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The use of the Frame Theoretic Assembly as a unifying construct

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This is a version of the MSc Thesis Submitted to the then Victoria University of Manchester in September 2000.

This version we prepared by Harold Simmons, and doesn't use the silly spacing and formatting required by the University regulations. A few minor corrections have been made.

Contents

Chapter 1
Introduction

When studying a topological space, there are two things to consider: the points, and the open sets. In a standard account of topology, the emphasis is mostly on the points, while the open sets are there just to add some structure. However, some results in topology refer only to the open sets and the points seem not to be needed. For example, the standard definition of continuity is a point-free construction. The study of frames allows us to concentrate on just the open sets, giving us point-free analogues of many topological concepts. In this dissertation, we use the concept of frames to investigate several different ways of creating new topologies from old ones.

Let Top be the category of topological spaces and continuous maps. This is the environment we use for the standard point-sensitive results. We also construct the category Frm of frames and frame morphisms which allows us to transfer many results to a point-free context via a contravariant adjuction between Top and Frm.

Although we use the category of frames to study topological spaces, not every object of Frm is the topology of some space. In fact, there are many frames which are not at all like topologies. We examine the internal algebraic structure of the frames as well as their connection with topological spaces.

The original aim of this dissertation was to provide an account of papers [2] and [3]. However, in researching and presenting the background material, new results appeared and it was decided to concentrate on these instead.

Given any frame A, we show how to construct an associated frame, NA, which we call the assembly. There is a frame embedding

$$A \longrightarrow NA$$

which is only an isomorphism in certain limited cases. In general, the object NA is much larger than the original frame A, and is a very complicated object. In this dissertation, we study some aspects of NA, and in particular a number of subframes of NA and their properties. Much of the content is standard, but some of the results are quite new. The new material is due to Harold Simmons and consists partly of previously known but unpublished results, together with some material which has been discovered during the course of this dissertation.

Here is an outline of the contents of each chapter.

Chapter 2 We introduce the idea of a frame. We also present some standard results about frames, including the construction of the point space of a frame which allows us to move from a frame to a topology on a space. We state and prove the "frame separation principle" and various consequences of it; these results allow us to deduce the existence of points with certain properties and is used later on in several constructions. We also introduce the concept of nuceli on a frame, and lay the foundations for the construction of the assembly by exhibiting the infima and suprema of collections of nuclei. We look in particular at a certain type of nuclei on frames of open sets and show that many of the nuclei we wish to examine are of this type. The material in this chapter is almost entirely standard, except possibly the contents of Section 2.5.

Chapter 3 We construct and begin to analyse the assembly, NA, of a frame. We begin by showing that the assembly is itself a frame, and then present the Theorem which states that the passage from a frame to its assembly is functorial. We show that the point space of the assembly of a topology is the front topology on the original space. Again, most of this material is standard, but the contents of Sections 3.3 and 3.4 are not "well known".

Chapter 4 Here we look at some separation properties that we need. We begin with the standard definition of a regular space, and give a point free version. A minor modification of this gives us the definition of a "fit" frame; a separation property that turns out to be particularly connected with the assembly. We consider the relative strength of some of these separation properties and show that the property of fitness is incomparable with the standard separation property T_2.

Chapter 5 We consider the notion of admissible filters and fitted nuclei and how these concepts relate to the separation property fit discussed in the previous chapter. These concepts give a block structure on the assembly of a frame, and we state the result that the collection of fitted nuclei is a subframe of the assembly. We also discuss the relationship between open filters and admissible filters.

Chapter 6 We present the standard construction of the patch topology on a space, a topology lying between the original topology and the front topology. We also look at the functorial properties of this. We then obtain a closely related point-free patch construction and show that this acts as a functor on certain classes of morphisms. We discuss the relationship between these two constructions. Most of the material in the latter half of this chapter is new or not readily available in the literature.

Chapter 7 We examine the continuous assembly on a frame. This construct has been developed over the past eighteen months or so by other people, and we wish to investigate it, in particular in relation to the patch construction of the previous chapter. The continuous assembly has some similar properties to the point free patch construction of the previous chapter, but we will show that in general they are not the same and that the continuous assembly does not have the same nice functorial properties.

We will now give aqn example of a topology on a space together with its assembly and patch assembly.

1.1 Example Let S be the space of real numbers, $\mathbb{R}$. We have the following three topologies on S.

1. $O_l S$: the collection of all sets of the form $(-\infty, a)$.
2. $O_m S$: the metric topology.
3. $O_n S$: the topology generated by all sets of the form $[a, b)$.

We know that $O_n S$ is a refinement of $O_m S$, which is a refinement of $O_l S$, and hence

$$O_l S \hookrightarrow O_m S \hookrightarrow O_n S$$

holds. It turns out that we can actually construct $O_m S$ and $O_n S$ from $O_l S$; the topology $O_n S$ is the assembly of the frame $O_l S$ and the metric topology is the "patch" assembly (Chapter 6). In this case it is also equal to the continuous assembly (Chapter 7). ■

Required Background and Notation

In discussing topological spaces, we refer to the standard separation properties T_0, T_1 and T_2. In the case of T_2 (Hausdorff) spaces, many of the constructions which we consider are be trivial. We are more concerned with cases where the space is T_0 or T_1.

We will refer to the specialisation order on a space S. This is the ordering $\leq_S$ given by

$$p \leq_S q \iff \{p\}^- \subseteq \{q\}^-$$

and is always a pre-order. It is a partial order if and only if S is T_0 and it is equality if and only if S is T_1. The upper sections in this ordering form the Alexandroff topology, ΥS, which contains the original topology since every open set is an upper section.

We also use the notion of a topological space being "sober". The definition of a sober space is a space which is T_0 and every closed irreducible set is a point closure. Intuitively, we may think of a sober space as one in which every point can be uniquely specified by the open sets which it is a member of, and every such collection of open sets specifies a point; in other words, a sober space is one which doesn't "see double" and doesn't "see" points which aren't there - hence the name.

If E is a subset of a space S, we use the standard notation E', E^- and E° for the complement, closure and interior of E respectively.

Chapter 2
Frames

2.1 Basic Material

In this chapter we set down the rudimentary properties of the category Frm. Almost everything that is done (apart from Section 2.5) can be found in [10]. However, sometimes we will develop the material in a slightly different order.

We begin with the definition of a frame and a frame morphism.

2.1 Definition 1. A *frame* is a structure

$$(A, \leq, \wedge, \bigvee, \top, \bot)$$

such that

- $(A, \leq)$ is a complete poset
- $(A, \leq, \wedge, \top)$ is a $\wedge$-semilattice
- $(A, \leq, \bigvee, \bot)$ is a $\bigvee$-semilattice

and the *Frame Distributive Law*

$$a \wedge \bigvee X = \bigvee \{a \wedge x \mid x \in X\}$$

holds for each $a \in A$ and $X \subseteq A$.

2. A *frame morphism*

$$A \xrightarrow{\quad f \quad} B$$

between frames A and B is a function

$$f \colon A \longrightarrow B$$

which preserves $\leq, \wedge, \bigvee, \top$ and $\bot$.

■

It is easy to check that this gives us a category, Frm of frames and frame morphisms.

Notice that, while we can show that any frame, A, will necessarily also posess arbitrary infima, they do not have a corresponding distributive law, and they are not in general preserved by frame morphisms.

We now come to the motivating example for our study of frames.

2.2 Example For any topological space, S, its collection of open sets, OS, is a frame under the usual operations $\subseteq, \cap, \bigcup$. ■

In fact, we have the following result.

2.3 Lemma *There is a contravariant functor*

$$O\colon \mathsf{Top} \longrightarrow \mathsf{Frm}$$

with object assignment and arrow assignment

$$S \longmapsto OS \qquad\qquad \phi \longmapsto \phi^*$$

respectively for each space S and map ϕ. Here OS is the topology of S, and ϕ^* is the restriction of the inverse image function $\phi^{\leftarrow}$ to open sets.

Proof. If

$$\phi\colon T \longrightarrow S$$

is a continuous map between spaces, then the inverse image map

$$\phi^*\colon OS \longrightarrow OT$$

is a frame morphism. It is easy to check that O defined in this way is a functor. ■

We will return to this functor later on, and eventually show that it is part of a contravariant adjunction.

2.4 Lemma *Every frame morphism has a right adjoint.*

Proof. Suppose that

$$f\colon A \longrightarrow B$$

is a frame morphism. We wish to show that there exists a monotone function

$$g\colon B \longrightarrow A$$

such that

$$fx \le y \iff x \le gy$$

holds for all $x \in A, y \in B$.

We define

$$gy = \bigvee\{x \mid fx \le y\}$$

for $y \in B$. We will show that the morphism g has the required property.

Clearly [1]

$$fx \le y \Longrightarrow x \le gy$$

by the definition of g. Conversely

$$x \le gy \Longrightarrow x \le z$$

for some z, where $fz \le y$. However, f is a frame morphism so $fx \le fz \le y$, which gives the result. ■

Notice that the morphism g need not itself be a frame morphism. It does preserve $\le$ and arbitrary infima, but it need not preserve binary suprema.

Conversely, we can show that any monotone function with a right adjoint passes across arbitrary suprema.

2.5 Lemma *If f is a monotone function from A to B with right adjoint g, then f passes across arbitrary suprema.*

Proof. Trivially we have

$$f\left(\bigvee X\right) \ge \bigvee f[X]$$

since f is monotone. We show the converse inequality.

We know that $fx \le \bigvee f[X]$ for every $x \in X$. This gives

$$x \le g\left(\bigvee f[X]\right)$$

for each $x \in X$ by the adjoint property, and so

$$\bigvee X \le g\left(\bigvee f[X]\right)$$

holds. Hence we have

$$f\left(\bigvee X\right) \le \bigvee f[X]$$

and so f preserves arbitrary suprema. ■

Since

$$\phi^* \colon OS \longrightarrow OT$$

is a frame morphism, it has a right adjoint by Lemma 2.4. We will call it ϕ_*.

---

[1] The following argument is nonsense. Since f is a frame morphism $x \le gy$ gives

$$fx \le f(gy) = f(\bigvee\{x \mid fx \le y\}) = \bigvee\{fx \mid fx \le y\} \le y$$

as required.

2.6 Lemma *The right adjoint of ϕ^* is given by*

$$\phi_* W = \phi[W']^{-\prime}$$

for every $W \in OS$.

We can use the right adjoint of a frame morphism when looking at the image of a filter on a frame.

2.7 Definition For $f\colon A \longrightarrow B$ a frame morphism, define

$$fF = \uparrow\{fa \mid a \in F\}$$

for every filter F on A. ■

We make this definition, because the direct image of a filter under a frame morphism may not itself be upwards closed, so may not be a filter. However, fF is always a filter providing that F is.

Since we are working with frame morphisms which will always have a right adjoint, we can express this definition in a way which will sometimes be easier to work with.

2.8 Lemma *If we have*

$$A \underset{g}{\overset{f}{\rightleftarrows}} B$$

where A and B are frames, f is a frame morphism and g is its right adjoint; then for $F \subseteq A$ and $G \subseteq B$ filters on A and B respectively then

$$a \in gG \Longrightarrow fa \in G$$
$$b \in fF \Longrightarrow gb \in F$$

where fF is as in Definition 2.7 and gG is a filter on A.

Proof. Suppose that f, g as above, and $y \in fF$. Then $y \geq fx$ for some $x \in F$, which gives $gy \geq x$ for some $x \in F$, and so $gy \in F$ since F is a filter. The converse is proved similarly.

We can check that gG is also a filter since f preserves $\leq$ and $\wedge$. ■

We will now define the kernel of an arbitrary frame morphism.

2.9 Definition If f is an arbitrary frame morphism, the *kernel* of f is given by

$$\ker(f) = f_* \circ f^*$$

where $f^* = f$ and f_* is its right adjoint. ■

Notice that if f is a frame morphism then

$$fx \leq fy \Longrightarrow x \leq \ker(f)y$$

holds.

Later on, we will see that this gadget has additional properties.

2.10 Definition If A is a frame, we define the functions $\neg$ (negation) and $\supset$ (implication) to be such that

- $\neg a = \bigvee\{x \mid a \wedge x = \bot\}$
- $x \leq (a \supset b) \Longleftrightarrow a \wedge x \leq b$

hold for every $a, b, x \in A$. ■

From this definition, we can make some observations which will be useful.

- $\neg a = a \supset \bot$
- $a \leq b \Longrightarrow (a \supset b) = \top$
- $(\top \supset a) = a$

It is clear that the negation operation is well-defined on a frame. However, we need to prove that there exists an implication operation such that

$$x \leq (a \supset b) \Longleftrightarrow a \wedge x \leq b$$

holds. In fact, we will prove a stronger result.

2.11 Theorem *If A is a complete lattice then A is a frame if and only if A carries an implication.*

Proof. Suppose A is a frame. Consider the set X defined by

$$x \in X \Longleftrightarrow a \wedge x \leq b$$

for fixed $a, b \in A$. Let $c = \bigvee X$, so that

$$a \wedge x \leq b \Longrightarrow x \leq c$$

holds and

$$a \wedge c = a \wedge \bigvee X = \bigvee\{a \wedge x \mid x \in X\} \leq b$$

by the frame distributive law. So we have

$$y \leq c \Longrightarrow a \wedge y \leq a \wedge c \leq b$$

holds for every $y \in A$, and the converse is immediate from the definition of c. Hence

$$a \wedge x \leq b \Longleftrightarrow x \leq c$$

and so $c = a \supset b$.

Conversely, suppose that $(A, \leq, \wedge, \bigvee, \top, \bot)$ is a complete lattice with an implication. We wish to show that the distributive law holds. The inequality

$$a \wedge \bigvee X \geq \bigvee \{a \wedge x \mid x \in X\}$$

is trivial. Now let

$$b = \bigvee \{a \wedge x \mid x \in X\}$$

so that $a \wedge x \leq b$ for all $x \in X$, so $x \leq a \supset b$ for $x \in X$ and hence $\bigvee X \leq a \supset b$ which gives $a \wedge \bigvee X \leq b$ as required. ■

2.2 Nuclei

In this section we will define the concept of nuclei and then state without proof a number of standard results which we will need to use later on.

2.12 Definition

1. An *inflator* on A is a function

$$j \colon A \longrightarrow A$$

which is inflationary and monotone, that is

$$x \leq jx \qquad x \leq y \Longrightarrow jx \leq jy$$

hold for all $x, y \in A$.

2. A *pre-nucleus* is an inflator j such that

$$j(x \wedge y) = jx \wedge jy$$

holds for all $x, y \in A$.

3. A *nucleus* on A is a pre-nucleus j such that $j^2 = j$.

■

We can now look at the collections of these gadgets.

2.13 Definition Let IA be the collection of all inflators on the frame A, PrA the collection of all pre-nuclei and NA the collection of all nuclei. ■

Notice that the objects IA, PrA and NA can be regarded as posets with the ordering on each given by the pointwise order. Do they have any additional structure?

We will now show that they each have binary meets and arbitrary suprema.

2.14 Definition For $F \subseteq A$, the *pointwise infimum* is the function

$$\bigwedge F \colon A \longrightarrow A$$

given by

$$\left(\bigwedge F\right)x = \bigwedge\{fx \mid f \in F\}$$

for each $x \in A$. ■

It can easily be shown that $\bigwedge F$ itself is in IA, and that $\bigwedge F$ is the infimum of F in IA. Furthermore,

$$F \subseteq PrA \Longrightarrow \left(\bigwedge F\right) \in PrA$$

and

$$F \subseteq NA \Longrightarrow \left(\bigwedge F\right) \in NA$$

hold.

Since NA has arbitrary infima, it must also have arbitrary suprema, but these are much harder to calculate. As before, we can define the pointwise supremum.

2.15 Definition For *directed* $F \subseteq IA, PrA$ or NA, the *pointwise supremum* is the function

$$\dot{\bigvee} F \colon A \longrightarrow A$$

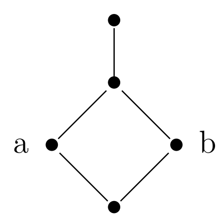
given by

$$\left(\dot{\bigvee} F\right)x = \bigvee\{fx \mid f \in F\}$$

for each $x \in A$. ■

This construction does not have the same nice properties as the pointwise infimum. Notice that this time, we require F to be a directed set. We could easily use the same definition for arbitrary $F \subseteq IA$, and it is straightforward to check that $\dot{\bigvee} F$ is the supremum of F in IA. However, this does not work for arbitrary $F \subseteq PrA$ as an example will show.

2.16 Example Consider the frame, A:



with nuclei j and k given by

$$jx = a \supset x \qquad kx = b \supset x$$

for every $x \in A$. We will later see more nuclei of this form, and prove some useful results with them. Notice that

$$(j\dot{\vee}k)(a \wedge b) = j\bot \vee k\bot = a \vee b \neq \top$$

however, we can see that

$$(j\dot{\vee}k)a \wedge (j\dot{\vee}k)b = (\top \vee b) \wedge (a \vee \top) = \top$$

and hence $j\dot{\vee}k$ does not preserve meets and is therefore not a pre-nucleus. ■

If we require F to be directed, we can ensure that PrA is closed under pointwise suprema.

2.17 Lemma *The collection PrA of pre-nuclei on A is closed under pointwise suprema of directed sets.*

Proof. We will show that the pointwise supremum of a directed set of pre-nuclei is also a pre-nucleus.

Note that

$$\begin{aligned} \dot{\bigvee} Jx \wedge \dot{\bigvee} Jy &= \bigvee\{jx \mid j \in J\} \wedge \bigvee\{jy \mid j \in J\} \\ &= \bigvee\{jx \wedge ky \mid j, k \in J\} \end{aligned}$$

holds. But J is directed, so for any $j, k \in J$ there exists $l \in J$ such that $l \geq j, k$ and hence

$$jx \wedge ky \leq lx \wedge ly$$

and so

$$\begin{aligned} \dot{\bigvee} Jx \wedge \dot{\bigvee} Jy &= \bigvee\{jx \wedge jy \mid j \in J\} \\ &= \dot{\bigvee} J(x \wedge y) \end{aligned}$$

holds as required. ■

However, this does not completely resolve the problems, since it is still the case that NA is not closed under pointwise suprema. However, NA does have arbitrary infima, and therefore must also have arbitrary suprema. What are these?

2.18 Lemma *For each $f \in IA$ there is a least inflator, f^∞, such that f^∞ is idempotent and $f \leq f^\infty$. If $f \in PrA$ then $f^\infty \in NA$.*

The reason for calling this least idempotent inflator f^∞ will become clear from the following proof.

Proof. There are several proofs of this result, all of which provide different information. The most straightforward is to let G be the set of all idempotent inflators g with $f \le g$. Then $f^\infty = \bigwedge G$ will do.

However, this gives us no information about how to construct f^∞. The proof we present here will give us this.

As usual, we generate the ordinal iterates of f by

$$f^0 = id_A \qquad f^{\alpha+1} = f \circ f^\alpha \qquad f^\lambda = \dot{\bigvee}\{f^\alpha \mid \alpha < \lambda\}$$

for successor ordinals α and limit ordinals λ. We can use the pointwise supremum here since trivially

$$\alpha \le \beta \Longrightarrow f^\alpha \le f^\beta$$

and so the set $\{f^\alpha \mid \alpha < \lambda\}$ is directed.

We can check that $f^\alpha \in IA$ and

$$f \in PrA \Longrightarrow f^\alpha \in PrA$$

hold. Now there must be some ordinal, ∞, such that $f^\infty = f^{\infty+1}$ since we can take ∞ to have cardinality greater than the cardinality of IA. This f^∞ is idempotent.

If g is an idempotent inflator, then

$$f \le g \Longrightarrow f^\alpha \le g$$

can be shown inductively, and hence f^∞ is the least idempotent inflator greater than f. ∎

This suggests a method of computing suprema in NA.

2.19 Definition For $J \subseteq NA$, let J° be the compositional closure of J, ie. the set of all

$$j_1 \circ \cdots \circ j_m$$

for $j_1, \ldots, j_m \in J$. ∎

Notice that J° will not in general be a subset of NA, since for $j_1, j_2 \in NA$, $j_1 \circ j_2$ may not be idempotent. However, it is straightforward to show that J° is a subset of PrA. Notice also that $j_1 \circ j_2 \ge j_1, j_2$, which shows that J° is directed, and so we can take its pointwise supremum.

2.20 Lemma *For $J \in NA$,*

$$\bigvee J = \left(\dot{\bigvee} J^\circ\right)^\infty$$

is the supremum of J in NA.

Proof. We have already seen that $\left(\dot{\bigvee} J^\circ\right)^\infty \in NA$. It is also clear that

$$\left(\dot{\bigvee} J^\circ\right)^\infty \geq j$$

holds for all $j \in J$. Now suppose that $j \leq g$ for all $j \in J$, some $g \in NA$. Then $\left(\dot{\bigvee} J^\circ\right) \leq g$ in PrA, and hence $\left(\dot{\bigvee} J^\circ\right)^\infty \leq g$ in NA as required. ■

We will return to the structure of NA in Chapter 3 when we will show that it is a frame.

2.21 Definition For every nucleus, j on A, let A_j be the image $j[A]$. ■

2.22 Lemma *If j is a nucleus on a frame A, and A_j is defined as above, then*

$$x \in A_j \iff x = jx$$

holds for every $x \in A$.

Proof. The implication $(\Longleftarrow)$ is trivial. Suppose $p \in A_j$. Then $p = jx$ for some $x \in A$ and so $jp = p$ since j is idempotent. ■

It is easy to check that A_j forms a frame; however, it is not a subframe of A, since suprema in A_j are not equal to suprema in A. We will think of A_j as the quotient of A over the nucleus j. We will shortly give a factorisation result in the category of frames which will show that nuclei are analogous to normal subgroups in the category of groups or congruence relations in the category of algebras.

Recall that the kernel of a frame morphism f is given by

$$\ker(f) = f_* \circ f^*$$

where $f^* = f$ and f_* is the right adjoint of f.

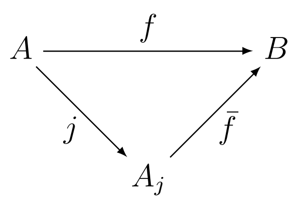
2.23 Lemma *If*

$$f\colon A \longrightarrow B$$

is a frame morphism and j is a nucleus such that $j \leq \ker(f)$ then there is a unique morphism

$$\bar{f}\colon A_j \longrightarrow B$$

such that the diagram



commutes.

In fact, we have a surjective frame morphism

$$\begin{array}{ccc} A & \longrightarrow & A_j \\ x & \longmapsto & jx \end{array}$$

the kernel of which is the original nucleus, j.

We now introduce some particular nuclei which are useful.

2.24 Definition We define the functions u_a, v_a and w_a so that:

$$u_a x = a \vee x \qquad v_a x = (a \supset x) \qquad w_a x = (x \supset a) \supset a$$

hold for all $x \in A$. ■

2.25 Lemma *The functions u_a, v_a and w_a are nuclei on A.*

It is almost trivial to prove this for the u_a and v_a; to prove that w_a is a nuclei takes a little more work. As a consequence, we can also show the following useful observation.

2.26 Lemma *For $a \in A$ and w_a as defined above,*

$$w_a(x \supset a) = (x \supset a)$$

for every $x \in A$.

When A is Boolean frame, we can show that every nucleus is of the form u_a.

2.27 Lemma *Let A be a Boolean frame. The nucleus j is given by*

$$jx = j\bot \vee x$$

for every $x \in A$.

Proof. For every $a \in A$ we have

$$\begin{aligned} j\bot \vee a &= j(a \wedge \neg a) \vee a \\ &= (ja \vee a) \wedge (j(\neg a) \vee a) \\ &= ja \wedge \top \\ &= ja \end{aligned}$$

as required. ■

This shows that the collection of nuclei on a Boolean frame is just the image of the frame itself.

We will now present a number of results about the nuclei u, v and w.

2.28 Lemma *The relations*

1. $u_a \vee u_b = u_{a\vee b} \qquad u_a \wedge u_b = u_{a\wedge b}$

2. $v_a \vee v_b = v_{a\wedge b} \qquad v_a \wedge v_b = v_{a\vee b}$

hold for every $a\, b \in A$. *Note that the far left suprema are taken in* NA.

The nuclei u_a and v_a are related, as we see in this next result.

2.29 Lemma *The nuclei* u_a *and* v_a *are complementary; in other words*

$$u_a \vee v_a = \top \qquad u_a \wedge v_a = \bot$$

for every $a \in A$.

We will very often want to compare an arbitrary nucleus to one of the nuclei u_a, v_a or w_a. The next results give various methods for doing this.

2.30 Lemma *The equivalences*

$$(1)\ u_a \leq j \Longleftrightarrow a \leq j\bot \qquad (2)\ v_a \leq j \Longleftrightarrow ja = \top \qquad (3)\ j \leq w_a \Longleftrightarrow ja = a$$

hold for all nuclei j *on* A.

This next general result will be particularly useful in analysing the continuous assembly in Chapter 7.

2.31 Lemma *If* $j \circ k \in NA$ *satisfy*

$$j \circ k \leq k \circ j$$

then

$$k \vee j = k \circ j$$

holds in NA.

Notice that although both $j \circ k$ and $k \circ j$ form an upper bound for j and k, it is not necessary for both of these to be a nucleus, which is why the condition $j \circ k \leq k \circ j$ is necessary. This result gives us the following Corollary.

2.32 Corollary *The relations*

1. $u_a \vee j = j \circ u_a$

2. $j \vee v_a = v_a \circ j$

hold for all $a \in A$, $j \in NA$.

2.33 Lemma *For each pair* $a \leq b$ *of elements,* $v_a \wedge u_b$ *is the least nucleus which collapses the interval* $[a, b]$.

The final result of this section shows us that we can use the u and v nuclei to represent an arbitrary nucleus j.

2.34 Lemma *For every nucleus j on A, the representations*

1. $j = \bigvee\{v_x \wedge u_{jx} \mid x \in A\}$
2. $j = \bigwedge\{w_{ja} \mid a \in A\}$

hold.

We can now start to examine the relationship between frames and topological spaces.

2.3 The Point Space

We have seen that if S is a topological space, then its collection of open sets, OS, is a frame, and that the construction of a frame from a topological space

$$S \longmapsto OS$$

is functorial. It is natural to ask whether there is a converse; is it the case that every frame A is the topology of some space, S? If so, what does the space S look like?

Essentially, we are asking whether the functor O from the category of topological spaces to the category of frames has an inverse. The answer to this question turns out to be no; however, we will construct the "next best thing" - an adjoint functor from frames to topological spaces.

This construction is quite general and has analogues in several different categories. For example, there is a very similar construction on the category of distributive lattices, which we call the *spectrum* of a lattice.

We begin by constructing the *point space* of a frame A.

2.35 Definition Given $A \in \mathsf{Frm}$ a *character* of A is a frame morphism to the 2-element object $\mathbf{2}$. Let $\mathsf{pt}A$ (the point space of A) be this set of characters. ■

A very similar definition can be used in different categories by considering the collection of all morphisms in the appropriate category to the 2-element object.

It is convenient to construct the point space using characters. However, these can be cumbersome to work with in practice. We will now show that we can think of the points of this space in several different ways, which will be easier to use in certain circumstances.

In a distributive lattice, there is a correspondence between the lattice characters, the prime filters and the prime ideals of the lattice. We can find a similar result in the case where p is a frame character. However the symmetry between filters and ideals breaks down because of the difference between joins and meets in a frame. It is not sufficient for the filters merely to be prime; we need a stronger constraint.

2.36 Definition Let A be a frame. We say that a proper filter F on A is

1. *prime* if
$$x \vee y \in \nabla \Longrightarrow x \in \nabla \text{ or } y \in \nabla$$
holds for every $x, y \in A$.

2. *completely prime* if
$$\bigvee X \in \nabla \Longrightarrow X \text{ meets } \nabla$$
holds for every subset X of A.

■

These two notions are closely related to another kind of filter which we will often make use of later on: open filters.

2.37 Definition Let A be a frame. A filter F on A is *(Scott)-open* if

$$\bigvee X \in F \Longrightarrow X \cap F \neq \emptyset$$

holds for each directed $X \subseteq A$. ■

Notice that this is almost the same definition as for a filter to be completely prime except that in the definition of open filters we require the subset X to be directed, and also that a completely prime filter is necessarily proper, while an open filter need not be. We will examine open filters in more detail in Section 5.3 where (5.12) gives examples of open filters on a linear and a Boolean frame. We have the following relationship between prime, completely prime and open filters.

2.38 Lemma *A filter ∇ is completely prime if and only if it is both prime and open.*

Proof. The implication

$$\text{completely prime} \Longrightarrow \text{prime} + \text{open}$$

is trivial.

For the other direction, suppose that ∇ is prime and open and that $\bigvee X \in \nabla$ for some subset X of A. Then we can take the directed closure, X°, of X by adding $\bigvee Y$ for all *finite* subsets Y of X.

Now we have
$$\bigvee X^\circ = \bigvee X \in \nabla$$
and so since ∇ is open, there exists $x \in X^\circ$ such that $x \in \nabla$. But $x = \bigvee Y$ for some finite $Y \subseteq X$, but then Y meets ∇ since ∇ is prime. Thus X meets ∇ as required. ■

We are now ready to look at two other equivalent ways of describing the point space of a frame.

2.39 Lemma *For $A \in \mathsf{Frm}$, the gadgets*

1. *completely prime filters ∇ of A*
2. *characters p of A*
3. *$\wedge$-irreducible elements of A*

are in pairwise bijective correspondence.

Proof. Suppose ∇ is a completely prime filter. As before, define p so that

$$pa = 1 \Longleftrightarrow a \in \nabla$$

holds. We can check that if ∇ is a completely prime filter then p is a character. For example,

$$p(\bigvee X) = 1 \Longleftrightarrow \bigvee X \in \nabla \Longleftrightarrow X \text{meets } \nabla \Longleftrightarrow (\exists x \in X)[px = 1]$$

since ∇ is completely prime.

Similarly, if p is a character and $\nabla = \{a \in A \mid pa = 1\}$, then ∇ is a completely prime filter.

Now suppose a is a $\wedge$-irreducible element of A. Define a subset ∇ of A by

$$x \in \nabla(a) \Longleftrightarrow x \nleq a$$

for $x \in A$. It can easily be checked that $\nabla(a)$ is a completely prime filter.

Conversely, suppose ∇ is a completely prime filter, and consider $a = \bigvee(A - \nabla)$. Then $a \notin \nabla$ since trivially ∇ and $A - \nabla$ are disjoint. Since ∇ is a filter,

$$x \in \nabla \Longleftrightarrow x \nleq a$$

holds. We claim that a is $\wedge$-irreducible. For suppose $x \wedge y \leq a$. Then $x \wedge y \notin \nabla$, so that $x \notin \nabla$ or $y \notin \nabla$, and hence $x \leq a$ or $y \leq a$ as required. ■

These alternative ways of describing the points are useful in different circumstances. The first is used in the proof of separation results such as the Frame Separation Principle below. The second is useful in finding functorial properties of the map pt and the $\wedge$-irreducible elements are often used for calculation. When we refer to the "points" of the point space, we may be referring to any of these.

We can put a natural topology on the point space using a hull-kernel construction.

2.40 Definition For $S = \mathsf{pt}A$, we define $U_A(a) \subseteq S$ by

$$p \in U(a) \Longleftrightarrow pa = 1$$

for $a \in A$. We will often drop the subscript and just write $U(a)$ when it is clear which frame we are in. ■

Finite intersections and arbitrary unions of these sets are also of this form:

$$U(a) \cap U(b) = U(a \wedge b)$$

and

$$\bigcup\{U(a) \mid a \in X\} = U\big(\bigvee X\big)$$

hold for all $a, b \in A$, $X \subseteq A$. This shows that $\big\{U(a) \mid a \in A\big\}$ is a topology on S and that the following Lemma holds

2.41 Lemma *The morphism*

$$A \xrightarrow{U_A(\cdot)} OS$$

is a surjective frame morphism.

Of course, we can rephrase Definition 2.40 in terms of $\wedge$-irreducible elements to give

$$p \in U(a) \Longleftrightarrow a \nleq p$$

for every $p \in S, a \in A$.

Recall that every topological space S has a specialisation order where

$$p \leq_S q \Longleftrightarrow p^- \subseteq q^-$$

for $p, q \in S$. If S is the point space of A, then we have

$$p \leq_S q \Longleftrightarrow U(q) \subseteq U(p) \Longleftrightarrow q \leq p$$

and so the specialisation order is the *reverse* of the order inherited from A. In other words, we have

$$\uparrow p = \{q \in S \mid q \leq p\}$$

where $\leq$ is the ordering on A.

2.42 Lemma *For each $p \in S$ viewed as a $\wedge$-irreducible element,*

$$U(p) = p^{-\prime}$$

holds.

Proof. For every $p, q \in S$,

$$q \in U(p)' \Longleftrightarrow p \leq q \Longleftrightarrow q \leq_S p \Longleftrightarrow q \in p^-$$

holds. ■

The following result shows that if we have a morphism from A to a spatial frame, we can always factorise it through the point space.

2.43 Theorem *For each frame A with point space* pt$A = S$, *space T and frame morphism*

$$A \xrightarrow{\ f\ } OT$$

there is a unique continuous map

$$T \xrightarrow{\ \phi\ } S$$

such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\ f\ } & OT \\ & {\scriptstyle U_A}\searrow \quad \nearrow{\scriptstyle \phi^*} & \\ & OS & \end{array}$$

commutes.

2.44 Lemma *The point space, S, of any frame A is sober.*

Proof. First we show that S is T_0. Suppose $p \not\leq q \in A$. Then there exists a such that $pa = 1$ and $qa = 0$, and hence $p \in U(a), q \notin U(a)$ as required.

To show that every irreducible set is a point closure, suppose $X \subseteq A$ is irreducible. Define

$$p \colon A \longrightarrow \mathbf{2}$$

to be the morphism given by

$$pa = 1 \iff X \text{ meets } U(a)$$

for every $a \in A$.

It is easy to see that p is a character; the only non-trivial case is to check that $\wedge$ is preserved. For this we have

$$\begin{aligned} p(a \wedge b) = 1 &\iff X \text{ meets } U(a) \cap U(b) \\ &\iff X \text{ meets } U(a) \text{ and } X \text{ meets } U(b) \\ &\iff pa = pb = 1 \\ &\iff pa \wedge pb = 1 \end{aligned}$$

as required, where the second implication holds because of the irreducibility of X.

Now we see that

$$p \in U(a) \iff X \text{ meets } U(a)$$

holds for every $a \in A$. Hence

$$X = X^- = p^-$$

as required.

Therefore every irreducible set is a point closure, and S is sober as required.
■

This suggests that if A is the topology of some space S, then by taking the topology of the point space, we will not get back to the same space unless S is sober. It turns out that this is sufficient; if S is sober, then OS is equal to the topology of its point space.

2.4 The Frame Separation Principle

In this this section, we introduce a result which we will need later on. The following theorem allows us to construct points in a frame which have certain properties. The proof of this theorem requires Zorn's Lemma, and therefore is dependent on the axiom of choice.

2.45 Theorem *[The Frame Separation Principle]*

Let A be a frame, let $a \in A$ and let ∇ be a proper open filter of A with $a \notin \nabla$. Then there is a completely prime filter, P, on A such that $\nabla \subseteq P$, $a \notin P$.

From the correspondence between characters, completely prime filters and $\wedge$-irreducible elements discussed earlier, there are two other equivalent formulations of the frame separation principle.

1. For the situation above, there is a frame character, p, of A with $pa = 0$ and $px = 1$ for all $x \in \nabla$.
2. For the situation above, there is a $\wedge$-irreducible element, p, of A with $a \leq p$ and $p \notin \nabla$.

These results will be used particularly when we discuss open filters and the patch constructions in Chapter 6. Now we will look at a variation of the frame separation principle which allows us to prove some results about open filters. First, we need another definition.

2.46 Definition The *spatial nucleus* of A is the kernel, s, of the morphism

$$A \xrightarrow{U(\cdot)} OS$$

where S is the point space of A. ■

This nucleus has two key properties which we will make use of.

2.47 Lemma *1.* $U(x) \subseteq U(a) \Longleftrightarrow x \leq sa$, *and*

2. $U(a) = U(sa)$.

We can now rephrase the frame separation principle as follows.

2.48 Lemma *Let A be a frame with spatial nucleus s. Then*

$$sa \in F \Longrightarrow a \in F$$

holds for each open filter F on A and $a \in A$.

Proof. We will show the contrapositive. Suppose $a \in A - F$. By the frame separation principle, there is a completely prime filter P with $a \notin P$ and $F \subseteq P$. But then

$$P \in U(a) = U(sa)$$

and so $sa \notin P$, which gives $sa \notin F$ as required. ■

This result is useful, because it allows us to prove something about how the morphism $U(\cdot)$ interacts with filters. Each filter on A can be transfered to a filter

$$\nabla = UF = \{U(x) \mid x \in F\}$$

on S.

We have seen that in general, the direct image of a filter under an arbitrary frame morphism may not itself be a filter because it may not be upwards closed. In this case, we do not have this problem because $U(\cdot)$ is surjective.

The morphism $U(\cdot)$ also preserves certain properties of filters, in particular openness.

2.49 Lemma *Let A be a frame with point space $S = \mathsf{pt}A$. For each open filter F on A, the filter $\nabla = UF$ is open on OS.*

Proof. Suppose $\mathcal{U}$ is a directed subset of OS with $\bigcup \mathcal{U} \in \nabla$. We need to show that $\mathcal{U}$ meets ∇.

Define $X \subseteq A$ by

$$x \in X \Longleftrightarrow U(x) \in \mathcal{U}$$

for every $x \in A$. Since the frame morphism $U(\cdot)$ is surjective, we have

$$\mathcal{U} = \{U(x) \mid x \in X\}$$

and so the set X indexes $\mathcal{U}$, with possible repetitions. Notice that X is closed under the spatial nucleus, s, since $U(sx) = U(x)$ so

$$x \in X \Longrightarrow sx \in X$$

holds.

We show now that X is a directed subset of A. Suppose $x, y \in A$. Then

$$U(x), U(y) \in \mathcal{U}$$

and hence there exists $z \in X$ such that

$$U(x), U(y) \subseteq U(z) = U(sz)$$

holds, since $\mathcal{U}$ is directed. Hence $sz \in X$ and we have $x, y \leq sz$ by Lemma 2.47 above, so that x, y have an upper bound.

Let $a = \bigvee X$. Then

$$U(a) = \bigcup\{U(x) \mid x \in X\} = \bigcup \mathcal{U}$$

so $U(a) \in \nabla$. Then

$$U(a) = U(b)$$

for some $b \in F$. But then

$$sa = sb \in F$$

and so

$$\bigvee X = a \in F$$

by Lemma 2.48. Then X meets F since F is open so there exists $x \in X \cap F$; hence

$$U(x) \in \mathcal{U} \cap \nabla$$

as required. ■

2.5 Spatially Induced Nuclei

When A is a spatial frame, many of the nuclei that we are interested in will be of a particular form. We will be especially interested in this characterisation of u and v nuclei.

2.50 Definition Let S be a space with topology OS. Then define

$$\langle E \rangle U = (E \cup U)^{\circ}$$

for each $E \in \mathcal{P}S$. ■

It is straightforward to check that for any such E, the function $\langle E \rangle$ does indeed define a nucleus on OS.

This function from $\mathcal{P}S$ to NOS has some nice properties.

2.51 Lemma *The assignment*

$$\begin{aligned} \mathcal{P}S &\longrightarrow NOS \\ E &\longmapsto \langle E \rangle \end{aligned}$$

is a $\{\wedge, \top\}$*-morphism.*

Proof. We have $\langle S\rangle(U) = (S \cup U)^\circ = S$ and so $\langle S\rangle = \top_{NOS}$. Now

$$\begin{aligned} \langle E \cap F\rangle(U) &= \big((E \cap F) \cup U\big)^\circ = \big((E \cup U) \cap (F \cup U)\big)^\circ \\ &= (E \cup U)^\circ \cap (F \cup U)^\circ = \langle E\rangle(U) \cap \langle F\rangle(U) \end{aligned}$$

and so the assignment

$$E \longmapsto \langle E \rangle$$

preserves binary meets and $\langle\cdot\rangle$ is a $\{\wedge, \top\}$-morphism as required. ■

In general, this assignment is not injective, but we will find that it is injective on a significant collection of sets: the front open sets.

2.52 Definition The *front space* ${}^fS$ of a topological space S is the space which consists of the same points as S, and the topology generated by the collection

$$\{U \cap V' \mid U, V \in OS\}$$

so that O^fS is the smallest topology containing all open and closed sets on S. ■

When S is a T_1 space, the front topology is just the power set on the space. However, this is not the case for a T_0 space. For instance, in Example 1.1 the topology O_nS is the front topology of O_lS.

We will return to the front topology later on when we look at the point space of the assembly on a frame.

2.53 Definition For a set $E \in \mathcal{P}S$, we write $E^=$ for the front closure of E and $E^\square$ for the front interior of E. ■

2.54 Lemma *The equivalence*

$$\langle D\rangle = \langle E\rangle \iff D^\square = E^\square$$

holds for all $D, E \in \mathcal{P}S$.

Proof. ($\Longleftarrow$). We show that $\langle D\rangle = \langle D^\square\rangle$. The inequality $\langle D^\square\rangle \le \langle D\rangle$ is immediate from $D^\square \subseteq D$.

To get the other direction, suppose that $p \in \langle D\rangle U = (D \cup U)^\circ$ for $U \in OS$. We wish to show that $p \in (D^\square \cup U)^\circ$. For

$$p \in (D \cup U)^\circ \Longrightarrow p \in V \subseteq D \cup U$$

for some $V \in OS$. Either $p \in U$, in which case we are done, or else $p^- \subseteq U'$ and hence

$$p \in V \cap p^- \subseteq (D \cup U) \cap U' \subseteq D$$

to give $p \in D^\square$. This shows $p \in D^\square \cup U$. However, p is contained in an open set, and is therefore in the interior of $D^\square \cup U$ as required.[2]

($\Longrightarrow$). We just need to show that

$$\langle D \rangle \leq \langle E \rangle \Longrightarrow D^\square \subseteq E^\square$$

holds, which will prove the result.

Suppose $\langle D \rangle \leq \langle E \rangle$. However, the sets $U \cap p^-$ form a base for the front topology. Hence

$$\begin{aligned} p \in D^\square &\Longrightarrow (\exists U \in OS)[p \in U \cap p^- \subseteq D] \\ &\Longrightarrow (\exists U \in OS)[p \in U \subseteq D \cup p^{-\prime}] \\ &\Longrightarrow p \in \langle D \rangle p^{-\prime} \subseteq \langle E \rangle p^{-\prime} \subseteq E \cup p^{-\prime} \\ &\Longrightarrow p \in E \end{aligned}$$

holds, and so $D^\square \subseteq E$ to give $D^\square \subseteq E^\square$ as $D^\square$ is front open. ■

We will now determine explicitly the implication operation on a spatial frame.

2.55 Lemma *The implication on the spatial frame OS is given by*

$$W \supset M = (W' \cup M)^\circ$$

for every $W, M \in OS$.

Proof. For every $U, W, M \in OS$, we have

$$\begin{aligned} U \subseteq W \supset M &\Longleftrightarrow U \cap W \subseteq M \\ &\Longleftrightarrow U \subseteq (W' \cup M) \\ &\Longleftrightarrow U \subseteq (W' \cup M)^\circ \end{aligned}$$

since U is open. This gives the result. ■

Now we can show that the u and v nuclei on a spatial frame OS are of this form.

2.56 Lemma *For $W \in OS$,*

$$(1)\ u_W = \langle W \rangle \qquad (2)\ v_W = \langle W' \rangle$$

hold.

[2]This proof is a bit garbled.

Proof. 1. For $M \in OS$ we see that

$$u_W(M) = W \cup M = (W \cup M)^\circ = \langle W \rangle (M)$$

holds as required.

2. Suppose $M \in OS$. Then recall from the previous Lemma that the implication on OS is given by $W \supset M = (W' \cup M)^\circ$. Then

$$v_W(M) = (W' \cup M)^\circ = \langle W' \rangle M$$

as required. ∎

We can now describe the nucleus of the frame morphism

$$\phi^* \colon OS \longrightarrow OT$$

generated by the morphism

$$\phi \colon T \longrightarrow S$$

of topological spaces.

2.57 Lemma *For ϕ^* as above,*

$$\ker(\phi^*) = \langle E \rangle$$

where $E = S - \phi[T]$.

In fact, whenever E is a front open set, the spatially induced nucleus $\langle E \rangle$ always arises in this way.

2.58 Lemma *For each $E \in O^f S$ there is some ϕ with* $\ker(\phi^*) = \langle E \rangle$.

Proof. Let $T = S - E$ and let ϕ be the insertion

$$\phi \colon T \hookrightarrow S$$

so that $\phi[T] = S - E$ and so by the above Lemma, we have $\ker(\phi^*) = E$. ∎

We have now introduced all the basic concepts we need. In the next Chapter we will look at the relationship between a frame and its collection of nuclei.

Chapter 3
The Full Assembly

3.1 NA is a Frame

We have seen in Section 2.2 that the poset NA of nuclei on a frame A has arbitrary infima and suprema. We will now show that it is itself a frame. To do this we will use Theorem 2.11 which states that a complete lattice is a frame if and only if it carries an implication. First, we prove the following result.

3.1 Lemma *For each $f \in PrA, k \in NA$ there is some $l \in PrA$ such that*

$$f \wedge g \leq k \iff g \leq l$$

holds for all $g \in PrA$. Furthermore, $l \in NA$.

Proof. Suppose $f \in PrA, k \in NA$. Define G to be the set of all $g \in PrA$ such that $f \wedge g \leq k$. We show that G is closed under composition. For $g, h \in G$ and $x \in A$ we have

$$\begin{aligned}
(f \wedge (g \circ h))x &= fx \wedge g(hx) \\
&\leq f(kx) \wedge g(fx) \wedge g(hx) \ \text{ since } f(kx), g(fx) \geq fx. \\
&= f(kx) \wedge g(fx \wedge hx) \\
&\leq f(kx) \wedge g(kx) \\
&\leq k^2 x \\
&= kx
\end{aligned}$$

so that $g \circ h \in G$.

We know that $f, g \in PrA, f \circ g \geq f, g$ and hence any subset of PrA that is closed under composition is directed. Therefore

$$g \in G \Longrightarrow g^2 \in G$$

holds for all $g \in PrA$.

Now let $l = \dot{\bigvee} G$, so l is the supremum of G in PrA. However, l may not be in NA. But we know that for each $x \in A$

$$\begin{aligned}
(f \wedge l)x = fx \wedge lx &= fx \wedge \bigvee\{gx \mid g \in G\} \\
&= \bigvee\{fx \wedge gx \mid g \in G\} \\
&\leq kx
\end{aligned}$$

holds, and hence $l \in G$. Then $l^2 \in G$ since G is closed under composition, which gives $l^2 \leq l$. But l is inflationary, so $l^2 = l$ and $l \in NA$. This proves the lemma. ■

3.2 Theorem *For each frame A, the assembly NA is also a frame.*

Proof. For $f, k \in NA$ we can see that, using the notation of Lemma 3.1,

$$(f \supset k) = l$$

with $l \in NA$ and so NA carries an implication and therefore is a frame by Theorem 2.11. ■

The proof of this result in [10] uses a more explicit construction of the implication, which provides us with different information. The implication operation is defined as

$$(j \supset k)a = \bigwedge\{j(x) \supset k(x) \mid x \geq a\}$$

for every $a \in A$ and then it is proved that this is in fact an implication on the lattice NA.

3.2 N as a Functor

The frame NA, otherwise known as "The Assembly", can be an extremely complex object. In this section we will investigate some of its properties and its relationship with its parent frame, A. Of particular interest to us will be the fact that N acts as a functor on the category of frames.

This next result shows that NA provides complements for every element in the canonical image of A, and moreover, it does it in the most efficient way possible.

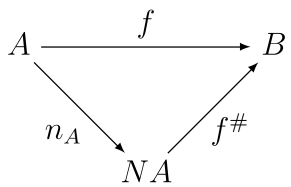
3.3 Theorem *The morphism*

$$A \xrightarrow{n_A} NA$$

universally solves the complementation problem for A. That is, for each morphism

$$A \xrightarrow{f} B$$

such that fa has a complement in B for every $a \in A$, there exists a unique morphism $f^{\#}$ such that the diagram



commutes.

The proof of this result is quite detailed and will be omitted. We can now prove the next main result.

3.4 Theorem *The assignment*

$$A \longmapsto NA$$

is the object part of a functor

$$\mathsf{Frm} \xrightarrow{\ N\ } \mathsf{Frm}$$

and the morphism

$$\begin{array}{ccc} A & \xrightarrow{\ n_A\ } & NA \\ a & \longmapsto & u_a \end{array}$$

is a natural transformation. In other words, for every $A, B \in \mathsf{Frm}$ *and every frame morphism* f*, the diagram*

$$\begin{array}{ccc} A & \xrightarrow{n_A} & NA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Nf} \\ B & \xrightarrow[n_B]{} & NB \end{array}$$

commutes for some unique morphism Nf*.*

Proof. In the above diagram, the image of every element of A under $n_B \circ f$ is complemented in NB, and so by Theorem 3.3 there is a unique morphism,

$$NA \xrightarrow{\ Nf\ } NB$$

which makes the above square commute.

We just need to check that this is a functor; that is, for

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

then $N(g \circ f) = Ng \circ Nf$. But

$$\begin{array}{ccc} A & \xrightarrow{n_A} & NA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Nf} \\ B & \xrightarrow{n_B} & NB \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle Ng} \\ C & \xrightarrow{n_C} & NC \end{array}$$

commutes, so $Ng \circ Nf$ must be the unique arrow which makes the outer square commute. Hence $N(g \circ f) = Ng \circ Nf$ as required. ■

This method of proving the functorality of N is different from that given in [10], which proves Theorem 3.4 directly and then goes on to give a modified version of Theorem 3.3.

3.5 Corollary *For f a frame morphism from A to B*

1. $(Nf)u_a = u_{fa}$
2. $(Nf)v_a = v_{fa}$

for every $a \in A$.

Proof. 1. Follows immediately from the above diagram.

2. We know that u and v are complementary in NA and that Nf is a frame morphism. Hence

$$u_{fa} \wedge (Nf)v_a = (Nf)(u_a \wedge v_a) = \bot$$

and

$$u_{fa} \vee (Nf)v_a = (Nf)(u_a \vee v_a) = \top$$

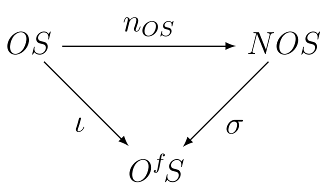
and so $(Nf)v_a$ is the complement in NA to u_{fa}. Hence $(Nf)v_a = v_{fa}$ as required. ■

3.3 The Fundamental Triangle of a Space

The remainder of this chapter will build up to a description of the point space of an assembly. In this section, we investigate the relationship between the assembly on a space, NOS, and the front space, O^fS.

The following Theorem is a special case of Theorem 3.3.

3.6 Theorem *For each space S there is a unique σ such that*



commutes.

Proof. Every $U \in OS$ has a complement in O^fS, so O^fS solves the complementation problem for OS. Therefore σ exists and is unique by Theorem 3.3. ■

This works for any space S, but in practice we almost always use it for the point space of a frame, which is sober. The morphism σ can be checked to be a natural transformation as S varies.

Notice that the natural transformation

$$OS \xrightarrow{n_{OS}} NOS$$

is given by

$$U \longmapsto \langle U \rangle$$

ie. the restriction of $\langle\cdot\rangle$ to OS.

What can we say about the morphism σ? We will now define a morphism, σ, and show that it is a frame morphism and makes the above triangle commute.

3.7 Definition For S as above let σ be the assignment given by

$$\sigma j = \bigcup\{(jW) - W \mid W \in OS\}$$

for all $j \in NOS$. ■

We will eventually show that this σ is a frame morphism. The following lemma also provides us with a characterisation of σ which will often be more useful than the definition given above.

3.8 Lemma *For σ defined as above,*

$$p \in \sigma j \Longleftrightarrow p \in jp^{-\prime}$$

holds for every $p \in S$, and σ is a $\wedge$-morphism.

Proof. If $p \in \sigma j$ then

$$p \in \bigcup\{(jW) - W \mid W \in OS\}$$

so that for some $W \in OS$, we have $p \in (jW)$ and $p \notin W$. Hence $W \subseteq p^{-\prime}$ and therefore $p \in jW \subseteq jp^{-\prime}$. Conversely, suppose that $p \in jp^{-\prime}$. Then seting $W = p^{-\prime}$, we have $p \in jW$, $p \notin W$ so that $p \in \sigma j$ as required.

Now we can use this to show that σ passes across binary meets. We have

$$\begin{aligned} p \in \sigma(j \wedge k) &\Longleftrightarrow p \in (j \wedge k)p^{-\prime} \\ &\Longleftrightarrow p \in jp^{-\prime} \cap kp^{-\prime} \\ &\Longleftrightarrow p \in \sigma j \cap \sigma k \end{aligned}$$

as required. ■

This next Lemma is needed to complete the proof that σ is a frame morphism.

3.9 Lemma *For σ as above, $\sigma\langle E\rangle = E^{\Box}$ holds for every $E \in \mathcal{P}S$.*

Proof. From the definition of σ, we see that for every $U \in OS$,

$$\langle E \rangle U - U = (E \cup U)^{\circ} - U \subseteq E^{\Box}$$

and so $\sigma\langle E \rangle \subseteq E^{\Box}$.

Now we use the same idea as in the proof of Lemma 2.54. Suppose $p \in E^{\Box}$. Then there exists $U \in OS$ such that

$$p \in U \cap p^{-} \subseteq E$$

which gives $p \in U \subseteq E \cup p^{-\prime}$ and so $p \in \langle E \rangle p^{-\prime}$. Hence

$$p \in \langle E \rangle p^{-\prime} - p^{-\prime}$$

and therefore $p \in \sigma\langle A \rangle$ as required. ■

3.10 Lemma *The morphism σ as defined above has a right adjoint, $\langle \cdot \rangle$, and σ is a frame morphism.*

Proof. We need to check that

$$\sigma j \subseteq E \Longleftrightarrow j \leq \langle E \rangle$$

holds for each $j \in NOS$ and $E \in O^{f}S$. Suppose $\sigma j \subseteq E$. Then for every $U \in OS$ and $p \in S$, we have

$$\begin{aligned} p \in jU &\Longrightarrow p \in (jU) - U \ \text{ or } p \in U \\ &\Longrightarrow p \in \sigma j \ \text{ or } p \in U \\ &\Longrightarrow p \in E \cup U \end{aligned}$$

and hence $jU \subseteq \langle E \rangle U$ as required.

Conversely, suppose that $j \leq \langle E \rangle$. Then $\sigma j \subseteq \sigma\langle E \rangle$. From Lemma 3.9, we know that $\sigma\langle E \rangle = E^{\Box}$ for all $E \in \mathcal{P}S$. This shows that $\sigma j \subseteq E^{\Box}$, which gives $\sigma j \subseteq E$, since σj is front open.

Now we know that σ has a right adjoint, and so by Lemma 2.5 it passes across arbitrary suprema.

It is trivial to check that $\sigma(\bot_{NOS}) = \bot_{O^{f}S}$ and $\sigma(\top_{NOS}) = \top_{O^{f}S}$ and so we have now shown that σ is a frame morphism. ■

Finally, we can prove that the σ we defined above is the morphism given by Theorem 3.6.

3.11 Lemma *The morphism σ defined in 3.7 makes the triangle of Theorem 3.6 commute.*

Proof. Suppose that $U \in OS$. Then

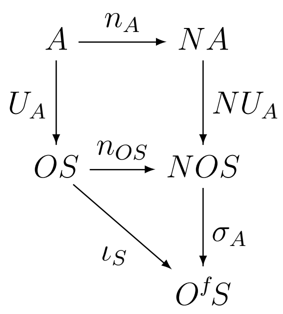
$$(\sigma \circ n_{OS})U = \sigma\langle U \rangle = U^{\Box} = U$$

again using Lemma 3.9, and so σ makes the above diagram commute. ■

3.4 The Point Space of an Assembly

We can now put together the diagrams we constructed in the previous sections.

3.12 Theorem *As usual, let A be a frame and $S = \mathsf{pt}A$. Then the diagram*



commutes.

Proof. Just apply Theorem 3.4 to the morphism

$$A \xrightarrow{\quad U \quad} OS$$

to give the square. Theorem 3.6 gives the triangle. ■

This on its own is not particularly useful or interesting. However, later in this section we will show that $O^f S$ is the topology on the point space of both NA and NOS.

Before we can prove this result, we need several lemmas.

This result first appeared in print as Lemma 3.2 of [12].

3.13 Lemma *For each $j \in NA$, the three conditions*

1. *j is $\wedge$-irreducible (in NA)*
2. *j is two-valued*
3. *$a = j\bot$ is $\wedge$-irreducible (in A) and $j = w_a$ are equivalent.*

Proof. By condition (2) we mean that every value of j is either $j\bot$ or $\top$. This is only possible if j is of the form

$$jx = \begin{cases} \top & \text{if } x \not\leq a \\ a & \text{if } x \leq a \end{cases}$$

where $j\bot = a$ and $a \neq \top$. This will only be a nucleus for certain values of a.

We will prove that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1).

(1) $\Longrightarrow$ (2). Suppose that j is irreducible, and let $a = j\bot$. We know that

$$v_x \wedge u_x = \bot_{NA} \leq j$$

holds for each $x \in A$. Therefore either $u_x \leq j$ or $v_x \leq j$.

Suppose that $u_x \leq j$. Then

$$x = u_x \bot \leq j \bot = a$$

and hence $jx = ja = a$. This is only possible when $x \leq a$. Now suppose that $x \nleq a$. Then we have

$$v_x \leq j \Longrightarrow \top = v_x x \leq jx$$

which gives

$$x \nleq a \Longrightarrow jx = \top$$

and hence j is two valued.

(2) $\Longrightarrow$ (3). Suppose j is two-valued, and $x \wedge y \leq a$ for some $x, y \in A$. Then we have

$$jx \wedge jy \leq j(x \wedge y) \leq a$$

by the form of j. Since $jx, jy \in \{a, \top\}$ then

$$jx = a \quad \text{or} \quad jy = a$$

which gives

$$x \leq a \quad \text{or} \quad y \leq a$$

again by the form of j. We know that $a \neq \top$, so this shows that a is $\wedge$-irreducible.

Now we need to show that w_a is of the form

$$w_a x = \begin{cases} \top & \text{if } x \nleq a \\ a & \text{if } x \leq a \end{cases}$$

whenever a is $\wedge$-irreducible. Suppose that $x \leq a$. Then

$$(x \supset a) \supset a = \top \supset a = a$$

by the properties of $\supset$. If $x \nleq a$, then

$$(x \supset a) \wedge ((x \supset a) \supset a) \leq a$$

but w_a is a nucleus, so $(x \supset a) \supset a \nleq a$. Hence, since a is $\wedge$-irreducible, $(x \supset a) \leq a$ and then

$$(x \supset a) \supset a = \top$$

as required.

(3) $\Longrightarrow$ (1). We wish to show that w_a is $\wedge$-irreducible whenever a is. We have already shown that when a is irreducible then w_a is two-valued in the proof of (2) $\Longrightarrow$ (3). We also have

$$\begin{aligned} k \wedge l \leq w_a &\Longrightarrow ka \wedge la \leq w_a a = a \\ &\Longrightarrow ka \leq a \quad \text{or} \quad la \leq a \\ &\Longrightarrow k \leq w_a \quad \text{or} \quad l \leq w_a \end{aligned}$$

since if $ka \leq a$ then $kx \leq a$ whenever $x \leq a$ and trivially $kx \leq w_a x = \top$ when $x \nleq a$. This shows w_a is $\wedge$-irreducible as required, which completes the proof. ■

This result gives us a lot of information about the point space of an assembly. Recall that if T is the point space of NA, then T can be regarded as the collection of $\wedge$-irreducible elements of NA. By the previous Lemma, we now know that these are the nuclei w_p where p is $\wedge$-irreducible in A. In other words

$$T = \{w_p \mid p \in S\}$$

where S is the point space of A.

By the previous Lemma, we have an inverse pair of bijections

$$S \underset{\psi}{\overset{\phi}{\rightleftarrows}} T$$

such that $\phi(p) = w_p$ and $\psi(P) = P\bot$ hold.

This shows that the point space of NA has essentially the same points as S, but a different topology. So what is the induced topology on S?

3.14 Lemma *For S, T, ϕ and ψ as defined above, the induced topology on S is the front topology $O^f S$.*

Proof. In this proof, we will regard the point space of a frame as its collection of $\wedge$-irreducible elements.

The typical open sets of S are

$$p \in U(x) \Longleftrightarrow x \nleq p$$

for $x \in A, p \in S$, and the open sets of T are

$$P \in U(j) \Longleftrightarrow j \nleq P$$

for $j \in NA, P \in T$.

Now for each $j \in NA$, we have

$$j = \bigvee \{v_x \wedge u_{jx} \mid x \in A\}$$

and hence the sets

$$U(u_x) \qquad U(v_x)$$

with $x \in A$ form a subbase of T, since $U(\cdot)$ is a frame morphism.

Now consider that

$$\begin{aligned} \phi^{\leftarrow} U(u_x) = \psi\big(U(u_x)\big) &= \{\psi P \mid u_x \nleq P\} \\ &= \{P\bot \mid u_x \nleq P\} \\ &= \{p \mid x \nleq p\} \\ &= U(x) \end{aligned}$$

since P is two-valued by the previous lemma and so

$$u_x \nleq P \iff x \nleq P\bot$$

holds.

We know that

$$v_x \wedge u_x = \bot \leq P \qquad v_x \vee u_x = \top$$

hold for every $x \in A, P \in T$ and so exactly one of

$$v_x \leq P \qquad u_x \leq P$$

holds. In other words

$$P \in U(v_x) \iff P \notin U(u_x)$$

for each $P \in T, x \in A$. Hence

$$\phi^{\leftarrow} U(v_x) = \psi\big(U(v_x)\big) = \psi\big(U(u_x)'\big) = U(x)'$$

since ψ is a bijective frame morphism.

Similarly, we have $\psi^{\leftarrow} = U(x)$ and $\psi^{\leftarrow} U(x)' = U(v_x)$ and hence the sets $U(x)$ and $U(x)'$ form a subbase for the induced topology on S, which must therefore be the front topology $O^f S$. ■

This finally gives the result we are aiming for.

3.15 Theorem *For each frame A the point space of the assembly is the front space of the point space of A.*

From this, provided S is sober, it follows immediately that $O^f S$ is the point space of the frame OS, since S is the point space of OS and so we can apply Theorem 3.15.

We conclude this chapter with some general results about the morphisms σ and NU which we will return to later on when discussing the continuous assembly.

Define Σ to be the composite map

$$\sigma \circ NU \colon NA \longrightarrow O^f S$$

from NA to its point space. We make two observations.

3.16 Lemma *For each frame A*

$$U(ja) \subseteq \big((NU)j\big)U(a)$$

holds for each $j \in NA$ and $a \in A$.

Proof. We have the usual representation

$$j = \bigvee\{v_x \wedge u_{jx} \mid x \in A\}$$

for j. This gives

$$NUj = \bigvee\{\langle U(x)' \cap U(jx)\rangle \mid x \in A\}$$

and so in particular $\langle U(a)' \cap U(ja)\rangle \leq NUj$. Applying both sides to $U(a)$ gives

$$U(ja) \leq \big((NU)j\big)U(a)$$

as required. ■

3.17 Lemma *For each frame A with point space S (viewed as $\wedge$-irreducibles)*

$$\Sigma j = S - A_j$$

holds for each $j \in NA$.

Proof. As before, we can use the usual representation of j to get

$$NUj = \bigvee\{\langle U(x)' \cap U(jx)\rangle \mid x \in A\}$$

and hence that

$$\begin{aligned}\Sigma j = \sigma\big((NU)j\big) &= \bigcup\{\langle U(x)' \cap U(jx)\rangle W - W \mid x \in A, W \in OS\}\\ &= \bigcup\{U(a)' \cap U(ja) \mid a \in A\}\end{aligned}$$

holds. Hence

$$p \in \Sigma j \iff p \in U(ja), p \notin U(a)$$

for some a. This gives

$$p \in \Sigma j \iff j \nleq p, a \leq p$$

for some a, or equivalently,

$$p \in \Sigma j \iff jp \neq p \iff p \notin A_j$$

since we know that $p \in A_j$ if and only if $jp = p$. This gives the result. ■

Chapter 4
Separation Properties

4.1 Regularity and Fitness

In this section we will discuss some separation properties on topological spaces, and their point-free equivalents on frames. Later on, we will see how these can be used to give us information about the properties of a frame.

4.1 Definition A topological space, S, is *regular* if for each $p \notin X \in CS$ there are $U, V \in OS$ such that

$$X \subseteq U \qquad p \in V \qquad U \cap V = \varnothing$$

hold. ■

It is straightforward to give a point-free version of this definition for frames.

4.2 Definition A frame A is *regular* if for each $a, b \in A$ with $a \not\leq b$ there are $x, y \in A$ such that

$$a \vee x = \top \qquad y \not\leq b \qquad x \wedge y = \bot$$

hold. ■

We need to show that these two definitions coincide when applied to a space S and its frame OS of open sets respectively.

4.3 Lemma *A space S is regular if and only if the frame OS is regular.*

Proof. Suppose the space S is regular, and $M, N \in OS$ with $M \not\subseteq N$. Then there exists p such that $p \in M, p \notin N$. Now set $X = M'$; then $p \notin X$, so we can apply Definition 4.1 to get $U, V \in OS$ such that

$$X \subseteq U \qquad p \in V \qquad U \cap V = \varnothing$$

hold. But then $M \cup U = S$,

$$p \in V, p \notin N \Longrightarrow V \not\subseteq N$$

and $U \cap V = \varnothing$ as required for OS to be regular.

Conversely, suppose that OS is regular and $p \notin X \in CS$. Then we have $X' \nsubseteq p^{-\prime}$ and so there are $U, V \in OS$ such that

$$X' \cup U = S \qquad V \nsubseteq p^{-\prime} \qquad U \cap V = \varnothing$$

hold. But then

$$X \subseteq U \qquad p \in V \qquad U \cap V = \varnothing$$

as required for S to be regular. ■

4.4 Definition We say that a space S has the separation property T_3 if it is both regular and T_0. ■

This gives us another separation property which fits into our usual hierarchy

$$T_3 \Longrightarrow T_2 \Longrightarrow T_1 \Longrightarrow T_0$$

in the natural way. We will later see that on its own, the property of being regular is independent of T_2.

We will now introduce a related separation property fit which we will need to refer to later on. This time we will go straight for a point-free definition.

4.5 Definition A frame is fit if for each $a, b \in A$ with $a \nleq b$ there are $x, y \in A$ such that

$$a \vee x = \top \qquad y \nleq b \qquad x \wedge y \leq b$$

hold. ■

Notice that this definition of fitness is very similar to the point-free definition of regularity, the only difference being the last clause where we only require $x \wedge y \leq b$ instead of the stronger $x \wedge y = \bot$. Trivially, then, we can see that regularity implies fitness. The following example will show that fitness is a strictly weaker condition than regularity.

4.6 Example Let $S = \mathbb{N}$ with OS the topology of cofinite sets. We claim that OS is fit but not regular. [1] ■

Proof. Suppose we have $A, B \in OS$ with $A \nsubseteq B$. Then we can write

$$A = \mathbb{N} - \{a_1, \ldots, a_n\} \qquad B = \mathbb{N} - \{b_1, \ldots, b_m\}$$

where

$$\{b_1, \ldots, b_m\} \nsubseteq \{a_1, \ldots, a_n\}$$

holds. Now there exists $p \in \{b_1, \ldots, b_m\}$, $p \notin \{a_1, \ldots, a_n\}$, which we will call b_m. Set

$$X = \mathbb{N} - \{b_m\} \qquad Y = \mathbb{N} - \{b_1, \ldots, b_{m-1}\}$$

[1] In fact, this example is wrong. The following proof doesn't consider the case where $B = \emptyset$.

so that

$$A \cup X = \mathbb{N} \qquad Y \nsubseteq B \qquad X \cap Y = \mathbb{N} - \{b_1, \dots, b_m\} = B$$

as required.

However, for $U, V \in OS$,

$$U \cap V = \varnothing \Longrightarrow U = \varnothing \text{ or } V = \varnothing$$

so that in general, given $p \notin X \in CS$ there are no open sets U and V such that

$$X \subseteq U \qquad p \in V \qquad U \cap V = \varnothing$$

holds. Hence S is not regular and therefore OS is not regular, as claimed. ■

We will say that a space S is fit if its frame of open sets is fit.

4.2 The Point Space of a Fit Frame

In this section we will look at some of the consequences of a frame being fit, and in particular what we can say about the point space of a fit frame.

4.7 Lemma *Let*

$$A \xrightarrow{\quad f \quad} B$$

be a surjective frame morphism. Then

$$\begin{array}{ccccc} A & \textit{regular} & \Longrightarrow & B & \textit{regular} \\ A & \textit{fit} & \Longrightarrow & B & \textit{fit} \end{array}$$

hold.

Proof. Let g be the right adjoint of f so that

$$fy \leq b \Longleftrightarrow y \leq gb$$

for every $y \in A, b \in B$. Since f is surjective, then

$$f \circ g = \mathrm{id}_B$$

holds.

We will prove that

$$A \text{ fit} \Longrightarrow B \text{ fit}$$

holds. Then implication

$$A \text{ regular} \Longrightarrow B \text{ regular}$$

follows similarly.

Suppose $a, b \in B$ with $a \not\leq b$. Then $ga \not\leq gb$ since $f \circ g = \mathrm{id}_B$ and f is a frame morphism. But A is fit, and so there exist $x, y \in A$ such that

$$ga \vee x = \top \qquad y \not\leq gb \qquad x \wedge y \leq gb$$

hold. Applying f to each of these gives

$$a \vee fx = \top \qquad fy \not\leq b \qquad fx \wedge fy \leq b$$

which gives the result. ■

4.8 Corollary *Let A be a frame with point space $S = \mathsf{pt}A$. Then*

$$\begin{array}{cccccc} A & \textit{regular} & \Longrightarrow & S & \textit{regular} \\ A & \textit{fit} & \Longrightarrow & S & \textit{fit} \end{array}$$

hold.

Proof. Recall from Lemma 2.41 that

$$A \xrightarrow{U(\cdot)} OS$$

is a surjective frame morphism. ■

Recall that the point space of a frame A can be regarded as the collection of $\wedge$-irreducible elements of A. We have the following general result. Recall that an element $a \in A$ is said to be maximal if

$$a < b \Longrightarrow b = \top$$

holds.

4.9 Lemma *If $a \in A$ is maximal, then $a \in \mathsf{pt}A$.*

Proof. We need to show that

$$a \text{ maximal } \Longrightarrow a \text{ is } \wedge\text{-irreducible}$$

holds. Suppose that a is maximal and $x \wedge y \leq a$. Then we have $a \vee (x \wedge y) = a$. Now

$$x \not\leq a \Longrightarrow a \vee x > a \Longrightarrow a \vee x = \top$$

holds, and so $\top \wedge (a \vee y) = a$ which gives $y \leq a$.

Hence either $x \leq a$ or $y \leq a$ which proves the result. ■

This Lemma applies to all frames; however, when a frame A is fit, the implication becomes an equivalence.

4.10 Lemma *If A is fit, then every point ($\wedge$-irreducible element) of A is maximal.*

Proof. Suppose $a \in A$ is $\wedge$-irreducible and $a < b$. Then $b \not\leq a$ and because A is fit, there exist $x, y \in A$ such that

$$b \vee x = \top \qquad y \not\leq a \qquad x \wedge y \leq a$$

hold. The last two conditions imply that $x \leq a$ since a is $\wedge$-irreducible. Hence

$$b \vee a \geq b \vee x = \top$$

and so $b = \top$ as required. ■

These results tell us that the point space of a fit frame is simply its collection of maximal elements. This gives us some immediate corollaries.

4.11 Corollary *If the frame A is fit, then $\mathsf{pt}A$ is T_1.*

Proof. Recall that the typical open sets of $\mathsf{pt}A$ are

$$U(x) = \{p \in \mathsf{pt}A \mid x \not\leq p\}$$

for every $x \in A$. Now suppose $p, q \in \mathsf{pt}A$. Then

$$U(q) = \mathsf{pt}A - \{q\}$$

and so $p \in U(q)$ but $q \notin U(q)$ which shows that $\mathsf{pt}A$ is T_1. ■

4.12 Corollary *If a space S is sober and fit, then S is T_1.*

Proof. We know that

$$OS \text{ fit} \Longrightarrow \mathsf{pt}(OS) \text{ is } T_1$$

and

$$S \text{ sober } \Longrightarrow S \cong \mathsf{pt}(OS)$$

and so the space S is T_1. ■

In the next section, we will investigate further the relationship between fitness and the other separation properties.

4.3 A Hierarchy of Separation Properties

At the end of the previous section, we saw that

$$\text{sober } + \text{fit} \Longrightarrow T_1$$

holds. We will now go further and show that in a fit frame, the properties T_0, T_1 and sober are all equivalent.

4.13 Theorem *Let the space S be fit. Then the three conditions*

1. *S is T_0*
2. *S is T_1*
3. *S is sober*

are equivalent.

Proof. In the previous section we saw that in a fit space

$$\text{sober} \Longrightarrow T_1 \Longrightarrow T_0$$

holds. To complete the equivalence, we will show that

$$\text{fit} + T_0 \Longrightarrow \text{sober}$$

holds.

Let S be T_0 and fit, and consider a closed irreducible subset $X \subseteq S$ and an element $p \in S$. Thus X' is $\wedge$-irreducible in OS. Then

$$p \in X \Longrightarrow p^- \subseteq X \Longrightarrow X' \subseteq p^{-\prime}$$

holds. But by Lemma 4.10, X' is maximal, and so we have $X' = p^{-\prime}$ and hence that $X = p^-$ holds.

Thus every closed irreducible set is a point closure, and therefore S is sober. ■

The next question we will consider is the relationship between fitness and the separation property T_2. It turns out that they are incomparable; we have

$$T_2 \not\Longrightarrow \text{fit} \qquad \text{fit} \not\Longrightarrow T_2$$

in general. In fact, the latter can be strengthened to

$$\text{fit} \not\Longrightarrow T_0$$

otherwise, by the above theorem every fit space would also be sober.

In Example 7.17 we give an example of a space which is both fit and sober, but not T_2. Here we will construct the converse: a space which is T_2 but not fit.

4.14 Example Let S be the real numbers, $\mathbb{R}$. Consider all subsets of the form

$$U \cup (V \cap \mathbb{Q})$$

where U and V are metric open sets on $\mathbb{R}$. This gives a topology, OS, which includes the metric topology and therefore is T_2. Notice that in this topology, $\mathbb{Q}$ is an open set. We will show that OS is not fit.

Suppose for a contradiction that OS is fit. Then since

$$Q \nsubseteq \varnothing$$

holds, we can find metric open sets U, V and W such that

1. $\mathbb{Q} \cup U = \mathbb{R}$
2. $V \cup (W \cap \mathbb{Q}) \neq \varnothing$
3. $U \cap \big(V \cup (W \cap \mathbb{Q})\big) = \varnothing$

hold. Then by (3), we can deduce that

$$U \cap V = U \cap W \cap \mathbb{Q} = \varnothing$$

holds. This shows that $U \cap W$ is empty, since every non empty metric open set contains a rational point. Hence, by (1) we must have $W \subseteq \mathbb{Q}$ but W is metric open and so $W = \varnothing$. This means that V cannot be empty, while

$$U \cap V = \varnothing$$

which gives a contradiction. ■

This example is taken from page 85 of [10] where it is used for a slightly different purpose.

Chapter 5
The Block Structure of an Assembly

5.1 Admissible Filters and Fitted Nuclei

In this chapter we will introduce the first example of a subframe of NA, the frame of fitted nuclei. Many of the ideas in this chapter will be revisited when we look at the patch constructions in the next chapter. The results in this Chapter also give us some information about the algebraic properties of the Assembly, NA. We start by defining the notion of a fitted nucleus.

5.1 Definition For a frame A, an element $a \in A$ and a nucleus $j \in NA$, we say that j *admits* a if and only if $ja = \top$. ■

Notice that the set $\nabla(j)$ of elements admitted by j is a filter in A.

5.2 Definition

1. We say a filter is *admissible* if it is of the form $\nabla(j)$ for some $j \in NA$. The equivalence classes of this equivalence relation are called *blocks*.
2. Two nuclei are *companions* if they admit the same elements.
3. A nucleus is *fitted* if it is the least member of its block.

■

As we will see from the following result, there is a one to one correspondence between blocks and their fitted nuclei.

5.3 Lemma *Each block has a least member.*

Proof. Suppose that ∇ is an admissible filter, and $B = \{j \mid j \in NA, \nabla(j) = \nabla\}$. Remembering that infima are computed pointwise in NA, let $k = \bigwedge B$. We claim that $k \in B$. For suppose $a \in \nabla$. Then $ja = \top$ for all $j \in B$. But then $(\bigwedge B)a = \top$, so $a \in \nabla(\bigwedge B)$, then $\nabla(\bigwedge B) = \nabla$ (since $\bigwedge B \leq j$ for all $j \in B$) so we see that $\bigwedge B \in B$. ■

There are two natural questions to ask.

1. Is every filter admissible?
2. Is every nucleus fitted?

The answer to both of these questions is no; we now give examples of non-admissible filters and non-fitted nuclei.

5.4 Example Consider the case where A is a Boolean frame. From Lemma 2.27, each nucleus on A has the form u_a where $a = j\bot$. But then

$$x \in \nabla(j) \Longleftrightarrow a \vee x = \top \Longleftrightarrow \neg\, a \le x$$

and so $\nabla(j)$ is principal. Thus the admissible filters on A are exactly the principal filters.

For an example of a non-principal filter on a Boolean frame, consider the filter of cofinite set on the power set of the natural numbers. ■

5.5 Example We now give an example of a frame where every filter is admissible but, with one exception, each block has lots of members.

Consider the frame $A = [0, 1]$ with the usual ordering. The filters are exactly the final sections

$$(m, 1] \qquad [m, 1]$$

for $m \in A$. Both are admissible, and in general each block has many members.

We can check that for every $a \in A$

$$u_a x = \begin{cases} x & \text{if } a \le x \\ a & \text{if } x < a \end{cases} \qquad v_a x = \begin{cases} 1 & \text{if } a \le x \\ x & \text{if } x < a \end{cases} \qquad w_a x = \begin{cases} 1 & \text{if } a < x \\ a & \text{if } x \le a \end{cases}$$

hold for all $x \in A$. This gives us that

$$v_m \wedge w_m = \begin{cases} 1 & \text{if } m < x \\ x & \text{if } x \le m \end{cases}$$

so that

$$\nabla(v_m \wedge w_m) = \nabla(w_m) = (m, 1]$$

and these are distinct members of this block. It is possible to produce many others.

For filters of the form $\nabla = [m, 1]$, we have $\nabla(v_m) = \nabla$. If $m = 0$, then this nucleus is the top element of NA and the only member of its block. Otherwise, we have

$$(v_m \circ u_a)x = \begin{cases} 1 & \text{if } m \le x \\ x & \text{if } a \le x < m \\ a & \text{if } x \le a \end{cases}$$

for every $x \in A$. Then we have

$$\nabla(v_m \circ u_a) = [m, 1]$$

for each $a \in A$, giving us many distinct members of this block. ■

There is a handy characterisation of fitted nuclei which will be very useful to us later on. First we need a definition.

5.6 Definition Suppose F is a filter in A. Then let

$$v_F = \bigvee\{v_a \mid a \in F\}$$

where the supremum is taken in NA; ie. $v_F = f^\infty$ where $f = \dot{\bigvee}\{v_a \mid a \in F\}$. ■

Notice that we can do the same thing for an arbitrary subset X of A; however, this does not give us any additional nuclei since $v_X = v_F$ where F is the filter generated by X.

5.7 Lemma *A nucleus is fitted if and only if it is of the form*

$$\bigvee\{v_a \mid a \in F\}$$

for some filter $F \subseteq A$.

Proof. We claim that v_F is the least nucleus which admits F.

With f as above, we have $fa = \top$ for all $a \in F$ so $v_F a$ admits F. Now suppose that j admits F. Let $y = v_a x = (a \supset x)$ so that $a \wedge y \leq x$. Then

$$y \leq jy = jy \wedge ja = j(y \wedge a) = jx$$

holds for all $x \in A$, and hence $v_a \leq j$ for all $a \in F$. So we have $f \leq j$ and also $v_F = f^\infty \leq j$ as required. ■

5.8 Theorem *The collection of fitted nuclei on a frame, A, is a subframe of the assembly NA.*

Proof. It is easy to check that

1. $v_F \wedge v_G = v_{F\cap G}$
2. $v_F \vee v_G = v_{F\vee G}$
3. $\bigvee\{v_F \mid F \in \mathcal{F}\} = v_{\bigcup\mathcal{F}}$

hold for all filters F, G and directed families $\mathcal{F}$ of filters. ■

If F is an arbitrary filter, then the nucleus v_F is the least nucleus which admits F. However, if F is not an admissible filter, then v_F will also admit elements of A which are not in F.

In addition to a least element, some blocks also have a greatest element. In fact, we have the following result.

5.9 Lemma *For each $a \in A$ the nucleus w_a is the greatest member of its block.*

Proof. Suppose j is a companion of w_a. It suffices to show that $ja = a$, for then $j \leq w_a$ by Lemma 5.9.[????]

Let $x = ja$ and $y = (x \supset a)$, so that $w_a y = y$ by Lemma 2.26. Then we have

$$(y \vee x) \supset a = (y \supset a) \wedge (x \supset a) = (y \supset a) \wedge y = a$$

so that $w_a(y \vee x) = \top$. But then $j(y \vee x) = \top$ since j and w_a are companions, and hence

$$j(y \vee a) = j(y \vee ja) = j(y \vee x) = \top$$

to give $w_a(y \vee a) = \top$. Hence

$$(x \supset a) = y = w_a y = w_a(y \vee a) = \top$$

and $x \leq a$ as required. ■

5.2 The Separation Property fit Revisited

Recall the separation property fit from Section 4.1. In this section we will investigate the relationship between this property on a frame A, and the fitted nuclei on the frame. First, we need the following Lemma.

5.10 Lemma *Suppose $j \in NA$ is fitted. Then*

$$j \leq k \Longleftrightarrow \nabla(j) \subseteq \nabla(k)$$

holds for all $k \in NA$.

Proof. ($\Longrightarrow$) is trivial.

($\Longleftarrow$). Suppose $a \in \nabla(j) \subseteq \nabla(k)$ and $x \in A$. Set $y = v_a x$ so that $a \wedge y \leq x$. Hence

$$y \leq ky = ka \wedge ky = k(a \wedge y) \leq kx$$

which shows that $v_a \leq k$. Thus

$$j = \bigvee\{v_a \mid a \in \nabla(j)\} \leq k$$

as required. ■

We are now ready to prove the main result of this section.

5.11 Theorem *For each frame A, the four conditions*

1. *A is fit*
2. *each nucleus on A is fitted*
3. *each u-nucleus on A is alone*
4. *each u-nucleus on A is minimal in its block*

are equivalent.

Proof. (1) $\implies$ (2). Suppose A is fit, and suppose there are unfitted nuclei, so there exist companions j and k with $j \not\leq k$. Then $jc \not\leq kc$ for some $c \in A$. Let $a = jc, b = kc$. Now, since A is fit, we can find x and y such that

$$a \vee x = \top \qquad y \not\leq b \qquad x \wedge y \leq b$$

hold. Now define $z = (y \supset b)$ so that $x \leq z$ and $c \leq b \leq z$. Now

$$a \leq jc \leq jz \qquad x \leq z \leq jz$$

so $jz \geq a \vee x = \top$ and then $kz = \top$ since j and k are companions. But since $y \wedge z \leq b$ then $ky \leq kb = b$ and so $y \leq ky \leq b$, which contradicts $y \not\leq b$.

(2) $\implies$ (3) $\implies$ (4) are trivial.

(4) $\implies$ (1). Suppose (4) holds for A, and $a \not\leq b \in A$. Then $u_a \not\leq w_b$ (since $u_a(\bot) = a$ but $w_b(\bot) = b$). By assumption, u_a is a fitted nucleus, so by Lemma 5.10 we know that

$$\nabla(u_a) \not\subseteq \nabla(w_b)$$

holds. Then there exists some $x \in A$ such that

$$a \vee x = \top \qquad w_b x \neq \top$$

holds. Now let $y = x \supset b$. Then

$$w_b x = y \supset b \neq \top$$

so that $y \not\leq b$. Then also $x \wedge y \leq b$ as required. ■

The notion of a fitted frame was introduced by Isbell in [8] where he took property 2 of Theorem 5.11 as the definition. The characterisation as a separation property is due to Macnab in [11].

5.3 Open Filters

In this section we examine the connection between open filters and admissible filters. Recall from Definition 2.37 that a filter ∇ is open if it is proper and

$$\bigvee X \in F \implies X \cap F \neq \emptyset$$

holds for each directed $X \subseteq A$.

We will begin by giving some examples of open filters.

5.12 Example 1. When $A = [0, 1]$, the filters are just the sets of the form $[a, 1]$ and $(a, 1]$. Every subset of A is directed, so F is open if and only if it is of the form $(a, 1]$ for some $a \in A$.

2. If A is a Boolean frame, then the open filters are the principal filters generated by a compact elements of A.

■

In general, we have the following result.

5.13 Lemma *Every principal filter is admissible.*

Proof. Let ∇ be the filter

$$\nabla = \{x \mid x \geq a\}$$

for some $a \in A$. Then the nucleus $j = v_a$ admits ∇. ■

We will now state and prove the main result of this section.

5.14 Theorem *Each open filter is admissible.*

Proof. Suppose F is an open filter. Let

$$f = \dot{\bigvee}\{v_a \mid a \in F\}$$

so that $v_F = f^\infty$ is the least nucleus which admits F for some sufficiently large ordinal ∞. We need to show that $\nabla(f^\infty) \subseteq F$. We show first that

$$fx \in F \Longrightarrow x \in F \qquad (\dagger)$$

holds. For

$$fx = \bigvee\{v_a x \mid a \in F\}$$

is a directed supremum in A, and F is open, so if $fx \in F$ then

$$a \supset x = v_a x \in F$$

holds for some $a \in F$. But then

$$x \geq a \wedge (a \supset x) \in F$$

and so $x \in F$ as required.

Now we show by induction that

$$f^\alpha x \in F \Longrightarrow x \in F$$

holds for each ordinal α.

The case $\alpha = 0$ is trivial. The induction step $\alpha \longmapsto \alpha + 1$ follows from $(\dagger)$. Now when λ is a limit ordinal,

$$f^\lambda x = \bigvee\{f^\alpha x \mid \alpha < \lambda\}$$

holds by definition. So

$$f^\lambda x \in F \Longrightarrow (\exists \alpha < \lambda)[f^\alpha x \in F]$$

since $f^\lambda x$ is a directed supremum and F is open. But by the induction hypothesis, this implies that $x \in F$.

Hence

$$f^\infty x \in F \Longleftrightarrow x \in F$$

for all $x \in A$. In particular

$$f^\infty x = \top \Longrightarrow f^\infty x \in F \Longrightarrow x \in F$$

so $\nabla(f^\infty) \subseteq F$ as required. ■

The above Lemma is Lemma 2.4(ii) of [9]. Part (i) of this result is Lemma 5.17 below.

Although every open filter is admissible, the converse is not true; it is not the case that every admissible filter is open.

5.15 Example Consider the filter on $\mathcal{P}(\mathbb{N})$ generated by the set of even numbers. This is a principal filter, and therefore admissible by Lemma 5.13 but it is not open. ■

So which nuclei do give rise to open filters? We need another definition.

5.16 Definition

1. An element $a \in A$ is compact if

$$a \leq \bigvee \Delta \Longrightarrow a \in \Delta$$

holds for each ideal Δ of A.

2. A frame A is compact if its top element, $\top_A$ is compact.

■

Notice that we can modify this definition to give a notion of compactness in terms of directed sets instead of ideals. If a is compact and X is a directed subset of A, then

$$a \leq \bigvee X \Longrightarrow a \leq x$$

for some $x \in X$. In particular, the top element $\top_A$ is compact if and only if

$$\top_A = \bigvee X \Longrightarrow \top_A \in X$$

holds for every directed set X.

5.17 Lemma *The filter $\nabla(j)$ is open if and only if the frame A_j is compact.*

Proof. We will write $\top_j$ for the top element of A_j. This is the same as the top element of A, but we label it differently to make it clear which frame we are working in. We also write $\bigvee_j$ for the supremum in A_j which is not the same as the supremum in A.

Now suppose $\nabla(j)$ is open. Then

$$\bigvee X \in \nabla(j) \Longrightarrow X \text{ meets } \nabla(j)$$

holds for every directed $X \subseteq A$. This gives

$$j\big(\bigvee X\big) = \top \Longrightarrow (jx = \top) \text{ for some } x \in X$$

or in other words

$$j\big(\bigvee X\big) = \top \Longrightarrow \top \in j[X]$$

holds for every directed $X \subseteq A$. But recall that $\bigvee_j X = j(\bigvee X)$ for $X \subseteq A_j$, and so

$$\bigvee\nolimits_j Y = \top_j \Longrightarrow \top_j \in Y$$

holds for every directed set $Y \subseteq A_j$. This is what we need for $\top_j$ to be compact in A_j.

Conversely, suppose that $\top_j$ is compact. Then

$$\bigvee\nolimits_j Y = \top_j \Longrightarrow \top_j \in Y$$

holds for every directed $Y \subseteq A_j$. But suppose $X \subseteq A$ is directed. Then

$$\begin{aligned} j\left(\bigvee X\right) = \top &\Longrightarrow \bigvee\nolimits_j j[X] = \top_j \\ &\Longrightarrow \top_j \in j[X] \\ &\Longrightarrow jx = \top \end{aligned}$$

holds for some $x \in X$, and so

$$\bigvee X \in \nabla(j) \Longrightarrow X \text{ meets } \nabla(j)$$

as required. ■

We finish this section with a result which will be used in the next chapter to give us some information about the patch construction on a regular frame. The following result is Proposition 1.2(iii) of [10], but we have simplified the proof.

5.18 Lemma *Suppose A is a regular frame, and $\nabla(j)$ is open for some nucleus j on A. Then*

$$j = u_a$$

where $a = j\bot$.

Proof. Since A is regular, it is fit, so it suffices to show that j and u_a are companions. We have $u_a \leq j$, so it suffices to show that

$$jx = \top \Longrightarrow a \vee x = \top$$

holds.

Consider any $x \in A$ with $jx = \top$. Since A is regular, we have

$$x = \bigvee \{y \mid (\exists z)[z \wedge y = \bot \text{ and} z \vee x = \top]\}$$

and this is a directed supremum. Since $\nabla(j)$ is open, we have

$$jy = \top \qquad z \wedge y = \bot \qquad z \vee x = \top$$

for some $y, z \in A$. But now

$$a = j\bot = jz \wedge jy = jz$$

and hence

$$a \vee x \geq jz \vee x \geq z \vee x = \top$$

as required. ■

Chapter 6
Two Patch Constructions

6.1 The Point-sensitive Patch Construction

As usual, we let S be a space and OS its topology. In this section we will construct the patch topology on S, a larger topology containing OS. We will see that the patch topology on a space is a topology lying between the original topology and the front topology. The construction really only works properly when S is a sober space, but some of it works in more general cases.

We know that in a T_2 space every compact set is closed. This is not true for spaces with weaker separation properties, and it is these that we want to work with. The point-sensitive patch topology adds extra open sets to the original topology to make some of the compact sets closed. If the original space is T_1 then all the original compact sets become closed.

The construction of this section is taken from page 261 of [4]. At first sight this looks messy and arbitrary. However, it can be useful in some circumstances, and later on we will see that there is something deeper going on.

6.1 Example Finding the Boolean algebra generated by an arbitrary lattice, A, is tricky to do algebraically. However, the patch topology of the spectrum of A turns out to be the spectrum of Boolean closure of A. ■

To begin the construction, recall the specialisation order, $\leq$, on a space S and the Alexandroff topology, ΥS, of upper sections with respect to this ordering.

6.2 Definition 1. A *saturated set* is an upper section in the specialisation ordering on S. In other words, E is saturated if and only if $E \in \Upsilon S$.

2. For each $E \subseteq S$, the *saturation* $\uparrow E$ of E, is given by

$$\uparrow E = \{s \in S \mid (\exists a \in E)[a \leq s]\}$$

in other words the smallest saturated set that includes E.

■

From the definition of the specialisation order, we see that for each $A \subseteq S$

$$\uparrow A = \bigcap \{U \in OS \mid A \subseteq U\}$$

holds. It follows easily that $OS \subseteq \Upsilon S$, in other words, every open set is saturated.

6.3 Definition Let QS be the collection of compact saturated subsets of S. ■

6.4 Lemma *If K is compact, then $\uparrow K \in QS$.*

Proof. We need to show that if K is compact, then so is $\uparrow K$.

Suppose $\mathcal{U}$ is a directed open cover of $\uparrow K$. Then $\mathcal{U}$ is a directed open cover of K, since $K \subseteq \uparrow K$, so there exists $U \in \mathcal{U}$ such that $K \subseteq U$. But then $\uparrow K \subset U$ since U is an open set containing K. Hence $\uparrow K$ is compact as required. ■

6.5 Lemma *The union of two compact saturated sets is compact saturated.*

Proof. The union of two compact sets is compact, and it is trivial to check that the union of two saturated sets is saturated. ■

Now consider the collection

$$\mathsf{pbase} = \{U \cap Q' \mid U \in OS, Q \in QS\}$$

of sets. This is closed under binary intersections, since the union of two compact saturated sets is also compact and saturated, and so forms a base for a topology.

6.6 Definition The *patch space*, ${}^{p}S$ of S has the same points as S, but is furnished with the topology generated by pbase. ■

In other words, $O^{p}S$ is the smallest topology containing all the original open sets and also the complement of every compact saturated set.

6.7 Lemma *1. The patch space of a T_0 space is T_1.*

2. The patch space of a T_2 space is itself.

Proof. 1. Suppose $p, q \in S$. Since S is T_0 by assumption then there is some $U \in OS$ such that either $p \in U, q \notin U$ or $q \in U, p \notin U$. Without loss of generality, assume the former. Hence $q \notin \uparrow\{p\}$. Now $\{p\}$ is a compact set and so $\uparrow\{p\} \in QS$. However, $\left(\uparrow\{p\}\right)' \in O^{p}S$ by definition, so this is a patch open set containing q but not p. This shows that for any $x, y \in S$, there exists U such that $x \in U, y \notin U$, and so S is T_1 as required.

2. All sets are saturated and every compact set is closed, so

$$\{Q' \mid Q \in QS\} \subseteq OS$$

and hence the two topologies are equal. ■

6.8 Lemma *Let S be a space. Every patch open is front open.*

Proof. We need to show that for every compact saturated set Q, its complement is front open. Now Q' is a lower section in the speciatlisation order, so

$$p \in Q' \Longrightarrow p^{-} \subseteq Q'$$

holds. Hence

$$Q' = \bigcup\{p^{-} \mid p \notin Q\}$$

holds. But for each p, its closure is an original closed set and therefore front open. So in the front topology, Q' is the union of a collection of open sets, therefore it is open. ■

This shows that the patch topology is indeed an intermediate topology sitting between the original and the front topologies, in other words we have that

$$\mathcal{O}S \hookrightarrow \mathcal{O}^{p}S \hookrightarrow \mathcal{O}^{f}S$$

holds. Later on we will see more of the properties of this construction, and how it relates to other topologies which we introduce. However, there are still questions we cannot answer, notably:

6.9 Question
1. Is the patch space of a sober space sober?
2. What is the patch space of a patch space? ■

For the point-sensitive patch topology to be a sensible construct to work with, it *should* be the case that when we apply it to a sober space we get a sober space back, and when we apply it to itself it stays the same. However, at the moment we are unable to prove these results.

6.2 Compact Sets and Open Filters

The patch construction of the previous section revolves around compact saturated sets. In this section we will discuss the link between sets of this kind on a space, S, and open filters. These will play a major role in the rest of this chapter.

Recall the definition of an open filter (2.37); we say that F is open if for every directed subset X of A, X meets F whenever $\bigvee X \in F$.

We will now show that when S is a sober space the collection of open filters is in bijective correspondence with the collection $\mathcal{Q}S$ of compact saturated sets.

6.10 Definition For S a space, $E \subseteq \mathcal{O}S$ define $\nabla(E)$ to be the filter on $\mathcal{O}S$ given by

$$U \in \nabla \Longleftrightarrow E \subseteq U$$

for $U \in \mathcal{O}S$. We say that $\nabla(E)$ is the neighbourhood filter of E. ■

It is trivial to check that $\nabla(E)$ is indeed a filter. The next result constructs open filters from compact saturated sets.

6.11 Lemma *The filter $\nabla(E)$ is open exactly when E is compact.*

Proof. Assume E is compact. Suppose $\mathcal{U} \subseteq OS$ is directed and $\bigcup\mathcal{U} \in \nabla(E)$. Then $E \subseteq \bigcup\mathcal{U}$, but E is compact, so that $E \subseteq U$ holds for some $U \in \mathcal{U}$. Hence $\mathcal{U}$ meets ∇ as required.

Now suppose $\nabla(E)$ is open and $\mathcal{U}$ is a directed open cover of E so that $E \subseteq \bigcup\mathcal{U}$. Then $\bigcup\mathcal{U} \in \nabla(E)$ by definition, so $\mathcal{U}$ meets $\nabla(E)$. So there exists some $U \in \mathcal{U}$ such that $E \subseteq U$. Hence E is compact as required. ■

We have seen how to construct a filter from a subset of S. If, on the other hand, we are given a filter that has been constructed in this way, is it possible for us to recover the original set, E? We see in the next result that this is only possible when E is saturated.

6.12 Lemma *For E and $\nabla(E)$ as above, $\uparrow E = \bigcap\nabla(E)$.*

Proof. We know that $E \subseteq \bigcap\nabla(E)$, and so $\uparrow E \subseteq \bigcap\nabla(E)$, since $\uparrow E$ is contained in every open set containing E.

Now suppose that $\uparrow E \neq \bigcap\nabla(E)$. Then there exists x such that $x \in \bigcap\nabla(E)$ but $x \notin \uparrow E$. However

$$(\forall U \in OS)[\uparrow E \subseteq U \Longrightarrow x \in U]$$

so that

$$C \subseteq (\uparrow E)' \Longrightarrow x \notin C$$

holds for every closed set C. Hence $x^- \nsubseteq (\uparrow E)'$ and therefore $x^- \cap \uparrow E \neq \emptyset$. Now, there exists y such that $y \in x^-$, $y \in \uparrow E$. Then $y^- \subseteq x^-$, so that $y \leq x$ in the specialisation order. But $\uparrow E$ is saturated, so $x \in \uparrow E$, contradicting our initial assumption. ■

This immediately gives us the following result, which we will need later.

6.13 Corollary *For $Q \in QS$, with $\nabla(Q)$ as above, $Q = \bigcap\nabla(Q)$.*

We have now shown that any compact saturated set gives rise to an open filter. Is it the case that every open filter arises in this way? If so, Lemma 6.13 suggests how we might recover a compact saturated set from an arbitrary filter.

The following is the Hofmann-Mislove characterisation given as Lemma 2.13 of [7].

6.14 Theorem *Let S be a sober space, ∇ an open filter on OS. Then there exists a unique compact saturated set Q such that $\nabla = \nabla(Q)$.*

Proof. Let $Q = \bigcap\nabla$ as above. We show first that $\nabla = \nabla(Q)$. Clearly $U \in \nabla$ implies $Q \subseteq U$, so it remains to show that

$$Q \subseteq U \Longrightarrow U \in \nabla$$

holds. In fact, we will prove

$$U \notin \nabla \Longrightarrow Q \nsubseteq U$$

by constructing a point p such that $p \in Q$, $p \notin U$.

We may assume that ∇ is a proper open filter (otherwise the result is trivial). Suppose $U \notin \nabla$. By the frame separation principle (Theorem 2.45), there exists a $\wedge$-irreducible element P of OS such that $U \subseteq P$ and

$$V \in \nabla \Longrightarrow V \nsubseteq P$$

holds. Since S is sober, we have $P = p^{-\prime}$ for some $p \in S$. Now

$$U \subseteq P \Longrightarrow p \notin U$$

and

$$V \in \nabla \Longrightarrow V \nsubseteq P \Longrightarrow p \in V$$

and so $p \in Q$ as required.

Now Lemmas 6.11 and 6.12 show that Q is compact and saturated. ■

6.3 Functorial Properties of the Point-sensitive Patch

Suppose that

$$T \xrightarrow{\ \ \phi\ \ } S$$

is a continuous map between topological spaces. Then we have the following diagram.

$$\begin{array}{ccc} OS & \xrightarrow{\phi^*} & OT \\ \downarrow & & \downarrow \\ O^f S & & O^f T \end{array}$$

Is there a natural way of applying the point-sensitive patch construction to maps so that we get a functor? We cannot do this in general, but we can restrict ourselves to a particular class of maps which are already patch continuous. There appears to be no simple characterization of patch continuity; however, we can restrict ourselves further to a particular kind of patch continuous maps.

6.15 Definition We say that the continuous map

$$T \xrightarrow{\ \ \phi\ \ } S$$

converts compact saturated sets if $\phi^*(Q) \in T$ is compact saturated whenever $Q \in S$ is. ■

This gives us a sufficient condition for ϕ to be patch continuous.

6.16 Lemma *If*

$$T \xrightarrow{\ \phi\ } S$$

converts compact saturated sets then ϕ is patch continuous.

However, this is not always an easy condition to check. Later on, we will look at other conditions which are easier to handle.

In general it is not true that for a filter F, fF is open whenever F is. What happens, though, if we just consider morphisms which do have this property?

6.17 Definition For $f\colon A \longrightarrow B$ a frame morphism, we say that f *converts open filters* if

$$F \text{ open on } A \Longrightarrow fF \text{ open on } B$$

holds. ■

So which morphisms convert open filters? We have the following two results.

6.18 Lemma *If f is a frame morphism with a continuous right adjoint, g, then f converts open filters.*

Proof. Suppose f is a frame morphism from A to B with a continuous right adjoint, g. Suppose also that F is an open filter on B and $Y \subseteq B$ is a directed set such that $\bigvee Y \in fF$. Then $g\big(\bigvee Y\big) \in F$ by Lemma 2.8, and so $\bigvee g[Y] \in F$ by the continuity of g . Then $g[Y]$ meets F, since F is open, and so $gy \in F$ for some $y \in Y$. But then $y \in fF$ and hence Y meets fF which shows that fF is an open filter as required. ■

In fact, in the spatial case, this is an equivalence. The following result is part of the Hofmann-Lawson characterisation given in [6]. We don't need the full characterisation.

6.19 Lemma *If*

$$T \xrightarrow{\ \phi\ } S$$

is a continuous morphism and

$$OS \underset{\phi_*}{\overset{\phi^*}{\rightleftarrows}} OT$$

the induced frame morphism and its right adjoint, then ϕ^* converts open filters if and only if ϕ_* is continuous.

Proof. We just need to show that

$$\phi^* \text{ converts open filters } \Longrightarrow \phi^* \text{ continuous}$$

holds.

Suppose ϕ^* converts open filters. Consider a directed set $\mathcal{U} \subseteq OT$. Let

$$V = \phi_*(\bigcup \mathcal{U})$$

so we need to show that

$$V \subseteq \bigcup \phi_*[\mathcal{U}]$$

holds. Suppose $x \in V$. Let F be the neighbourhood filter of x, given by

$$U \in F \Longleftrightarrow x \in U$$

for every $U \in OS$. It is easy to check that this filter is open, and hence so is $\phi^* F$ in OT. But we have

$$W \in \phi^* F \Longleftrightarrow \phi_* W \in F \Longleftrightarrow x \in \phi_* W$$

for each $W \in OT$. In particular, $\bigcup \mathcal{U} \in \phi^* F$ and hence $\phi^* F$ meets $\mathcal{U}$; hence there is some $W \in \mathcal{U}$ such that $W \in \phi^* F$ Thus we have

$$x \in \phi_* W \subseteq \bigcup \phi_*[W]$$

as required. ■

It is not clear whether this equivalence holds in general. We have the following open question.

6.20 Question Does every frame morphism which converts open filters have a continuous right adjoint?

This next result is crucial to the relationship between the two patch constructions.

6.21 Lemma *The morphism*

$$U(\cdot)\colon A \longrightarrow OS$$

(where S is the point space of A) converts open filters.

Proof. This is just a rephrasing of Lemma 2.49. ■

This discussion of morphisms which convert open filters will be useful when we come to look at the point-free patch construction and its functorial properties.

6.4 The Point-free Patch Construction

This is another construction, which is closely related to the point-sensitive patch construction. However, instead of using the topological space S, this construction is performed entirely on a frame, A. It is in this sense that the construction is "point free".

In the previous section, we saw that compact saturated sets on a space, S, corresponded to open filters on its topology, OS. These compact saturated sets were used in the point-sensitive patch construction. However, we can work with the open filters entirely within the frame without having to refer to the original space. This suggests that they will be useful when looking for a point-free version of the construction.

We now prove certain closure properties on the collection of open filters which we need later on.

6.22 Lemma *1. F, G open $\Longrightarrow F \cap G$ open.*

2. If $\mathcal{F}$ is a directed family of open filters, then the filter $\bigcup \mathcal{F}$ is open.

Proof. 1. Suppose F and G are open filters, $X \subseteq A$ is directed and $\bigvee X \in F \cap G$. Then

$$\bigvee X \in F \Longrightarrow X \cap F \neq \emptyset$$

$$\bigvee X \in G \Longrightarrow X \cap G \neq \emptyset$$

hold, so $\exists a, b \in X, a \in F, b \in G$. But because X is directed, $\exists c \in X$ such that $a \leq c, b \leq c$. So $c \in F \cap G$ and hence $X \cap (F \cap G) \neq \emptyset$.

2. Suppose $\mathcal{F}$ is a directed family of open filters, $X \subseteq A$ is directed and $\bigvee X \in \bigcup \mathcal{F}$. Then $\bigvee X \in F$ for some $F \in \mathcal{F}$. But then $X \cap F \neq \emptyset$ so $X \cap \bigcup \mathcal{F} \neq \emptyset$ as required. ∎

There is an obvious case missing from this result: what happens when $\mathcal{F}$ is a collection of filters which is *not* directed. Recall that we can only take the union of $\mathcal{F}$ when $\mathcal{F}$ is directed. Otherwise, we have to form $\bigvee \mathcal{F}$, the filter generated by the elements of $\mathcal{F}$. This operation does not necessarily preserve open filters:

$$F, G \text{ open } \Longrightarrow F \vee G \text{ open}$$

is not true in general. In fact, we can show the following result.

6.23 Lemma *If S is a sober space, $P, Q, R \in QS$ and*

$$\nabla(P) \vee \nabla(Q) = \nabla(R)$$

then

$$P \cap Q = R$$

holds.

Proof. We show first that $R \subseteq P \cap Q$. Consider any $p \in R$. Suppose that $p \notin P \cap Q$, say $p \notin P$. Then, since P is saturated, $p \notin U$ for some $P \subseteq U \in OS$. But then

$$U \in \nabla(P) \subseteq \nabla(R)$$

and so $R \subseteq U$, contradicting $p \in R, p \notin U$.

We now show that $P \cap Q \subseteq R$. Suppose $p \in P \cap Q, p \notin R$. Now, since R is saturated, R is the intersection of all open sets containing it. Hence there exists some $W \in OS$ such that $R \subseteq W, p \notin W$. Then

$$W \in \nabla(R) = \nabla(Q) \vee \nabla(Q)$$

so that $W \supseteq U \cap V$ where $P \subseteq U$ and $Q \subseteq V$. Therefore

$$p \in P \cap Q \subseteq U \cap V \subseteq W$$

holds, contradicting $p \notin W$. ■

This shows that open filters are closed under $\vee$ only when QS is closed under binary intersections. In Example 7.17 and Lemma 7.18 we will see a space for which this is not the case.

Now consider the set

$$\mathsf{PBase} = \{u_a \wedge v_F \mid a \in A, F \text{ an open filter } \}$$

of nuclei on A.

6.24 Lemma *The set* PBase *is closed under binary meets.*

Proof. Recall that

$$v_F \wedge v_G = v_{F \cap G}$$

and so

$$(u_a \wedge v_F) \wedge (u_b \wedge v_G) = (u_{a \wedge b} \wedge v_{F \cap G})$$

by Lemma 6.22. ■

This result is analagous in the point-sensitive patch construction to pbase boing closed under binary intersections. As in that case, we now use this base to generate a new frame.

6.25 Definition Let PA be the set of all nuclei of the form $\bigvee J$ for some $J \subseteq \mathsf{PBase}$. ■

We showed in Section 6.1 that the point-sensitive patch topology was intermediate between the original topology and the front topology. In this case, we will show that the point-free patch construction is a frame which sits in between the original frame and the assembly. Later on, we will investigate the relationship between these two constructions.

6.26 Theorem *For each frame A, the set PA is a subframe of NA which includes the canonical image of A.*

Proof. Clearly $u_a \in PA$ for all $a \in A$. Now

$$\bot_{NA} = u_\bot \in PA$$

and

$$\top_{NA} = u_\top \in PA$$

so it just remains to show that PA is closed under $\wedge$ and $\bigvee$.

For each i, j let r_i, s_j be nuclei of the form $u_a \wedge v_F$ as above. Now

$$\begin{aligned} \bigvee\{r_i \mid i \in I\} \wedge \bigvee\{s_j \mid j \in J\} &= \bigvee\left\{s_j \wedge \bigvee\{r_i \mid i \in I\} \mid j \in J\right\} \\ &= \bigvee\{s_j \wedge r_i \mid i \in I, j \in J\} \end{aligned}$$

by two applications of the frame distributive law on NA. But for each i, j, $s_j \wedge r_i$ is again of this form by Lemma 6.24, so PA is closed under binary meets.

The supremum of a collection of elements of PA is again the supremum of elements of the form $u_a \wedge v_F$ and so is in PA. Hence PA is also closed under $\bigvee$ and is therefore a subframe of NA. ■

We have seen that PA lies between A and NA. Under what conditions is the embedding

$$A \longrightarrow PA$$

an isomorphism? We have the following consequence to Lemma 5.18.

6.27 Theorem *Suppose A is regular. Then for each $j \in PA$,*

$$j = u_a$$

where $a = j\bot$. In particular, the canonical embedding

$$A \longrightarrow PA$$

is an isomorphism.

Proof. By Lemma 5.18 we know that when $F = \nabla(v_F)$ is open, then v_F is of the form u_b for some $b \in A$. Hence

$$\mathsf{PBase} = \{u_a \mid a \in A\}$$

and PA is just the canonical image of A. ■

6.5 Functorial Properties of the Point-free Patch

We now have the embeddings

$$A \longrightarrow PA \hookrightarrow NA$$

from a frame A to its point-free patch, to its assembly. We already know from Section 3.2 that N is a functor. Is it also the case that P is a functor? In other words, given a morphism

$$A \xrightarrow{\ f\ } B$$

is there a morphism

$$PA \xrightarrow{\ Pf\ } PB$$

such that the following diagram commutes?

$$\begin{array}{ccccc} A & \longrightarrow & PA & \hookrightarrow & NA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Pf} & & \downarrow{\scriptstyle Nf} \\ B & \longrightarrow & PB & \hookrightarrow & NB \end{array}$$

Unfortunately, the answer to this is no; in general, P is not a functor on the full category of frame morphisms. However, P does interact like a functor with certain kinds of morphisms.

6.28 Theorem *Let $f\colon A \longrightarrow B$ be a frame morphism which converts open filters. Then*

$$j \in PA \Longrightarrow (Nf)j \in PB$$

and the restriction

$$(Pf)\colon PA \longrightarrow PB$$

of (Nf) to PA is a frame morphism.

Proof. Suppose $j \in PA$. Then j is of the form

$$j = \bigvee\{u_a \wedge v_F \mid a \in X, F \in \mathcal{F}\}$$

for some $X \subseteq A$, $\mathcal{F}$ a collection of open filters. Then

$$(Nf)j = \bigvee\{(Nf)u_a \wedge (Nf)v_F \mid a \in X, F \in \mathcal{F}\}$$

holds, since (Nf) is a frame morphism. But $(Nf)u_a = u_{fa}$ and $(Nf)v_F = v_{fF}$ by Corollary 3.5, and so $(Nf)j \in PB$ providing fF is an open filter.

That (Pf) is a frame morphism follows immediately. ■

This last result will be crucial. In particular, it tells us that the diagram

$$\begin{array}{ccccc} A & \xrightarrow{n_A} & PA & \hookrightarrow & NA \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle PU} & & \downarrow{\scriptstyle NU} \\ OS & \xrightarrow[n_{OS}]{} & POS & \hookrightarrow & NOS \end{array}$$

commutes. By the results in Chapter 3 and Section 6.1, we can add O^fS to this as the point space of NA and NOS and add the point sensitive patch between OS and O^fS.

$$\begin{array}{ccccc} A & \xrightarrow{n_A} & PA & \hookrightarrow & NA \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle PU} & & \downarrow{\scriptstyle NU} \\ OS & \xrightarrow{n_{OS}} & POS & \hookrightarrow & NOS \\ & \searrow & & & \downarrow{\scriptstyle \sigma} \\ & & O^pS & \hookrightarrow & O^fS \end{array}$$

The obvious question to ask is whether there is a natural map from PA to O^pS.

In fact, we can show the following result.

6.29 Lemma *The restriction of the frame morphism σ to POS gives a frame morphism, π, from PA to O^pS. Furthermore, the morphism π is surjective.*

Proof. The frame POS is generated by spatial nuclei of the form

$$\langle W\rangle \wedge v_F$$

where $W \in OS$, F is an open filter on OS and

$$v_F = \bigvee\{v_M \mid M \in F\}$$

as before. We need to check what σ does to these nuclei.

We already know from Lemma 3.9 that $\sigma\langle W'\rangle = W'$. By Lemma 6.14, F is generated by some $Q \in QS$ so that $F = \nabla(Q)$. We will prove that $\sigma v_F = Q'$.

The filter F is open and therefore admissible and v_F admits F, so

$$v_F = S \iff U \in F$$

holds for all $U \in OS$. But then

$$U \in F \iff U \in \nabla(Q) \iff Q \subseteq U \iff Q' \cup U = S$$

and so

$$v_F U = S \iff \langle Q' \rangle U = S$$

holds and therefore $\langle Q' \rangle$ and v_F are companions. We know that v_F is fitted by Lemma 5.7, so $v_F \leq \langle Q' \rangle$ and

$$\sigma v_F \subseteq \sigma \langle Q' \rangle = Q'$$

holds.

Conversely, if $p \in Q'$ then $p^- \subseteq Q'$ and so $Q \subseteq p^{-\prime}$ and $p^{-\prime} \in F$. Hence

$$p \in (p^- \cup p^{-\prime}) \subseteq v_F p^{-\prime}$$

and so $p \in \sigma v_F$. Thus we have

$$Q' \subseteq \sigma v_F$$

as required.

If $j \in POS$, then

$$j = \bigvee \left\{ \langle W \rangle \wedge v_F \mid W \in \mathcal{W}, F \in \mathcal{F} \right\}$$

for some subsets $\mathcal{W} \subseteq OS$ and $\mathcal{F}$ a collection of open filters. Then, because σ is a frame morphism,

$$\begin{aligned} \sigma j &= \bigvee \left\{ \sigma\langle W \rangle \wedge \sigma v_F \mid W \in \mathcal{W}, F \in \mathcal{F} \right\} \\ &= \bigvee \left\{ W \cap Q \mid W \in \mathcal{W}, Q \in \mathcal{Q} \right\} \end{aligned}$$

for some collection $\mathcal{Q}$ of compact saturated sets.

But O^pS is generated by sets of the form

$$W \cap Q'$$

where $W \in OS$, $Q \in QS$ and hence $\sigma j \in O^pS$ and we are done.

Now, if M is a patch open set, then M is of the form

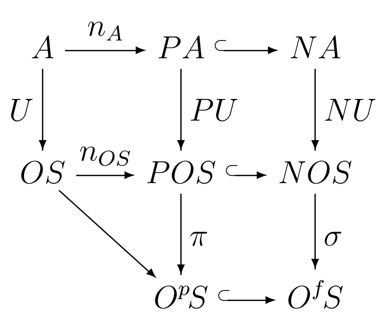
$$\bigcup \{ W \cap Q' \mid W \in \mathcal{W}, Q \in \mathcal{Q} \}$$

for some sets $\mathcal{W} \subseteq OS$ and $\mathcal{Q} \subseteq QS$. But then we get

$$M = \pi\left(\bigvee \{ \langle W \rangle \wedge v_{\nabla(Q)} \mid W \in \mathcal{W}, Q \in \mathcal{Q} \} \right)$$

and hence π is surjective. ■

We can now give the full patch diagram diagram



where as before, PU is the restriction of NU to the subframe PA and π is the restriction of σ to the subframe POS.

Chapter 7
The Continuous Assembly

7.1 The Construction

7.1 Definition An inflator f on a frame A is *continuous* (or in full *Scott continuous*) if

$$f(\bigvee X) = \bigvee f[X]$$

holds for each directed subset X of A. ■

The family of continuous nuclei on a frame A turns out to have some useful properties. We will show that it is a subframe of NA which includes the image of A.

Notice that trivially, for every $a \in A$, the nucleus u_a is continuous. However, the nucleus v_a is not continuous in general.

7.2 Lemma *1. The composition of two continuous inflators is continuous.*

2. The pointwise infimum of two continuous inflators is continuous.

Proof. 1. Suppose f and g are continuous inflators. Then

$$f \circ g(\bigvee X) = f(\bigvee g[X]) = \bigvee (f \circ g)[X]$$

holds as required.

2. Suppose f and g are continuous inflators and X is a directed subset of A. Then

$$\begin{aligned}(f \wedge g)(\bigvee X) &= f(\bigvee X) \wedge g(\bigvee X)\\ &= \bigvee f[X] \wedge \bigvee g[X]\\ &= \bigvee \{fy \wedge gx \mid y, x \in X\}\\ &= \bigvee \{fz \wedge gz \mid z \in X\}\end{aligned}$$

holds since X is directed. ■

Notice that this proof will not work for arbitrary infima, since we cannot use the distributive law in that case.

In Chapter 3 we saw that in general the pointwise supremum of a family of nuclei is not necessarily a nucleus, and that suprema in NA are in general very hard to compute. When dealing with continuous nuclei, however, this is a much simpler problem, since we can show that the pointwise supremum of a directed family of continuous nuclei is itself a continuous nucleus.

We start with the following result.

7.3 Lemma *The pointwise supremum $\dot{\bigvee}F$ of a directed family F of continuous inflators is continuous.*

Proof. Let $g = \dot{\bigvee}F$. It is easy to check that g is an inflator. Now suppose that $X \subseteq A$ is directed. Then

$$\begin{aligned} g\big(\bigvee X\big) &= \bigvee \{f\big(\bigvee X\big) \mid f \in F\} \\ &= \bigvee \Big\{ \bigvee \{fx \mid x \in X\} \mid f \in F \Big\} \\ &= \bigvee \{gx \mid x \in X\} \\ &= \bigvee g[X] \end{aligned}$$

holds as required. ■

Recall that to compute the supremum of a family J of (not necessarily continuous) nuclei, we first form the compositional closure

$$J^\circ = \{j_1 \circ \cdots \circ j_n \mid j_i \in J, n \in \mathbb{N}\}$$

of J, then take the pointwise supremum, and then iterate it to a 'sufficiently large' ordinal. The following lemma shows that when dealing with continuous nuclei, we can omit this last step.

7.4 Lemma *Let J be a set of continuous nuclei on A. Then the pointwise supremum $\dot{\bigvee}J^\circ$ is a continuous nucleus, and hence is $\bigvee J$ in NA.*

Proof. We have already seen that for any directed family of nuclei, the pointwise supremum is a pre-nucleus. It remains to show that when the nuclei are continuous, it is also idempotent.

For any given $x \in A$, the set $\{fx \mid f \in J^\circ\}$ is directed, and so

$$\big(j \circ \dot{\bigvee} J^\circ\big)x = j\big(\bigvee \{fx \mid f \in J^\circ\}\big) = \{(j \circ f)x \mid f \in J^\circ\}$$

for each continuous $j \in NA$, and in particular for each $j \in J$. However, $j \circ f \in J^\circ$, so that $j \circ \dot{\bigvee} J^\circ = \dot{\bigvee} J^\circ$ for each $j \in J$, and hence $f \circ \dot{\bigvee} J^\circ = \dot{\bigvee} J^\circ$ for each $f \in J^\circ$.

Setting $y = \big(\dot{\bigvee} J^\circ\big)x$, we have $fy = y$ for each $f \in J^\circ$, so

$$\big(\dot{\bigvee} J^\circ\big)^2 x = \big(\dot{\bigvee} J^\circ\big)y = \bigvee \{fy \mid f \in J^\circ\} = y = \big(\dot{\bigvee} J^\circ\big)x$$

as required. ■

This gives the immediate corollary that if J is a directed set of continuous nuclei on the frame A, then $\bigvee J = \dot{\bigvee} J$. In other words, suprema of directed sets can be computed pointwise. This gives us a result analagous to Theorem 6.26.

7.5 Theorem *For each frame A, the set MA is a subframe of NA which includes the canonical image of A.*

Proof. Immediate from Lemma 7.4 and Lemma 7.3. ■

7.2 A Composition Result

We will show in this section that if the frame A is fitted, then its continuous assembly is just the canonical image of A.

7.6 Theorem *If A is any frame, $j \in NA$ fitted and $k \in NA$ continuous, then*

$$j \vee k = j \circ k$$

holds.

Proof. Let $F = \nabla(j)$ and set

$$f = \dot{\bigvee}\{v_a \mid a \in F\}$$

so that $j = f^\infty$, since we know that

$$j = \bigvee\{v_a \mid a \in F\}$$

holds. Then for every $x \in A$, we have

$$fx = \bigvee\{v_a x \mid a \in F\}$$

which is a directed supremum. But k is continuous, and so we have

$$\begin{aligned}(k \circ f)x &= k\big(\bigvee\{v_a x \mid a \in F\}\big)\\ &= \bigvee\{(k \circ v_a)x \mid a \in F\}\\ &\leq \bigvee\{v_a \circ k)x \mid a \in F\}\\ &= (f \circ k)x\end{aligned}$$

for every $x \in A$. Thus $k \circ f \leq f \circ k$ holds.

We use the usual induction on the ordinals to show that

$$k \circ f^\alpha \leq f^\alpha \circ k$$

holds for each ordinal α.

The case $\alpha = 0$ is immediate. For the induction step $\alpha \longmapsto \alpha + 1$ we have

$$k \circ f^{\alpha+1} = k \circ f \circ f^{\alpha} \le f \circ k \circ f^{\alpha} \le f \circ f^{\alpha} \circ k = f^{\alpha+1} \circ k$$

using the above result and the induction hypothesis to give the required result.

When λ is a limit ordinal, for each $x \in A$ we have

$$f^{\lambda}x = \bigvee\{f^{\alpha}x \mid \alpha < \lambda\}$$

where this is a directed supremum. Thus by the continuity of k, whe have

$$(k \circ f^{\lambda})x = \bigvee\{(k \circ f^{\alpha})x \mid x < \lambda\} \le \bigvee\{(f^{\alpha} \circ k)x \mid x < \lambda\} = (f^{\lambda} \circ k)x$$

by the induction hypothesis.

Thus we have

$$k \circ j \le j \circ k$$

by setting $\alpha = \infty$. Hence we have

$$j \vee k = j \circ k$$

by Lemma 2.31 as required. ■

This is what we need to prove the main result of this section.

7.7 Theorem *Suppose A is fit. Then for each $j \in MA$,*

$$j = u_a$$

where $a = j\bot$. In particular, the canonical embedding

$$A \longrightarrow MA$$

is an isomorphism.

Proof. By Lemma 2.32 we know that

$$k \vee u_x = k \circ u_x$$

holds for every $k \in NA$ and $x \in A$. But A is fit, and so the nucleus u_x is fitted, and so by Lemma 7.6 we also have

$$k \vee u_x = u_x \vee k = u_x \circ k$$

for every continuous nucleus k and $x \in A$. Hence

$$k \circ u_x = u_x \circ k$$

holds for every $k \in MA, x \in A$. This gives

$$k(x \vee y) = x \vee ky$$

for every $x, y \in A$; in particular by setting $y = \bot$ and $a = k\bot$ we get

$$kx = x \vee a$$

and hence $k = u_a$ as required. ■

7.3 The Functor N on Continuous Nuclei

We now have a diagram

$$\begin{array}{ccccc} A & \longrightarrow & MA & \hookrightarrow & NA \\ {\scriptstyle u_A}\downarrow & & & & \downarrow{\scriptstyle NU_A} \\ OS & \longrightarrow & MOS & \hookrightarrow & NOS \end{array}$$

which looks similar to the diagram we constructed for the point free patch.

As with the patch construction, we can ask whether M is a functor. Can we apply M to morphisms in a natural way? This would give us an arrow MU_A to add to the diagram. It appears that the answer is no; we do not know of any such functor. However, we can still find an arrow which we will call MU_A that makes the above diagram commute, although we do it by a different method.

In fact, we will also construct a space ${}^mS$ such that the diagram

$$\begin{array}{ccccc} A & \longrightarrow & MA & \hookrightarrow & NA \\ {\scriptstyle U_A}\downarrow & & \downarrow{\scriptstyle MU_A} & & \downarrow{\scriptstyle NU_A} \\ OS & \longrightarrow & MOS & \hookrightarrow & NOS \\ & {\scriptstyle \iota}\searrow & \downarrow{\scriptstyle \mu_S} & & \downarrow{\scriptstyle \sigma_S} \\ & & O^mS & \hookrightarrow & O^fS \end{array}$$

commutes.

Any arrow MU_A must be a restriction of NU_A to MA and so we need to check that such a restriction will make the appropriate squares commute, ie. that

$$j \in MA \Longrightarrow (NU)j \in MOS$$

holds.

Before we can show this result, we need some more background.

7.8 Lemma *Suppose $j \in MA$ for an arbitrary frame A. Suppose $a \le p$ where $ja = a$ and p is $\wedge$-irreducible (in A). Then there exists some $\wedge$-irreducible $q \in A$ such that*

$$jq = q \qquad a \le q \le p$$

hold.

Proof. Consider the set

$$Z = \{z \in A \mid jz = z \le p\}$$

for p as above. Then, in particular, $a \in Z$. This set is closed under directed suprema (since j is continuous), and so we can use Zorn's Lemma to show there is some maximal element $q \in Z$ with $a \le q$.

We show that q is $\wedge$-irreducible. Suppose that $x \wedge y \leq q$. Then

$$(q \vee x) \wedge (q \vee y) = q \vee (x \wedge y) = q$$

holds. Applying j to each side, we get

$$j(q \vee x) \wedge j(q \vee y) = jq = q \leq p$$

by the properties of q. But then, one of $j(q \vee x)$ or $j(q \vee y)$ is less than p, say $j(q \vee x)$. But then

$$q \leq j(q \vee x) \in Z$$

and so

$$q = j(q \vee x)$$

by the maximality of q. Then

$$x \leq j(q \vee x) = q$$

to give the required result. ■

Recall that for $p \in S$, the saturation of p is given by

$$\uparrow p = \{q \in S \mid q \leq p\}$$

since the specialisation order on a frame is the reverse of the original ordering.

7.9 Lemma *For each frame A,*

$$\uparrow p \subseteq (\Sigma j) \cup U(a) \Longrightarrow p \in U(ja)$$

holds for each $j \in MA$, $a \in A$ and $p \in S$.

Proof. Consider j, a, p such that

$$\uparrow p \subseteq (\Sigma j) \cup U(a)$$

and suppose that $p \notin U(ja)$. Then $ja \leq p$ and so by Lemma 7.8 there exists some $q \in S$ such that

$$a \leq ja \leq q \leq p \qquad jq = q$$

hold. Then $q \in \uparrow p$ (since $q \leq p$) and so either $q \in \Sigma j = S - A_j$ or $q \in U(a)$. But this is impossible by the above properties of q. ■

7.10 Lemma *For each frame A,*

$$\langle \Sigma j \rangle U(a) = U(ja)$$

holds for each $j \in MA$, $a \in A$.

Proof. We know that

$$(NU)j \leq \langle \Sigma j \rangle$$

since $\Sigma = (\sigma \circ NU)$ and the map

$$E \longmapsto \langle E \rangle$$

is the right adjoint to σ by Lemma 3.10. Hence by Lemma 3.16 we get

$$U(ja) \subseteq \langle \Sigma j \rangle U(a)$$

so we need only check that the converse inclusion holds.

Suppose that $p \in S$ is such that

$$p \in \langle \Sigma j \rangle U(a)$$

holds. Then

$$\uparrow p \subseteq \langle \Sigma j \rangle U(a) \subseteq (\Sigma j) \cup U(a)$$

and so $p \in U(ja)$ by the previous Lemma. This gives the required result. ■

We can now prove that the restriction of NU_A to MA behaves in an appropriate manner.

7.11 Theorem *For each frame A and $j \in MA$ we have*

$$(NU)j = \langle \Sigma j \rangle$$

and this is a continuous nucleus on OS.

Proof. As before, we know that

$$(NU)j \leq \langle \Sigma j \rangle$$

holds. We show the converse inequality.

Consider an open set of S. This has the form $U(a)$ for some $a \in A$. Then by Lemma 7.10 and Lemma 3.16 we get

$$\langle \Sigma j \rangle U(a) = U(ja) \subseteq \big((NU)j\big)U(a)$$

which is sufficient to show that

$$(NU)j = \langle \Sigma j \rangle$$

holds.

We need to show that this nucleus is continuous. Consider a directed subset $\mathcal{U}$ of OS. Using the same trick as in Lemma 2.49 we index this as

$$\mathcal{U} = \{U(x) \mid x \in X\}$$

where, X is a directed subset of A. As before we have

$$\bigcup \mathcal{U} = U(a)$$

and by continuity of j we get

$$ja = \bigvee j[X] \qquad U(ja) = \bigcup \{U(jx) \mid x \in X\}$$

since $U(\cdot)$ is a frame morphism. Thus

$$\begin{aligned}\langle \Sigma j \rangle \big(\bigcup \mathcal{U}\big) = \langle \Sigma j \rangle U(a) &= U(ja) \\ &= \bigcup \{U(jx) \mid x \in X\} \\ &= \bigcup \{\langle \Sigma j \rangle U(x) \mid x \in X\}\end{aligned}$$

using two applications of Lemma 7.10. This gives the result. ■

We now have an arrow

$$MA \xrightarrow{MU_A} MOS$$

where MU_A is the restriction of NU_A to MA; the previous theorem shows that this morphism makes the appropriate diagram commute. We do not, however, know that this arrow is surjective, unlike the corresponding arrow

$$PA \xrightarrow{PU_A} POS$$

from the patch assembly diagram (see Lemma 6.29).

7.4 Quilted Topologies

We will now construct a space ${}^{m}S$ with the same points as S and a topology

$$OS \subseteq O^{m}S \subseteq O^{f}S$$

between OS and its front topology.

Consider the following triangle.

$$\begin{array}{ccc} MOS & \hookrightarrow & NOS \\ & \searrow & \downarrow \sigma \\ & & O^{f}S \end{array}$$

We will show that this composite morphism from MOS to the front space of S is an embedding. This will enable us to use the image of MOS under σ as the topology $O^{m}S$.

7.12 Definition Let S be a sober space.

1. A subset $E \subseteq S$ is a *tessel* if $E \in O^f S$ and for every $p \in S$ both

$$\uparrow p \subseteq E \Longrightarrow p \in E^\circ \qquad \text{(the interior property of } E\text{)}$$

   and

$$\uparrow p - E \text{ is compact} \qquad \text{(the compactness property of } E\text{)}$$

   hold.

2. A subset $E \subseteq S$ is a *uniform tessel* if $E \cup U$ is a tessel for each $U \in OS$.

■

We will show that this concept gives a characterisation of the image of MOS under σ. This will allow us to use uniform tessels to construct the topology $O^m S$.

7.13 Theorem *Let S be a sober space, let $j \in MOS$ and put $E = \sigma j$. Then $j = \langle E \rangle$ and E is a uniform tessel.*

Proof. When S is a sober space, we have

$$\mathsf{pt}(OS) \cong S$$

and so in this case the morphisms U_{OS}, NU_{OS} and MU_{OS} are all the identity maps on the respective frames. As a special case of Theorem 7.11 we get that

$$j = \langle \sigma j \rangle = \langle E \rangle$$

for each $j \in MOS$.

By Lemma 7.9 we get that

$$\uparrow p \subseteq E \cup U \Longrightarrow p \in jU \Longrightarrow p \in (E \cup U)^\circ$$

and so the interior property holds.

We need to check that $E \cup U$ has the compactness property for every $U \in OS$. Suppose we have $p \in S$ and $\mathcal{V}$ a directed family of open sets such that

$$\uparrow p \cap (E \cup U)' \subseteq \bigcup \mathcal{V}$$

holds. Define

$$\mathcal{U} = \{U \cup V \mid V \in \mathcal{V}\}$$

so that $\mathcal{U}$ is another directed subset of OS. We have

$$\uparrow p \subseteq E \cup U \cup \bigcup \mathcal{V} = E \cup \bigcup \mathcal{U}$$

and so by the interior property of $E \cup \bigcup \mathcal{U}$ we see that

$$p \in \left(E \cup \bigcup \mathcal{U}\right)^\circ = \langle E \rangle \left(\bigcup \mathcal{U}\right)$$

holds. But $\langle E\rangle = j$ is continuous, and so

$$p \in \langle E\rangle\big(\bigcup\mathcal{U}\big) = \bigcup\{\langle E\rangle U \mid U \in \mathcal{U}\}$$

to give

$$p \in \langle E\rangle(U \cup V) = (E \cup U \cup V)^\circ$$

for some $V \in \mathcal{V}$. But then

$$\uparrow\! p \subseteq E \cup U \cup V$$

and so $\uparrow\! p \cap (E \cup U)' \subseteq V$ for some $V \in \mathcal{V}$ which shows that

$$\uparrow\! p - (E \cup U) = \uparrow\! p \cap (E \cup U)'$$

is compact as required. ■

This result shows that the morphism

$$\begin{array}{ccc} MOS & \longrightarrow & O^f S \\ j & \longmapsto & \sigma j \end{array}$$

is an embedding, and that the image of every continuous nucleus is a uniform tessel. We will now show that all uniform tessels arise in this way.

7.14 Lemma *Let S be a sober space, and let E be a uniform tessel on S. Then $\langle E\rangle$ is continuous and $Q - E$ is compact for each $Q \in QS$.*

Proof. To check continuity, consider any directed subfamily $\mathcal{U}$ of OS and any $p \in \langle E\rangle\big(\bigcup\mathcal{U}\big) = (E \cup \bigcup\mathcal{U})^\circ$. Since this set is open, we have

$$\uparrow\! p \subseteq E \cup \bigcup\mathcal{U}$$

and hence by compactness

$$\uparrow\! p \subseteq E \cup U$$

for some $U \in \mathcal{U}$. Now, the interior property of $E \cup U$ gives $p \in \langle E\rangle U$. This shows that

$$\langle E\rangle\big(\bigcup\mathcal{U}\big) = \bigcup\{\langle E\rangle U \mid U \in \mathcal{U}\}$$

and so $\langle E\rangle$ is continuous as required.

To get the stronger compactness result, suppose Q is a compact saturated set such that

$$Q \subseteq E \cup \bigcup\mathcal{U}$$

where, as usual, $\mathcal{U}$ is a directed family of open sets. Then for each $q \in Q$ we have

$$\uparrow\! q \subseteq Q \subseteq E \cup \bigcup\mathcal{U}$$

and so

$$q \in \left(E \cup \bigcup \mathcal{U}\right)^{\circ} = \bigcup\{\langle E\rangle U \mid U \in \mathcal{U}\}$$

by the interior property of $E \cup \bigcup \mathcal{U}$ and continuity of E. This gives

$$Q \subseteq \bigcup\{\langle E\rangle U \mid U \in \mathcal{U}\}$$

and hence, since Q is compact,

$$Q \subseteq \langle E\rangle U \subseteq E \cup U$$

for some $U \in \mathcal{U}$. Thus

$$Q - E \subseteq U$$

and $Q - E$ is compact as required. ■

The second part of this Lemma is a strengthening of the second property of tessels. It shows that for a uniform tessel, we not only know that $\uparrow p - E$ is compact for every $p \in S$, but that $Q - E$ is compact for every compact saturated set Q.

The first part of this Lemma allows us to prove the following strengthening of Theorem 7.13.

7.15 Theorem *Let S be a sober space. The assignment*

$$\begin{array}{ccc} MOS & \longrightarrow & O^{f}S \\ j & \longmapsto & \sigma j \end{array}$$

is a frame embedding where the range is exactly the family of uniform tessels on the space S.

Proof. Follows immediately since

$$\sigma\langle E\rangle = E^{\Box} = E$$

for every uniform tessel E. ■

We have now shown that the family of uniform tessels forms a topology since it is the image of a frame under σ and is therefore itself a frame. We use this to define the topology $O^{m}S$.

7.16 Definition For a sober space, S, let $O^{m}S$ be the topology of all uniform tessels on S. This is the *quilted topology* on S, and we call ${}^{m}S$ the *quilted space* of S. ■

This gives us the full diagram from Section 7.3.

7.5 The Patch and Continuous Assemblies can be Different

We have now constructed two subframes of NA; the patch assembly PA and the continuous assembly MA. We have noted that when A is regular, the embedding

$$A \longrightarrow PA$$

is an isomorphism, and that when A is fit, the embedding

$$A \longrightarrow MA$$

is an isomorphism.

We have also seen that

$$\text{fit} \not\Longrightarrow \text{regular}$$

so this suggests that there should be examples of frames for which MA is the canonical image of A, but the patch assembly PA is larger. Here, we give such an example. This will also provide an example of a space which is T_1 and fit but not T_2 and in which the intersection of compact saturated sets is not necessarily compact saturated.

7.17 Example Consider a set

$$S = T \cup \mathbb{N}$$

where T is a set disjoint from the set $\mathbb{N}$ of natural numbers which contains at least two elements.

Consider the topology OS on S generated by all sets of natural numbers, together with all the sets of the form

$$B(t, m) = \{t\} \cup [m, \infty)$$

for $t \in T$ and $m \in \mathbb{N}$. This family is a base for OS. However, we will want to refer to the smaller collection, $\mathcal{B}$ which is the family of all sets of the form

$$\{m\} \qquad B(t, m)$$

for $t \in T$ and $m \in \mathbb{N}$. It is easy to see that any set in OS can be formed from the union of sets in $\mathcal{B}$, and hence $\mathcal{B}$ is a base for OS.

We will show that the space S has the following properties.

1. There are non-closed compact sets, and hence S is not T_2.

2. S is fit and T_1 and hence sober.

3. We have $MOS = OS$ (since S is fit), but ${}^{p}S \neq S$ (since there are non-closed compact sets) and in particular we have $POS \neq OS$.

■

Proof. *Each member of $\mathcal{B}$ is compact.* Trivially we can see that $\{m\}$ is compact for every $m \in \mathbb{N}$. Suppose we have an open covering, $\mathcal{U}$ of $B(t, m)$ for some $t \in T, m \in \mathbb{N}$. Then $\mathcal{U}$ must contain some set $B(t, n)$ which either covers $B(t, m)$ or leaves only a finite set $[m, n)$ which can be covered by finitely many members of $\mathcal{U}$.

The set $B(t, 0)$ is not closed for any $t \in T$. We write $K(t)$ for $B(t, 0)$. We will show that $K(t)$ is dense, and therefore cannot be closed.

The complement of $K(t)$ is

$$K(t)' = T - \{t\}$$

which is a subset of T. Suppose $s \in K(t)'$ and $s \in U \in \mathcal{B}$. Then U contains infinitely many elements of $\mathbb{N}$, and so $U \nsubseteq K(t)'$. Hence

$$K(t)'^{\circ} = \varnothing$$

to give

$$K(t)^{-} = K(t)'^{\circ\prime} = S$$

and therefore the closure of $K(t)$ is the whole set. Since T has at least two members, $K(t)$ cannot be the whole set, and hence the set $B(t, 0)$ is not closed.

We have now constructed a compact set which is not closed.

The space S is T_1. For each $m \in \mathbb{N}$ and $t \in T$ we have

$$\{m\}' = [0, m-1] \cup T \cup [m+1, \infty)$$
$$\{t\}' = \big(T - \{t\}\big) \cup \mathbb{N}$$

both of which are unions of basic open sets. Hence, for every $p, q \in S$ we have

$$p \in \{q\}' \qquad q \notin \{q\}'$$

with $\{q\}'$ an open set. This gives the required separation property.

The space S is fit. Suppose $A, B \in OS$ with $A \nsubseteq B$. We will construct open sets U and V such that

$$A \cup U = S \qquad V \nsubseteq B \qquad U \cup V \subseteq B$$

hold. Suppose there exists $m \in \mathbb{N} \cap (A - B)$. Then

$$U = S - \{m\} \qquad V = \{m\}$$

will do.

Otherwise, we have $A \cap \mathbb{N} \subseteq B$. Then, since $A \nsubseteq B$, we have $t \in T \cap (A - B)$. Therefore we have

$$t \in B(t, m) \subseteq A$$

for some $m \in \mathbb{N}$ and

$$U = S - \{t\} \qquad V = B(t, m)$$

will do: by construction we have

$$A \cup U = S \qquad V \nsubseteq B$$

and

$$U \cap V = [m, \infty) \subseteq A \cap \mathbb{N} \subseteq B$$

by our earlier assumption.

Thus S is fit, and so $MOS = OS$, and there are non-closed compact sets, and so $POS \neq OS$ as claimed; hence the continuous assembly and the patch assembly may be different. ■

We also use this example to complete our proof in Section 6.4 that the join of two open filters is not necessarily open. We had reduced the problem via Lemma 6.23 to showing that the intersection of two elements of QS is not necessarily in QS.

7.18 Lemma *The intersection of two compact saturated sets is not necessarily compact saturated. In fact, the above example is taken from Exercise 5(b) on page 142 of [1] where it is used for the following.*

Proof. Consider the space S of Example 7.17. The space is T_1 and therefore all sets are saturated. We show that the intersection of two compact sets is not necessarily compact.

Consider the sets $K(s)$, $K(t)$ as defined above for distinct $s, t \in T$. Then we have

$$K(s) \cap K(t) = \mathbb{N}$$

which is not compact since the collection of all singletons

$$\Big\{ \{m\} \mid m \in \mathbb{N} \Big\}$$

forms an open cover of $\mathbb{N}$ which has no finite subcover. ■

In [3] Escardó considers the category SLoc of stably locally compact frames and a certain subcategory BReg of boundedly regular frames. He shows that the continuous assembly M provides a reflection from SLoc to BReg. However, each $A \in$ SLoc is spatial, $A = OS$, and

$$MA = PA = O^{p}S = O^{m}S$$

hold. In other words, all the constructions we consider in this thesis agree on SLoc. It is not clear which reflection properties are contributed by which construction. This topic requires further investigation.

Bibliography

[1] N. Bourbaki: *General Topology*, part 1, (Addison-Wesley, 1966).

[2] M. H. Escardó: Properly injective spaces and function spaces, *Topology and its Appications*, 89 (1998) 75-120

[3] M. H. Escardó: The regular-locally-compact coreflection of a stably locally compact locale, *J. Pure and Appl. Alg.* (to appear).

[4] G. Gierz *et al: A Compendium of Continuous Lattices* (Springer, 1980).

[5] R-E. Hoffmann (editor): *Continuous Lattices and Related Topics*, Mathematik Arbeitspapiere Nr. 27, Universität Bremen, 1982

[6] K. H. Hofmann and J. D. Lawson: On the order theoretical foundations of a theory of quasicompactly generated spaces without separation, pp 143-160 of [5]

[7] K. H. Hofmann and M. W. Mislove: Local compactness and continuous lattices, pp 207-148 of *Continuous Lattices*, (Springer LNM vol 871).

[8] J. R. Isbell: Atomless parts of spaces, *Math. Scand.* 32 (1972) 5-32

[9] P. T. Johnstone: The Vietoris monad on the category of locales, pp 162 - 179 of [5]

[10] P. T. Johnstone: *Stone spaces* (Cambridge University Press, 1982).

[11] D. S. Macnab: An algebraic study of modal operators on heyting algebras with applications to topology and sheafification, Doctoral Thesis, University of Aberdeen, 1976.

[12] S. B. Niefield and K. I. Rosenthal: Spatial sublocales and essential primes, *Topology and its Applications*, 26 (1987) 263-269.