Posets and their properties: Exercises

1.1  (a) Think of a discrete horizontal set. Each element is a maximal.

(b) Let $S$ be a poset and consider a subset $X$ of $S$.
An element $a \in X$ is minimal in $X$ if
\[ x \leq a \implies x = a \]
for all $x \in X$.
An element $a \in X$ is a minimum in $X$ if
\[ a \leq x \]
for all $x \in X$.

(c)(i) The real interval $(0, 1]$.
(c)(ii) Think of a discrete horizontal set and put an extra element above. ■

1.2  (a) Let $a \in X$ be the maximum of $X$. Then $x \leq a$ for each $x \in X$, so $a$ is an upper bound for $X$. Now consider any upper bound $b$ for $X$. Then $x \leq b$ for each $x \in X$. In particular $a \leq b$ since $a \in X$. Thus $a$ is the least upper bound.

(b) Think of a poset which is a discrete horizontal set with one extra element $\top$ above this. Let $X$ be the horizontal set. Each element of $X$ is maximal and $\top$ is the supremum of $X$.

(c) Think of $\mathbb{N}$ going upwards together with one extra element at the side. That extra element is maximal, but the whole poset has no supremum.

(d) Consider the real interval $[0, 1]$ as a poset and let $X = [0, 1)$. Then $X$ has a supremum, namely 1, but no maximal. ■

1.3  We find an example using subsets of $\mathbb{R}$ as the posets, but we draw these lying down (to give them a rest).
Let $T = [0, 3]$  $S = [0, 1) \cup [2, 3]$  $X = [0, 1)$
where $S \subseteq T$ as linear sets. Also $X$ is a subset of both $S$ and $T$. In $T$ the supremum of $X$ is 1, but in $S$ the supremum of $X$ is 2. ■

1.4  (a) This is a routine induction on the size of $X$.

(b) Think of $\mathbb{N}$ as the poset and the directed subset. ■

1.5  Let us write $\alpha, \beta, \gamma, \ldots$ for the chains in $S$.

(a) This is immediate.

(b) Let $\mathcal{A}$ be any directed family of chains. As a collection of sets the union $A = \bigcup \mathcal{A}$ makes sense. We show this is a chain in $S$, and extends each $\alpha \in \mathcal{A}$.
We can certainly restrict the comparison on $S$ to one on $A$. Thus $A$ is a poset, and almost trivially we have $\alpha \subseteq A$, as posets, for each $\alpha \in \mathcal{A}$. Thus it suffices to show that $A$ is linear.

Consider $a, b \in A$. We must show that $a, b$ are comparable one way or the other. By the construction of $A$ we have

$$a \in \alpha \in \mathcal{A} \quad b \in \beta \in \mathcal{A}$$

for some chains $\alpha, \beta$. But $\mathcal{A}$ is directed, so that

$$a \in \alpha \subseteq \gamma \in \mathcal{A} \quad b \in \beta \subseteq \gamma \in \mathcal{A}$$

for some chain $\gamma$. Working in $\gamma$ we have

$$a \leq b \quad \text{or} \quad b \leq a$$

for the required result. \hfill \blacksquare

**Posets and their properties: Problems**

1.6 (a) If we think of the partial orders on $S$ as sets of ordered pairs, subsets of $S \times S$, then this is more or less trivial.

(b) Let $\mathcal{C}$ be any directed family of partial orders on $S$. Thus for each pair $\leq_1, \leq_2$ of members of $\mathcal{C}$ there is some member $\leq_3$ of $\mathcal{C}$ with

$$x \leq_i y \implies x \leq_3 y$$

for each $i = 1, 2$ and each $x, y \in S$. Consider the relation $\sqsubseteq$ on $S$ given by

$$x \sqsubseteq y \iff (\exists \text{ some } \leq \text{ in } \mathcal{C})[x \leq y]$$

for $x, y \in S$. We show first that this $\sqsubseteq$ is a partial order, namely that it is reflexive, transitive, and antisymmetric.

(ref) Since $\mathcal{C}$ is directed there is at least one $\leq$ in $\mathcal{C}$. But now for each $x \in S$ we have $x \leq x$, and hence $x \sqsubseteq x$.

(trans) Suppose

$$x \sqsubseteq y \sqsubseteq z$$

for some $x, y, z \in S$. Thus

$$x \leq_1 y \leq_2 z$$

for some members $\leq_1$ and $\leq_2$ of $\mathcal{C}$. By the directed property this gives

$$x \leq_3 y \leq_3 z$$

for some $\leq_3$ in $\mathcal{C}$. Then $x \leq_3 z$, and hence $x \sqsubseteq z$, as required.

(anti) Suppose

$$x \sqsubseteq y \sqsubseteq x$$

for some $x, y \in S$. Thus

$$x \leq_1 y \leq_2 x$$

for some $x, y \in S$. Thus
for some members $\leq_1$ and $\leq_2$ of $C$. By the directed property this gives

$$x \leq_3 y \leq_3 x$$

for some $\leq_3$ in $C$. Then $x = y$, as required.

Trivially, for $x, y \in S$ we have

$$x \leq y \implies x \sqsubseteq y$$

for each member $\leq$ of $C$. Thus $\sqsubseteq$ is an upper bound for $C$. We must show that $\sqsubseteq$ is the least upper bound for $C$.

To this end suppose the partial order $\preceq$ is an upper bound for $C$, that is

$$x \leq y \implies x \preceq y$$

for each member $\leq$ of $C$ and each $x, y \in S$. We require

$$x \sqsubseteq y \implies x \preceq y$$

for each $x, y \in S$. But if $x \sqsubseteq y$ then $x \preceq y$ for some member $\leq$ of $C$, and hence $x \preceq y$, as required. $\blacksquare$

1.7 (a) Trivially we have

$$x \leq y \implies x \sqsubseteq y$$

for $x, y \in S$. Thus it suffices to show that $\sqsubseteq$ is reflexive, transitive, and antisymmetric.

(ref) This is immediate.

(trans) Suppose $x \sqsubseteq y \sqsubseteq z$, that is

$$x \leq y \quad \text{and} \quad y \leq z$$

or

$$x \leq a \& b \leq y \quad \text{and} \quad y \leq a \& b \leq z$$

hold. We consider the four possible combinations to get $x \sqsubseteq z$.

If $x \leq y$ and $y \leq z$ then we certainly have $x \leq z$, to give $x \sqsubseteq z$.

Next suppose $x \leq y$ and $y \leq a$ and $b \leq z$. Then

$$x \leq a \text{ and } b \leq z$$

to give $x \sqsubseteq z$.

The other diagonal possibility is dealt with in the same way.

Finally consider the bottom two possibilities. These give

$$b \leq y \leq a$$

so that $b \leq a$, which does not hold. Thus this case can not arise.

(anti) Suppose $x \sqsubseteq y \sqsubseteq x$, that is

$$x \leq y \quad \text{and} \quad y \leq x$$

or

$$x \leq a \& b \leq y \quad \text{and} \quad y \leq a \& b \leq x$$

3
hold. We consider the four possible combinations to get \( x = y \).

If \( x \leq y \) and \( y \leq x \) then we certainly have \( x = y \), to give \( x \sqsubseteq z \).

Next suppose \( x \leq y \) and \( y \leq a \) and \( b \leq x \). Then

\[
    b \leq x \leq y \leq a
\]

so that \( b \leq a \), which does not hold. Thus this case can not arise.

The other diagonal possibility is dealt with in the same way.

Finally consider the bottom two possibilities. These give

\[
    b \leq y \leq a
\]

so that \( b \leq a \), which does not hold. Thus this case can not arise.

(b) By way of contradiction suppose the maximal comparison \( \leq \) is not linear. Then we have

\[
    a \nless b \quad b \nless a
\]

for some \( a, b \in S \). By part (a) there is some extension \( \sqsubseteq \) of \( \leq \) with \( a \sqsubseteq b \). Since \( \leq \) is maximal in the poset of all comparisons on \( S \) we see that \( \leq \) and \( \sqsubseteq \) are the same. Thus \( a \leq b \), which the contradiction.

\[\square\]

Zorn’s Lemma and some uses: Exercises

2.1 Let \( G \) be an arbitrary group and let \( \mathcal{A} \) be its poset of abelian subgroups. By Lemma 2.2 \( \mathcal{A} \) is closed under directed unions. Thus we have \( Y(\mathcal{A}) \). A use of YL now gives the required result.

2.2 (a) A subset \( L \subset V \) is linearly independent if for each finite non-repeating list \( l_1, \ldots, l_m \) from \( L \), and each matching list \( k_1, \ldots, k_m \) from \( K \), if

\[
k_1l_1 + \cdots + k_ml_m = 0
\]

then \( k_1 = \cdots = k_m = 0 \).

(b) Being linearly independent has finite character, so this is more or less trivial. In detail, suppose \( \mathcal{L} \) is a directed family of linearly independent subsets of \( V \). Let \( l_1, \ldots, l_m \) be any non-repeating list from the union of \( \mathcal{L} \). Then this list lives in at least one of the members of \( \mathcal{L} \), and hence is linearly independent.

(c) We must show that every member \( v \in V \) can be written as a linear combination of some members of \( B \) (with multipliers taken from \( K \)). By way of contradiction suppose there is some \( v \in V \) which can not by written in this way. We show that \( B \cup \{v\} \) is linearly independent, which contradicts the maximality of \( B \).

Suppose

\[
k_1b_1 + \cdots + k_mb_m + kv = 0
\]

where \( b_1, \ldots, b_m \) is a non-repeating list from \( B \), and \( k_1, \ldots, k_m, k \in K \). If \( k = 0 \) then

\[
k_1b_1 + \cdots + k_mb_m = 0
\]
which can not be since \( b_1, \ldots, b_m \) are linearly independent. Thus \( k \neq 0 \), and we may divide by \( k \) to obtain
\[
v = k_1' b_1 + \cdots + k_m' b_m
\]
for suitable \( k_1', \ldots, k_m' \in K \). Again this can not be by the nature of \( v \).

\[\square\]

2.3 For an arbitrary poset \( S \) let \( CS \) be its family of chains. By Exercise 1.5 we know that \( CS \) is a poset that is closed under directed unions. Thus we have \( Y(CS) \), and hence \( YL \) gives \( K(CS) \), which is the required property.

\[\square\]

2.4 (a) This is just Exercise 2.3.

(b) Actually, using KL we can get \( ZL^+ \).

Consider any poset \( S \) with \( K(S) \). Consider any \( a \in S \). We must produce a maximal element above \( a \). The singleton \( \{a\} \) is a chain and hence, by KL, we have \( a \in \mu \) for some maximal chain \( \mu \). By \( Z(S) \) this chain has an upper bound \( b \), say. In particular we have \( a \leq b \). We show that \( b \) is a maximal element of \( S \).

Consider any \( b \leq c \). Then \( \mu \cup \{b \leq c\} \) is a chain that extends \( \mu \). But \( \mu \) is a maximal chain, so that \( b, c \in \mu \). Since every member of \( \mu \) is below \( b \) we have \( c \leq b \), and hence \( b = c \), to show that \( b \) is a maximal element.

\[\square\]

2.5 Consider any poset \((S, \leq)\) in which each chain has a lower bound. Then \( S \) has a minimal element.

To prove this we may apply \( ZL \) to the opposite \((S, \leq^{op})\) of the given poset.

\[\square\]

Zorn's Lemma and some uses: Problems

2.6 Consider any poset \( S \) and its family \( P(OS) \) of partial orderings. By Problem 1.6 we know the poset \( P(OS) \) is closed under directed suprema. In particular, we have \( Y(P(OS)) \). A use of \( YL \) may show that each partial ordering of \( S \) is included in a maximal partial ordering. By Problem 1.7(b) any such maximal partial ordering is linear.

\[\square\]

2.7 (a) A simple induction gives
\[
f(m) = mc
\]
for \( m \in \mathbb{N} \). Thus, for the base case, \( m = 0 \), we have
\[
f(0) + c = f(0) + f(1) = f(0 + 1) = f(1) = c
\]
so that \( f(0) = 0 \). For the induction step, \( m \mapsto m + 1 \), we have
\[
f(m + 1) = f(m) + f(1) = mc + c = (m + 1)c
\]
for the required result.

Next we have
\[
f(-x) + f(x) = f(-x + x) = f(0) = 0
\]
so that \( f(-x) = -f(x) \). In particular
\[
f(m) = mc
\]
for all $m \in \mathbb{Z}$.

Finally for this part, consider any rational number $q = (n/d)$ with $d > 0$. We have

$$df(q) = df(n/d) = f(n/d) + \cdots + f(n/d) = f(n) = nc$$

so that

$$f(q) = nc/d = qc$$

to show that $f$ is uniquely determined on $\mathbb{Q}$.

(b) Suppose $f$ is continuous. Each real $x \in \mathbb{R}$ is a limit of a rational sequence, and then $f(x)$ is the limit of the corresponding values. Thus $f(x) = xc$.

(c) Let $B = (b_i \mid i \in I)$ be a basis for $\mathbb{R}$ over $\mathbb{Q}$. Thus for each $x \in \mathbb{R}$ has a unique representation

$$x = \sum\{x(i)b_i \mid i \in I\}$$

where the function

$$x(\cdot) : I \longrightarrow \mathbb{Q}$$

is zero almost everywhere, that is it has only finitely many non-zero outputs. Also, for each index $i \in I$ we have

$$x(i) + y(i) = (x + y)(i)$$

for each $x, y \in \mathbb{R}$. Thus for any particular index $\bullet \in I$ we may set

$$f(x) = x(\bullet)$$

for each $x \in \mathbb{R}$ to obtain a function $f$ of the required kind.

2.8 We consider the set of all partial linear functions from $W$ to $T$ where the source includes $U$. That is we consider all the pairs

$$(g, V)$$

where $U \subseteq V \subseteq W$ is an intermediate vector space, with

$$g : V \longrightarrow T$$

a linear function which extends $f$ and with $g[V] = f[U]$. By extending $f$ we mean that

$$g_{|U} = f \quad \text{that is} \quad g(x) = f(x)$$

for all $x \in U$. We partially order this family by extension. Thus $(f, U)$ is the least member of this poset. We show that the poset is closed under directed unions.

Consider any directed subfamily $\mathcal{G}$ of this family. Let

$$L = \bigcup \{V \mid (g, V) \in \mathcal{G}\}$$

to obtain a subset of $W$. Consider any two members $x_1, x_2 \in L$. We have

$$x_1 \in V_1 \quad x_2 \in V_2$$
for two members \((g_1, V_1), (g_2, V_2)\) of \(\mathcal{G}\). By \(\mathcal{G}\) is directed, so that
\[
(g_1, V_1), (g_2, V_2) \leq (g, V)
\]
for some \((g, V) \in \mathcal{G}\). This gives \(x_1, x_2 \in V\), so that
\[
x_1 + x_2 \in V \subseteq L
\]
to show that \(L\) is closed under addition. A simpler argument shows that \(L\) is closed under the action of \(K\) on \(L\), and hence \(L\) is a subspace of \(W\).

Next consider any \(x \in L\). This belongs to at least one \(V\) with \((g, V) \in \mathcal{G}\). It may belong to many. Suppose have
\[
x \in V_1 \quad x \in V_2
\]
for two members \((g_1, V_1), (g_2, V_2)\) of \(\mathcal{G}\). By \(\mathcal{G}\) is directed, so that
\[
(g_1, V_1), (g_2, V_2) \leq (g, V)
\]
and then
\[
g_1(x) = g(x) = g_1(x)
\]
to show that \(x\) is sent to a unique member of \(W\). Thus we may define \((h, L)\) by
\[
h(x) = g(x) \text{ for any } (g, V) \in \mathcal{G} \text{ with } x \in V
\]
to obtain
\[
h : L \longrightarrow W
\]
which is linear with \(h[L] = f[U]\). In other words \((h, L)\) is an upper bound of \(\mathcal{G}\) in the family of partial functions under consideration. (In fact, it is the least upper bound of \(\mathcal{G}\).)

A use of ZL now gives us a maximal member \((g, V)\) of the family of partial linear functions. It suffices to show that \(V = W\).

By way of contradiction suppose \(V \neq W\). Consider any \(w \in W - V\), and look at the subspace of \(W\) generated by \(V\) and \(w\). Each member of this larger space has the form
\[
v + rw
\]
for some \(v \in V\) and \(r \in K\). Furthermore, this decomposition is unique. We may set
\[
h(v + rw) = g(v)
\]
to obtain an extension of \((g, V)\). This contradiction the maximality of \((g, V)\).  

\[\square\]

**The Krull Separation Lemma: Exercises**

4.1 (a) Let \(M\) be a maximal ideal and by way of contradiction suppose \(M\) is not prime. Thus we have
\[
ab \in M \quad a \notin M \quad b \notin M
\]
for two elements \( a, b \). The two ideals
\[
M + \langle a \rangle \quad M + \langle b \rangle
\]
are strictly larger than \( M \), so each is \( R \) by the maximality of \( M \). Each must contain 1, so that
\[
1 = m + \sum lar \quad 1 = n + \sum sbt
\]
for some \( m, n \in M \) and some finite sums involving \( a \) and \( b \), as shown. But now, since
\[
(1 - m)(1 - n) = 1 - m - n + mn
\]
we have
\[
1 = (mn - m - n) + \sum uavbw
\]
where the \( u, v, w \) are obtained. The first component \( (mn - m - n) \) is in \( M \) since \( m, n \in M \). The sum is in \( M \) since each \( avb \in M \). Thus \( 1 \in M \), which is the contradiction.

(b) Let \( M \) be an ideal that is maximally disjoint from \( X \), and by way of contradiction suppose \( M \) is not prime. Thus we have
\[
ab \in M \quad a \notin M \quad b \notin M
\]
for two elements \( a, b \). The two ideals
\[
M + \langle a \rangle \quad M + \langle b \rangle
\]
are strictly larger than \( M \), so each meets \( X \). Thus we have
\[
x = m + \sum lar \quad y = n + \sum sbt
\]
for some \( x, y \in X \), some \( m, n \in M \) and some finite sums involving \( a \) and \( b \), as shown. But now, since we have
\[
xy = (mn - nx - my) + \sum uavbw
\]
where the \( u, v, w \) are obtained. The first component \( (mn - ny - mx) \) is in \( M \) since \( m, n \in M \). The sum is in \( M \) since each \( avb \in M \). Thus \( xy \in M \). But \( xy \in X \) since \( X \) is \( x \)-closed, and this is the contradiction. ■

4.2 (a) Consider any directed subfamily \( J \subseteq I \). We must first show that \( U = \bigcup J \) is an ideal. To this end consider any \( x, y \in U \). We have \( x \in J \subseteq I \) and \( y \in K \subseteq J \) for some ideals \( J, K \). Since \( J \) is directed we have \( J, K \subseteq L \subseteq J \) for some ideal \( L \). But now
\[
x \pm y \in L \subseteq U
\]
to obtain two of the required conditions. The other required condition is immediate.

Next we must show that \( X \cap U = \emptyset \). By way of contradiction suppose we have \( a \in X \cap U \). By the construction of \( U \) we have \( a \in J \) for some \( J \in J \). But now
\[
a \in X \cap J = \emptyset
\]
which is the contradiction.
(b) This is a consequence of the KSL for the case $X = \{1\}$.

4.3 (a) Consider first the required closure under $\pm$, that is

$$a, b \in \sqrt{I} \implies a \pm b \in \sqrt{I}$$

holds. The hypothesis gives

$$a^m, b^n \in I$$

for some $m, n \in \mathbb{N}$. We have

$$(a \pm b)^k = \sum \left\{ \binom{k}{r} a^r b^s \mid r + s = k \right\}$$

where this depends heavily of the commutativity of $R$. When $k$ is large enough, $k \geq m + n$, one of $a^r, b^s$ is in $I$, so that $(a \pm b)^k \in I$, to give $(a \pm b) \in \sqrt{I}$.

The other required property is easier.

(b) For convenience let

$$Q = \bigcap \{ P \mid I \subseteq P \text{ where } P \text{ is prime} \}$$

so that $\sqrt{I} = Q$ is required. We obtain this via two inclusions.

Consider first any prime ideal $P$ with $I \subseteq P$. For any $a \in \sqrt{I}$ we have $a^m \in I \subseteq P$, and hence $a \in P$, since $R$ is commutative. Thus $\sqrt{I} \subseteq Q$.

For the converse consider any $a \in R - \sqrt{I}$. We must find some prime ideal $P$ with $I \subseteq P$ and $a \notin P$. The set

$$X = \{ a^m \mid m \in \mathbb{N} \}$$

is $\times$-closed and $X \cap I = \emptyset$. Thus, by KSL, there is some prime ideal $P$ with $I \subseteq P$ and $X \cap P = \emptyset$. In particular, $a \notin P$, to give the required result.

(c) This is the particular case $I = O$ of (a, b).

4.4 (a) Let $\mathcal{P}$ be a chain of prime ideals. The intersection of any family of ideals is an ideal, so it suffices to show that $\bigcap \mathcal{P}$ is prime. By way of contradiction suppose it is not prime. Thus we have

$$ab \in \bigcap \mathcal{P} \quad a \notin \bigcap \mathcal{P} \quad b \notin \bigcap \mathcal{P}$$

for some elements $a, b \in R$. The second two give

$$a \notin P_a \quad b \notin P_b$$

for some $P_a, P_b \in \mathcal{P}$. The given property of $\mathcal{P}$ ensures that $P \subseteq P_a \cap P_b$ for some $P \in \mathcal{P}$. But now $a, b \notin P$ which since $P$ is prime, is the contradiction.

(b) Exactly the same argument works if $\mathcal{P}$ is downward directed, not just a chain.

(c) This is a consequence of ZL. We look at the poset of prime ideals partially ordered by reverse inclusion $\leq$-say. Part (b) show that we may apply ZL to verify that each prime ideal is $\leq$-below a $\leq$-maximal prime ideal. That is each prime ideal includes a minimal prime ideal.

(d) Consider two members $xu, yv$ of $X$, that is $x, y \notin Z$ and $u, v \notin P$. A simple argument show that $xy \notin Z$ and $uv \notin P$ since $P$ is prime. Hence

$$xuyv = xyuv \in X$$
as required.

(e) The set \( X \) is \( \times \)-closed and hence, by KSL, we have
\[
X \cap Q = \emptyset
\]
for some prime ideal. By (c) we may suppose that \( Q \) is a minimal prime ideal.

Now observe that
\[
R - Z \subseteq X \quad R - P \subseteq X
\]
by taking \( x \in R - Z, u = 1 \) and \( x = 1, u \in R - P \), respectively. Thus we have
\[
(R - Z) \cap Q = \emptyset \quad (R - P) \cap Q = \emptyset
\]
and these translate into
\[
Q \subseteq Z \quad Q \subseteq P
\]
to give the required result.

(f) Suppose \( P \) is a minimal prime. By part (e) we have
\[
Q \subseteq Z \cap P
\]
for some prime, and so
\[
P = Q \subseteq Z
\]
by the minimality of \( P \).

(g) When \( R \) is an integral domain then \( O \) is the only minimal prime, so (c, d, e, f) are trivial.

The Krull Separation Lemma: Problems

4.5 We fix the \( m \)-closed set \( X \) and the ideal \( M \) with
\[
X \cap M = \emptyset
\]
and \( M \) is maximally so. By way of contradiction suppose the \( M \) is not prime. Thus
\[
aRb \subseteq M \quad a \notin M \quad b \notin M
\]
for two elements \( a, b \). The two ideals
\[
M + \langle a \rangle \quad M + \langle b \rangle
\]
are strictly larger than \( M \), so each meets \( X \). Thus we have
\[
x = m + p \text{ where } p = \sum lar \quad y = n + q \text{ where } q = \sum sbt
\]
for some \( x, y \in X \), some \( m, n \in M \) and some finite sums \( p, q \) involving \( a \) and \( b \), as shown.

Since \( X \) is \( m \)-closed we have
\[
xyz \in X
\]
for some \( z \in R \). We now expand \( xyz \), but with some care.
We have
\[ xyz = (m + p)z(n + q) = mzn + pzn + mzq = pzq \]
and the first three components are in \( M \), by the occurrence of \( m \) or \( n \). We also have
\[ pzq = \sum_{u,v,w} uavbw \]
for appropriate \( u, v, w \in M \). Since \( aRb \subseteq M \) we have \( avb \in M \), and hence \( uavbw \in M \) for each \( u, v, w \). This gives \( pzq \in M \), so that
\[ pzq \in X \cap M \]
which is the contradiction. ■

4.6 As with the earlier argument, the family of ideals that are disjoint from \( X \) is closed under directed unions. Thus by ZL, each ideal which is disjoint from \( X \) is included in an ideal which is maximally disjoint from \( X \). By Exercise 4.5 each such ideal is prime. ■

4.7 By KSP we know there is an ideal \( P \) with
\[ I \cap P = O \]
and is maximally so. We also know that \( P \) is prime. We show that
\[ E = I + P \]
is essential in \( R \).
To this end consider any ideal \( J \) with
\[ E \cap J = O \]
so that \( J = O \) is required. We show that \( J \subseteq P \), so that
\[ J \subseteq E \cap J = O \]
to give the required result.
To show that \( J \subseteq P \) we use the maximal property of \( P \). To do that we show that the ideal
\[ I \cap (P + J) \]
is trivial. Consider any member \( a \) of this intersection. We have \( a \in I \) and
\[ a = b + c \]
for some \( b \in P, c \in J \). But now
\[ c = a - b \in I + P = E \]
so that \( c \in E \cap J = O \), and hence \( c = 0 \). But now
\[ a = b \in I \cap P = O \]
so that \( a = b = c = 0 \), as required. ■