stacking height of the corresponding ordinal.

This kind of result often uses the Grzegorczyk hierarchy. Consider the G jump operator

\[ G : \mathbb{N}'' \]
(as in Example 2.2(b)) given by
\[ Gf y = f^{\circ 2} \]
for each \( f : \mathbb{N}' \) and \( y \in \mathbb{N} \). The hierarchy
\[ \alpha \mapsto G^\alpha S \]
has the same complexity as the R-P hierarchy at each ordinal. The function
\[ B(x, y) = G^x S y \]
is given by
\[
\begin{align*}
B(0, y) &= y + 1 \\
B(x', 0) &= 2 \\
B(x', y') &= B(x, B(x'y))
\end{align*}
\]
for \( x, y \in \mathbb{N} \).

For the record, the original Ackermann 3-placed function is given by
\[
A(0, y, z) = y + z \\
A(1, y, z) = y \times z \\
A(2, y, z) = z^y \\
A(x', 0, z) = 0, 1, z (?) \\
A(x', y', z) = A(x, A(x'y, z), z)
\]
for \( x, y, z \in \mathbb{N} \). Information about this can be found in Chapters 9 and 10 of [10].

There is no explicit clause for \( A(x', 0, z) \). We are told to take any ‘suitable’ value. The inclusion of \( A(1, y, z) \) as \( y \times z \) messes up the structure of the hierarchy at its lower end. A similar error was (and often still is) included in the original Grzegorczyk hierarchy.

The book [10] could have been written by Charles Dickens. Its the best of books, its the worst of books. If I could rewrite the book it would be a far better thing than I have ever done.

The last paragraph of the Preface of the original 1950 version of [10] is also worth noting.

References


[18] H. Simmons: *Derivatives for ordinal functions and the Schütte brackets*, can be found at http://www.cs.man.ac.uk/~hsimmons under the link ‘Papers-and-Notes’ as item (04P).

[19] H. Simmons: The three papers in http://www.cs.man.ac.uk/~hsimmons under the link ‘Papers-and-Notes’ as items (05P), (04P), (03P) were originally written as a single set of notes. A version of those notes can be found at http://www.cs.man.ac.uk/~hsimmons/TEMP as IterTemplates.pdf, but be warned, there are mistakes.
