Several λ-calculi for ordinal notations

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Actually, this should be

iteration notations

because iterations don't combine in the same way as ordinals do. See later for an example.

Applied λ-calculi

Recall what λ-calculi and related syntactic systems do.

There is a derivation system for producing correctly formed terms in context.

$$\Gamma \vdash t : \tau$$

There is a computation mechanism for reducing terms.

$$t^{-} \rhd\!\!\rhd t^{+}$$

A system is ‘applied’ if certain gadgets have an intended interpretation.

The system may be enriched by some equational reasoning

$$\Gamma \vdash t_1 \approx t_2$$

and perhaps further proof-theoretic mechanisms.

$\boldsymbol{\lambda}\mathsf{G}$, Gödel's T

There is one atom (basic type) $\mathcal{N}$ (to name $\mathbb{N}$).

All other types are formed by

$$\frac{\rho \quad \sigma}{\rho \times \sigma} \qquad \frac{\rho \quad \sigma}{\rho \to \sigma}$$

There are constants (basic terms) to handle pairing. These won't appear explicitly here.

There are constants

$$\mathsf{0} : \mathcal{N} \qquad \mathsf{S} : \mathcal{N}'$$

to name zero and the successor function.

There are certain other constants.

A bit of notation

For each type σ we let

$$\sigma' = (\sigma \to \sigma)$$

Thus

$$\sigma'' = (\sigma' \to \sigma') = (\sigma \to \sigma) \to (\sigma \to \sigma)$$

$$\sigma''' = (\sigma'' \to \sigma'') = (\sigma' \to \sigma') \to (\sigma' \to \sigma') = \text{quite long}$$

We also let

$$\sigma^{(0)} = \sigma \qquad \sigma^{(s+1)} = \sigma^{(s)\prime}$$

We also use this with concrete sets.

Iteration gadgets

For a selected family of types σ we have a constant

$$\mathsf{I}_\sigma : \mathcal{N} \to \sigma'' \qquad \begin{array}{c} \mathsf{I}_\sigma \mathsf{0} t s \rhd s \\ \mathsf{I}_\sigma (\mathsf{S} r) t s \rhd t(\mathsf{I}_\sigma r t s) \end{array} \qquad \text{``}\mathsf{I}_\sigma r t s = t^r s\text{''}$$

with indicated one-step reduction properties.

Within such a system many numeric gadgets can be named.

Furthermore, the syntax used gives a coarse measure of the complexity of the gadget.

Ackermann stuff

$$\mathsf{ack} := \lambda y : \mathcal{N}', x : \mathcal{N} \,.\, \mathsf{I}_{\mathcal{N}}(\mathsf{S}x)yx \qquad \text{“}\mathsf{ack}\,fx = f^{x+1}x\text{”}$$

$$\mathsf{Ack} := \lambda f : \mathcal{N}', i, r : \mathcal{N} \,.\, \mathsf{I}_{\mathcal{N}}(\mathsf{S}r)(\mathsf{I}_{\mathcal{N}'}i\,\mathsf{ack}) \qquad \text{“}\mathsf{Ack}\,fir = (\mathsf{ack}^i f)^{r+1}\text{”}$$

$$\mathsf{ack} : \mathcal{N}'' \qquad \mathsf{Ack} : \mathcal{N}' \to \mathcal{N} \times \mathcal{N} \to \mathcal{N}'$$

$$\begin{aligned} A_f(0,0,x) &= f(x) \\ A_f(i',0,x) &= A_f(i,x,x) \\ A_f(i,r',x) &= A_f(i,0,y) \\ \text{where } y &= A_f(i,r,x) \end{aligned} \qquad (\mathsf{Ack}\,f)irx = A_f(i,r,x)$$

Observe the types and iterators used.

Higher level jump functions

For each index $s \in \mathbb{N}$ let B_s be

$$\lambda z : \mathcal{N}^{(s+2)}, y : \mathcal{N}^{(s+1)}, u_s : \mathcal{N}^{(s)}, \ldots, u_1 : \mathcal{N}', x : \mathcal{N} \,.\, \mathrm{Body}$$

where Body is $\mathsf{I}_{\mathcal{N}^{(s+1)}} xzyu_s \cdots u_1 x$

$$\mathsf{B}_0 F f x = F^x f x \qquad \mathsf{B}_{s+1} F f \cdots x = F^x f \cdots x$$

Theorem. *For each list of natural numbers*

$$e(s), e(s-1), \ldots, e(0), r$$

there is an ordinal notation β such that

$$((\cdots(\mathsf{B}_s^{e(s)} \mathsf{B}_{s-1})^{e(s-1)} \cdots \mathsf{B}_0)^{e(0)} F)^r = F^\beta$$

$$\begin{aligned} \alpha(s) &= e(s) \\ \alpha(s-1) &= \omega^{\alpha(s)} \cdot e(s-1) \\ \alpha(s-2) &= \omega^{\alpha(s-1)} \cdot e(s-2) \\ &\vdots \\ \alpha(0) &= \omega^{\alpha(1)} \cdot e(0) \\ \beta &= \omega^{\alpha(0)} \cdot r \end{aligned}$$

Peano structures

A peano structure is a triple

$$(\mathbb{A}, a, A)$$

where $\mathbb{A}$ is a set, $a \in \mathbb{A}$, $A \in \mathbb{A}'$.

These form the objects of a category Peano where the comparison arrows are the obvious functions.

Theorem. (Dedekind) *The category* Peano *has an initial object*

$$(\mathbb{N}, 0, S)$$

that is, for each peano structure $(\mathbb{A}, a, A)$ *there is a unique arrow*

$$(\mathbb{N}, 0, S) \xrightarrow{\phi} (\mathbb{A}, a, A)$$

Limit structures

A limit structure is a a peano structure

$$(\mathbb{A}, a, A, \mathcal{A})$$

furnished with a ‘limit’ creator $\mathcal{A} : (\mathbb{N} \to \mathbb{A}) \to \mathbb{A}$. ‘collator’

An arrow between two such structure

$$(\mathbb{B}, b, B, \mathcal{B}) \xrightarrow{\phi} (\mathbb{A}, a, A, \mathcal{A})$$

is a peano arrow with an extra property.

$$\begin{tikzcd}
\mathbb{N} \arrow[d, "q"'] & \mathbb{N} \arrow[d, "\phi \circ q"] \\
\mathbb{B} \arrow[r, "\phi"] & \mathbb{A}
\end{tikzcd}$$

$$\phi \circ \mathcal{B}(q) = \mathcal{A}(\phi \circ q)$$

These objects and arrows form the category Limit.

Tree ordinals

Theorem. *The category* Limit *has an initial object*

$$\mathfrak{O} = (\mathbb{O}, 0, S, Lim)$$

that is, for each limit structure $(\mathbb{A}, a, A, \mathcal{A})$ *there is a unique arrow*

$$\begin{tikzcd}\mathfrak{O} \arrow[r, "\Phi"] & (\mathbb{A}, a, A, \mathcal{A})\end{tikzcd}$$

This arrow Φ satisfies

$$\Phi(0) = a \qquad \Phi(S\alpha) = A(\Phi(\alpha)) \qquad \Phi(Lim(q)) = \mathcal{A}(\Phi \circ q)$$

for each $\alpha \in \mathbb{O}$ and $q : \mathbb{N} \to \mathbb{O}$.

Example

For arbitrary $f : \mathbb{N}', F : \mathbb{N}''$ consider the limit structure

$$(\mathbb{N}', f, F, \Delta) \quad \text{where} \quad \Delta(p)x = pxx$$

for each $p : \mathbb{N} \to \mathbb{N}'$ and $x \in \mathbb{N}$. Then

$$\Phi(\alpha) = F^{\alpha} f$$

for each $\alpha \in \mathbb{O}$.

For instance, suppose $\eta = Lim(q)$. Then

$$F^{\eta} f x = \Phi(\eta)x = \Phi(Lim(q))x = (\Phi \circ q)xx = \Phi(q(x))x$$

so that

$$F^{\eta} f x = \Phi(q(x))x = F^{q(x)} f x = F^{\eta[x]} f x$$

as expected.

$\boldsymbol{\lambda}\mathsf{H}$, the term calculus of Howard's system

Two atoms $\mathcal{N}$ (to name $\mathbb{N}$) and $\mathcal{O}$ (to name $\mathbb{O}$).

Constants for pairing gadgets.

Five constants

$$\underline{0} : \mathcal{N} \quad \underline{\mathsf{S}} : \mathcal{N}' \qquad \overline{0} : \mathcal{O} \quad \overline{\mathsf{S}} : \mathcal{O}' \quad \mathsf{Lim} : (\mathcal{N} \to \mathcal{O}) \to \mathcal{O}$$

Iterators

$$\mathsf{I}_\sigma : \mathcal{N} \to \sigma'' \qquad \mathsf{J}_\sigma : \mathcal{O} \to \mathcal{L}(\sigma) \to \sigma''$$

where $\mathcal{L}(\sigma) = (\mathcal{N} \to \sigma) \to \sigma$.

$$\begin{array}{ll} \mathsf{J}_\sigma \overline{0} lts & \rhd s \\ \mathsf{J}_\sigma (\overline{\mathsf{S}} \alpha) lts & \rhd t(\mathsf{J}_\sigma \alpha lts) \\ \mathsf{J}_\sigma (\mathsf{Lim}\, p)) lts & \rhd l(\lambda x : \mathcal{N} \,.\, \mathsf{J}_\sigma (px) lts) \end{array}$$

A far more powerful system

In **λG** each ordinal $< \epsilon_0$ can be simulated.

In **λH** each ordinal below the Howard ordinal can be named.

Furthermore, the syntax needed to name the ordinal (ordinal notation, iteration gadget) gives us a measure of its complexity.

Some canonical examples

$$\mathsf{Fix} := \lambda f : \mathcal{O}', \alpha : \mathcal{O} \,.\, (\mathsf{J}_{\mathcal{O}}\omega\mathsf{Lim}) f (\overline{\mathsf{S}}\alpha) \qquad \mathsf{Fix} : \mathcal{O}''$$

This corresponds to a function

$$\boldsymbol{Fix} : \mathbb{O}'' \qquad \boldsymbol{Fix}\, f\zeta = \text{least } \eta \text{ with } \zeta < \eta = f(\eta)$$

provided f is sufficiently nice.

$$\boldsymbol{Next} = \boldsymbol{Fix}\omega^{\bullet} : \mathbb{O}' \qquad \epsilon_\alpha = \boldsymbol{Next}^{1+\alpha}\omega$$

How do we enumerate the δ with $\epsilon_\delta = \delta$?

$[\,0\,] : \mathbb{O}''$ $\qquad$ $[\,0\,]h = \boldsymbol{Fix}(\alpha \mapsto h^\alpha\omega)$ $\qquad$ $[\,0\,]\boldsymbol{Next}$ does the job

Going further

$$\begin{array}{ll} [\,1\,] : \mathbb{O}^{(3)} & [\,1\,]Hh \quad\;\; = \boldsymbol{Fix}(\alpha \mapsto H^{\alpha}h\omega) \\ [\,2\,] : \mathbb{O}^{(4)} & [\,2\,]Hhg \;\; = \boldsymbol{Fix}(\alpha \mapsto H^{\alpha}hg\omega) \\ [\,3\,] : \mathbb{O}^{(5)} & [\,3\,]Hhgf = \boldsymbol{Fix}(\alpha \mapsto H^{\alpha}hgf\omega) \end{array}$$

and so on

With these we find that

$$\Delta[0] = \omega \quad \Delta[1] = \boldsymbol{Next}\omega \quad \Delta[l+2] = [\,\imath\,]\cdots[\,0\,]\boldsymbol{Next}\omega$$

is a fundamental sequence for the Howard ordinal using higher and higher types as it proceeds.

Of course, proving this is not entirely straight forward.

What next

This can be carried out in $\boldsymbol{\lambda}\mathsf{2}$, the second order λ-calculus.

Of course that is a leap too far.

However, $\boldsymbol{\lambda}\mathsf{2}$ can be stratified (and severely weakened).

$\boldsymbol{\lambda}\mathsf{H}$ requires only a couple of the lower layers, so there are plenty left.

I have no idea what happens there.

Comparison with a standard $\psi : \Omega^+ \to \Omega$

For each $\xi < \Omega^+$ define $\{\!|\ \xi\ |\!\}_l : \mathbb{O}\mathrm{rd}^{(l+2)}$ by

$$\begin{aligned}
\{\!|\ 0\ |\!\}_l &= \boldsymbol{id}^{(l+2)} \\
\xi &= \Omega^{\xi(s)} \cdot \alpha(s) + \cdots + \Omega^{\xi(0)} \cdot \alpha(0) \\
\{\!|\ \xi\ |\!\}_l &= \big(\{\!|\ \xi(0)\ |\!\}_{l+1}[\,\imath\,]\big)^{\alpha(0)} \circ \cdots \circ \big(\{\!|\ \xi(s)\ |\!\}_{l+1}[\,\imath\,]\big)^{\alpha(s)}
\end{aligned}$$

Then

$$\psi\big(\Omega^{\xi} \cdot (1+\beta)\big) = \big(\{\!|\ \xi\ |\!\}_0 \boldsymbol{Next}\big)^{1+\beta}\omega$$

$1+\omega \neq \omega$

Consider the jump function $F : \mathbb{N}''$ given by

$$Ffx = f(x+1)$$

for all $f : \mathbb{N}', x : \mathbb{N}$. Then

$$F^r fx = f(x+r)$$

for all $f : \mathbb{N}', r, x : \mathbb{N}$, and so

$$F^\omega fx = F^{\omega[x]} fx = f(x+\omega[x])$$

using the fundamental sequence for ω. But now

$$F^{1+\omega} fx = F^\omega (Ff)x = Ff(x+\omega[x]) = f(1+x+\omega[x])$$

so that

$$F^{1+\omega} \neq F^\omega$$

Recursors from iteration

in the manner of Kleene, Bernays, ….
Consider a standard recursor.

$$R : \mathcal{N} \to (\mathcal{N} \to \sigma') \to \sigma' \qquad \begin{aligned} R0fs &= s \\ R(Sr)fs &= fr(Rrfs) \end{aligned}$$

For an input function $f : \mathcal{N} \to \sigma'$ look at

$$F : \mathcal{N} \times \sigma \to \mathcal{N} \times \sigma \quad \text{where} \quad F(x, s) = (x + 1, fxs)$$

Then

$$F^x(r, Rrfs) = (r + x, R(r + x)fs)$$

so that

$$F^x(0, s) = (x, Rxfs)$$