The back-and-forth technique

Our aim is to prove the following.

THEOREM. (4.2) Each two lines \( A, B \) are potentially isomorphic, \( A \cong_p B \).
This is for lines = DLOWEPs. We need to set up \( \cong_p \).

THEOREM. (4.3) We have
\[
A \cong_p B \implies A \cong B
\]
for all countable linear sets \( A, B \).
This is for arbitrary linear sets.

THEOREM. (4.4) Each two countable lines are isomorphic. In particular, each countable line is isomorphic to \( \mathbb{Q} \).
This is the characterisation of \( \mathbb{Q} \).
It follows from the previous two results.
A back-and-forth system for $A, B$

is a non-empty set $\mathcal{P}$ of partial isomorphisms $f$ from $A$ to $B$ with certain properties.

$$f : U \to V \text{ to } f^+ : U \cup \{x\} \to V \cup \{y\}$$

(back) For each member $f$ of $\mathcal{P}$ and each element $y$ of $B$, there is an element $x$ of $A$ together with an extension of $f$ to $f^+$ in $\mathcal{P}$.

(forth) For each member $f$ of $\mathcal{P}$ and each element $x$ of $A$, there is an element $y$ of $B$ together with an extension of $f$ to $f^+$ in $\mathcal{P}$.

We write

$$A \cong_p B$$

and say $A$ and $B$ are potentially isomorphic if there is at least one back-and-forth system for the pair $A, B$. 
Proof of (4.2)

We have the earlier result for a linear set and a line.

**LEMMA.** Consider the situation on the left where $A$ is a linear set, $B$ is a line, and $f$ is a partial isomorphism between the finite sets $U$ and $V$. Consider also any element $a \in A$.

\[
\begin{array}{ccc}
A & \rightarrow & B \\
U & \rightarrow & V \\
\end{array}
\]

Then there is an element $b \in B$ together with a partial isomorphism $g$ which extends $f$ and with $g(a) = b$.

We apply this twice to two lines, forwards and backwards. The set of all finite partial isomorphisms is a b&f system.
Proof of (4.3)

Suppose $A, B$ are countable linear sets. Then

$$A \cong_p B \iff A \cong B$$

We are given a b&f system $\mathcal{P}$ for the pair. Each is countable, so each can be enumerated

$$(a_i \mid i \in \mathbb{N}) \quad (b_i \mid i \in \mathbb{N})$$

where these have nothing to do with the comparisons on $A, B$. Using $\mathcal{P}$ we produce an increasing chain of partial isomorphisms

$$U_n \xrightarrow{f_n} V_n$$

where

$$a_0, \ldots, a_n \in U_n \quad b_0, \ldots, b_n \in V_n$$

for each $n \in \mathbb{N}$. 4
More to come?

There is more material in Document (C), namely Section 5.

We may go back to that later.

Until then it is not part of the official syllabus.