The rationals and the reals as linearly ordered sets

We know that both $\mathbb{Q}$ and $\mathbb{R}$ are something special. When we think about either of these we usually view it as a field, or at least some kind of algebraic structure. In this document we consider each merely as linearly ordered set. Even in this seemingly narrow context they are still something special.

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1 The characterizing results

As indicated in the preamble, in these notes we consider the characterization of $\mathbb{Q}$ and $\mathbb{R}$ as linear sets. We prove look at two main results and these can be stated here. We will prove Theorem 1.1 but leave the proof of Theorem 2.2 for later.

Of course, in the statements of these results there are one or two technical words which have not yet been defined. I will indicate these, but leave the formal definitions until later.

In the following a line is a linear set with a certain extra property that both $\mathbb{Q}$ and $\mathbb{R}$ clearly have.

1.1 THEOREM. The rationals $\mathbb{Q}$ as a linear set has the following properties.

(i) $\mathbb{Q}$ is a countable line.

(ii) $\mathbb{Q}$ is embedded in every line.

(iii) Each countable linear set is embedded in $\mathbb{Q}$.

(iv) Each pair of countable lines are isomorphic.

This is the characterization of $\mathbb{Q}$. We also obtain a similar characterization of $\mathbb{R}$. In the following separable and complete are two technical properties that $\mathbb{R}$ has.

1.2 THEOREM. The reals $\mathbb{R}$ as a linear set has the following properties.

(i) $\mathbb{R}$ is a separable complete line.

(ii) $\mathbb{R}$ is embedded in every complete line.
(iii) Each separable line is embedded in \( \mathbb{R} \).

(iv) Each pair of separable complete lines are isomorphic.

It is worth comparing these two characterizations. To do that let’s state a cod result. In it sizable and string are used in a pseudo-technical sense.

1.3 THEOREM. The gadget \( \mathcal{G} \) as a linear set has the following properties.

(i) \( \mathcal{G} \) is a sizable string.

(ii) \( \mathcal{G} \) is embedded in every string.

(iii) Each sizable linear set is embedded in \( \mathcal{G} \).

(iv) Each pair of sizable strings are isomorphic.

We can see that Theorems 1.1 and 1.2 have the form of Theorem 1.3. We interpret the cod words as follows.

\[ (1.1) \text{ countable line} \]
\[ (1.2) \text{ separable complete line} \]
\[ (1.3) \text{ sizable string} \]
\[ (1.4) \text{ c.c.c. complete line} \]

In both cases the word ‘string’ is a property that a linear ordered set may or may not have. However, there seem to be a disparity between the two meanings of ‘sizable’. For \( \mathbb{Q} \) it is a cardinality condition, namely being countable. But for \( \mathbb{R} \) it is a topological condition. This will be explained in more detailed later. But why do we need a topological condition? Can we replace being separable by some cardinality condition?

There is an obvious condition to look at, that of being c.c.c (having the countable chain condition). This is weaker than being separable

\[ \text{separable} \implies \text{c.c.c.} \]

and making this change gives the following.

1.4 THEOREM. The reals \( \mathbb{R} \) as a linear set has the following properties.

(i) \( \mathbb{R} \) is a c.c.c. complete line.

(ii) \( \mathbb{R} \) is embedded in every complete line.

(iii) Each c.c.c. line is embedded in \( \mathbb{R} \).

(iv) Each pair of c.c.c. complete lines are isomorphic.

Here part (i) is a consequence of the corresponding part of Theorem 1.2, and part (ii) is just the same. However, parts (iii, iv) are stronger than the corresponding parts of Theorem 1.2.

Now the world becomes weirder. It is know that parts (iii, iv) of Theorem 1.4 can never be proved or disproved without some extra hypothesis. The two statements are independent of the standard axioms of Axiomatic Set Theory. There are certain set theoretic universes in which the results are true, and other set theoretic universes in which the results are false.

We will not look at these independence results. The methods used are much to complicated to explain here. However, later we may look at the two conditions of being separable and having c.c.c., and observe that one implies the other almost trivially.
2 Basic properties

These notes are concerned with linearly ordered sets in general, and the important role played by \( Q \) and \( R \) in this context. We need to set down the basic notions.

2.1 DEFINITION. A linear set (linearly ordered set) is a set \( A \) with a specified linear comparison \( \leq \), that is a binary relation on \( A \) which is

(Reflexive) \( a \leq a \)

(Transitive) \( a \leq b \leq c \implies a \leq c \)

(Antisymmetric) \( a \leq b \land b \leq a \implies a = b \)

(Linear) \( a \leq b \) or \( b \leq a \)

for all \( a, b, c \in A \).

We sometimes use the strict version \( < \) of \( \leq \), that is \( a < b \iff a \leq b \) and \( a \neq b \) for \( a, b \in A \). ■

The term ‘linear set’ is not standard, but it is useful since it is shorter than ‘linearly ordered set’.

Let’s look at some examples. Some of these are obvious and some not. The verification of some aspects are left as exercises.

2.2 EXAMPLE. (a,b) Each of \( N, Z, Q, R \) is a linear set. There is a clear difference between the pair \( N, Z \) and the pair \( Q, R \). There is also a difference between \( Q \) and \( R \) which is not so obvious.

(c) Let \( A \) be the set of eventually zero members of \([N \to \mathbb{2}]\). Thus each member of \( A \) is a function \( f : N \to \mathbb{2} \) with \( f(m) = 0 \) for all sufficiently large \( m \). Consider the following two relations on \( A \).

(last) \( f \leq g \iff f = g \lor (\exists m)[f(m) < g(m) \land (\forall n > m)[f(n) = g(n)]] \)

(first) \( f \leq g \iff f = g \lor (\exists m)[f(m) < g(m) \land (\forall n < m)[f(n) = g(n)]] \)

These are the principles of last difference and first difference, respectively.

These two comparisons make \( A \) into a linear set in two different ways. In fact, these are two well-known linear sets in disguise.

(d) Let \( A \) be the set of all functions \([N \to \mathbb{2}]\) compared by first difference. This is a linear set and does have a more concrete description.

Notice that it doesn’t make sense to compare all such function by last difference. However, if we consider those function which are eventually constant, then we can use the last difference comparison.

(e) Let \( A \) be any linear set. A lower section, initial section, is a subset \( X \subseteq A \) such that

\[ y \leq x \in X \implies y \in X \]

holds. Let \( \mathcal{L}A \) be the family of all lower sections. This is a linear set. ■
As with any kind of structure, we need to decide how linear sets should be compared. One of the following gives the correct notion. The other is also useful at times.

2.3 DEFINITION. Let $A, B$ be a pair of linear sets. A function

$$f : A \longrightarrow B$$

is monotone antitone if

$$a_1 \leq a_2 \implies f(a_1) \leq f(a_2) \quad a_1 \leq a_2 \implies f(a_2) \leq f(a_1)$$

for all $a_1, a_2 \in A$.

The standard comparison, a morphism, between linear sets is a monotone function. But antitone function are useful occasionally. Here we use a more restricted kind of comparison function.

2.4 DEFINITION. (a) An embedding from one linear set to another

$$A \xrightarrow{f} B$$

is a monotone function that is also injective. In other words, it is a function, as indicated, where

$$a_1 \leq a_2 \iff f(a_1) \leq f(a_2)$$

for all $a_1, a_2 \in A$.

(b) An isomorphism from one linear set to another

$$A \xrightarrow{f} B$$

is a monotone function which is also a bijection for which is inverse function

$$A \xleftarrow{g} B$$

is also monotone.

In this document we mostly consider only embeddings between linear sets. However, monotone functions do arise in other parts of the whole collection of documents. Also in this document we often restrict out attention to a special kind of linear set.

2.5 DEFINITION. Let $A$ be a linear set.

This set is dense if for each comparable pair $a < b$ of distinct elements there is a strictly intermediate elements, that is $a < x < b$ for some element $x$.

The set is without end points if for each element $a$ we have $y < a < z$ for some elements $y, z$.

A line is a dense linear set without end points.
The use of ‘line’ in this sense is not standard. In the literature you will often see ‘dense linearly ordered set without end points’ which is a bit long. The word ‘line’ does seem to capture the intuitive idea a bit better.

Let’s have another look at the Examples 2.2.

2.6 EXAMPLES. (a) Neither \( \mathbb{N} \) nor \( \mathbb{Z} \) is a line, mainly because neither is dense.

(b) Both \( \mathbb{Q} \) and \( \mathbb{R} \) are lines.

(c) Let \( A \) be the set of eventually zero sequences.

When compared by last difference the linear set \( A \) is nowhere near a line. This is considered in one of the exercises.

When compared by first difference this is not quite a line. It is dense and has no last point, but does have a first point. Again this is considered in one of the exercises.

(d) Let \( A \) be the set of all functions \( f : \mathbb{N} \rightarrow 2 \), compared by first difference. This has a first point, the constant 0 function. It also has a last point, the constant 1 function. Between these the linear set can look rather odd.

(e) As a particular example consider the linear set \( \mathcal{L} \mathbb{Q} \) of initial sections of \( \mathbb{Q} \). This has end points, namely the empty set and the whole set. For each \( a \in \mathbb{Q} \) the set

\[
\downarrow a = \{ x \in \mathbb{Q} \mid x \leq a \}
\]

is a member of \( \mathcal{L} \mathbb{Q} \). The assignment

\[
\begin{array}{ccc}
\mathbb{Q} & \longrightarrow & \mathcal{L} \mathbb{Q} \\
\downarrow a & \longmapsto & \downarrow a
\end{array}
\]

is an embedding.

Think about what else might occur in \( \mathcal{L} \mathbb{Q} \). There is a copy of \( \mathbb{R} \) and the rationals have a rather split personality. ■

Linear sets are not quite as simple as we might think when we first see them. Here we look at some of these aspects but concentrate on \( \mathbb{Q} \) and \( \mathbb{R} \).

2.1 Exercises: Basic properties

2.1 (a) Consider three linear sets, and a composable pair of functions

\[
A \xrightarrow{f} B \xrightarrow{g} C
\]

between them. Suppose each of the functions is either monotone or antitone (so there are four cases). What property does the composite have?

(b) Show that for each antitone function, if we take the opposite of the source or the target then we get a monotone function. You will need to sort out what the opposite of a linear set is.

2.2 (a) Consider the two versions of ‘embedding’ as given in Definition 2.4. Show that they are equivalent.

(b) Show that a monotone function between to lines is an isomorphism precisely when it is a surjective embedding.
2.3 (a) Show that, as linear sets, each two open ended real intervals are isomorphic. And remember that \( \pm \infty \) can be end points.

(b) Show there are infinitely many pairwise non-isomorphic lines of cardinality \( \aleph_1 \).

(c) Recall that the dyadic rationals are those of the form \( n/d \) where the denominator \( d \) is a power of 2. Show that the dyadic rationals form a line.

2.4 Let \( A \) be the set of eventually zero members of \( \mathbb{N} \to 2 \), and consider the following two relations on \( A \).

\[
\text{(last)} \quad f \leq g \iff f = g \text{ or } (\exists m)[f(m) < g(m) \& (\forall n > m)[f(n) = g(n)]
\]

\[
\text{(first)} \quad f \leq g \iff f = g \text{ or } (\exists m)[f(m) < g(m) \& (\forall n < m)[f(n) = g(n)]
\]

These are the principles of last difference and first difference, respectively.

Show that each of these linearly orders \( A \), and describe a more natural example of the corresponding order type.

2.2 Problems: Basic properties

2.5 Consider the set \( \mathbb{N} \to 2 \). For a typical member \( x(\cdot) \) of this set of functions let

\[
x = \sum_{r=0}^{\infty} x(r)2^{-r}
\]

to obtain a real number \( x \).

(a) Show that \( 0 \leq x \leq 2 \).

(b) Show that the assignment

\[
\begin{array}{ccc}
\mathbb{N} \to 2 & \longrightarrow & [0, 2] \\
x(\cdot) & \longmapsto & x
\end{array}
\]

is surjective.

(c) Show that

\[
x \leq y \iff (\forall r)[x(r) \leq y(r)]
\]

for all \( x, y \in [0, 2] \).

(d) Show that

\[
\begin{array}{ccc}
[0, 2] & \longrightarrow & [0, 3] \\
\phi & \longmapsto & 2 \sum_{r=0}^{\infty} x(r)3^{-r}
\end{array}
\]

is a linear embedding, and describe its range.

(e) As a linear set, where have you seen the range of \( \phi \) before.

2.6 Think about the function

\[
\begin{array}{ccc}
\mathbb{Q} & \longrightarrow & \mathcal{LQ} \\
f & \longmapsto & x
\end{array}
\]

given by Exercise 2.6(e).

(a) Show that this is an embedding.

(b) Give a concrete description of the linear set \( \mathcal{LQ} \) as a modified version of \( \mathbb{R} \).
3 Embedding properties

How might we prove Theorem 1.2? Part (i) is trivial (for those who can count). In this section we prove parts (ii) and (iii). To do that we set up an iteration procedure which produces an embedding bit by bit. To do that we need a notion of a ‘partial isomorphism’.

3.1 DEFINITION. Let $A, B$ be a pair of linear sets. A partial isomorphism from $A$ to $B$ is a pair of subsets

$$U \subseteq A \quad V \subseteq B$$

together with a bijection

$$f : U \longrightarrow V$$

which is an isomorphism when $U, V$ are viewed as linear sets, that is

$$x \leq a \iff f(x) \leq f(a)$$

for $a, x \in U$. ■

An few examples never goes amiss. Also part of the following will be useful later.

3.2 EXAMPLE. Notice that a full isomorphism

$$A \longrightarrow B$$

is also a partial isomorphism with $U = A$ and $V = B$.

A finite partial isomorphism from $A$ to $B$ is a pair of lists

$$a_1 < \cdots < a_m \quad b_1 < \cdots b_m$$

from the two sets where these are the same length and the bijection

$$a_i \longmapsto b_i$$

for $1 \leq i \leq m$ is a partial isomorphism.

We make use of finite partial isomorphism later. ■

A partial isomorphism is an embedding as far as it can be. In the notion above, the partial isomorphism $f$ embeds $U$ into $B$. The trick is to find a way of extending such a partial isomorphism a little bit further, and then repeat the trick a lot.

Here is the one step extension method.

3.3 LEMMA. Consider the situation on the left where $A$ is a linear set, $B$ is a line, and $f$ is a partial isomorphism between the finite sets $U$ and $V$. Consider also any element $a \in A$.

$$A \quad B$$

$$\downarrow \quad \downarrow$$

$$f \quad f$$

$$U \quad V$$

Then there is an element $b \in B$ together with a partial isomorphism $g$ which extends $f$ and with $g(a) = b$. 7
Proof. If \( a \in U \) then we take \( g = f \). Thus we may assume \( a \notin U \). This element \( a \) divides \( U \) into two parts \( X, Y \) given by

\[
x \in X \iff x < a \quad \text{and} \quad a < y \iff y \in Y
\]

for \( x, y \in U \). One of these parts may be empty, but it does no harm to assume that both are non-empty.

Since \( U \) is finite we see that \( X \) has a largest member \( l \), and \( Y \) has a smallest member \( s \). Since \( f \) is an isomorphism from \( U \) to \( V \) we have

\[ f(l) < f(s) \]

and these two elements split \( V \) into two parts. Since \( B \) is a line there is at least one element \( b \) with \( f(l) < b < f(s) \). With this we may set

\[ g(a) = b \]

for the required extension.

When one of \( X, Y \) is empty a minor variant of this construction works. \( \square \)

With this we can obtain a result that covers both parts (ii) and (iii) of Theorem 1.2.

3.4 THEOREM. Let \( A \) be a countable linear set, and let \( B \) be a line. Then there is an embedding \( f : A \rightarrow B \).

Proof. We first produce an increasing chain

\[ f_0 \leq f_1 \leq f_2 \leq \cdots \leq f_i \leq \cdots \quad i < \omega \]

where

\[ U_i \xrightarrow{f_i} V_i \]

for each index \( i < \omega \), and where

\[ A = \bigcup \{ U_i \mid i < \omega \} \quad \text{so that} \quad f = \bigvee \{ f_i \mid i < \omega \} \]

is the required embedding. In more detail we have

\[ U_i \subseteq U_{i+1} \quad f_i = f_{i+1}|_{U_i} \]

for each index \( i \).

To produce this chain we first select and

\[ a_0 \in A \quad b_0 \in B \]

and set

\[ U_0 = \{ a_0 \} \quad V_0 = \{ b_0 \} \quad \text{with} \quad f_0(a_0) = b_0 \]

and then we generate the rest of the chain by a certain recursion.
Since $A$ is countable it has an enumeration

$$(a_i \mid i < \omega)$$

where the order of this has nothing whatsoever to do with the given comparison in $A$. We use the first element $a_0$ to set up $f_0$, as above.

Now suppose we have

$$U_i \xrightarrow{f_i} V_i$$

where $U_i = \{a_0, \cdots, a_i\}$

at stage $i$. We then set

$$U_{i+1} = U_i \cup \{a_{i+1}\}$$

and use Lemma 3.3 to obtain $f_{i+1}$.

Since we use an enumeration of the whole of $A$ we see that the resulting $f$ is an embedding of $A$ into $B$. $\blacksquare$

Two particular cases of this result gives us parts (ii, iii) of Theorem 1.2. You should work out how this is achieved.

We haven’t yet obtained part (iv) of Theorem 1.2. To obtain that we need a further refinement of this embedding technique. That is the topic of the next section.

### 3.1 Exercises: Embedding properties

3.1 Using Theorem 3.4, how much of Theorem 1.1 can you prove.

3.2 A linear set $A$ is **homogeneous** if for each finite linear set $F$ and each pair of embeddings

$$F \xrightarrow{f} A \xleftarrow{g}$$

from $F$, there is an automorphism $h$ of $A$ with $f = h \circ g$.

(a) Show that each non-trivial homogeneous linear set $A$ is a line.

(b) Show that both $\mathbb{Q}$ and $\mathbb{R}$ are homogeneous

(c) Can you show that each line is homogeneous?

In the following two exercise we consider **equi-structures**

$$(A, \approx)$$

where $\approx$ is an equivalence relation on the set $A$. Recall that an equivalence relation is a binary relation that is reflexive, transitive, and symmetric. For each $a \in A$ we let

$$[a] = \{x \in A \mid x \approx a\}$$

be the block (equivalence class) in which $a$ lives.

Given two such equi-structures

$$(A, \approx) \quad (B, \approx)$$
a partial isomorphism from $A$ to $B$ is a bijection

$$f : U \rightarrow V$$

from some $U \subseteq A$ to some $V \subseteq B$ such that

$$a_1 \approx a_2 \iff f(a_1) \approx f(a_2)$$

for all $a_1, a_2 \in A$.

- If $U = A$ then this is an embedding of $A$ into $B$.
- If $U = A$ and $V = B$ then this is a (full) isomorphism from $A$ to $B$.

Finally, we say such a equ-structure $(A, \approx)$ is chubby if it has infinitely many blocks, and for each $a \in A$ the block $[a]$ is infinite.

### 3.3 Consider a partial isomorphism from a equ-structure $A$ to a chubby equ-structure $B$, as on the left. And suppose $U$ and $V$ are finite.

![Diagram of a partial isomorphism](image)

Show that for each $a \in A$ there is a partial isomorphism $g$ from $A$ to $B$ which extends $f$ and takes in $a$, as on the right.

### 3.4 By making use of Exercise 3.3, show that each countable equ-structure is embeddable in to each chubby equ-structure.

### 3.2 Problems: Embedding properties

#### 3.5 Let $A, B$ be a pair of linear sets and let

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
g & & \\
& & \\
\end{array}$$

be a pair of embeddings where

- $f[A]$ is an initial section of $B$
- $g[B]$ is a final section of $A$

respectively. By refining the proof of the CSB-result, show that $A \cong B$. 

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4 The back-and-forth technique

In this section we extend the embedding methods of Section 3 to produce a method of showing that two linear sets are isomorphic. In fact, this method can be used on many different algebraic systems, but here we concentrate on linear sets.

The general idea is to use a whole family of partial isomorphisms which interact in a certain way.

4.1 Definition. Let $A, B$ be a pair of linear sets.

(a) A back-and-forth system for the pair $A, B$ is a non-empty set $P$ of partial isomorphisms from $A$ to $B$ with the following two properties.

(back) For each member of $P$ 

$$f : U \longrightarrow V$$

and each element $y$ of $B$, there is an element $x$ of $A$ together with an extension

$$f^+ : U \cup \{x\} \longrightarrow V \cup \{y\}$$

of $f$ in $P$.

(forth) For each member of $P$

$$f : U \longrightarrow V$$

and each element $x$ of $A$, there is an element $y$ of $B$ together with an extension

$$f^+ : U \cup \{x\} \longrightarrow V \cup \{y\}$$

of $f$ in $P$.

We often abbreviate ‘back-and-forth’ to ‘b&f’.

(b) We write $A \sim_p B$ and say $A$ and $B$ are potentially isomorphic if there is at least one back-and-forth system for the pair $A, B$. ■

The name ‘back-and-forth’ system is the common terminology. However, Robin Gandy used to say ‘to-and-fro’ system, which perhaps is a better name. Robin Gandy was a student of Alan Turing, and had a big influence on developing Mathematical Logic in Manchester.

Notice that for each isomorphism

$$f : A \longrightarrow B$$

the singleton $\{f\}$ is a b&f system. This ensures that

$$A \cong B \implies A \cong_p B$$

holds. However, we are more interested in those systems $P$ where each component is a small part of $A, B$. Mostly we use finite partial isomorphisms.

A potential isomorphism does not require that the two linear sets have the same cardinality. Here is an important example of this.
4.2 **THEOREM.** Each two lines $A, B$ are potentially isomorphic.

**Proof.** Given the two lines $A, B$ let $\mathcal{P}$ be the family of all finite partial isomorphism from $A$ to $B$. We show that this $\mathcal{P}$ has the b&f property. In fact, by symmetry, it suffices to show that $\mathcal{P}$ has the forth property. This is precisely what Lemma 3.3 does. ■

As an example of this we have

$$Q \cong_p R$$

and these two lines do not have the same cardinality.

Trivially, each two isomorphic linear sets are potentially isomorphic but, as we have just seen, the converse is not true. However, when we restrict to the countable case the two notions are the same.

4.3 **THEOREM.** We have

$$A \cong_p B \implies A \cong B$$

for all countable linear sets $A, B$.

**Proof.** We are given a b&f system $\mathcal{P}$ for the two countable linear sets. We use $\mathcal{P}$ to generate a full isomorphisms $f$ from $A$ to $B$.

Since both $A$ and $B$ are countable there are enumerations

$$(x_i \mid i < \omega) \quad (y_i \mid i < \omega)$$

of these sets. These enumerations can be quite arbitrary with repetitions allowed. We produce an ascending chain

$$f_0 \subseteq f_1 \subseteq \cdots \subseteq f_i \subseteq \cdots \quad i < \omega$$

of partial isomorphism

$$f_i : U_i \longrightarrow V_i$$

in $\mathcal{P}$ and we arrange that

$$x_i \in U_{i+1} \quad y_i \in V_{i+1}$$

for each $i < \omega$. The union

$$f = \bigcup \{f_i \mid i < \omega\}$$

is then the required isomorphism.

Let $f_0$ be any member of $P$.

Suppose we have produced $f_i$. By first going forth (to deal with $x_i$) and then coming back (to deal with $y_i$) we obtain $f_{i+1}$. ■

Theorems 4.2 and 4.3 gives us the remaining part of Theorem 1.1.

4.4 **THEOREM.** Each two countable lines are isomorphic. In particular, each countable line is isomorphic to $Q$. 

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Proof. Let $A, B$ be two countable lines. By Theorem 4.2 we have $A \cong_p B$, and then Theorem 4.3 gives $A \cong B$, as required. \[\square\]

This back-and-forth technique is useful in several different areas. What I find odd is that it seems to have been devised quite recently in the 1950s or 1960s (which is not very long ago in terms of the development of Mathematics). It would be nice to find out when it first appeared. (Hint: don’t trust Wikipedia. I’m not even sure they can spell their own name.)

In the Problems for this section there are two more examples of the use of this technique.

4.1 Exercises: The back-and-forth technique

4.1 This exercise extends Exercises 3.3 and 3.4.

(a) Show that each pair of chubby equ-systems are potentially isomorphic (that is there are least one back-and-forth system between the pair).

(b) Show that each two countable and chubby equ-systems are isomorphic.

(c) Can you suggest a picture of the unique chubby (up to isomorphism) equ-system, say carried by $\mathbb{N} \times \mathbb{N}$.

4.2 Problems: The back-and-forth technique

In the context of the next two problems a graph is a set $A$ which carries a notion of ‘being mates’. We refer to the members of $A$ as nodes (but some fancy-pants prefer to call them vertices). For two nodes $x, y$ we write
to indicate that
respectively. We impose two conditions on being mates.

The relation is symmetric, that is
for all $x, y \in A$. The relation is irreflexive, that is
for all $x \in A$, so no node is a mate of itself.

Given two such graphs $A, B$ a partial isomorphism from $A$ to $B$ is a bijection
between subsets $U \subseteq A$ and $V \subseteq B$ such that

\[
x_1 \rightarrow x_2 \iff f(x_1) = f(x_2)
\]

\[
x_1 \leftarrow x_2 \iff f(x_1) \leftarrow f(x_2)
\]

for all $x_1, x_2 \in U$. Of course, by taking contrapostives each of these equivalences implies the other.

As usual, an embedding of $A$ into $B$ is a partial isomorphism where $U = A$. A full isomorphism is a partial isomorphism with $U = A$ and $V = B$.

We are interested in a special kind of graph. Thus a graph $A$ is random if it is infinite (or at least non-empty) and for each disjoint pair $L, R \subseteq A$ of finite sets of nodes we have

\[(\forall x \in L)[x \rightarrow a] \quad (\forall y \in R)[a \leftarrow y]\]

for at least one node $a \notin L \cup R$. It is sometimes convenient to write

\[L \rightarrow a \leftarrow R\]

for this condition.

4.2 Consider a finite partial isomorphism $f$, as above, between two graphs $A$ and $B$ where $B$ is random. Consider any $a \in A$.

(a) Show there is some $b \in B$ such that sending $a \longleftrightarrow b$ produces a partial isomorphism that extends $f$.

(b) Show that every countable graph is embedded in every random graph.

(c) Show that each pair of countable random graphs are isomorphic.

4.3 This exercise produces a concrete example of a countable random graph. It makes use of the hereditary finite sets that occur in other places in these various documents.

Let

\[V(0) = \emptyset \quad V(r + 1) = \mathcal{P}V(r)\]

for each $r < \omega$. This produces an ascending chain

\[V(0) \subseteq V(1) \subseteq \cdots \subseteq V(r) \subseteq \cdots\]

of finite sets. We let

\[V = \bigcup \{V(r) \mid r < \omega\}\]

to produce a family of sets. Each member $x \in V$ is finite, each member of that $y \in x \in V$ is finite, each member of that $z \in Y \in X \in V$ is finite, and so on.

For $x, y \in V$ we let

\[x \rightarrow y \iff x \in y \text{ or } y \in x \quad x \leftarrow y \iff x \notin y \text{ and } y \notin x\]

to produce a mate/non-mate relation on $V$.

Let $L, R$ be a disjoint pair of finite subsets of $V$. We show that

\[L \rightarrow a \leftarrow R\]

for infinitely many $a \in V$. To do that we consider $a = L \cup b$ for certain $b \in V$. 

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(a) Show that $L \rightarrow a$ for each such $b$.
(b) Show that if $R \cap b = \emptyset$ then
\[ y \in R \implies y \notin a \]
holds.
(c) Show there are only finitely many $b$ such that
\[ a \in y \in R \]
for at least one $y \in V$.
(d) Show that $a \not\rightarrow R$ for infinitely many $b \in V$.
(e) Show that $V$ is random.

5 The characteristic properties of $\mathbb{R}$

In this section we look at the characterization of $\mathbb{R}$ as given by Theorem 1.2. We can’t quite prove that result. Not because it is too difficult, but because the proof is a bit long and we have other things to do.

Theorem 1.2 contains two new notions that a linear ordered set may or may not have, that of being separable and that of being complete. The first of these is easy to understand. It is a refinement of being dense. To emphasize that we repeat the definition of being dense here.

5.1 DEFINITION. Let $A$ be a linear set (a linearly ordered set).

The linear set $A$ is dense if for each pair of elements $l < r$ there is some element $s \in A$ with $l < s < r$.

The linear set $A$ is separable if there is a countable subset $S \subseteq A$ such that for each pair of elements $l < r$ there is some element $s \in S$ with $l < s < r$. ■

Trivially, each separable linear set is dense. Also a countable linear set is dense precisely when it is separable. However, in general being dense does not ensure being separable. The reals $\mathbb{R}$ are separable since each pair of real numbers is separated by a rational.

Informally, being complete means that each ‘limiting process’ does produce a ‘limit’, in some sense or other. To make this precise there is a bit of a nuisance we have to sort out. Do we want our linear sets to have end points or not? Since here we are analysing $\mathbb{Q}$ and $\mathbb{R}$ it is best to insist that end points do not exists. To emphasize that we introduce a new word which is not standard terminology.

5.2 DEFINITION. A linswep is a linear set (linearly ordered set) without end points. ■

Each linear set we deal with here will be a linswep.

We come now to the crucial notion of being complete.

5.3 DEFINITION. Let $A$ be a linswep.

A cut of $A$ is a pair $L, U$ of non-empty subsets $L, U$ of $A$ such that
\[ L < U \]
that is $x \leq y$ for all $x \in L$ and $y \in U$.

The linswep $A$ is **complete** (or D-complete, or Dedekind-complete) if for each cut $L < U$ we have

$$ L \leq a \leq U \quad \text{that is} \quad L \subseteq \downarrow a \quad U \subseteq \uparrow a $$

for some $a \in A$. ■

This gives us the two new properties of Theorem 1.2. They are designed to characterize $\mathbb{R}$ so we certainly need the following.

5.4 **THEOREM.** As a linswep the reals $\mathbb{R}$ is complete.

We won’t prove this here but I will mention two possible proofs.

If you are Dedekind then this is the way $\mathbb{R}$ is defined. It is the unique ‘completion’ of $\mathbb{Q}$. I will explain what that means in more detail later, if we have time.

If you are Cauchy then you prove it is complete. You show that each cauchy sequence has a limit, and you are done.

This is not a conflict. We may define $\mathbb{R}$ as the completion of $\mathbb{Q}$, and then derive various general properties of $\mathbb{R}$, as we might do here. However, for various particular calculations, in real analysis, use of differential equations, and so on, this definition is not very helpful. In such situations we work in terms of sequences and use cauchy completeness.

We know that $\mathbb{Q}$ is inserted in $\mathbb{R}$

$$ \mathbb{Q} \xrightarrow{i} \mathbb{R} $$

and we are aiming to determine the crucial properties of this function. To do that we need the following notions.

5.5 **DEFINITION.** Let $A$ and $B$ be a pair of linsweps.

An **embedding** of $A$ into $B$ is a function

$$ f : A \longrightarrow B $$

such that

$$ a_1 \leq a_2 \iff f(a_1) \leq f(a_2) $$

for all $a_1, a_2 \in A$.

An embedding $f$, as above, is **dense** if for each $b_1 < b_2$ in $B$ we have

$$ b_1 < f(a) < b_2 $$

for some $a \in A$. ■

Observe that an embedding is an injection. If it also a surjection then it is an isomorphism. In fact, an embedding is an isomorphism to its range. Observe also that if there is a dense embedding of $A$ into $B$ then $B$ is dense (as a linear order). If $A$ is countable the $B$ is separable.

Of course, the standard example of such a dense embedding is the rationals $\mathbb{Q}$ in the reals $\mathbb{R}$.

As we will see, the following generalizes the properties of this embedding.
$A, B, C$ are linsweps

where

$f, g$ are embeddings

Table 1: The wedge and the factorization problem

The problem is to factorize $g$ through $f$, that is to produce an embedding

$$B \xrightarrow{h} C$$

for which $g = h \circ f$.

5.6 DEFINITION. Let $A$ be any linswep. A completion of $A$ is a dense embedding

$$A \longrightarrow DA$$

into a complete linswep.

Observe that a completion of a linswep $A$ is not just a special kind of linswep. It is such a linswep $DA$ and a dense embedding to connect the two. As a particular example the embedding

$$\mathbb{Q} \xleftarrow{i} \mathbb{R}$$

is a completion of $\mathbb{Q}$.

We ought to sort out the existence and uniqueness of a completion. We can’t go into the details, but we can state the appropriate results. The following result deals partly with the required uniqueness and existence.

5.7 LEMMA. Consider the wedge of Table 1

(a) Suppose the embedding $g$ is dense. Then there is at most one embedding $h$ with $g = h \circ f$.

(b) Suppose the embedding $f$ is dense, and the linswep $C$ is complete. Then there is at least one embedding $h$ with $g = h \circ f$.

These help us to produce the following important result.

5.8 THEOREM. Let $A$ be a linswep

(Existence) There is at least one dense embedding

$$A \longrightarrow DA$$

into a complete linswep. That is $A$ does have a completion.
(Uniqueness) Consider the situation of Table 1. Suppose that both \( f, g \) are completions of \( A \). Then there is a unique embedding

\[
B \xrightarrow{h} C
\]

with \( g = h \circ f \). Furthermore, this embedding is an isomorphism. Thus \( A \) has a unique completion up to isomorphism.

With these general results we can verify the four parts of Theorem 1.2.

\( \mathbb{R} \) is a separable complete line

By Theorem 5.4 we observed that \( \mathbb{R} \) is complete. The embedding of \( \mathbb{Q} \) into \( \mathbb{R} \) ensures that \( \mathbb{R} \) is separable.

\( \mathbb{R} \) is embedded in every complete line

Let \( C \) be any complete line. We first set up a diagram as follows.

\[
\begin{array}{ccc}
Q & \xrightarrow{f} & \mathbb{R} \\
\downarrow{g} & & \downarrow{h} \\
C & & \mathbb{R}
\end{array}
\]

At the top we use the canonical embedding \( f \) of \( \mathbb{Q} \) into \( \mathbb{R} \). We know this is a dense embedding, since \( \mathbb{Q} \) is dense in \( \mathbb{R} \). Since \( C \) is a line, Theorem 1.1(ii) gives us an embedding \( g \). With these Lemma 5.7(b) gives an embedding

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{h} & C \\
\end{array}
\]

with \( g = h \circ f \). This is what we want.

Each separable line is embedded in \( \mathbb{R} \)

Let \( B \) be any separable line. We first produce a diagram

\[
\begin{array}{ccc}
Q & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{} \\
\mathbb{R} & & \mathbb{R}
\end{array}
\]

and then apply Theorem 5.8 to obtain an embedding of \( B \) into \( \mathbb{R} \). Here \( g \) is the canonical embedding of \( \mathbb{Q} \) into \( \mathbb{R} \). The problem is to produce a dense embedding \( f \), and this is where the separability of \( B \) is used.

By definition, since \( B \) is separable, there is a countable subset \( X \subseteq B \) which is dense in \( B \), that is

\[
a < b \implies (\exists x \in X)[a < x < b]
\]
for all elements \( a, b \in B \). The set \( X \) is certainly a linear set (using the restriction of the comparison on \( B \)). The separable property applied to elements of \( X \) shows that it is a line. But now \( X \) is a countable line, and hence isomorphic to \( \mathbb{Q} \) by Theorem 1.1(iv). This gives the required dense embedding \( f \).

Each pair of separable complete lines are isomorphic

As we have just observed a line \( C \) being separable means there is an embedding

\[
\mathbb{Q} \longrightarrow C
\]

and so Theorem 5.8 ensures that any two complete and separable lines are isomorphic.

At this point we will leave the analysis of \( \mathbb{Q} \) and \( \mathbb{R} \). Later we may come back to fill in the missing proofs to show how an arbitrary linswep can be completed.