Point-sensitive and point-free patch constructions

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Abstract

Using the category of frames we consider various generalizations of the patch space of a topological space. Some of these constructions are old and some are new. We consider how these variants interact and under what circumstances they agree.

Contents

1 Introduction

We are concerned with a certain 'desirable' property of topological spaces, namely being packed. Put differently, not being packed may be regarded as a defect that ought to be corrected. To explain this notion we need a bit of a preamble.

Each space S carries its specialization order, a comparison of its points obtained from inclusion of point closures. The saturated sets are the upper sections of this comparison. Every open set is saturated, but there can be many non-open saturated sets. In particular, if the space is T_1 then every subset is saturated.

We say a topological space S is packed if each of its compact saturated subsets is closed. Each T_0 space that is packed is automatically T_1 (and so the saturated aspect disappears). Each T_2 space is packed and sober but there are T_0+sober+packed spaces that are not T_2. The maximal compact topology, Example 99 of [21], gives such a space.

As an attempt to correct the defect of a non-packed space S we consider a larger topology on the same carrying set S where we declare that certain compact sets should become closed. The patch topology on S is the smallest topology which includes the original topology $\mathcal{O}S$ and for which each compact saturated set (of the original space) is now closed. This gives us the patch space ${}^{p}S$ of S.

A space S is packed precisely when ${}^{p}S = S$. When S is not packed the space ${}^{p}S$ goes some way towards correcting the defect. As we see in Section 6, the space ${}^{p}S$ need not be packed, but a second application of the construction does produce a packed space ${}^{pp}S$. However, if we also require the correction to be sober, then we can be a long way short.

This patch construction is point-sensitive. That is, it makes explicit reference to the points of the space. Quite a lot of topology can be done in point-free fashion by concentrating on the algebraic properties of the topology (as a lattice). So is there a point-free analogue of this patch construction?

A frame is a particular kind of complete lattice. These form the objects of a category $\boldsymbol{Frm}$ (using the appropriate arrows). For each space S the topology $\mathcal{O}S$ is a frame, and this is the object assignment of a contravariant functor

$$\boldsymbol{Top} \longrightarrow \boldsymbol{Frm} \tag{1}$$

from the category $\boldsymbol{Top}$ of spaces (and continuous maps) to $\boldsymbol{Frm}$. There is also a contravariant functor in the opposite direction which converts each frame into its point space.

A decent amount of the analysis of $\boldsymbol{Top}$ can be done using $\boldsymbol{Frm}$, and is often easier this way. However, there are many objects of $\boldsymbol{Frm}$ which are not topologies. Indeed, there are objects in $\boldsymbol{Frm}$ for which it is unwise to even think about in spatial terms.

Frames have a reasonably straightforward universal algebra. Each quotient of a frame A is codified by a certain operator, a nucleus, on A. The set NA of all such nuclei is partially ordered in a natural way, and forms a frame. There is a canonical embedding

$$A \longrightarrow NA \tag{2}$$

which is an isomorphism exactly when A is boolean. In general, NA is rather complicated.

It has been known for many years that for a spectral space S the patch topology $\mathcal{O}^pS$ is embedded in $N\mathcal{O}S$ in a canonical fashion. In [12] Karazieris isolated the range of this embedding as the Scott continuous nuclei. Building on this, in [3] and [4], Escardó showed that for any frame A the set MA of continuous nuclei is a subframe of NA, and termed this 'the patch frame of A'. In [4] he showed that for the category of stably locally compact frames, the construction MA coincides with the topology of the standard patch space of the point space of A, and M provides a reflection into a certain category of regular frames.

In this paper we describe a more general point-free construction $A \longmapsto PA$. This mimics quite closely the point-sensitive construction; it agrees with $A \longmapsto MA$ on the stably locally compact frames of [4]; and it has some general functorial properties.

Many of the results given here are are not constructive. There are several explicit uses of Zorn's Lemma, and some hidden uses as well.

This work arose from a detailed reading of [3] and [4], and from various conversions with Escardó. We thank him for that.

We also thank the referee of an earlier version of this paper. He indicated several crucial improvements. He also pointed out that some 'well-known' results are not as well known as they ought to be.

Preliminary versions of some of these results are contained in [15], and are developed further in [16]. Some were presented at a PSSL held in Cambridge in November 2000.

2 Basic material

We set down the basic information we require about frames. Most of the missing details can be found in [11]. A more leisurely and up to date account can be found in [20].

A frame is a complete lattice

$$(A, \leq, \top, \wedge, \bot, \bigvee)$$

which satisfies a certain distributive law. Equivalently, a frame is a complete lattice which carries a certain binary operation $(\cdot \supset \cdot)$, its implication operation.

A morphism (that is, a frame morphism)

$$A \xrightarrow{\ f\ } B \tag{3}$$

between frames A and B, is function f, as indicated, which respects the distinguished attributes. In particular, f preserves arbitrary suprema, but need not preserve arbitrary infima (although it will preserve finitary infima).

This gives us the category ***Frm*** of frames and morphisms.

As a monotone map each morphism $f = f^*$ has a right adjoint f_*

$$A \underset{f_*}{\overset{f^*}{\rightleftarrows}} B \tag{4}$$

where this need not be a morphism. It will preserve arbitrary infima, but need not preserve even binary suprema. When we are concerned with a morphism and its adjoint $f^* \dashv f_*$, we use an affix to indicate which is which. When we are concerned only with the morphism f, we often drop the affix.

Let ***Top*** be the category of (topological) spaces and (continuous) maps. For each space S its topology $\mathcal{O}S$ is a frame (with set theoretical attributes). Each map

$$S \xleftarrow{\ \phi\ } T \tag{5}$$

produces a morphism

$$\mathcal{O}S \underset{\phi_*}{\overset{\phi^*}{\rightleftarrows}} \mathcal{O}T \tag{6}$$

between the topologies given by

$$\phi^*(U) = \phi^{\leftarrow}(U) \qquad \phi_*(W) = \phi^{\rightarrow}(W')^{-\prime}$$

for each $U \in \mathcal{O}S$ and $W \in \mathcal{O}T$. Here $\phi^{\leftarrow}(\cdot)$ is the inverse image function across ϕ and $\phi^{\rightarrow}(\cdot)$ is the direct image function across ϕ. We write

$$(\cdot)^- \quad (\cdot)^\circ \quad (\cdot)'$$

for the closure operation, interior operation, and complementation operation on a space. Notice that to calculate ϕ_* we need an internal and an external complementation.

This sets up a contravariant functor (1). Its contravariant adjoint is a bit more involved.

A filter F on a frame A is prime if it is proper (does not contain $\bot$) and satisfies

$$a \vee b \in F \Longrightarrow a \in F \text{ or } b \in F$$

(for each $a, b \in A$). We need two other kinds of filters F, both determined by a property

$$\bigvee X \in F \Longrightarrow X \text{ meets } F \tag{7}$$

for certain subsets X of A. A filter F is open if (7) holds for all directed subsets X of A. A filter F is completely prime if it is proper and (7) holds for all subsets X of A. We allow the improper filter to be open, but a completely prime filter must be proper. Thus a filter is completely prime if and only if it is prime and open.

The word 'open' here is an abbreviation of 'Scott open', in other words it means open in the Scott topology carried by the frame. We will make other uses of Scott topologies and invariably we will drop the qualifier 'Scott'.

Two filters F and G can be combined to give the meet $F \cap G$ which is just the intersection, and the join $F \vee G$ which is more that just the union. If F and G are open, then so is $F \cap G$, but $F \vee G$ need not be. Given a directed set $\mathcal{F}$ of filters, the union $\bigcup \mathcal{F}$ is a filter. If each member of $\mathcal{F}$ is open, then so is $\bigcup \mathcal{F}$.

Filters can be transfered across a morphism $f^* \dashv f_*$, as in (4), in either direction.

2.1 DEFINITION. Consider a morphism (4). Given filters F on A and G on B we use

$$b \in f^*F \Longleftrightarrow f_*(b) \in F \qquad a \in f_*G \Longleftrightarrow f^*(a) \in G$$

(for $a \in A$ and $b \in B$) to obtain filters f^*F on B and f_*G on A, respectively. ■

Notice that f^*F is just the filter on B generated by the direct image of F across f^*.

Almost trivially, we have implications

$$G \text{ prime} \Longrightarrow f_*G \text{ prime} \qquad G \text{ open} \Longrightarrow f_*G \text{ open} \qquad G \text{ c. prime} \Longrightarrow f_*G \text{ c. prime}$$

and less trivially these are not equivalences,

2.2 DEFINITION. Consider a morphism $f^* \dashv f_*$, as in (4).

(a) The morphism f^* converts open filters if f^*F is open on B whenever F is an open filter on A.

(b) The right adjoint f_* is continuous if

$$f_*\left(\bigvee Y\right) = \bigvee f_*^{\rightarrow}(Y)$$

holds for each directed subset $Y \subseteq B$. ■

These two properties are related. The following is proved by a routine argument.

2.3 LEMMA. *(a) If the right adjoint f_* of a morphism is continuous, then the morphism f^* converts open filters.*

(b) Let ϕ be a map, as in (5). The right adjoint ϕ_* is continuous exactly when the left adjoint ϕ^* converts open filters.

This result gives a nice illustration of how misleading the point-sensitive view can be. In Theorem 3.1 and its associated Example 3.2 we see that the canonical frame morphism (8) below always converts open filters but need not have a continuous right adjoint.

In Proposition 3.3 of [8], Hofmann and Lawson characterize those maps ϕ between sober spaces for which the right adjoint ϕ_* is continuous. We don't need that result here.

Each frame A has a point space $S = \mathsf{pt}(A)$ with a nominated surjective morphism

$$A \xrightarrow{\ U_A\ } \mathcal{O}S \tag{8}$$

indexing the topology. The construction $A \longmapsto \mathsf{pt}(A)$ is functorial, and the morphism U_A is natural for variation of A. In fact, U_A reflects A into a topology.

The points of $S = \mathsf{pt}(A)$ have three different guises. Each can be viewed as a character of A, that is a morphism $A \longrightarrow 2$ to the 2-element frame, or as a completely prime filter P on A, or as a $\wedge$-irreducible element p of A. The use of characters ensures that the whole set up is schizophrenically induced, and so makes most of the functorial properties routine. However, when calculating in a particular frame the use of $\wedge$-irreducible elements, and occasionally completely prime filters, is more convenient. In these terms we have

$$p \in U_A(a) \Longleftrightarrow a \nleq p \qquad P \in U_A(a) \Longleftrightarrow a \in P$$

for each $a \in A$, $\wedge$-irreducible element p, and completely prime filter P.

Here we routinely view the points of a frame A as its $\wedge$-irreducible elements. Each space S carries its specialization ordering $\sqsubseteq$ given by

$$s \sqsubseteq t \Longleftrightarrow s \in t^-$$

for $s, t \in S$. When $S = \mathsf{pt}(A)$ we have

$$s \sqsubseteq t \Longleftrightarrow t \leq s$$

that is the specialization order on $\mathsf{pt}(A)$ is the reverse of the comparison inherited from A.

Each morphism $f^* \dashv f_*$, as in (4), induces a map, as in (5), between the point spaces $S = \mathsf{pt}(A)$ and $T = \mathsf{pt}(B)$. This is given by

$$\phi(q) = f_*(q) \qquad \phi(Q) = f_*Q$$

for each $\wedge$-irreducible $q \in B$ or completely prime filter Q on B.

There are times when we need to know that a frame has enough points (to be spatial or do a similar job). In general, this requires a choice principle. More often than not the following result provides the points we need.

2.4 THEOREM. (The frame separation principle) *Let $a \in A - F$ where F is an open filter on the frame A. Then there is a completely prime filter P on A with $a \notin P$ and $F \subseteq P$.*

This, of course, is a choice principle. Its proof is a simple application of Zorn's Lemma. We use Theorem 2.4 several times, and there are other uses of ZL.

3 The assembly

An inflator on a frame A is a monotone and inflationary function $f : A \longrightarrow A$. A pre-nucleus on A is an inflator f such that

$$f(a \wedge b) = f(a) \wedge f(b)$$

for each $a, b \in A$. A **nucleus** is a pre-nucleus j which is idempotent, that is $j^2 = j$.

For each morphism $f^* \dashv f_*$ the kernel

$$\ker(f^* \dashv f_*) = f_* \circ f^*$$

is a nucleus on A, and every nucleus arises in this way. Given any nucleus j on a frame A, the set A_j of elements fixed by j can be structured as a frame together with a morphism $A \longrightarrow A_j$ for which j is the kernel.

The set of all inflators on A is partially ordered by the pointwise comparison. For each set F of inflators, the pointwise infimum $\bigwedge F$ of F given by

$$\left(\bigwedge F\right)(x) = \bigwedge\{f(x) \mid f \in F\}$$

is an inflator, and is the infimum of F in the poset of all inflators. If F is a set of pre-nuclei, then $\bigwedge F$ is a pre-nucleus, and if F is a set of nuclei, then $\bigwedge F$ is a nucleus. Thus we have three complete lattices of gadgets associated with A. We are concerned mainly with the set NA of all nuclei on A. The other gadgets are useful computational devices.

The most important fact about NA is that it is a frame, as can be seen by exhibiting the implication on NA. We call NA the **assembly** of A.

Infima in NA are computed pointwise. However, suprema, even binary suprema, are a different matter. Whenever they can be calculated, suprema in NA seems to have more to do with composition than with the pointwise suprema. A simple exercise shows that

$$k \circ j \leq j \circ k \Longrightarrow j \vee k = j \circ k \tag{9}$$

holds for nuclei j, k. To go further we use inflators and pre-nuclei.

For a *directed* set F of inflators, the pointwise supremum $\dot{\bigvee} F$ of F given by

$$\left(\dot{\bigvee} F\right)(x) = \bigvee\{f(x) \mid f \in F\}$$

is an inflator, and is the supremum of F in the poset of inflators. If F is a set of pre-nuclei, then $\dot{\bigvee} F$ is a pre-nucleus. For each inflator f the ordinal iterates f^α are generated by

$$f^0 = id_A \qquad f^{\alpha+1} = f \circ f^\alpha \qquad f^\lambda = \dot{\bigvee}\{f^\alpha \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ. On cardinality grounds there is some ordinal ∞ such that f^∞ is idempotent, and then this is the least closure operation above f. When f is a pre-nucleus, each iterate is a pre-nucleus, and the closure is the least nucleus above f. The closure f^∞ has exactly the same fixed elements as f.

For an arbitrary family J of nuclei we have

$$\bigvee J = \left(\dot{\bigvee} J^\circ\right)^\infty$$

where J° is the compositional closure of J and ∞ is a suitable ordinal. This ordinal can get arbitrarily large and, of course, the members of J° need not be nuclei (only pre-nuclei).

For each $a \in A$ we set

$$u_a(x) = a \vee x \qquad v_a(x) = a \supset x \qquad w_a(x) = (x \supset a) \supset a$$

(for each $x \in A$) to obtain a triple u_a, v_a, w_a of nuclei, each of which has a role to play.

We have

$$j \circ v_a \leq v_a \circ j \qquad u_a \circ j \leq j \circ u_a$$

and hence (9) gives

$$v_a \vee j = v_a \circ j \qquad j \vee u_a = j \circ u_a$$

for each $j \in NA$. In particular, for each $a \in A$ the nuclei u_a, v_a are complementary elements of NA. For each $j \in NA$ we have

$$j = \bigvee \{ u_{j(a)} \wedge v_a \mid a \in A \} \tag{10}$$

so that these simple nuclei generate NA. The w-nuclei will be used later.

The assignment

$$\begin{array}{ccc} A & \xrightarrow{\;n_A\;} & NA \\ a & \longmapsto & u_a \end{array}$$

is the embedding (2). It is an injective epic and provides a neat characterization of NA.

We say a frame morphism f, as in (3), solves the complementation problem for A if for each $a \in A$ the element $f(a) \in B$ has a complement in B. Since the nuclei u_a, v_a arising from $a \in A$ are complementary in NA, we see that the embedding n_A solves the complementation problem for A. More importantly, by a more delicate argument, it *universally* solves the complementation problem for A. That is, for each other solution f there is a unique morphism

$$NA \xrightarrow{\;g\;} B$$

with $f = g \circ n_A$.

As a consequence of this for each morphism f, as in (3), there is a commuting square

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ {\scriptstyle n_A}\downarrow & & \downarrow{\scriptstyle n_B} \\ NA & \xrightarrow[N(f)]{} & NB \end{array} \tag{11}$$

for some unique morphism $N(f)$, as indicated, In particular, N is a functor on ***Frm*** and the embedding n_A is natural for variation of A.

The behaviour of $N(f)$ on a nucleus $j \in NA$ can be described in several ways. Here it suffices to know that

$$N(f)(u_a) = u_{f(a)} \qquad N(f)(v_a) = v_{f(a)} \tag{12}$$

for each $a \in A$. The left hand identity is just the naturality of n_A, and the right hand identity follows since v_a is the complement of u_a.

Using (10) we have

$$N(f)(j) = \bigvee \{ u_{f(j(a))} \wedge v_{f(a)} \mid a \in A \} \tag{13}$$

for each $j \in NA$.

3.1 THEOREM. *For a frame A with point space $S = \mathsf{pt}(A)$, the morphism U_A of (8) converts open filters. However, it need not have a continuous right adjoint.*

Proof. Let F be an open filter on A, and let $\nabla = U_A^* F$. We must show that ∇ is open in $\mathcal{O}S$. To this end consider any directed family $\mathcal{U}$ of open subsets of S with $\bigcup \mathcal{U} \in \nabla$. We must show that $\mathcal{U}$ meets ∇. Consider $X \subseteq A$ given by

$$x \in X \Longleftrightarrow U_A(x) \in \mathcal{U}$$

(for $x \in A$). This X indexes $\mathcal{U}$ (with some repetition). We show that X meets F, and hence $\mathcal{U}$ meets ∇.

Let s be the kernel of U_A. Thus

$$y \leq s(x) \Longleftrightarrow U_A(y) \subseteq U_A(x)$$

and

$$U_A(x) = U_A(s(x))$$

for all $x, y \in A$. In particular, we have

$$x \in X \Longleftrightarrow s(x) \in X$$

for $x \in A$. Using these facts we first check that X is directed.

Consider $x, y \in X$. We have $U_A(x), U_A(y) \in \mathcal{U}$, and hence, since $\mathcal{U}$ is directed, we have

$$U_A(x), U_A(y) \subseteq U_A(z) = U_A(s(z))$$

for some $z \in X$. In particular, we have $s(z) \in X$ and the characterizing property of s gives $x, y \leq s(z)$, to produce the required upper bound of x, y in X.

Now let $a = \bigvee X$. Since X is directed and F is open, it suffices to show that $a \in F$.

By way of contradiction, suppose $a \notin F$. A use of the frame separation principle, Theorem 2.4, gives some completely prime filter P with $a \notin P$ and $F \subseteq P$. In particular, we have $P \notin U_A(a) = U_A(s(a))$, so that $s(a) \notin P$.

Since $a = \bigvee X$, we have $U_A(a) = \bigcup \mathcal{U} \in \nabla = U_A^* F$, so that $U_A(a) = U_A(b)$ for some $b \in F$. But now $U_A(b) \subseteq U_A(a)$ to give $b \leq s(a)$ so that $s(a) \in F \subseteq P$, which is the contradiction.

This proves the positive part. The negative part deserves a number of its own. ■

To show that the spatial reflection morphism need not have a continuous right adjoint we use a non-trivial frame with no points. Such frames are easy to come by, for instance any atomless complete boolean algebra will do. There are also some quite exotic examples.

3.2 EXAMPLE. Let B be any non-trivial frame with no points. Let b be the bottom element of B. Dangle from b a tail T consisting of an ascending ω-chain of elements with supremum b. This gives a new frame A. Let $U^* = U_A$ be the spatial reflection morphism of A, and let U_* be its right adjoint.

By way of contradiction, suppose U_* is continuous. Then the kernel $s = U_* \circ U^*$ of U_A is continuous (since the left adjoint U^* is always continuous). Thus we have

$$s\left(\bigvee X\right) = \bigvee s^{\rightarrow}(X)$$

for each directed subset X of A.

Each $t \in T$ is a point of A, so that $s(t) = t$, and hence we have the contradictory

$$b = \bigvee T = \bigvee s^{\rightarrow}(T) = s\left(\bigvee T\right) = s(b) = \top$$

where the right hand equality holds since B has no points. ■

4 The functorial diagram

We can attach to each frame A several other frames: the assembly NA of all nuclei on A, the topology $\mathcal{O}S$ of the point space $S = \mathsf{pt}(A)$ of A, the assembly $N\mathcal{O}S$ of this topology, and the topologies $\mathcal{O}\mathsf{pt}(NA)$ and $\mathcal{O}\mathsf{pt}(N\mathcal{O}S)$ of the point space of each of the two assemblies. Each of these five constructions is the object assignment of an endo-functor on ***Frm***, and there are several connecting natural transformations. In this section we gather together all the relevant information. Most of this comes from [18], [19], and [14]. A more coherent and fuller account can be found in [20].

What is the point space $\mathsf{pt}(NA)$ of the assembly of A? We look first at a restricted case, the point space $\mathsf{pt}(N\mathcal{O}S)$ of the assembly of the topology of a (sober) space.

Recall that the embedding

$$\mathcal{O}S \xrightarrow{\;n_{\mathcal{O}S}\;} N\mathcal{O}S$$

universally solves the complementation problem for $\mathcal{O}S$. There is also a rather crude way to solve this problem.

The space S can be re-topologized by declaring that each originally open set $U \in \mathcal{O}S$ should become clopen. This gives the front space ${}^{f}S$ of S (on the same set of points) with its front topology $\mathcal{O}^{f}S$ (which is sometimes called the Skula topology on S). We have $\mathcal{O}S \subseteq \mathcal{O}^{f}S$ and the set of all

$$U \cap s^{-}$$

(for $U \in \mathcal{O}S$ and $s \in S$) forms a base for ${}^{f}S$. We write

$$(\cdot)^{=} \qquad (\cdot)^{\square}$$

for the front closure and front interior, respectively. Notice that ${}^{f}S$ is discrete precisely when S is T_D, a separation property that seems to crop up all over the place.

Trivially, the insertion $\mathcal{O}S \hookrightarrow \mathcal{O}^{f}S$ solves the complementation problem for $\mathcal{O}S$, and so factors uniquely through the embedding $n_{\mathcal{O}S}$. We can write down the appropriate map.

4.1 DEFINITION. For a space S and nucleus $j \in N\mathcal{O}S$ let $\sigma_S(j) \subseteq S$ be given by

$$s \in \sigma_S(j) \iff s \in j(s^{-\prime})$$

(for $s \in S$). ■

A few simple calculations give

$$\sigma_S(j) = \bigcup\{j(W) \cap W' \mid W \in \mathcal{O}S\}$$

and so we have an assignment

$$NOS \xrightarrow{\sigma_S} \mathcal{O}^f S$$

which is a $\wedge$-semilattice morphism. We exhibit the right adjoint of this monotone map, and hence show that σ_S is a frame morphism.

Each frame A carries two families u_a, v_a of simple nuclei. When A is a topology $\mathcal{O}S$ these have a common extension.

4.2 DEFINITION. For each subset $E \subseteq S$ of a space S let

$$[E](U) = (E \cup U)^\circ$$

for each $U \in OS$ to produce a spatially induced nucleus $[E]$ on OS. ■

It is routine to check that each $[E]$ is a nucleus on $\mathcal{O}S$. For each $W, U \in \mathcal{O}S$ we have

$$[W](U) = (W \cup U)^\circ = (W \cup U) \qquad [W'](U) = (W' \cup U)^\circ = (W \supset U)$$

so that

$$[W] = u_W \qquad [W'] = v_W \tag{14}$$

to illustrate the earlier claim.

Each map ϕ between spaces, as in (5), gives us an adjoint pair $\phi^* \dashv \phi_*$ between the topologies, as in (6). It can be checked that

$$\ker(\phi^* \dashv \phi_*) = \phi_* \circ \phi^* = [E]$$

where $E = S - \phi^{\rightarrow}(T)$. Furthermore, each nucleus $[E]$ arises in this way from a map, for we simply take the subspace topology on $S - E$.

The following goes back to Lemma 14 of [2].

4.3 LEMMA. *The equivalence*

$$[E] = [F] \Longleftrightarrow E^\square = F^\square$$

holds for all subsets E, F of the space S.

We have $\sigma([E]) = E^\square$ for each $E \subseteq S$, and a few simple calculations gives the following.

4.4 THEOREM. *For each space S the assignments*

$$NOS \underset{[\,\cdot\,]}{\overset{\sigma_S}{\rightleftarrows}} \mathcal{O}^f S$$

form a morphism and its adjoint, $\sigma \dashv [\,\cdot\,]$, where the right adjoint is injective, and the composite $\sigma \circ [\,\cdot\,]$ is the identity on $\mathcal{O}^f S$.

For an arbitrary frame A let $S = \mathsf{pt}(A)$. From the naturality of $n_\bullet$ we obtain a commuting diagram

$$\begin{array}{ccc} A & \xrightarrow{n_A} & NA \\ {\scriptstyle U_A}\downarrow & & \downarrow{\scriptstyle N(U_A)} \\ \mathcal{O}S & \hookrightarrow & N\mathcal{O}S \\ & & \downarrow{\scriptstyle \sigma_S} \\ & {\scriptstyle \iota_S} & \mathcal{O}^f S \end{array}$$

where the unnamed arrow is n_{OS} and ι_S is the insertion. Each horizontal arrow and ι_S is an embedding, and each vertical arrow is surjective. This is the full assembly diagram for A. For convenience let

$$\Sigma_A = \sigma_S \circ N(U_A)$$

so that Σ_A is right hand composite of the assembly diagram. The subscripting here indicates that these arrows are natural. We can, of course, drop the suffix when there is only one parent gadget around.

What is the significance of the space ${}^f S$? To answer that we use the w-nuclei introduced in Section 3. The following appears as Lemma 3.2 in [14].

4.5 LEMMA. *For each frame A and nucleus $\ell \in NA$ the following are equivalent.*
(i) ℓ is a point of NA (viewed as a $\wedge$-irreducible).
(ii) ℓ is 2-valued, that is $\ell(\bot) \neq \top$ and these are the only two values of ℓ.
(iii) $\ell = w_s$ for some $s \in \mathsf{pt}(A)$.

In other words, we have a bijection

$$\begin{array}{ccc} \mathsf{pt}(A) & \longrightarrow & \mathsf{pt}(NA) \\ s & \longmapsto & w_s \end{array}$$

between the two sets of points. The canonical topology on $\mathsf{pt}(NA)$ induces a topology on $\mathsf{pt}(A)$ to convert the bijection into a homeomorphism. We find that ${}^f\mathsf{pt}(A)$ is the new space on the old points of A, and with a bit more work obtain the following.

4.6 THEOREM. *For each frame A with point space $S = \mathsf{pt}(A)$, the front space ${}^f S$ is the point space of the assembly NA. Furthermore, the composite Σ_A is just the indexing morphism U_{NA}. In particular, for each sober space S, the front space ${}^f S$ is the point space of the assembly $N\mathcal{O}S$, and σ_S is the indexing morphism.*

Let us check that Σ_A is the indexing morphism U_{NA}. By Lemma 4.5 we have

$$s \in U_{NA}(j) \Longleftrightarrow j \not\leq w_s$$

for each $s \in S$ and $j \in NA$. In particular

$$s \in (U_{NA} \circ n_A)(a) \Longleftrightarrow u_a \not\leq w_s \Longleftrightarrow a \not\leq s \Longleftrightarrow s \in U_A(a) \Longleftrightarrow s \in (\iota_S \circ U_A)(a)$$

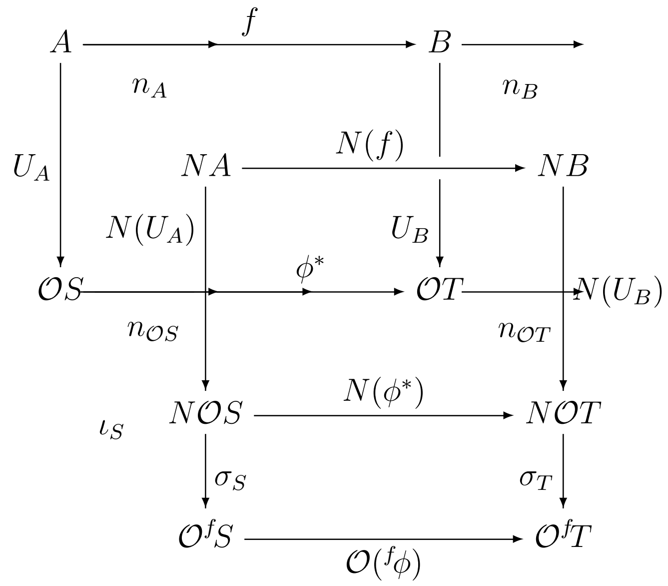

Table 1: The functorial diagram

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for each $s \in S$ and $a \in A$. Thus, using the commuting diagram, we have

$$U_{NA} \circ n_A = \iota_S \circ U_A = \Sigma_A \circ n_A$$

and hence $U_{NA} = \Sigma_A$ since n_A is epic.

We need a couple more observations about this diagram.

We have

$$\left.\begin{aligned} j &= \bigvee\{u_{j(a)} \wedge v_a \mid a \in A\} \\ N(U_A)(j) &= \bigvee\Big\{[U_A(j(a)) \cap U_A(a)'] \mid a \in A\Big\} \\ \Sigma_A(j) &= \bigcup\Big\{U_A(j(a)) \cap U_A(a)' \,\Big|\, a \in A\Big\} \end{aligned}\right\} \tag{15}$$

for each $j \in NA$. The first of these is just (10). For the second let $f = U_A$. Then

$$\begin{aligned} N(U_A)(j) &= N(f)(j) \\ &= \bigvee\{u_{f(j(a))} \wedge v_{f(a)} \mid a \in A\} \\ &= \bigvee\{[f(j(a))] \wedge [f(a)'] \mid a \in A\} \\ &= \bigvee\{[f(j(a)) \cap f(a)'] \mid a \in A\} \quad = \bigvee\{[U_A(j(a)) \cap U_A(a)'] \mid a \in A\} \end{aligned}$$

by (13), (14), the $\wedge$-preserving property of $[\cdot]$, and the definition of f. The third identity is now immediate. These show that each nucleus on $\mathcal{O}S$ is a supremum of spatially induced ones, and exhibit $\Sigma_A(j)$ as a front-open set. We have

$$\Sigma_A(j) = S - A_j \tag{16}$$

and simple calculations show that both

$$N(U_A)(j) \leq [\Sigma_A(j)] \qquad U_A(j(a)) \subseteq \Big(N(U_A)(j)\Big)(U_A(a))$$

hold (for each $a \in A$).

The full assembly diagram attaches to each frame A four objects and six arrows. How do these change as the parent frame varies along an arrow?

Consider a morphism f between frame A, B, as in (3), let $S = \mathsf{pt}(A)$ and $T = \mathsf{pt}(B)$, and let ϕ be the map induced by f, as in (5). Using the various constructions we obtain the functorial diagram of A, as in Table 1.

4.7 THEOREM. *For each morphism f, all 10 cells of the functorial diagram commute.*

5 Block structure

It is not surprising that much of the analysis of separation properties can be done in terms of frames. In fact, by doing this we obtain further insight into these properties.

5.1 DEFINITION. A frame A is, respectively, regular or fit if for each pair $a \not\leq b$ of elements, there are elements x, y such that $a \vee x = \top$ and $y \not\leq b$ and with

$$x \wedge y = \bot \qquad x \wedge y \leq b$$

accordingly. ■

A space S is regular in the usual point-sensitive sense if and only if its topology $\mathcal{O}S$ is regular in the above point-free sense. What about the fitness of a space?

Trivially, each regular frame is fit. This implication can not be reversed. Observe that in a fit frame the $\wedge$-irreducible elements are precisely the maximal elements.

5.2 LEMMA. *Let S be a T_0 space with a fit topology. Then S is T_1 and sober.*

A space is T_3 if it is T_0 and regular. Thus each T_3 space is T_2 with a fit topology. These two consequences are independent.

5.3 EXAMPLE. There is a T_2 space which is not fit. There is a T_1 and fit space which is not T_2. We describe the first space here. The second space will appear as Example 12.1.

Let S be the reals furnished with the topology of all sets

$$U \cup \mathbb{Q} \cap V$$

where $U \subseteq V$ are Euclidean open sets. This space is T_2, and has $\mathbb{Q}$ as an open set. By way of contradiction, suppose the topology is fit. Since $\mathbb{Q} \neq \emptyset$, we have

$$\mathbb{Q} \cup X = \mathbb{R} \qquad Y \neq \emptyset \qquad X \cap Y = \emptyset$$

where

$$X = U \cup \mathbb{Q} \cap Z \qquad Y = V \cup \mathbb{Q} \cap W$$

for Euclidean open sets $U \subseteq Z$ and $V \subseteq W$. These give

$$\mathbb{Q} \cup U = \mathbb{R} \qquad V \cup (\mathbb{Q} \cap W) \neq \emptyset \qquad U \cap (V \cup (\mathbb{Q} \cap W)) = \emptyset$$

so that

$$U \cap V = \emptyset = U \cap \mathbb{Q} \cap W$$

by the third equality, and hence

$$U \cap V = \emptyset = U \cap W$$

since every non-empty Euclidean open set contains a rational point. Thus

$$V \cup W \subseteq U' \subseteq \mathbb{Q}$$

and hence $V = \emptyset = W$, which leads to a contradiction. ■

A nucleus j on a frame A admits an element x if $j(x) = \top$. The set $\nabla(j)$ of all elements admitted by j is its admissible filter. Not every filter on a frame is admissible, and a nucleus need not be determined by its admissible filter. Two nuclei j, k are companions if $\nabla(j) = \nabla(k)$. This puts an equivalence relation on the assembly NA. Each block of this equivalence relation is closed under arbitrary infima, and so has a least member. These are the fitted nuclei. Given a filter F on a frame A we set

$$v_F = \bigvee \{ v_a \mid a \in F \} \tag{17}$$

to obtain a nucleus on A.

5.4 LEMMA. *Let A be a frame.*

(a) For each $a \in A$ the nucleus w_a is the maximum member of its block.

(b) For each filter F on a frame A, the nucleus v_F is the least one which admits each member of F. The filter $\nabla(v_F)$ is the smallest admissible filter that includes F. The nucleus v_F is the minimum member of its block.

In other words, the filters on A index the fitted nuclei, but with some repetitions. Note that if $F = \nabla(j)$ then v_F is the least companion of j. Note also that not every block has a maximum member, and some don't even have maximal members.

5.5 THEOREM. *For each frame A the following are equivalent.*

(i) A is fit.

(ii) Each nucleus is fitted.

(iii) Each u-nucleus is alone in its block.

(iv) Each u-nucleus is minimal in its block.

Filters on a frame can be combined to produce other filters. These combinations transfer to the indexed nuclei.

5.6 LEMMA. *Let F, G be filters and let $\mathcal{F}$ be a directed family of filters on a frame A. Then*

$$v_F \wedge v_G = v_{F \cap G} \qquad v_F \vee v_G = v_{F \vee G} \qquad \bigvee \{ v_F \mid F \in \mathcal{F} \} = v_{\bigcup \mathcal{F}}$$

where $F \vee G$ is the join of F and G in the lattice of all filters on A.

We know how to transfer a filter in either direction across a morphism. This leads to a transfer (in one direction) of fitted nuclei. Let f be a morphism, and let F be a filter on the source frame A. Then

$$N(f)(v_F) = v_{f^* F} \tag{18}$$

where $f^* F$ is the transfer of F across f. This follows using (17) and (12).

Some filters are admissible on general grounds. The following is Lemma 2.4(ii) of [6].

5.7 LEMMA. *Each open filter on a frame is admissible.*

Each open filter F on the frame A gives a fitted nucleus v_F on A. By Lemmas 5.7 and 5.4(b) we have $\nabla(v_F) = F$. Thus we have a bijective correspondence between the open filter on A and certain fitted nuclei. We develop this further in Section 7.

6 The point-sensitive patch construction

The most general point-sensitive patch construction is described in Definition 5.11 on page 261 of [5]. We use a minor variant of that.

Each space S carries its specialization order, $\sqsubseteq$, which generates the Alexandroff (upper section) topology ΥS on S. We have $\mathcal{O}S \subseteq \Upsilon S$ but, in general ΥS is larger than $\mathcal{O}S$. In this context sets $E \in \Upsilon S$ are said to be saturated. Notice that $\Upsilon S \subseteq \mathcal{C}^f S$, that is each saturated set is f-closed. For each subset $E \subseteq S$ the upward closure $\uparrow E$ is the smallest saturated set that includes E. The neighbourhood filter $\nabla(E)$ of E is given by

$$U \in \nabla(E) \Longleftrightarrow E \subseteq U$$

for each $U \in \mathcal{O}S$. We have $\nabla(E) = \nabla(\uparrow E)$ and $\uparrow E = \bigcap \nabla(E)$ for each $E \subseteq S$.

6.1 DEFINITION. For each space S let $\mathcal{Q}S$ be the family of subsets Q which are both compact (relative to $\mathcal{O}S$) and saturated. ■

For each compact subset K of the space S, the saturation $\uparrow K$ is in $\mathcal{Q}S$. The family $\mathcal{Q}S$ is closed under binary unions but need not be closed under binary intersections. As mentioned in the Introduction, we say a space S is packed if each $Q \in \mathcal{Q}S$ is closed. As an attempt to correct a non-packed space we extend the topology by declaring that each $Q \in \mathcal{Q}S$ should now be closed.

6.2 DEFINITION. For a space S, let pbase be the family of all sets

$$U \cap Q'$$

for $U \in \mathcal{O}S$ and $Q \in \mathcal{Q}S$. The patch space ${}^{p}S$ of S has the same points but carries the larger topology $\mathcal{O}^{p}S$ generated by pbase. ■

The family pbase is closed under binary intersections, and hence does form a base of a topology. Thus each patch open subset of S is a union of members of pbase. Note that

$$\mathcal{O}S \subseteq \mathcal{O}^{p}S \subseteq \mathcal{O}^{f}S$$

(since each saturated set is front-closed). A space S is packed precisely when $S = {}^{p}S$, as spaces, that is when $\mathcal{O}S = \mathcal{O}^{p}S$.

The basic properties of this point-sensitive patch construction are easy to sort out.

6.3 THEOREM. *Let S be an arbitrary space.*

(a) If S is T_0 then ${}^{p}S$ is T_1.
(b) If S is T_1 then ${}^{pp}S = {}^{p}S$.
(c) If S is T_1+sober then ${}^{p}S$ is T_1+sober.
(d) If S is T_2 then ${}^{p}S = S$.
(e) If S is T_0 then ${}^{ppp}S = {}^{pp}S$.

Proof. (a) Consider distinct points s, t of the T_0 space S. We must separate these both ways using patch open sets. Since S is T_0, by symmetry we may suppose $s \not\sqsubseteq t$. This gives us some $U \in \mathcal{O}S$ with $s \in U$ and $t \notin U$. This is one half of the required separation (and is done with an open set). For the other half notice that $\{s\}$ is compact, and hence $Q = \uparrow\{s\} \in \mathcal{Q}S$. The set $W = Q'$ is patch open with $t \in W$ and $s \notin W$.

(b) Suppose the space S is T_1. By (a) the patch space ${}^{p}S$ is also T_1. In particular, each subset of S is saturated in S and in ${}^{p}S$. Trivially, each subset of S which is compact in ${}^{p}S$ is also compact in S. Thus each compact saturated subset of ${}^{p}S$ is a compact saturated subset of S, and hence is closed in ${}^{p}S$. Thus ${}^{pp}S = {}^{p}S$.

(c) Suppose the space S is T_1+sober. By (a) the patch space ${}^{p}S$ is also T_1, so it suffices to show that ${}^{p}S$ is sober. Consider any subset F which is closed irreducible in ${}^{p}S$. We may check that the closure F^- in S is irreducible in S. Thus $F^- = s^- = \{s\}$ for some point s. Since $s \in F^-$, for each $U \in \mathcal{O}S$ we have

$$s \in U \Longrightarrow F \cap U \cap s^- = F \cap U \neq \emptyset$$

and hence $s \in F^=$. But F is patch closed, and hence front closed, so that

$$s \in F^= = F \subseteq F^- = \{s\}$$

to give the required result. ■

The patch space of a sober space need not be sober, which leads to a curious observation.

6.4 EXAMPLE. Let S be an uncountable set and let $\mathcal{O}S$ be the co-countable topology. This space is not sober, but is missing just one point, the generic point for the whole space. Let ${}^{+}S$ be the sober reflection of S. Thus, as sets ${}^{+}S = S \cup \{\star\}$ where $\star$ is the missing point. It can be checked that ${}^{p+}S$ is the cocountable topology on the set ${}^{+}S$, and hence is not sober. Furthermore, the spaces S and ${}^{p+}S$ are homeomorphic (but not canonically so), and hence the construction ${}^{p+}(\cdot)$ can be iterated indefinitely without stabilizing. ■

Given a continuous map ϕ between spaces, when can we view the same function as a continuous map ${}^{p}\phi$ between the corresponding patch spaces? Here is a sufficient condition.

6.5 DEFINITION. A map ϕ between spaces, as in (5), converts compact saturated sets if $\phi^{\leftarrow}(Q)$ is compact in T for each $Q \in \mathcal{Q}S$. (Observe that the inverse image of a saturated set is automatically saturated.) ■

If ϕ converts compact saturated sets then ${}^{p}\phi$ is patch continuous. However, this notion is not entirely satisfactory. Luckily, there is a nicer subfamily of such morphisms. The following is one component of the proof of the Hofmann-Lawson characterization.

6.6 LEMMA. *Let ϕ be a map, as in (5), where the space T is sober. If the right adjoint ϕ_* is continuous, then ϕ converts compact saturated sets, and so ϕ is patch continuous.*

Proof. Suppose ϕ_* is continuous, and consider any $Q \in \mathcal{Q}S$. The filter $F = \nabla(Q)$ is open, and hence, by Lemma 2.3(a), the image ϕ^*F is open (on $\mathcal{O}T$). The set $\bigcap(\phi^*F)$ is compact (in T), and hence it suffices to show that

$$\bigcap(\phi^*F) = \phi^{\leftarrow}(Q)$$

holds. But ϕ^*F is generated by the set of all $\phi^{\leftarrow}(U)$ for $U \in F$. ■

7 The patch assembly

How might we produce a point-free analogue of the point-sensitive patch construction $S \longmapsto {}^{p}S$? Of course, we try to mimic the construction $\mathcal{O}S \longmapsto \mathcal{O}^{p}S$ by replacing the topology $\mathcal{O}S$ by an arbitrary frame A. We need an analogue of Definition 6.2. We work inside the assembly NA of A. The typical open set U is replaced by the nucleus u_a arising from a typical $a \in A$. For the analogue of $Q \in \mathcal{Q}S$ we use a slight extension of the Hofmann-Mislove characterization given as Lemma 2.13 (or Corollary 2.14) of [9].

7.1 THEOREM. *For an arbitrary frame A let $S = \mathsf{pt}(A)$ viewed as the set of $\wedge$-irreducible elements of A. There is a bijective correspondence*

$$F \longleftrightarrow Q$$

between open filters F on A and $Q \in \mathcal{Q}S$ given by

$$a \in F \iff (\forall s \in Q)[a \nleq s] \qquad (\forall a \in F)[a \nleq s] \iff s \in Q$$

for $a \in A$ and $s \in S$. ■

Proof. We view the relation $\cdot \nleq \cdot$ as an orthogonality between A and S. As such it generates a galois connection between filters on A and saturated subsets of S. (Remember that the specialization order on S is the opposite to the comparison inherited from A.) This connection sets up a bijection between the galois closed parts of A and S. The result asserts that these galois closed parts are precisely the open filters of A and the compact saturated subsets of S.

Most of this follows from the routine symbol shuffling for galois connections. The only non-trivial part is to show that each open filter F arises as the galois transpose of some $Q \in \mathcal{Q}S$. This requires a choice principle.

Let F be an open filter on A, let Q be the galois transpose of F, and let G be the galois transpose of Q. The galois properties give $F \subseteq G$, so it suffices to show that $G \subseteq F$.

Consider any $a \in G$. We have

$$(\forall s \in Q)[a \nleq s]$$

since G is the transpose of Q. By way of contradiction suppose $a \notin F$. By Theorem 2.4 we have

$$F \subseteq P \qquad a \notin P$$

for some completely prime filter P. Thus gives us some $s \in S$ with

$$x \in P \iff x \nleq s$$

for $x \in A$. In particular, we have $a \leq s$ so that $s \notin Q$. But $F \subseteq P$, so that

$$(\forall x \in F)[x \nleq s]$$

to give $s \in Q$, which is the contradiction. ■

The galois transpose of an open filter and of a compact saturated set can be described in different ways.

Suppose $Q \in \mathcal{Q}S$ arises from the open filter F. Then

$$s \notin Q \iff (\exists a \in F)[a \leq s] \iff s \in F$$

for each $s \in S$. Thus

$$S - Q = S \cap F \qquad S - F = Q \tag{19}$$

and the right hand side gives a succinct description of the correspondence.

Suppose the open filter F arises from $Q \in \mathcal{Q}S$. Then

$$a \in F \iff (\forall s \in Q)[a \nleq s] \iff Q \subseteq U_A(a)$$

(for $a \in A$) to show that F indexes the filter $\nabla(Q)$ of open neighbourhoods of Q. The original version of the Hofmann-Mislove result has $A = \mathcal{O}S$ for a sober space S, and then $F = \nabla(Q)$. More information is given in Section 3 of [17].

After this preamble we can return to producing a point-free analogue of pbase. We replace the topology $\mathcal{O}S$ by an arbitrary frame A. We work inside NA. We replace the open set $U \in \mathcal{O}S$ by a nucleus u_a (for $a \in A$). By Theorem 7.1 each $Q \in \mathcal{Q}S$ corresponds to an open filter on A. By (17), Lemmas 5.4 and 5.6, and the remarks at the end of Section 5, each open filter F on A has a least admitting nucleus v_F from which F can be retrieved. We use such v_F as the analogue of Q.

7.2 DEFINITION. For a frame A let PBase be the set of all nuclei

$$u_a \wedge v_F$$

for $a \in A$ and F an open filter on A. Let PA be the set of those nuclei on A which are suprema of subsets of PBase. We call PA the patch assembly of A. ■

By construction PA is a subset of the full assembly NA of A. We know that

$$v_F \wedge v_G = v_{F \cap G}$$

for filters F, G on A. Furthermore, if F and G are open then so is $F \cap G$. Thus PBase is closed under binary meets. Since $v_A = \top_{NA}$ we see that PBase includes the canonical image of A in NA. With this a simple calculation proves the following.

7.3 THEOREM. *For each frame A the patch assembly PA is a subframe of NA which includes the canonical image of A.*

This gives us a factorization

$$A \longrightarrow PA \hookrightarrow NA$$

of the canonical embedding $A \longrightarrow NA$. There are some extreme positions for PA. The following result is essentially Proposition 1.2(iii) on page 81 of [11].

7.4 THEOREM. *Let A be a regular frame.*

(a) For each $j \in NA$, if $\nabla(j)$ is open then $j = u_a$ where $a = j(\bot)$.

(b) The canonical embedding $A \longrightarrow PA$ is an isomorphism.

In fact this is a rather poor result, since both (a) and (b) hold under conditions that are much weaker than regularity. It is stated here in this form to provide a direct comparison with Theorem 10.6 later. There is an ordinal indexed hierarchy of frame separation properties

$$\cdots \Longrightarrow \alpha\text{-tidy} \Longrightarrow \alpha\text{-regular} \Longrightarrow (\alpha+1)\text{-tidy} \Longrightarrow (\alpha+1)\text{-regular} \Longrightarrow \cdots$$

which become progressively weaker as the ordinal α increases. For a frame A we have

$$A \text{ is 0-tidy} \Longleftrightarrow A \text{ is trivial} \qquad A \text{ is 0-regular} \Longleftrightarrow A \text{ is regular}$$

and thereafter the properties seem not to have common names. For a T_0 space S we have

$$\mathcal{O}S \text{ is 1-tidy} \Longleftrightarrow S \text{ is } T_2$$

but 1-tidiness seems not to be connected with the other frame theoretic Hausdorff properties. It can be shown that the embedding $A \longrightarrow PA$ is an isomorphism precisely when the frame A is α-regular for some (and hence for any sufficiently large) ordinal α.

This hierarchy is described in [16]. The topological content is described in [17].

Theorem 7.4 is the general result for the case $\alpha = 0$, which is rather poor.

We know that $N(\cdot)$ is a functor on ***Frm***, but what about $P(\cdot)$? We need to restrict the family of arrows involved.

7.5 LEMMA. *If a morphism f converts open filters, then $N(f)(j) \in PB$ for each $j \in PA$.*

Proof. For $a \in A$ and an open filter F on A let

$$b = f(a) \qquad G = fF$$

so that G is an open filter on B. Then

$$N(f)(u_a) = u_b \qquad (fN)(v_F) = v_G$$

to show that both of these are in PB. For each $j \in PA$ we have

$$j = \bigvee \{ u_{a(i)} \wedge v_{F(i)} \mid i \in I \}$$

where $a(i) \in A$ and $F(i)$ is an open filter on A, for each $i \in I$. Since $N(f)$ is a morphism, we see that $N(f)(j) \in PB$, as required. ■

With this result we can turn P into a functor, of sorts.

7.6 DEFINITION. For each morphism f, as in (3), which converts open filters, let

$$PA \xrightarrow{P(f)} PB$$

be the restriction of $N(f)$ to PA. ■

Thus we allow P to act only on those arrows f which convert open filters.

In the next section we will compare the two patch constructions. In anticipation of that consider the commuting square of morphisms induced by a morphism f

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ {\scriptstyle U_A}\downarrow & & \downarrow{\scriptstyle U_B} \\ \mathcal{O}S & \xrightarrow[\ \phi^*\ ]{} & \mathcal{O}T \end{array}$$

making use of the associated map ϕ between the point spaces.

7.7 LEMMA. *For each morphism f, as above, if f converts open filters, then so does the associated morphism ϕ^*. In other word, the right adjoint ϕ_* is continuous.*

Proof. Consider any open filter ∇ on $\mathcal{O}S$. We must show that the filter $\phi^*\nabla$ is open on $\mathcal{O}T$. Recall that $\phi^*\nabla$ is the filter generated by the direct image of ∇ across ϕ^*.

To show that $\phi^*\nabla$ is open we go via f. Using the right adjoint of U_A we convert ∇ into some open filter F on A. This is given by

$$a \in F \Longleftrightarrow U_A(a) \in \nabla$$

(for each $a \in A$). By the assumed property of f, this converts into some open filter G on B, and this is generated by the direct image of F across f^*. Finally, by Theorem 3.1, the filter G converts into an open filter Γ on $\mathcal{O}T$. Since U_B is surjective, this is just the direct image of G across U_B. We show that $\phi^*\nabla = \Gamma$.

Consider typical $V \in \mathcal{O}S$ and $W \in \mathcal{O}T$. These have the form $U_A(a)$ and $U_B(b)$ for $a \in A$ and $b \in B$, respectively. Remember also that

$$\phi^*(U_A(a)) = U_B(f(a))$$

for each $a \in A$. With this, for each $W \in \mathcal{O}T$, we have

$$\begin{aligned} W \in \Gamma \iff & (\exists b \in G)[U_B(b) \subseteq W] \\ \iff & (\exists b \in B, a \in F)[f(a) \leq b \text{ and } U_B(b) \subseteq W] \\ \iff & (\exists a \in F)[U_B(f(a)) \subseteq W] \\ \iff & (\exists a \in F)[\phi^*(U_A(a)) \subseteq W] \\ \iff & (\exists V \in \nabla)[\phi^*(V) \subseteq W] \qquad \iff \quad W \in \phi^*\nabla \end{aligned}$$

to complete the proof. ■

Lemmas 7.7, 2.3(b), and 6.6 give the following.

7.8 THEOREM. *If the morphism f of (3) converts open filters, then the associated map*

$${}^{p}S \xleftarrow{\ {}^{p}\phi\ } {}^{p}T$$

is patch continuous.

8 Point-sensitive or point-free patch

How are these two patch constructions related?

Let A be a frame with point space $S = \mathsf{pt}(A)$. We produce a commuting diagram

$$
\begin{array}{ccccc}
A & \longrightarrow & PA & \hookrightarrow & NA \\
{\scriptstyle U_A}\downarrow & & \downarrow{\scriptstyle P(U_A)} & & \downarrow{\scriptstyle N(U_A)} \\
\mathcal{O}S & \hookrightarrow & P\mathcal{O}S & \hookrightarrow & N\mathcal{O}S \\
 & & \downarrow{\scriptstyle \pi_S} & & \downarrow{\scriptstyle \sigma_S} \\
 & & \mathcal{O}^pS & \hookrightarrow & \mathcal{O}^fS
\end{array}
$$

where the three right hand horizontal arrows and the left hand diagonal arrow are insertions, and the two left hand horizontal arrows are the canonical embeddings. Most of this patch assembly diagram is known already or easy to construct. Only the arrows $P(U_A)$ and π_S need some discussion.

By Theorem 3.1, the arrow U_A converts open filters, and hence Lemma 7.5 and Definition 7.6 give the top two squares. The composite of the two lower cells is just the bottom cell of the full assembly diagram. What can the arrow π_S be? If bottom right hand square commutes, then π_S must be the restriction of σ_S to $P\mathcal{O}S$. If this exists, as an arrow $P\mathcal{O}S \longrightarrow \mathcal{O}^pS$, then the left hand cell will commute. So we know what we have to do.

Consider any sober space S, and consider any open filter F on $\mathcal{O}S$. What is the nucleus v_F on $\mathcal{O}S$ and what is $\sigma(v_F)$ on $\mathcal{O}^fS$? Since S is sober, by the spatial version of Hofmann-Mislove correspondence, Theorem 7.1, we know that $F = \nabla(Q)$ for some unique $Q \in \mathcal{Q}S$. Look at the spatially induced nucleus $[Q']$. For each $U \in \mathcal{O}S$ we have

$$[Q'](U) = S \iff Q' \cup U = S \iff Q \subseteq U \iff U \in F$$

so that v_F and $[Q']$ are companions, and hence

$$v_F \leq [Q'] \tag{20}$$

(since v_F is the least member of its block).

8.1 LEMMA. *Let S be a sober space and let $Q \in \mathcal{Q}S$. Then*

$$\sigma(v_F) = Q'$$

where $F = \nabla(Q)$.

Proof. Since $v_F \leq [Q']$ (and $Q' \in \mathcal{O}^fS$) we have

$$\sigma(v_F) \subseteq \sigma([Q']) = Q'$$

and hence we require the converse inclusion.

Consider any point $s \in Q'$. Then $s^- \subseteq Q'$ (since Q is saturated) so that $s^{-\prime} \in F$, and hence $s \in v_F(s^{-\prime}) = S$, to give the required result. ■

Using this result we can look at the behaviour of σ on the subframe $P\mathcal{O}S$ of $N\mathcal{O}S$.

8.2 LEMMA. *Let S be a sober space. Then $\sigma(j) \in \mathcal{O}{}^{p}S$ for each $j \in P\mathcal{O}S$.*

Proof. Using (14), the nucleus j is the supremum of a family of nuclei

$$[W] \wedge v_F$$

for various $W \in \mathcal{O}S$ and open filters $F = \nabla(Q)$ on $\mathcal{O}S$. For each one of these we have

$$\sigma\Big([W] \wedge v_F\Big) = \sigma\Big([W]\Big) \cap \sigma(v_F) = (W \cap Q') \in \mathcal{O}{}^{p}S$$

and hence $\sigma(j)$ is a union of basic open sets of ${}^{p}S$, to give the required result. ■

This completes the construction the patch assembly diagram of a frame A.

8.3 THEOREM. *Consider the patch assembly diagram of a frame A.*
(h) Each horizontal arrow is an embedding, and each right hand one is an insertion.
(v) Each vertical arrow is surjective.

Proof. (h) This is immediate.

(v) It suffices to consider the two central arrows. We look at these in turn.

To show that $P(U)$ (that is $P(U_A)$) is surjective it suffices to show that each canonical generator

$$[W] \wedge v_\nabla$$

of $P\mathcal{O}S$ is a value of $P(U)$. Here $W \in \mathcal{O}S$ and ∇ is an open filter on $\mathcal{O}S$.

We have $W = U(a)$ for some $a \in A$, and then

$$P(U)(u_a) = [U(a)] = [W]$$

to exhibit $[W]$ as a value of $P(U)$.

Since S is sober, we have $\nabla = \nabla(Q)$ for some $Q \in \mathcal{Q}S$. Consider the open filter F on A given by

$$a \in F \iff Q \subseteq U(a)$$

(for $a \in A$). We have $v_F \in PA$, so that

$$P(U)(v_F) = N(U)(v_F) = v_{UF}$$

where UF is the image of F on OS. The image is nothing more than

$$\{U \in \mathcal{O}S \mid Q \subseteq U\} = \nabla(Q) = \nabla$$

and hence $P(U)(v_F) = v_\nabla$ to exhibit v_∇ as a value of $P(U)$.

To show that the arrow $\pi = \pi_S$ is surjective it suffices to show that each generator

$$W \cap Q'$$

of $\mathcal{O}{}^{p}S$ is a value of the morphism. Here $W \in \mathcal{O}S$ and $Q \in \mathcal{Q}S$. But

$$[W] \wedge v_\nabla \in P\mathcal{O}S$$

where $\nabla = \nabla(Q)$, and hence, using Lemma 8.1, we have

$$\sigma\Big([W] \wedge v_\nabla\Big) = \sigma\Big([W]\Big) \cap \sigma(v_\nabla) = W \cap Q'$$

which gives the required result. ■

As a frame varies along a morphism, its full assembly diagram reacts naturally. A similar naturality holds for the patch assembly diagram, but we have to restrict the class of arrows along which the parent frame varies.

Consider the functorial diagram of Table 1, and suppose the parent morphism f converts open filters. The right adjoint ϕ_* of the associated map ϕ is continuous. The patch assembly diagram of A interpolates a column into the two left hand cells. Similarly, the patch assembly diagram of B interpolates a column into the two right hand cells. We now observe there are commuting diagrams

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & B \\ \downarrow & & \downarrow \\ PA & \xrightarrow{P(f)} & PB \\ \downarrow & & \downarrow \\ P\mathcal{O}S & \xrightarrow[P(\phi^*)]{} & P\mathcal{O}T \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{O}S & \xrightarrow{\ \phi^*\ } & \mathcal{O}T \\ \downarrow & & \downarrow \\ P\mathcal{O}S & \xrightarrow{P(\phi^*)} & P\mathcal{O}T \\ \downarrow & & \downarrow \\ \mathcal{O}^pS & \xrightarrow[\mathcal{O}({}^p\phi)]{} & \mathcal{O}^pT \end{array}$$

to show that the whole patch assembly diagram varies naturally along f.

In the left hand diagram the top cell is essentially the definition of $P(f)$. The bottom cell is the result of hitting the naturality diagram for $U_\bullet$ with P. In the right hand diagram the top cell is essentially the definition of $P(\phi^*)$. Finally, a direct calculation gives the bottom cell. The crucial observation is that for each $Q \in \mathcal{Q}S$ we have $P(\phi^*)(v_F) = v_G$ where $F = \nabla(Q)$ and $G = \phi^* F = \nabla(\phi^{\leftarrow}(Q))$ are the two open filters involved.

9 The point space of the patch assembly

Each frame A gives us two spaces

$${}^p\mathsf{pt}(A) \qquad \mathsf{pt}(PA)$$

using the point-sensitive and the point-free patch constructions. How are these related?

Recall that from Lemma 4.5 we have a bijection

$$\begin{array}{ccc} \mathsf{pt}(A) & \longrightarrow & \mathsf{pt}(NA) \\ s & \longmapsto & w_s \end{array}$$

between the two indicated sets of points. Here the target space arises from the full assembly, not the patch assembly. When $\mathsf{pt}(A)$ carries its front topology this bijection is a homeomorphism. We look for a similar relationship between ${}^p\mathsf{pt}(A)$ and $\mathsf{pt}(PA)$.

As usual we view each point of PA as a $\wedge$-irreducible element. Thus these points are certain nuclei on A. We have met quite a lot of these already.

9.1 LEMMA. *For each frame A we have $w_s \in PA$ for each $s \in \mathsf{pt}(A)$.*

Proof. Consider any $s \in \mathsf{pt}(A)$. By Lemma 5.4(a) we know that w_s is maximum in its block. We look for its least companion. Let $F = \nabla(w_s)$, so that

$$x \in F \Longleftrightarrow x \nleq s$$

for each $x \in A$, and F is the corresponding completely prime filter of the point s of A. In particular, F is open. Let

$$f_s = \dot{\bigvee}\{v_x \mid x \in F\}$$

so that f_s^∞ is the least companion of w_s, and is a member of PA. A few calculations give

$$w_s = u_s \vee f_s^\infty = f_s \circ u_s$$

and hence $w_s \in PA$. ■

The nucleus w_s (for $s \in \mathsf{pt}(A)$) is $\wedge$-irreducible in NA, and remains so in PA. Thus we have an injection

$$\begin{array}{ccc} \mathsf{pt}(A) & \xrightarrow{\ \alpha\ } & \mathsf{pt}(PA) \\ s & \longmapsto & w_s \end{array} \tag{21}$$

which, as yet, we do not know is surjective. However, the canonical topology on $\mathsf{pt}(PA)$ does induce a topology on $\mathsf{pt}(A)$. This is generated by the sets

$$(\alpha^{\leftarrow} \circ U_{PA})(u_a) \qquad (\alpha^{\leftarrow} \circ U_{PA})(v_F)$$

for $a \in A$ and open filters F on A. For such an open filter F, by (19), we see that $Q = \mathsf{pt}(A) - F$ is the member of $\mathcal{Q}\mathsf{pt}(A)$ that determines F. We may also check that

$$w_s \in U_{PA}(u_a) \Longleftrightarrow s \in U_A(a) \qquad w_s \in U_{PA}(v_F) \Longleftrightarrow s \in Q'$$

for each $s \in \mathsf{pt}(A)$. For the right hand equivalence observe that Theorem 7.1 gives

$$w_s \in U(v_F) \Longleftrightarrow v_F \nleq w_s \Longleftrightarrow (\exists a \in F)[v_a \nleq w_s] \qquad (\exists a \in F)[a \leq s] \Longleftrightarrow s \in Q'$$

and

$$v_a \nleq w_s \Longleftrightarrow a \leq s$$

follows by a few simple calculations with the implication on A.

This gives the following.

9.2 THEOREM. *For a frame A the insertion (21) exhibits ${}^p\mathsf{pt}(A)$ as a subspace of $\mathsf{pt}(PA)$.*

To what extent does ${}^p\mathsf{pt}(A)$ determine $\mathsf{pt}(PA)$? Consider the case where $A = \mathcal{O}S$ for some sober space S. We have an embedding

$$^pS \longrightarrow \mathsf{pt}(P\mathcal{O}S)$$

where the target space $\mathsf{pt}(P\mathcal{O}S)$ is sober. There are two observation to be made. Firstly, the assembly $P\mathcal{O}S$ need not be spatial. We look at some examples of this at the end of this section. Secondly, the source space ^pS need not be sober (as we saw Example 6.4). For such a space the embedding can not be a homeomorphism. This leads to the major open question concerning the construction P.

9.3 QUESTION. Is it the case that for an arbitrary frame A the space $\mathsf{pt}(PA)$ is just the sober reflection of ${}^{p}\mathsf{pt}(A)$ under the canonical embedding?

There are some cases where the question has a positive answer. A few simple calculations gives the following.

9.4 LEMMA. *Let A be a frame and let $\ell \in \mathsf{pt}(PA)$. Then $\ell(\bot) \in \mathsf{pt}(A)$, and is the unique $a \in A$ with $u_a \leq \ell \leq w_a$.*

Each point ℓ of PA is monitored by some point $s = \ell(\bot)$ of A. The tame points are those at the top of the range, $\ell = w_s$, but in general there can also be wild points with $u_s \leq \ell < w_s$. This monitoring of the points of PA gives us the following.

9.5 THEOREM. *Let A be a frame such that $u_s = w_s$ for each $s \in \mathsf{pt}(A)$. Then PA has no wild points and the two spaces ${}^{p}\mathsf{pt}(A), \mathsf{pt}(PA)$ are canonically homeomorphic.*

Each maximal element s of a frame A is a point and then $u_s = w_s$ (by a simple calculation). Thus we have the following.

9.6 COROLLARY. *Let A be a frame for which each point is a maximal element. Then PA has no wild points.*

For instance, if A is fit or is the topology of a T_1+sober space, then the patch assembly PA has no wild points. However, things are not always so nice.

9.7 EXAMPLE. Let S be a T_1+sober space. Thus, by Corollary 9.6, the point space $\mathsf{pt}(P\mathcal{O}S)$ is essentially the usual patch space ${}^{p}S$. However, the assembly $P\mathcal{O}S$ need not be spatial. To see this observe that for such a space the morphism

$$P\mathcal{O}S \xrightarrow{\ \pi\ } \mathcal{O}^{p}S$$

is the spatial reflection of $P\mathcal{O}S$ (with ${}^{p}S$ sober). Consider any $Q \in \mathcal{Q}S$ and let $F = \nabla(Q)$ on $\mathcal{O}S$. By (8) we have $v_F \leq [Q']$ and these are companions with

$$\pi(v_F) = Q' = \pi([Q'])$$

by Lemma 8.1. If $P\mathcal{O}S$ is spatial then π is injective, to give $v_f = [Q']$. However, there are many T_1+sober spaces S where this is not so. ■

In contrast to this, if the topology $\mathcal{O}S$ is fit, or if the space S is T_2, then we have $v_F = [Q']$ for all related pairs F, Q, and hence $P\mathcal{O}S$ is spatial. There is a long hierarchy of separation properties between T_1+sober and T_2. This is analysed in [17].

9.8 EXAMPLE. Let S be an uncountable set, and let $A = \mathcal{O}S$ be the co-countable topology on S. Each member of S gives us a point s of A, and we have $u_s = w_s$. However, the space S is not sober, and is missing just one point. We find that this is the bottom $\bot$ of A, and this corresponds to double negation $w_\bot = (\neg\neg)$ as a point of PA. We have

$$\mathsf{id}_S = u_\bot < w_\bot = (\neg\neg)$$

and a few calculations, as in [16], shows that id_A is the only other point of PA. ■

10 The continuous assembly

Building on the earlier work [12] of Karazeris, in [3] and [4] Escardó produced a somewhat different point-free version of the patch construction.

10.1 DEFINITION. Let A be a frame. An inflator f on a A is continuous if

$$f\left(\bigvee X\right) = \bigvee f^{\rightarrow}(X)$$

for each directed subset X of A. Let MA be the set of all continuous nuclei on A. We call MA the continuous assembly of A. ■

Each u-nucleus is continuous. The composite and the meet of two continuous inflators is continuous. Simple manipulations of a nested pair of directed suprema gives the following.

10.2 LEMMA. *Let A be a frame. The pointwise supremum $\dot{\bigvee} F$ of a directed family F of continuous inflators on A is itself continuous.*

As indicated in Section 3, for an arbitrary family J of nuclei we have

$$\bigvee J = \left(\dot{\bigvee} J^{\circ}\right)^{\infty}$$

where J° is the compositional closure of J and ∞ is a suitable ordinal which, in general, can be arbitrarily large. However, for certain J this closure ordinal is always small. The following result is taken from Lemma 3.1.8 of [3].

10.3 LEMMA. *Let J be a set of continuous nuclei on the frame A.*
The pointwise supremum $\dot{\bigvee} J^{\circ}$ is a nucleus, and hence is $\bigvee J$ (in NA).
If J is directed set then $\bigvee J = \dot{\bigvee} J$.

Lemma 10.3 gives the following, which should be compared with Theorem 7.3.

10.4 THEOREM. *For each frame A the continuous assembly MA is a subframe of NA which includes the canonical image of A.*

This gives us a second factorization

$$A \longrightarrow MA \hookrightarrow NA$$

of the canonical embedding $A \longrightarrow NA$. As with PA, the assembly MA can take up an extreme position. To isolate when this can occur, we prove a composition result.

10.5 LEMMA. *For a frame A, let $j \in NA$ be fitted and let $k \in MA$. Then $j \vee k = j \circ k$.*

Proof. Let $F = \nabla(j)$ and set

$$f = \dot{\bigvee}\{v_a \mid a \in F\}$$

so that $j = f^\infty$. For each $x \in A$ we have

$$f(x) = \bigvee\{v_a(x) \mid a \in F\}$$

and this is a directed supremum. Thus, since k is continuous, we have

$$\begin{aligned}(k \circ f)(x) &= k\big(\bigvee\{v_a(x) \mid a \in F\}\big)\\ &= \bigvee\big\{(k \circ v_a)(x) \mid a \in F\big\}\\ &\le \bigvee\big\{(v_a \circ k)(x) \mid a \in F\big\} \quad = \quad (f \circ k)(x)\end{aligned}$$

using the standard comparison at the third step. Thus $k \circ f \le f \circ k$. We now show

$$k \circ f^\alpha \le f^\alpha \circ k$$

for each ordinal α.

The base case, $\alpha = 0$, is immediate.

For the induction step, $\alpha \mapsto \alpha + 1$, we have

$$k \circ f^{\alpha+1} = k \circ f \circ f^\alpha \le f \circ k \circ f^\alpha \le f \circ f^\alpha \circ k = f^{\alpha+1} \circ k$$

using the above result and the induction hypothesis.

For the induction leap to a limit ordinal λ, for each $x \in A$ we have

$$f^\lambda(x) = \bigvee\{f^\alpha(x) \mid \alpha < \lambda\}$$

where this is a directed supremum. Thus the continuity of k gives

$$(k \circ f^\lambda)(x) = \bigvee\{(k \circ f^\alpha)(x) \mid \alpha < \lambda\} \le \bigvee\{(f^\alpha \circ k)(x) \mid \alpha < \lambda\} = (f^\lambda \circ k)(x)$$

as required. Here the comparison uses the induction hypothesis.

Finally, we take $\alpha = \infty$ to get $k \circ j \le j \circ k$ and hence $j \vee k = j \circ k$ as required. ■

As a consequence of this, if a frame has a modicum of separation, then there are no interesting continuous nuclei. The following should be compared with Theorem 7.4.

10.6 THEOREM. *Let A be a fit frame.*

(a) For each $k \in NA$, if k is continuous then $k = u_a$ where $a = k(\bot)$.

(b) The canonical embedding $A \longrightarrow MA$ is an isomorphism.

Proof. In general, we have $k \vee u_x = k \circ u_x$ for all $k \in NA$ and $x \in A$. But here A is fit so u_x is fitted, and hence

$$u_x \circ k = u_x \vee k = k \vee u_x = k \circ u_x$$

(by Lemma 10.5). This shows that

$$x \vee k(y) = k(x \vee y)$$

for $x, y \in A$. In particular, with $y = \bot$ and $a = k(\bot)$, we have $x \vee a = k(x)$ as required. ■

After Theorem 7.4 we indicated that the result holds under conditions much weaker than regularity. We suspect that a similar strengthening of Theorem 10.6 can be obtained by replacing the assumed fitness by an appropriate weaker property. However, we have not been able to determine such a 'weak-fitness'.

As far as we can discern, the construction M has no general functorial properties.

11 The quilted topology

The patch assembly diagram connects the new point-free patch assembly construction with the old point-sensitive patch space construction. We also have the point-free continuous assembly construction but, as yet, no corresponding point-sensitive construction.

Let A be a frame with $S = \mathsf{pt}(A)$. We will construct the diagram

$$
\begin{array}{ccccc}
A & \longrightarrow & MA & \hookrightarrow & NA \\
{\scriptstyle U_A}\downarrow & & \downarrow{\scriptstyle M(U_A)} & & \downarrow{\scriptstyle N(U_A)} \\
OS & \hookrightarrow & M\mathcal{O}S & \hookrightarrow & N\mathcal{O}S \\
 & & \downarrow{\scriptstyle \mu_S} & & \downarrow{\scriptstyle \sigma_S} \\
 & & \mathcal{O}^m S & \hookrightarrow & \mathcal{O}^f S
\end{array}
$$

where the three right hand horizontal arrows and the diagonal arrow are insertions, and the two left hand horizontal arrows are the canonical embeddings. The space ${}^mS$ and its topology will be constructed shortly.

How do we obtain the arrow $M(U_A)$? In spite of the notation, we can not get this by hitting U_A with a certain functor M (for we don't know of such a functor). We have to take a round about route. If there is such an arrow, then it can only be the restriction of $N(U_A)$ to MA. If there is such a restriction, then it will certainly make both the top two squares commute. Thus we must show that

$$j \in MA \Longrightarrow N(U_A)(j) \in M\mathcal{O}S$$

holds. For this we need a bit of preparation.

Consider any continuous nucleus $j \in MA$ on the frame A. Look at the fixed set A_j of j. The continuity of j ensures that A_j is closed under directed suprema (as computed in A). Thus A_j is suitable for applications of Zorn's Lemma.

11.1 LEMMA. *Suppose $j \in MA$ for a frame A. Suppose $a \leq s$ where $j(a) = a$ and s is $\wedge$-irreducible (in A). Then there is a $\wedge$-irreducible $t \in A$ with $j(t) = t$ and $a \leq t \leq s$.*

Proof. The following subset of A_j

$$Z = \{z \in A \mid j(z) = z \leq s\}$$

contains a and is closed under directed suprema (computed in A). By ZL, there is some maximal member $t \in Z$ with $a \leq t$. A short calculation shows that t is $\wedge$-irreducible. ■

A simple application of this result leads to a proof of Theorem 3.1.4 of [3].

11.2 THEOREM. *Suppose $j \in MA$ for a spatial frame A. Then the quotient A_j is spatial.*

To analyse the behaviour of $N(U_A)$ on continuous nuclei we make use of the full assembly diagram and the composite arrow $\Sigma = \sigma \circ N(U)$ down the right hand side. (Here we may omit the subscripts A and S since these objects are held fixed).

The point space $S = \mathsf{pt}(A)$ carries its specialization order, and this give the saturation $\uparrow s$ of each point $s \in S$. However, we must remember that this specialization order is the *reverse* of the order inherited from A. In other words

$$\uparrow s = \{t \in S \,|\, t \leq s\}$$

where $\leq$ is the parent comparison on A.

11.3 LEMMA. *For each frame A we have*

$$\uparrow s \subseteq \Sigma(j) \cup U(a) \Longrightarrow s \in U(j(a)) \qquad [\Sigma(j)](U(a)) = U(j(a))$$

for each $j \in MA$, $a \in A$, and $s \in S$.

Proof. We make use of the information given just after Theorem 4.6.

For the implication, suppose

$$\uparrow s \subseteq \Sigma(j) \cup U(a) = (S - A_j) \cup U(a)$$

(for the indicated j, a, s). The equality follows by (16). By way of contradiction, suppose $s \notin U(j(a))$. Since $j(a) \leq s$, Lemma 11.1 gives some $t \in S$ such that $j(t) = t$ and $a \leq j(a) \leq t \leq s$. Since $t \leq s$, we have $t \in \uparrow s$, and hence one of

$$t \in \Sigma(j) = S - A_j \qquad t \in U(a)$$

must hold. The given properties of t contradict both of these.

For the equality, since $N(U)(j) \leq [\Sigma(j)]$, we have

$$U(j(a)) \subseteq [\Sigma(j)](U(a))$$

by the third identity of (15), so it suffices to verify the converse inclusion. To this end consider any point $s \in [\Sigma(j)](U(a))$. We have

$$\uparrow s \subseteq [\Sigma(j)](U(a)) \subseteq (\Sigma(j)) \cup U(a)$$

so that $s \in U(j(a))$, by the implication, and this leads to the required result. ■

With this we are in a position to describe the action of $N(U_A)$ on a continuous nucleus.

11.4 THEOREM. *For each frame A and $j \in MA$, we have*

$$N(U_A)(j) = [\Sigma_A(j)]$$

and this is a continuous nucleus on $\mathcal{O}S$.

Proof. The comparison $N(U)(j) \leq [\Sigma(j)]$ holds by (15), so we require the converse. Consider any open set of S. This has the form $U(a)$ for some $a \in A$. Then Lemma 11.3 and a general comparison give

$$[\Sigma(j)](U(a)) = U(j(a)) \subseteq \Big(N(U)(j)\Big)(U(a))$$

as required to show $N(U)(j) = [\Sigma(j)]$.

To show this nucleus on $\mathcal{O}S$ is continuous consider any directed family $\mathcal{U}$ taken from $\mathcal{O}S$. Using the frame morphism $U(\cdot)$ we may index this as

$$\mathcal{U} = \{U(x) \mid x \in X\}$$

where X is a directed subset of A. Let $a = \bigvee X$ so that

$$\bigcup \mathcal{U} = U(a) \qquad j(a) = \bigvee j[X] \qquad U(j(a)) = \bigcup \Big\{ U(j(x)) \mid x \in X \Big\}$$

by two uses of the morphism $U(\cdot)$ and one use of the continuity of j. With these and with $k = [\Sigma(j)]$ we have

$$k\Big(\bigcup \mathcal{U}\Big) = k(U(a)) = U(j(a)) = \bigcup \Big\{ U(j(x)) \mid x \in X \Big\} = \bigcup \Big\{ k(U(x)) \mid x \in X \Big\}$$

to give the required result. The second and fourth equality follow by Lemma 11.3. ■

This result gives us the required arrow $M(U_A)$ which makes the top two cells of the continuous assembly diagram commute. We now turn to the bottom two cells.

Let S be any sober space. The space ${}^{m}S$ has the same points as S, but with a topology

$$\mathcal{O}S \subseteq \mathcal{O}^{m}S \subseteq \mathcal{O}^{f}S$$

sitting between the parent topology $\mathcal{O}S$ and its front topology $\mathcal{O}^{f}S$. We locate this new topology by analysing the composite

$$M\mathcal{O}S \hookrightarrow N\mathcal{O}S \xrightarrow{\ \sigma\ } \mathcal{O}^{f}S$$

from the continuous assembly of $\mathcal{O}S$ to the front topology

11.5 DEFINITION. Let S be a sober space.

(a) A subset $E \subseteq S$ is a tessel if $E \in \mathcal{O}^{f}S$ and both

(the interior property)	$\uparrow s \subseteq E \Longrightarrow s \in E^{\circ}$
(the compactness property)	$\uparrow s - E$ is compact

hold for each $s \in S$.

(b) A subset $E \subseteq S$ is a uniform tessel if $E \cup U$ is a tessel for each $U \in \mathcal{O}S$. ■

We make use of Theorem 11.4 for the case $A = \mathcal{O}S$. Thus $\Sigma = \sigma$ and $j = [\sigma(j)]$ for each $j \in M\mathcal{O}S$. Furthermore, Lemma 11.3 gives us some information about this set $\sigma(j)$.

11.6 LEMMA. *Let S be a sober space, let $j \in M\mathcal{O}S$, and let $E = \sigma(j)$. Then $j = [E]$ and E is a uniform tessel.*

Proof. Theorem 11.4 gives $j = [E]$, and Lemma 11.3 gives

$$\uparrow s \subseteq E \cup U \Longrightarrow s \in j(U) = (E \cup U)^{\circ}$$

to verify the uniform interior property.

It remains to check that $E \cup U$ has the compactness property (for arbitrary $U \in \mathcal{O}S$). To this end consider any $s \in S$ and any directed family $\mathcal{V} \subseteq OS$ such that

$$\uparrow s \cap (E \cup U)' \subseteq \bigcup \mathcal{V}$$

holds. Let

$$\mathcal{U} = \{U \cup V \mid V \in \mathcal{V}\}$$

so that $\mathcal{U}$ is a second directed family from $\mathcal{O}S$. We have

$$\uparrow s \subseteq E \cup U \cup \bigcup \mathcal{V} = E \cup \bigcup \mathcal{U}$$

and hence

$$s \in \left(E \cup \bigcup \mathcal{U}\right)^{\circ} = [E]\left(\bigcup \mathcal{U}\right)$$

by the interior property of $E \cup \bigcup \mathcal{U}$. But $[E] = j$ is continuous, and hence

$$s \in [E]\left(\bigcup \mathcal{U}\right) = \bigcup \left\{[E](U) \mid U \in \mathcal{U}\right\}$$

which gives

$$s \in [E](U \cup V) = (E \cup U \cup V)^{\circ}$$

for some $V \in \mathcal{V}$. In particular

$$\uparrow s \subseteq E \cup U \cup V$$

and hence

$$\uparrow s \cap (E \cup U)' \subseteq V$$

to give the required result. ■

This shows that the composite morphism

$$\begin{array}{rcl} MOS & \longrightarrow & \mathcal{O}^f S \\ j & \longmapsto & \sigma(j) \end{array} \tag{22}$$

is an embedding, and each value $E = \sigma(j)$ is a uniform tessel. We need a converse.

11.7 LEMMA. *Let S be a sober space, and let E be a uniform tessel on S. Then the nucleus $[E]$ is continuous and the difference $Q - E$ is compact for each $Q \in \mathcal{Q}S$.*

Proof. To show that $[E]$ is continuous consider any directed subfamily $\mathcal{U}$ of OS, and consider any $s \in [E]\left(\bigcup \mathcal{U}\right)$. Since this set is open we have $\uparrow s \in E \cup \bigcup \mathcal{U}$ and hence, by the compactness property of E, we have $\uparrow s \subseteq E \cup U$ for some $U \in \mathcal{U}$. But $E \cup U$ is a tessel, so that $s \in [E](U)$, as required.

To verify the stronger compactness property consider any $Q \in \mathcal{Q}S$ with $Q \subseteq E \cup \bigcup \mathcal{U}$ (where $\mathcal{U}$ is as above). Since $E \cup \bigcup \mathcal{U}$ is a tessel, this set has the interior property. For each $t \in Q$ we have $\uparrow t \subseteq Q \subseteq E \cup \bigcup \mathcal{U}$ to give

$$t \in \left(E \cup \bigcup \mathcal{U}\right)^{\circ} = \bigcup \left\{[E](U) \mid U \in \mathcal{U}\right\}$$

using the continuity of $[E]$ verified above. This shows that

$$Q \subseteq \bigcup \Big\{ [E](U) \mid U \in \mathcal{U} \Big\}$$

and hence, since Q is compact,

$$Q \subseteq [E](U) \subseteq E \cup U$$

for some $U \in \mathcal{U}$, as required. ■

These results give an improvement of Lemma 11.6.

11.8 THEOREM. *Let S be a sober space. The assignment (22) is a frame embedding where the range is exactly the family of uniform tessels on S.*

This shows that the family of uniform tessels on a sober space re-topologizes the space.

11.9 DEFINITION. For a sober space S, let $\mathcal{O}^m S$ be the topology of all uniform tessels on S. This is the quilted topology on S, and ${}^m S$ is the quilted space of S. ■

To conclude this section we have the analogue of Theorem 8.3.

11.10 THEOREM. *Consider the continuous assembly diagram of a frame A.*
(h) Each horizontal arrow is an embedding, and each right hand one is an insertion.
(v) Each vertical arrow, except possibly the top central one, is surjective. Furthermore, the bottom central arrow is an isomorphism.

The crucial omission here is that we do *not* know that the arrow $M(U_A)$ is surjective.

12 Patch or quilt

In this final section we make some observations on Escardó's use of the construction M in [4]. Before that let's check that the two constructions P and M are different.

We know that for a frame A each embedding

$$A \longrightarrow PA \qquad\qquad A \longrightarrow MA$$

is an isomorphism if A is regular or A is fit, respectively.

12.1 EXAMPLE. We produce a space S which is T_1 and fit (and hence sober), which has a base of compact open sets (and hence is locally compact), but with the following.

- There are many compact open sets which are dense.
- The intersection of two compact open sets need not be compact.
- The space is not T_2 (since there are non-closed compact sets).
- Although $M\mathcal{O}S = \mathcal{O}S$ (since S is fit), we have ${}^p S \neq S$ (since the complement of each compact open set belongs to $\mathcal{O}^p S$). In particular $P\mathcal{O}S \neq \mathcal{O}S$.

This example is based on [1], Exercise 5(b) on page 142.

Let $S = T \cup \mathbb{N}$ where $T \cap \mathbb{N} = \emptyset$ and T has at least two members. Consider the family of all subsets of $\mathbb{N}$ together with all sets

$$B(t,m) = \{t\} \cup [m, \infty)$$

for $t \in T$ and $m \in \mathbb{N}$. This family is closed under binary intersections, and so forms a base for a topology $\mathcal{O}S$ on S. However, it is convenient to use a smaller base.

Let $\mathcal{B}$ be the family of all singleton sets $\{m\}$ together with all the sets $B(t,m)$ for $t \in T$ and $m \in \mathbb{N}$. A few moment's thought shows that $\mathcal{B}$ is a base for S, but is not closed under binary intersections.

(a) The space is T_1. For each $m \in \mathbb{N}$ and $t \in T$ we have

$$\{m\}' = \{0, \ldots, m-1\} \cup T \cup [m+1, \infty) \qquad \{t\}' = (T - \{t\}) \cup \mathbb{N}$$

both of which are unions of basic open sets.

(b) The space S is fit. Consider open sets A, B with $A \not\subseteq B$. We require open sets U, V such that

$$A \cup U = S \quad V \not\subseteq B \qquad U \cap V \subseteq B$$

hold.

Suppose first that there is some $m \in \mathbb{N}$ with $m \in A - B$. Then

$$U = S - \{m\} \qquad V = \{m\}$$

will do.

It remains to deal with the case where $A \cap \mathbb{N} \subseteq B$. Since $A \not\subseteq B$ there is some $t \in T$ with $t \in A - B$. There is some $m \in \mathbb{N}$ with $t \in B(t,m) \subseteq A$. Let

$$U = S - \{t\} \qquad V = B(t,m)$$

to produce two open sets. The behaviour of t ensures that

$$A \cup U = S \qquad V \not\subseteq B$$

and

$$U \cap V = [m, \infty) \subseteq A \cap \mathbb{N} \subseteq B$$

as required.

(c) Each member of $\mathcal{B}$ is compact. Every finite set is compact. In particular, $\{m\}$ is compact for each $m \in \mathbb{N}$. Consider any set $B(t,m)$ for $t \in T$ and $m \in \mathbb{N}$. Any open covering of $B(t,m)$ by members of $\mathcal{B}$ must contain at least one set which contains t. Thus there is some $n \in \mathbb{N}$ such that $B(t,n)$ is a member of the covering. But now $B(t,m) - B(t,n)$ is a finite subset of $\mathbb{N}$, and this can be covered by finitely many members of the given cover.

(d) For each $t \in T$ the set $K(t) = B(t,0)$ is in $\mathcal{B}$ and so, by (c), is compact. The complement

$$K(t)' = T - \{t\}$$

is a subset of T. Consider any $s \in K(t)'$. Any $U \in \mathcal{B}$ with $s \in U$ must contain infinitely many natural numbers, and hence $U \not\subseteq K(t)'$. Thus $K(t)'^{\circ} = \emptyset$ to show that $K(t)$ is dense.

(e) For distinct $s, t \in T$, we have $K(s) \cap K(t) = \mathbb{N}$ which is not compact. This is why we assume that T has at least two members.

It can be checked that the patch topology $\mathcal{O}^{p}S$ is generated by $\mathcal{O}S$ and the family of cofinite subsets of T. In particular, if T is finite, then ${}^{p}S$ is discrete. ■

In [4] Escardó uses the construction M to obtain a reflection from the category of stably locally compact frames to a certain category of regular frames. In fact, on closer inspection of the proofs in [4] we see that many are actually using the properties of the construction P. Although not stated explicitly in [4] the following result is obtained.

12.2 THEOREM. *If the frame A is stably locally compact, then $PA = MA$.*

Proof. By Lemma 5.1 of [4], for each open filter F on A, the nucleus v_F is continuous. Thus $PA \subseteq MA$. Conversely, by Lemma 5.4(1) of [4], each $j \in MA$ is a supremum of certain nuclei $u_a \wedge v_F$ for $a \in A$ and open filters F on A. Thus $MA \subseteq PA$. ■

The use of stability here is crucial. The space S of Example 12.1 is locally compact but not stably so, and $S = {}^{m}S \neq {}^{p}S$.

12.3 THEOREM. *For a locally compact and sober space S, the following are equivalent.*
(i) $\mathcal{O}^{p}S \subseteq \mathcal{O}^{m}S$
(ii) The set $\mathcal{Q}S$ is closed under binary intersections.
(iii) The way below relation is stable.
(iv) $P\mathcal{O}S = M\mathcal{O}S$
(v) ${}^{p}S = {}^{m}S$

Proof. In this proof we need to use the way below relation $\ll$ on $\mathcal{O}S$.
(i)⇒(ii). Consider any $Q, R \in \mathcal{Q}S$. Then, assuming (i), we have

$$E = R' \in \mathcal{O}^{p}S \subseteq \mathcal{O}^{m}S$$

and hence E is a uniform tessel. By Lemma 11.7 the set $Q \cap R = Q - E$ is compact, and hence in $\mathcal{Q}S$.
(ii)⇒(iii). Consider any situation

$$U \ll V \qquad U \ll W$$

where $U, V, W \in \mathcal{O}S$. Since S is sober, we have

$$U \subseteq Q \subseteq V \qquad U \subseteq R \subseteq W$$

for some $Q, R \in \mathcal{Q}S$. But now

$$U \subseteq Q \cap R \subseteq V \cap W$$

and the assumption (ii) gives $Q \cap R \in \mathcal{Q}S$, so that

$$U \ll V \cap W$$

holds.
(iii)⇒(iv). Assuming (iii), the locally compact frame $\mathcal{O}S$ is stable, and hence Theorem 12.2 gives (iv).
(iv)⇒(v). When $P\mathcal{O}S = M\mathcal{O}S$ the two surjective morphisms

$$P\mathcal{O}S \longrightarrow \mathcal{O}^{p}S \qquad M\mathcal{O}S \longrightarrow \mathcal{O}^{m}S$$

coincide to give (v).
(v)⇒(i). This is trivial. ■

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