B-Cardinal comparison of sets

[Page 2, Definition 1.1]

For two sets $A, B$ we write

\[ A \preceq B \quad A \approx B \]

if there is at least one function $f : A \rightarrow B$ which is

an injection \quad a bijection

respectively. These are the

cardinal comparison \quad cardinal equivalence

relations, respectively.

[In Pussy/SetTheoryLectures/02   Written 25 September 2012]
We write \([S, T]\) for the set of all functions from \(S\) to \(T\). This notation is not standard but better than these usual awful notations.

injection surjection bijection
Some simple properties

[Page 2, Lemma 1.2]

**LEMMA.** Let $A, B, C$ be arbitrary sets. The following hold.

1. $A \approx B \implies A \preceq B$
2. $A \approx A$
3. $A \approx B \implies B \approx A$
4. $A \approx B \approx C \implies A \approx C$
5. $A \preceq B \preceq C \implies A \preceq C$

Notice that we do not yet have

$$A \preceq B \preceq A \implies A \approx B$$

This is the Cantor-Schröder-Bernstein result we look at later.
Some standard examples

\( \mathbb{N} \): 0, 1, 2, 3, 4, \ldots

\( \mathbb{Z} \): 0, ±1, ±2, ±3, ±4, \ldots

\( \mathbb{Q} \): All pairs \( n/d \) where \( d > 0 \) and the pair have no common factors and \ldots

- The above three are equipotent. Dealing with \( \mathbb{Q} \) needs a little bit of work.

\( \mathbb{R} \): This is strictly bigger than countable. This follows from the diagonal argument and a bit more work.
Alephs and . . .

[B page 4 and 5]
After much work we produce an infinite list of cardinal numbers.

\[ \aleph_0 < \aleph_1 < \ldots < \aleph_n < \ldots < \aleph_\omega < \ldots \]

Each infinite set \( A \) is equivalent to precisely one of these. We write \( |A| \) for that cardinal number.

In particular \( |\mathbb{N}| = \aleph_0 \).

These alephs are indexed by the ordinal numbers and as yet we know almost nothing about these.
...and Beths

There is also a sub-list of particular cardinal numbers.

\[ \aleph_0 = \beth_0 < \beth_1 < \ldots < \beth_n < \ldots < \beth_\omega < \ldots \]

If a set \( A \) has cardinality \( \aleph \), that is \( |A| = \aleph \), then we often write

\[ 2^{\aleph} \]

for the cardinality of \( \mathcal{P}A \)

The beths are produced by iterating this process.

\[ \beth_0 = \aleph_0 \quad \beth_1 = 2^{\beth_0} \quad \beth_2 = 2^{\beth_1} \quad \beth_3 = 2^{\beth_2} \quad \text{and so on} \]

We have \( \beth_1 \leq |\mathbb{R}| \) but not much else is known, except it cannot be \( \aleph_\omega \), and more generally \( \aleph_\lambda \) for a countable limit ordinal \( \lambda \).