Generating ordinal notations from below
with a non-recursive construction of the Schütte brackets

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Abstract

I re-work Veblen's ideas to show how hierarchies of normal functions can be generated merely by iterating certain higher order gadgets. As an illustration I show that an application of a Schütte bracket can be evaluated without the need of an intricate recursion.

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Preamble

Some 100 years ago Veblen, in [7], began the investigation of what are now known as the normal functions from ordinals to ordinals. He was, of course, building on the work of Cantor, and in part looking to answer the question raised by Hardy in [3]. How can we name each countable ordinal? Over time, 50 years or more, in the hands of several investigators, this developed into the topic of ordinal notations, but some of Veblen's suggested methods seem to have been put to one side.

The current technique as described in [1], say, names countable ordinals from above. By that I mean uncountable ordinals are used to index the processes by which the countable ordinals are generated. In contrast to this Veblen's method names countable ordinals from below. First generate some initial stretch of countable ordinals, then use this to index a process to generate a longer stretch of countable ordinals, then use this . . . , and so on. Of course, the 'and so on' is not exactly straight forward.

Veblen pointed out the analogy between normal functions and certain functions on the reals. Perhaps he was suggesting that we keep this analogy in mind when developing the analysis of normal functions.

Here I work in the spirit of Veblen. I consider several classes of ordinal functions, several operators (functions from ordinal functions to ordinal functions), and one or two constructors (functions from operators to operators). Using these I produce, I believe, a more coherent account of Veblen's method and its later developments. I show how to generate ordinal notations up to a certain level from below.

I think it is fair to say that the latter part of Veblen's account is not easy reading. There is a whole family of highly nested processes which have to be indexed. It is not entirely clear how this is done. Here I show that by using higher order gadgets this indexing can be achieved in a straight forward manner by a method that is little more than a Cantor normal form lifted up a level or two.

In [4] Schütte produced, amongst other things, a more systematic method of indexing the Veblen processes. He developed the Schütte bracket, or Klammersymbol in German. Each such bracket is an array of two rows and several columns of ordinals. When combined with a normal function each such bracket returns an ordinal value. However, the evaluation algorithm is given by a rather intricate recursion in which the size of the bracket can increase and decrease several times.

Roughly speaking each such bracket $(=)$ determines an operator F and an ordinal ζ. For each normal function f the value $(=)f$ is just $Ff\zeta$. Most of the intricate recursion is concerned with

$$(=) \longmapsto F$$

the process of converting the bracket into the operator. To illustrate the general methods I develop here I show that F can be read off directly from $(=)$ without any recursion. Each column of $(=)$ determines an operator, and these are combined using composition.

A nice survey of the history of this topic is given in [2].

This paper is more or less self contained, but is related to [5] and [6]. In those papers I show that, in principle, there is a system of naming ordinals from below that goes right up to the Howard ordinal. The idea is to use the type structure sitting above the set of countable ordinals together with various higher order functions on these levels. Here I describe in detail the techniques needed for the lower levels of this type structure. Only ordinals, functions, operators, and constructors are needed. Of course, these methods get nowhere near the Howard ordinal. The Ackermann ordinal is the upper bound of the ordinals that can be named in this way.

The paper is written in two parts. Sections 1 to 5 contain the general development. Sections 6 to 8 deal with the particular application to Schütte brackets.

Contents

1 Setting the scene

In this section I describe some of the general ideas, and outline a plan of attack. Some assertions are made without proof. These are justified in the later sections.

We are concerned entirely with the class of countable ordinals

$$\mathbb{O}\mathrm{rd}$$

so we usually omit the modifier ‘countable’. The global problem is to find methods of enumerating long initial stretches of critical ordinals, those ordinals ν satisfying $\nu = \omega^{\nu}$.

We follow Veblen in [7] and focus our attention on certain functions

$$f : \mathbb{O}\mathrm{rd} \longrightarrow \mathbb{O}\mathrm{rd}$$

which are now called normal. (Veblen didn't use that terminology.) Here it is convenient to assume also that

$$\omega^\zeta \le f\zeta$$

for each input ordinal ζ. Thus for us a normal function has this extra property.

Each normal function has unboundedly many fixed points, ordinals ν such that $\nu = f\nu$. Our extra restriction ensures that each such fixed point is critical.

Veblen suggests that we attach to each normal function f its derived function, the function that enumerates (in ascending order) the fixed points of f. This too is normal, and so we may repeat the trick. In fact, we may iterate this process through the ordinals.

For each normal function f the Veblen hierarchy $\phi[f]$ on f is generated by

$$\begin{aligned} \phi[f]0 &= f \\ \phi[f](\alpha+1) &= \text{enumeration of fixed points of } \phi[f]\alpha \\ \phi[f]\lambda &= \text{enumeration of common fixed points of } \phi[f]\alpha \text{ for all } \alpha < \lambda \end{aligned} \tag{1.a}$$

for each ordinal α and limit ordinal λ. We rephrase this construction in a compact form.

Consider the operator $\boldsymbol{Veb}$ which consumes, as input, a normal function f to output $\boldsymbol{Veb}f$ the normal function that enumerates the fixed points of f. We iterate $\boldsymbol{Veb}$ through the ordinals. For each normal function f we set

$$\begin{aligned} \boldsymbol{Veb}^0 f &= f \\ \boldsymbol{Veb}^{\alpha+1} f &= \boldsymbol{Veb}(\boldsymbol{Veb}^\alpha f) \\ \boldsymbol{Veb}^\lambda f &= \bigvee\{\boldsymbol{Veb}^\alpha f \mid \alpha < \lambda\} \end{aligned} \tag{1.b}$$

for each ordinal α and limit ordinal λ. At limit stages we take the pointwise supremum of the family of earlier functions. This produces an expanded version of the Veblen hierarchy.

The pointwise supremum of a family of normal functions need not be normal. It can be constant for long periods. To handle this we use a slightly larger class $\mathbb{F}$ of function, the fruitful functions (so called because each has many fixed points and each of these is a critical ordinal). These are discussed in Section 2. For each $f \in \mathbb{F}$ the enumerating function $\boldsymbol{Veb}f$ is normal. Furthermore, the pointwise supremum $\bigvee\mathcal{F}$ of a family $\mathcal{F}$ of fruitful functions is itself fruitful.

Starting with any fruitful function we may iterate $\boldsymbol{Veb}$ through the ordinals to produce an ascending chain through $\mathbb{F}$. At each successor stage the function obtained is normal.

This construction fills out the Veblen hierarchy. We find that

$$\phi[f](1+\alpha) = \boldsymbol{Veb}^{\alpha+1} f \tag{1.c}$$

for each ordinal α. Notice that this description misses out the limit stages $\boldsymbol{Veb}^\lambda f$, so even if we start from a fruitful function, each generated function is normal.

The flips here

$$1+\alpha \mapsto \alpha+1$$

are not a mistake. We will see several such flips. They are usually a consequence of some kind of stutter in a construction around limit ordinals.

Suppose we use (1.a) or its expanded version (1.b) to generate ordinal notations from below. The golden rule is that in any notation for a new ordinal we may use only ordinals that have been named earlier. How far can we go?

We call an ordinal ν such that

$$\nu = \phi[f]\nu 0$$

a barrier ordinal for $\phi[f]$. There are unboundedly many such barrier ordinals, and the least one is the boundary of the ordinals that can be named using $\phi[f]$.

How do we get beyond this boundary?

The (or perhaps, a) diagonal limit of the hierarchy $\phi[f]$ is the function ${}^*f$ given by

$${}^*f\alpha = \phi[f](1+\alpha)0$$

for each ordinal α. (The 1+ here makes some later analysis a bit neater.) It turns out that ${}^*f$ is normal, and the barrier ordinals of $\phi[f]$ are the fixed points of ${}^*f$. For example, consider $f = \omega^\bullet$. Then $\boldsymbol{Veb}f$ enumerates the critical ordinals $\epsilon_\bullet$, whereas $\boldsymbol{Veb}{}^*f$ enumerates the strongly critical ordinals $\Gamma_\bullet$.

We use ${}^*f$ as the base function for the next hierarchy $\phi[{}^*f]$ after $\phi[f]$. We repeat (1.a, 1.b) with f replaced by ${}^*f$. This gives a faster hierarchy of normal functions.

After that it is clear we what should attempt. We should iterate the construction

$$f,\ {}^*f,\ {}^{**}f,\ {}^{***}f,\ \ldots$$

and extend this process through the ordinals. When that runs out of steam we should take another diagonal limit, and so on. Clearly, this will require quite a bit of work to sort out. We also have to devise some method of indexing the various layers of nestings of the process. Veblen does this in the latter part of [7], but I think it is fair to say that the description is not exactly crystal clear.

Let's use that excellent technique, hindsight, to suggest what Veblen perhaps should have done. We generalize the idea of the operator $\boldsymbol{Veb}$.

A derivative is an operator $\boldsymbol{D}$ which converts each fruitful function f into a faster fruitful function $\boldsymbol{D}f$, a function whose fixed points form a sparse subset of the fixed points of f. In Section 4 we produce a whole family $\mathfrak{Derv}$ of such derivatives.

Suppose we start with

$$f \in \mathbb{F} \qquad \boldsymbol{D} \in \mathfrak{Derv}$$

a fruitful function and a derivative. We iterate $\boldsymbol{D}$ to obtain

$$\begin{aligned} \boldsymbol{D}^0 f &= f \\ \boldsymbol{D}^{\alpha+1} f &= \boldsymbol{D}(\boldsymbol{D}^\alpha f) \\ \boldsymbol{D}^\lambda f &= \bigvee\{\boldsymbol{D}^\alpha f \mid \alpha < \lambda\} \end{aligned} \tag{1.d}$$

for each ordinal α and limit ordinal λ. After some preliminary work to sort out the details we see that this gives an ascending chain of fruitful functions, and we may enumerate the fixed points of each one. Thus for each ordinal α we may set

$$\Phi[\boldsymbol{D}, f]\alpha = \boldsymbol{Veb}\big(\boldsymbol{D}^\alpha f\big) \tag{1.e}$$

to produce the $\boldsymbol{Der}$-induced Veblen-like hierarchy

$$\Phi[\boldsymbol{D}, f] \tag{1.f}$$

on the base function f. In particular we have

$$\phi[f](1+\alpha) = \Phi[\boldsymbol{Veb}, f]\alpha$$

by using $\boldsymbol{Veb}$ as the generating derivative.

The hierarchy (1.f) has its barrier ordinals

$$\nu = \Phi[\boldsymbol{D}, f]\nu 0$$

the least of which is the boundary of the ordinals that can be named using (1.f). What should we do to get beyond that boundary?

As above, an obvious idea is to jump the base function to produce a faster fruitful function. Let ${}^{\Uparrow}\!f$ be the function given by

$${}^{\Uparrow}\!f\alpha = \Phi[\boldsymbol{D}, f]\alpha 0$$

for each ordinal α. It can be checked that this is a fruitful function, and clearly $\boldsymbol{Veb}({}^{\Uparrow}\!f)$ enumerates the barrier ordinals of $\Phi[\boldsymbol{D}, f]$. We may use ${}^{\Uparrow}\!f$ as the new base function and generate a new hierarchy

$$\Phi[\boldsymbol{D}, {}^{\Uparrow}\!f]$$

which will go further than $\Phi[\boldsymbol{D}, f]$. After that we can use

$$f,\ {}^{\Uparrow}\!f,\ {}^{\Uparrow\!\Uparrow}\!f\ \ {}^{\Uparrow\!\Uparrow\!\Uparrow}\!f\ \ \ldots$$

as a succession of base functions.

There are two problems with this. The first is that it is not at all clear how we can take the iteration of ${}^{\Uparrow}(\cdot)$ into the transfinite. The second is that the construction

$$f \longmapsto {}^{\Uparrow}\!f$$

is dependent on the derivative $\boldsymbol{D}$. The second of these actually suggest what is perhaps is a better tactic. Rather that jump the base function we should jump the derivative.

Consider the constructor $(\Uparrow)$ given by

$$(\Uparrow)\boldsymbol{D}f\alpha = \Phi[\boldsymbol{D}, f]\alpha 0$$

for each derivative $\boldsymbol{D}$, each fruitful function f, and each ordinal α. Thus

$${}^{\Uparrow}\!f = (\Uparrow)\boldsymbol{D}f$$

is our previous construction. With some effort it can be shown that for each 'suitable' derivative $\boldsymbol{D}$ the resulting operator $(\Uparrow)\boldsymbol{D}$ is also a 'suitable' derivative. One of the aims of this paper is to show how we can take some of the effort out of this proof.

It has to be pointed out that the two hierarchies

$$\Phi[\boldsymbol{D}, {}^{\Uparrow}\!f] \qquad \Phi[(\Uparrow)\boldsymbol{D}, f]$$

are not the same. The left hand one is generated by iterating $\boldsymbol{D}$ whereas the right hand one is generated by iterating $(\Uparrow)\boldsymbol{D}$. Consequently the right hand one is much faster than the left hand one. We will give a direct comparison later in this section.

One of the benefits of this second approach is that it is easier to iterate the jump. In this way we may generate a super-hierarchy

$$\Phi[(\Uparrow)^i\boldsymbol{D}, f]\alpha$$

for ordinals i and α.

How are we going to handle these hierarchies? That is the novelty I want to bring to your attention.

The method is to put fruitful functions and derivatives to one side and use a class $\mathbb{H}$ of helpful functions, a class $\mathfrak{Help}$ of helpful operators, and a particular helpful constructor. The precise definitions and relevant properties of these classes are discussed is Sections 2 and 3. Here I will give an indication of how we use these classes.

We produce two operators

$$\boldsymbol{Fix} : \mathbb{F} \longrightarrow \mathbb{H} \qquad \boldsymbol{Enm} : \mathbb{H} \longrightarrow \mathbb{F}$$

to pass between $\mathbb{F}$ and $\mathbb{H}$. The left hand one is a fixed point extractor and the right hand one is an enumerating gadget. In particular we find that

$$\boldsymbol{Veb} = \boldsymbol{Enm} \circ \boldsymbol{Fix}$$

which agrees with our earlier description that $\boldsymbol{Veb} f$ enumerates the fixed points of f. The other composite

$$[\,\mathrm{o}\,] = \boldsymbol{Fix} \circ \boldsymbol{Enm}$$

is an important member of $\mathfrak{Help}$. The particular helpful constructor is named

$$[\,\mathrm{i}\,]$$

and the similarities between $[\,\mathrm{o}\,]$ and $[\,\mathrm{i}\,]$ will become apparent.

The crucial property of each $\nabla \in \mathfrak{Help}$ is that

$$\nabla h \in \mathbb{H}$$

for each $h \in \mathbb{H}$. Also, $\mathfrak{Help}$ is closed under composition and ordinal iteration. In particular

$$\alpha \longmapsto \nabla^{\alpha} h \tag{1.g}$$

is a hierarchy of helpful functions. By hitting each with $\boldsymbol{Enm}$ we obtain a hierarchy of normal functions. We will see how each hierarchy (1.f) can be generated in this way.

In the same way we have

$$[\,\mathrm{i}\,]\nabla \in \mathfrak{Help}$$

for each $\nabla \in \mathfrak{Help}$. We will see that this is the way we may jump from one hierarchy to the next. By combining these two constructions we obtain a description of the behaviour of Schütte brackets.

The operator

$$\boldsymbol{Veb} = \boldsymbol{Enm} \circ \boldsymbol{Fix}$$

is the best known example of a derivative. We find that

$$\boldsymbol{Enm} \circ \nabla \circ \boldsymbol{Fix}$$

is a derivative for each $\nabla \in \mathfrak{Help}$. We also use a different method of generating more powerful derivatives.

The main technique, as described in Section 5, is a shuffle relationship

$$\boldsymbol{D} \rightleftharpoons \nabla \tag{1.h}$$

between a derivative $\boldsymbol{D}$ and a helpful operator ∇. This holds precisely when

$$\boldsymbol{Fix} \circ \boldsymbol{D} = \nabla \circ \boldsymbol{Fix}$$

the fixed point operator can be shuffled from left to right. The existence of a helpful shuffling companion ensures that $\boldsymbol{D}$ is a 'suitable' derivative.

We obtain various properties of the relationship (1.h). In particular it ensures that

$$\boldsymbol{D}^{\alpha} \rightleftharpoons \nabla^{\alpha}$$

for each ordinal α. With this, for each fruitful function f and ordinal α we have

$$\begin{aligned} \Phi[\boldsymbol{D}, f]\alpha &= \boldsymbol{Veb}(\boldsymbol{D}^{\alpha} f) \\ &= (\boldsymbol{Enm} \circ \boldsymbol{Fix} \circ \boldsymbol{D}^{\alpha})f \\ &= (\boldsymbol{Enm} \circ \nabla^{\alpha} \circ \boldsymbol{Fix})f \;\; = \boldsymbol{Enm}(\nabla^{\alpha} h) \end{aligned} \tag{1.i}$$

where $h = \boldsymbol{Fix} f$. This shows how the helpful hierarchy (1.g) is the essential part of the generalized Veblen hierarchy (1.f).

Another consequence of (1.h) is

$$(\Uparrow)\boldsymbol{D} \rightleftharpoons [\,1\,]\nabla$$

which shows how the constructor $[\,1\,]$ comes into play. Using this we see that

$$\begin{array}{lcl} \Phi[\boldsymbol{D}, f] & \alpha \longmapsto & \boldsymbol{Enm}\big(\nabla^{\alpha} h\big) \\ \Phi[\boldsymbol{D}, {}^{\Uparrow}\!f] & \alpha \longmapsto & \boldsymbol{Enm}\big(\nabla^{\alpha}([\,1\,]\nabla h)\big) \\ \Phi[(\Uparrow)\boldsymbol{D}, f] & \alpha \longmapsto & \boldsymbol{Enm}\big(([\,1\,]\nabla)^{\alpha} h\big) \end{array}$$

are the helpful descriptions of the parent hierarchy, the slow next hierarchy, and the fast next hierarchy. Also it is easy to iterate the next hierarchy construction. We find that

$$(\Uparrow)^{i}\boldsymbol{D} \rightleftharpoons [\,1\,]^{i}\nabla$$

for each ordinal i. Thus the doubly indexed hierarchy

$$(i, \alpha) \longmapsto \big([\,1\,]^{i}\nabla\big)^{\alpha}$$

picks its way through the helpful part of a super-hierarchy. Notice how this description uncouples the base function from the construction.

Helpful operators are closed under composition. Given any selection of pairs of ordinals

$$\big((i(1), \alpha(1)\big), \ldots, \big(i(s), \alpha(s)\big)$$

we may compose the helpful operators to obtain a compound helpful operator.

$$\big([\,1\,]^{i(1)}\nabla\big)^{\alpha(1)} \circ \cdots \circ \big([\,1\,]^{i(s)}\nabla\big)^{\alpha(s)}$$

All these all fit into the super-hierarchy. The original Veblen super-hierarchy was generated using the helpful operator $[\,0\,]$, and so ordinal indexed compounds

$$\big([\,1\,]^{i(1)}[\,0\,]\big)^{\alpha(1)} \circ \cdots \circ \big([\,1\,]^{i(s)}[\,0\,]\big)^{\alpha(s)} \tag{1.j}$$

pick out paths through this super-hierarchy. We find that the Schütte brackets are a rather intricate way of generating such paths.

Of course, this is far from routine, there is plenty of work that has to be done. The important point is that with this idea we can generate a super-hierarchy of Veblen-like hierarchies by merely iterating the jump constructor ($\Uparrow$). The word 'merely' is a bit glib, but there is an idea here that is worth investigating. That is what I do in this paper.

That is the general plan of attack. I now introduce some notation and terminology. This will be augmented with more precise descriptions in the later sections.

We are concerned with the set $\mathbb{O}\mathrm{rd}$ of countable ordinals, and we use functions on three levels.

We may think of an ordinal, a member of $\mathbb{O}\mathrm{rd}$, as a 'function' on level 0.

A function on level 1 is merely a function from ordinals to ordinals.

$$g : \mathbb{O}\mathrm{rd} \longrightarrow \mathbb{O}\mathrm{rd}$$

A function requires an ordinal input ζ to output its value at ζ.

$$g\zeta$$

A function on level 2 is a function G from level 1 functions (of a certain kind) to level 1 functions. Thus G requires as an input a certain kind of function g on level 1 to output a function Gg on level 1. This requires an input ordinal ζ to output the eventual value via a 2-step evaluation.

$$Gg\zeta$$

We sometimes refer to such a level 2 function as an operator.

A function on level 3 is a function Γ from level 2 functions (of a certain kind) to level 2 function (of a certain kind). Thus Γ requires as an input a certain kind of function G on level 2 to output a function ΓG on level 2. This requires as an input a certain kind of level 1 function g to output a function ΓGg on level 1. This requires an input ordinal ζ to output the eventual value via a 3-step evaluation.

$$\Gamma Gg\zeta$$

We sometimes refer to such a level 3 function as an constructor.

We use two special classes of ordinal functions. The class $\mathbb{F}$ of fruitful functions and the class $\mathbb{H}$ of helpful functions. The class $\mathbb{F}$ is a mild extension of the class of normal functions. The class $\mathbb{H}$ has not yet received much attention.

As an informal convention it is useful to let

$$f \qquad\qquad\qquad g \qquad\qquad\qquad h$$

range over

$$\mathbb{F} \qquad \text{general ordinal functions} \qquad \mathbb{H}$$

respectively. Of course, there are times when this convention is broken.

We will look at two general kinds of level 2 functions, namely

$$\begin{array}{cc} \mathfrak{Derv} & \mathfrak{Help} \\ \boldsymbol{D} : \mathbb{F} \longrightarrow \mathbb{F} & \nabla : \mathbb{H} \longrightarrow \mathbb{H} \\ \text{derivatives} & \text{helpful operators} \end{array}$$

respectively. The class $\mathfrak{Help}$ is defined in Section 3 and the class $\mathfrak{Derv}$ in Section 4.

The notion of helpfulness can be lifted through all finite levels, but here we need only helpful functions on levels 1, 2, and 3. The higher levels are used in [5] and [6].

Several of the kinds of functions we use

$$g : \mathbb{O}\mathrm{rd} \longrightarrow \mathbb{O}\mathrm{rd} \qquad \boldsymbol{D} : \mathbb{F} \longrightarrow \mathbb{F} \qquad \nabla : \mathbb{H} \longrightarrow \mathbb{H} \qquad \Xi : \mathfrak{Help} \longrightarrow \mathfrak{Help}$$

can be iterated. Finite iterates are not a problem, but we also use ordinal iterates. With those we have to be a little careful. Given a set $\mathbb{S}$ of functions (of a suitable kind) we let

$$\mathbb{S}^{\ddagger} \quad \text{abbreviate} \quad (\mathbb{S} \longrightarrow \mathbb{S})$$

so that members of $\mathbb{S}^{\ddagger}$ can be iterated.

2 Fruitful and helpful functions

In this section we isolate the two classes

$\mathbb{F}$	$\mathbb{H}$
fruitful	helpful

of ordinal functions (functions on level 1). We also produce two operators $\boldsymbol{Fix}$ and $\boldsymbol{Enm}$.

We need some information about pointwise suprema and ordinal iterates.

2.1 DEFINITION. (a) For each (non-empty, countable) family $\mathcal{G}$ of ordinal functions (on level 1), the pointwise supremum $\bigvee \mathcal{G}$ is the function given by

$$(\bigvee \mathcal{G})\zeta = \bigvee\{g\zeta \mid g \in \mathcal{G}\}$$

for each ordinal $\zeta \in \mathbb{O}\mathrm{rd}$.

(b) For each function $g \in \mathbb{O}\mathrm{rd}^{\ddagger}$ the ordinal iterates g^{α} of g are generated by

$$g^0 = \boldsymbol{id} \qquad g^{\alpha+1} = g \circ g^{\alpha} \qquad g^{\lambda} = \bigvee\{g^{\alpha} \mid \alpha < \lambda\}$$

for each ordinal α and limit ordinal λ. (Here $\boldsymbol{id}$ is the identity function on $\mathbb{O}\mathrm{rd}$.) ■

A pointwise supremum $\bigvee \mathcal{G}$ need not be supremum simply because the whole family of ordinal functions is not partially ordered. We restrict to a subclass of ordinal functions.

2.2 DEFINITION. An ordinal function $g \in \mathbb{O}\mathrm{rd}^{\ddagger}$ is

inflationary if $\alpha \leq g\alpha$

strictly inflationary if $\alpha < g\alpha$

monotone if $\beta \leq \alpha \Longrightarrow g\beta \leq g\alpha$

strictly monotone if $\beta < \alpha \Longrightarrow g\beta < g\alpha$

for all ordinals $\alpha, \beta \in \mathbb{O}\mathrm{rd}$.

Let $\mathbb{IM}$ be the class of functions which are both inflationary and monotone. ■

Almost every ordinal function that we meet will be a member of $\mathbb{IM}$. This class $\mathbb{IM}$ is partially ordered by the pointwise comparison. Thus

$$f \le g \iff (\forall \alpha : \mathbb{O}\mathrm{rd})[f\alpha \le g\alpha]$$

for $f, g \in \mathbb{IM}$. With this, for each non-empty countable subset $\mathcal{G} \subseteq \mathbb{IM}$ the pointwise supremum $\bigvee \mathcal{G}$ is the actual supremum. We wish to iterate certain functions and stay within a nominated class. We say a class $\mathbb{S}$ of ordinal functions is smooth if it satisfies

$$f, g \in \mathbb{S} \Longrightarrow f \circ g \in \mathbb{S} \qquad \mathcal{G} \subseteq \mathbb{S} \Longrightarrow \bigvee \mathcal{G} \in \mathbb{S}$$

where, in the second clause, $\mathcal{G}$ is non-empty and countable.

The class $\mathbb{IM}$ is smooth. For a smooth class $\mathbb{S}$ and member $g \in \mathbb{S}$, the ordinal iterates

$$g \le g^2 \le \cdots \le g^\alpha \le \cdots$$

form an ascending chain through $\mathbb{S}$. (For technical reasons we omit the identity function $\boldsymbol{id} = g^0$.)

In a smooth class the arithmetic of ordinal iterates behaves as it should.

2.3 LEMMA. *Let* $g \in \mathbb{IM}$. *Then*

$$g^\alpha \circ g^\beta = g^{\beta+\alpha} \qquad (g^\beta)^\alpha = g^{\beta\cdot\alpha}$$

for all $\alpha, \beta \in \mathbb{O}\mathrm{rd}$.

To obtain fixed points of a function we use a topological property.

2.4 DEFINITION. A ordinal function $g \in \mathbb{O}\mathrm{rd}^\ddagger$ is continuous if

$$g(\bigvee A) = \bigvee g[A]$$

for each non-empty, countable set A of ordinals. ■

Here $g[A]$ is the direct image of the set A across g. Both suprema are taken in $\mathbb{O}\mathrm{rd}$.

If g is continuous then

$$g\lambda = \bigvee \{g\alpha \mid \alpha < \lambda\}$$

for each limit ordinal λ. This does not ensure continuity unless g is monotone. This is one reason for staying within $\mathbb{IM}$.

2.5 DEFINITION. (f) An ordinal function $f \in \mathbb{O}\mathrm{rd}^\ddagger$ is fruitful if it inflationary, monotone, continuous, and satisfies

$$\omega^\alpha \le f\alpha$$

for each ordinal α. Let $\mathbb{F}$ be the class of fruitful functions.

(h) A function $h \in \mathbb{O}\mathrm{rd}^\ddagger$ is helpful (on level 1) if it is strictly inflationary, monotone, and returns only critical values. Let $\mathbb{H}$ be the class of such functions. ■

We have $\mathbb{F}, \mathbb{H} \subseteq \mathbb{IM}$, and the proof of the following is more or less trivial.

2.6 LEMMA. *Each of the classes* $\mathbb{F}$ *and* $\mathbb{H}$ *is smooth.*

Recall that an ordinal function f is normal if it is strictly monotone and continuous. Here it is convenient to assume also that $\omega^\alpha \le f\alpha$ for each ordinal α. Thus each normal function (in our sense) is fruitful. Why do we use this larger class?

2.7 EXAMPLE. Let $g \in \mathbb{IM}$, and suppose λ is an additively critical limit ordinal. Then

$$g^\lambda(g^\alpha\zeta) = (g^\lambda \circ g^\alpha)\zeta = g^{\alpha+\lambda}\zeta = g^\lambda\zeta$$

for each ordinal ζ and ordinal $\alpha < \lambda$, and hence g^λ is constant between ζ and $g^\alpha\zeta$. The situation is even more dramatic when g is continuous. In this case we find that $g^\lambda = g^\mu$ for some ordinal μ which is considerably larger than λ. ■

Consider a family $\mathcal{F}$ of normal functions. We will see that the common fixed points of this family are precisely the fixed points of the single function $\bigvee \mathcal{F}$. This function $\bigvee \mathcal{F}$ is certainly fruitful, but Example 2.7 shows that it may not be normal. Pointwise suprema, in particular ordinal iterates, are a fundamental tool in our analysis. If we stick with normal functions then various stutters occur around iterates at limit ordinals. The use of fruitful functions gets rid of the hiccoughs.

A fruitful function f has lots of fixed points. Each such fixed point ν satisfies

$$\nu \le \omega^\nu \le f\nu = \nu$$

and hence is critical. We wish to harvest these fixed points.

2.8 DEFINITION. (fh) For each $f \in \mathbb{F}$ and $\zeta \in \mathbb{O}\mathrm{rd}$ let

$$\boldsymbol{Fix}\, f\zeta = f^\omega(\zeta+1) = \bigvee\{f^r(\zeta+1) \mid r < \omega\}$$

to produce an ordinal function $\boldsymbol{Fix}\, f$.

(hf) For each $h \in \mathbb{H}$ and $\alpha \in \mathbb{O}\mathrm{rd}$ let

$$\boldsymbol{Enm}\, h\alpha = h^{1+\alpha}0$$

to produce an ordinal function $\boldsymbol{Enm}\, h$. ■

Using the continuity of $f \in \mathbb{F}$ it is easy to check that

$$\boldsymbol{Fix}\, f\zeta = (\text{least } \nu \text{ with } \zeta < \nu = f\nu)$$

for each input ordinal ζ. In other words $\boldsymbol{Fix}$ is a fixed point extractor; it produces the next fixed point of a fruitful function f beyond a starting point ζ.

2.9 LEMMA. *(fh) For each fruitful function f the function $\boldsymbol{Fix}\, f$ is helpful.*

(hf) For each helpful function h the function $\boldsymbol{Enm} h$ is normal.

Proof. (fh) Let $h = \boldsymbol{Fix}\, f$ where $f \in \mathbb{F}$. For a $\zeta \in \mathbb{O}\mathrm{rd}$ let $\nu = h\zeta$ so that, by the remarks above, we have $\zeta < \nu = f\nu$ with a certain minimality on ν. In particular, this shows that h is strictly inflationary.

More generally, we have $\omega^\nu \le f\nu = \nu$, so that ν is critical.

Consider any $\zeta \leq \eta$. Let $\nu = h\zeta$ and $\mu = h\eta$. Then

$$\zeta \leq \eta < \mu = f\mu$$

and hence $\nu \leq \mu$ by the minimality of ν. This shows that h is monotone.

(hf) Let $f = \boldsymbol{Enm}h$ where $h \in \mathbb{H}$.

By construction for each ordinal α we have

$$f\alpha < h(f\alpha) = f(\alpha+1)$$

since h is strictly inflationary. Also by construction we have

$$f\lambda = \bigvee\{f\alpha \,|\, \alpha < \lambda\}$$

for each limit ordinal λ. But now, if $\alpha < \lambda$ then $\alpha < \alpha+1 < \lambda$ so that $f\alpha < f(\alpha+1) \leq f\lambda$ which is enough to show that f is strictly monotone.

Almost trivially, f is continuous.

For each ordinal α the output $f\alpha$ is either a value of h or a supremum of such values. Thus $f\alpha$ is critical. Hence

$$\omega^{\alpha} \leq \omega^{f\alpha} = f\alpha$$

as required. ■

This result shows that

$$\boldsymbol{Fix} : \mathbb{F} \longrightarrow \mathbb{H} \qquad \boldsymbol{Enm} : \mathbb{H} \longrightarrow \mathbb{F}$$

to enable us to pass between the two classes. The two composites are important gadgets.

2.10 DEFINITION. Let

$$\boldsymbol{Veb} = \boldsymbol{Enm} \circ \boldsymbol{Fix} \qquad [\,0\,] = \boldsymbol{Fix} \circ \boldsymbol{Enm}$$

to produce operators of types

$$\mathbb{F}^{\ddagger} = (\mathbb{F} \longrightarrow \mathbb{F}) \qquad \mathbb{H}^{\ddagger} = (\mathbb{H} \longrightarrow \mathbb{H})$$

respectively. ■

This operator $\boldsymbol{Veb}$ is the Veblen operator of Section 1. To see this observe that

$$\boldsymbol{Veb}f\alpha = \big(\boldsymbol{Fix}f\big)^{1+\alpha}0$$

for each $f \in \mathbb{F}$ and $\alpha \in \mathbb{O}\mathrm{rd}$. Thus $\boldsymbol{Veb}f$ enumerates the fixed points of f. By Lemma 2.9 this function $\boldsymbol{Veb}f$ is normal, and a simple calculation gives $f \leq \boldsymbol{Veb}f$.

As in (1.b) of Section 1 the operator $\boldsymbol{Veb}$ can be iterated through the ordinals. This produces an ascending chain of fruitful functions

$$f \leq \boldsymbol{Veb}f \leq \cdots \boldsymbol{Veb}^{\alpha}f \leq \cdots$$

most of which are normal. To prove (1.c) we need some information about $\boldsymbol{Fix}$.

Recall that a family $\mathcal{F} \subseteq \mathbb{F}$ is directed if it is non-empty, countable, and for each $f, g \in \mathcal{F}$ there is some $h \in \mathcal{F}$ with $f, g \leq h$.

2.11 LEMMA. *The operator* $\boldsymbol{Fix}$ *is monotone and Scott-continuous. In other words*

$$g \le f \Longrightarrow \boldsymbol{Fix}\, g \le \boldsymbol{Fix} f \qquad \boldsymbol{Fix}(\bigvee \mathcal{F}) = \bigvee \boldsymbol{Fix}[\mathcal{F}]$$

for all fruitful functions f, g *and all directed sets* $\mathcal{F}$ *of fruitful functions.*

Proof. Monotonicity is immediate, but the details of continuity are worth looking at.

Suppose first that $\mathcal{G}$ and $\mathcal{H}$ are families of functions with each $h \in \mathcal{H}$ continuous. Then we see that

$$\big(\bigvee \mathcal{G}\big) \circ \big(\bigvee \mathcal{H}\big) = \bigvee\{g \circ h \mid g \in \mathcal{G}, h \in \mathcal{H}\}$$

by evaluating the left hand side at an arbitrary ordinal ζ. Repeated use of this gives

$$\big(\bigvee \mathcal{F}\big)^r = \bigvee\{f_1 \circ \cdots \circ f_r \mid f_1, \dots, f_r \in \mathcal{F}\}$$

for each (non-zero) $r < \omega$. But $\mathcal{F}$ is directed so $f_1, \dots, f_r \le f$ for some $f \in \mathcal{F}$, and hence

$$\big(\bigvee \mathcal{F}\big)^r = \bigvee\{f^r \mid f \in \mathcal{F}\}$$

holds. Thus, for each ordinal $\zeta \in \mathbb{O}\mathrm{rd}$

$$(\bigvee \mathcal{F})^\omega \zeta = \bigvee\{(\bigvee \mathcal{F})^r \zeta \mid r < \omega\} = \bigvee\{f^r \zeta \mid f \in \mathcal{F}, r < \omega\} = \bigvee\{f^\omega \zeta \mid f \in \mathcal{F}\}$$

to give

$$\boldsymbol{Fix}(\bigvee \mathcal{F})\zeta = (\bigvee \mathcal{F})^\omega(\zeta + 1) = \bigvee\{f^\omega(\zeta + 1) \mid f \in \mathcal{F}\} = \bigvee\{\boldsymbol{Fix} f \zeta \mid f \in \mathcal{F}\}$$

as required. ■

This result has a useful consequence (which, in fact, could have been proved directly).

2.12 COROLLARY. *Let* $\mathcal{F}$ *be a directed family of fruitful functions. Then the common fixed points of the members of* $\mathcal{F}$ *are precisely the fixed points of the fruitful function* $\bigvee \mathcal{F}$.

With this (1.c) follows by induction on α. Only the induction leap to a limit ordinal λ is not immediate. The function $\phi[f]\lambda$ enumerates the common fixed points of the family of functions

$$\{\phi[f](1 + \alpha) \mid \alpha < \lambda\} = \{\, \boldsymbol{Veb}^{\alpha+1} f \mid \alpha < \lambda\}$$

already generated. By Corollary 2.12 these are the fixed points of the function

$$\boldsymbol{Veb}^{\lambda} f = \bigvee\{\, \boldsymbol{Veb}^{\alpha+1} f \mid \alpha < \lambda\}$$

and hence

$$\phi[f](1 + \lambda) = \phi[f]\lambda = \boldsymbol{Veb}(\,\boldsymbol{Veb}^{\lambda} f) = \boldsymbol{Veb}^{\lambda+1} f$$

as required.

By stripping off the outside use of $\boldsymbol{Veb}$ we have

$$\phi[f](1 + \alpha)\zeta = \Big(\big(\boldsymbol{Fix} \circ \boldsymbol{Veb}^{\alpha}\big) f\Big)^{1+\zeta} 0 \tag{2.a}$$

for each ordinal ζ. We use the operation $[\,0\,]$ of Definition 2.10 to reorganize this compound. We have

$$\boldsymbol{Fix} \circ \boldsymbol{Veb} = \boldsymbol{Fix} \circ \boldsymbol{Enm} \circ \boldsymbol{Fix} = [\,0\,] \circ \boldsymbol{Fix} \tag{2.b}$$

by the definition of $\boldsymbol{Veb}$ and $[\,0\,]$. The operator $[\,0\,]$ can be iterated through the ordinals, and a simple induction gives

$$\boldsymbol{Fix} \circ \boldsymbol{Veb}^{\alpha} = [\,0\,]^{\alpha} \circ \boldsymbol{Fix}$$

for each ordinal α. This leads to a refinement of (1.c, 2.a).

2.13 THEOREM. *For each $f \in \mathbb{F}$ with $h = \boldsymbol{Fix} f \in \mathbb{H}$ we have*

$$\phi[f](1+\alpha)\zeta = \big([\,0\,]^{\alpha} h\big)^{1+\zeta} 0$$

for each pair α, ζ of ordinals.

This shows that we may view the generation of the hierarchy $\phi[f]$ as an iteration

$$\alpha \longmapsto [\,0\,]^{\alpha} h$$

of the operator $[\,0\,]$. We will see that with this version it is much easier to produce extension of the hierarchy.

In the next section we analyse a family of operators of which $[\,0\,]$ is a kind of generic example. To prepare for this we need some information about helpful functions.

The following result is Lemma 3.12 of [6].

2.14 LEMMA. *Suppose $h \in \mathbb{H}$. Then*

(a) $\zeta + \alpha \le h^{\alpha}\zeta$

(b) $h^{\lambda}\zeta = h^{\lambda}0$

(c) $(\zeta < \nu = h^{\nu}0) \Longleftrightarrow (0 < \nu = h^{\nu}\zeta)$

for all ordinals α, ν, ζ, and (additively) critical ordinal $\lambda > \zeta$.

Proof. (a) We prove this by induction on α.

The base case, $\alpha = 0$, is trivial.

For the induction step, $\alpha \mapsto \alpha + 1$, since h is strictly inflationary we have

$$h^{\alpha+1}\zeta = h(h^{\alpha}\zeta) \ge h^{\alpha}\zeta + 1 \ge \zeta + \alpha + 1$$

using the induction hypothesis.

For the induction leap to a limit ordinal λ we have

$$h^{\lambda}\zeta = \bigvee\{h^{\alpha}\zeta \mid \alpha < \lambda\} \ge \bigvee\{\zeta + \alpha \mid \alpha < \lambda\} = \zeta + \bigvee\{\alpha \mid \alpha < \lambda\} = \zeta + \lambda$$

as required.

(b) The iterate h^{λ} is helpful, and hence monotone, so that $h^{\lambda}\zeta \ge h^{\lambda}0$. For the converse we have $\zeta \le h^{\zeta}0$ (by part (a)) and hence

$$h^{\lambda}\zeta \le h^{\lambda}(h^{\zeta}0) = h^{\zeta+\lambda}0$$

by Lemma 2.3. But $\zeta < \lambda$ and λ is (additively) critical so that $\zeta + \lambda = \lambda$, to give the required result.

(c) Let

$$\mu = h^{\nu}\zeta$$

for any pair ν, ζ of ordinals with $\nu \neq 0$. If $\nu = \alpha + 1$ then

$$\mu = h(h^{\alpha}\zeta)$$

so that μ is critical since it is a value of the helpful function h. If ν is a limit ordinal, then μ is a supremum of values of h, and so is again critical. In other words, μ is critical for all $\nu > 0$. Note also that $\zeta < \mu$, so that two uses of part (b) gives

$$\mu = h^{\nu}\eta$$

for each $\eta < \nu$. Further uses of part (b) give the two required implications. ■

This will give us a more detailed description of the behaviour of $[\,\mathrm{o}\,]$.

3 Helpful operators

The notion of helpfulness can be lifted through all levels of functions over $\mathbb{O}\mathrm{rd}$. In this section we investigate the helpful operators (functions on level 2), and catch a glimpse of a helpful constructor (function on level 3).

3.1 DEFINITION. A function $H \in \mathbb{H}^{\ddagger}$ is helpful (on level 2) if

$$h^2 \leq Hh \qquad Hf \leq Hg$$

for all $f, g, h \in \mathbb{H}$ with $f \leq g$. Let $\mathfrak{Help}$ be the class of such functions. ■

We have already seen one helpful operator.

3.2 THEOREM. *For each $h \in \mathbb{H}$ we have*

$$[\,\mathrm{o}\,]h\zeta = (\textit{least } \nu \textit{ with } \zeta < \nu = h^{\nu}0) = (\textit{least } \nu \textit{ with } 0 < \nu = h^{\nu}\zeta)$$

for each input ordinal ζ. In particular, $[\,\mathrm{o}\,] \in \mathfrak{Help}$.

Proof. We have

$$\begin{aligned} [\,\mathrm{o}\,]h\zeta &= \boldsymbol{Fix}\big(\boldsymbol{Enm}\, h)\zeta \\ &= (\text{least } \nu \text{ with } \zeta < \nu = \boldsymbol{Enm}\, h\nu) \\ &= (\text{least } \nu \text{ with } \zeta < \nu = h^{1+\nu}0) \end{aligned}$$

by the definitions of $\boldsymbol{Fix}$ and $\boldsymbol{Enm}$. The function h outputs only critical ordinals, so

$$\nu = h^{1+\nu}0 \Longrightarrow \nu = h^{\nu}0$$

since this ν must be infinite. With this Lemma 2.14(c) gives the first required result.

It remains to show that

$$h^2 \leq [\,0\,]h \qquad [\,0\,]f \leq [\,0\,]g$$

for all $f, g, h \in \mathbb{H}$ with $f \leq g$.

Such a function $h \in \mathbb{H}$ is strictly inflationary, and hence $h^2 \leq h^\nu$ for each infinite ordinal ν. Thus, for an arbitrary input ζ we have

$$[\,0\,]h\zeta = \nu = h^\nu\zeta \geq h^2\zeta$$

for some ν, to give the left hand comparison.

Consider $f, g \in \mathbb{H}$ with $f \leq g$. For an arbitrary input ζ let

$$\zeta < \nu = [\,0\,]f\zeta = f^\nu 0 \qquad \zeta < \mu = [\,0\,]g\zeta = g^\mu 0$$

where ν has a certain minimality. The comparison $f \leq g$ gives $f^\mu \leq g^\mu$, and hence

$$\mu \leq f^\mu\zeta \leq g^\mu\zeta = \mu$$

by Lemma 2.14(a). The minimality of ν gives $\nu \leq \mu$, for the right hand comparison. ■

As with $\mathbb{F}$ and $\mathbb{H}$ we partially order $\mathfrak{Help}$ using the pointwise comparison. Thus

$$H \leq K \iff (\forall h \in \mathbb{H})[Hh \leq Kh] \iff (\forall h \in \mathbb{H}, \zeta \in \mathbb{O}\mathrm{rd})[Hh\zeta \leq Kh\zeta]$$

for $H, K \in \mathfrak{Help}$. This is a lifting of the pointwise comparison on $\mathbb{H}$ which in turn is a lifting of the actual comparison on $\mathbb{O}$rd. For each non-empty, countable $\mathcal{H} \subseteq \mathfrak{Help}$, the pointwise supremum

$$\bigvee \mathcal{H} : \mathbb{H} \longrightarrow \mathbb{H}$$

is given by

$$(\bigvee \mathcal{H})h = \bigvee\{Hh \mid H \in \mathcal{H}\}$$

for each $h \in \mathbb{H}$, that is

$$(\bigvee \mathcal{H})h\zeta = \bigvee\{Hh\zeta \mid H \in \mathcal{H}\}$$

for each $\zeta \in \mathbb{O}$rd. This pointwise supremum is the actual supremum of $\mathcal{H}$ in $\mathfrak{Help}$.

3.3 LEMMA. *The class $\mathfrak{Help}$ is smooth. In other words $\mathfrak{Help}$ is closed under composition and pointwise suprema of non-empty countable $\mathcal{H} \subseteq \mathfrak{Help}$.*

Proof. We show first that $\mathfrak{Help}$ is closed under composition. Consider any $G, H \in \mathfrak{Help}$. We require

$$h^2 \leq G(Hh) \qquad G(Hf) \leq G(Hg)$$

for $f, g, h \in \mathbb{H}$ with $f \leq g$. But G, Hh, and H are helpful (on the appropriate level) so

$$G(Hh) \geq (Hh)^2 \geq Hh \geq h^2$$

to give the first comparison. Also if $f \leq g$ then $Hf \leq Hg$ and $G(Hf) \leq G(Hg)$ to give the second comparison.

To show that $\mathfrak{Help}$ is closed under pointwise suprema, consider a non-empty countable subset $\mathcal{H}$ of $\mathbb{H}$. For each $h \in \mathbb{H}$ we have

$$(\bigvee \mathcal{H})h = \bigvee\{Hh \mid H \in \mathcal{H}\} \in \mathbb{H}$$

since $\mathbb{H}$ is smooth. Thus $\bigvee \mathcal{H} \in \mathbb{H}^{\ddagger}$. Also $(\bigvee \mathcal{H})h \geq Hh \geq h^2$ by selecting any member H of $\mathcal{H}$. The implication

$$f \leq g \Longrightarrow (\bigvee \mathcal{H})f \leq (\bigvee \mathcal{H})g$$

follows in the same way. Thus $\bigvee \mathcal{H}$ is helpful. ■

A few moment's thought gives

$$G, H \leq G \circ H$$

for all $G, H \in \mathfrak{Help}$. This enables us to produce more and more powerful helpful operators.

In the standard way each $H \in \mathfrak{Help}$ can be iterated through the ordinals. Thus we set

$$H^0 h\zeta = h\zeta \qquad H^{\alpha+1}h\zeta = H(H^{\alpha}h)\zeta \qquad H^{\lambda}h\zeta = \bigvee\{H^{\alpha}h\zeta \mid \alpha < \lambda\}$$

for $h \in \mathbb{H}$, ordinals α and ζ, and limit ordinals λ. This gives an ascending chain

$$H \leq H^2 \leq \cdots \leq H^{\alpha} \leq \cdots$$

through $\mathfrak{Help}$. (The zero iterate H^0 is the identity operator on $\mathbb{H}$.)

The following result should be compared with Lemma 2.14.

3.4 LEMMA. *Suppose* $H \in \mathfrak{Help}$ *and* $h \in \mathbb{H}$. *Then*

(a) $h^{2^{\alpha}} \leq H^{\alpha}h$

(b) $H^{\alpha}h\alpha \leq H^{\alpha+1}h0$

(c) $H^{\lambda}h\zeta = H^{\lambda}h0$

(d) $(\zeta < \nu = H^{\nu}h0) \Longleftrightarrow (0 < \nu = H^{\nu}h\zeta)$

for all ordinals α, ν, ζ *and limit ordinal* λ *with* $\zeta < \lambda$.

Proof. (a) We proceed by induction on α.

Since $2^0 = 1$, the base case, $\alpha = 0$, is immediate.

For the induction step, $\alpha \mapsto \alpha + 1$, we have

$$h^{2^{\alpha+1}} = h^{2^{\alpha}} \circ h^{2^{\alpha}} = (h^{2^{\alpha}})^2 \leq (H^{\alpha}h)^2 \leq H(H^{\alpha}h) = H^{\alpha+1}h$$

as required. At the penultimate step we remember that $H \in \mathfrak{Help}$ and $H^{\alpha}h \in \mathbb{H}$.

For the induction leap to a limit ordinal λ we have

$$h^{2^{\lambda}} = \bigvee\{h^{\beta} \mid \beta < \lambda\} = \bigvee\{h^{2^{\alpha}} \mid \alpha < \lambda\} \leq \bigvee\{H^{\alpha}h \mid \alpha < \lambda\} = H^{\lambda}h$$

as required.

(b) A simple argument shows that

$$\alpha \leq H^{\alpha}h0$$

and hence

$$H^{\alpha}h\alpha \leq (H^{\alpha}h)^2 0 \leq H(H^{\alpha}h)0 = H^{\alpha+1}h0$$

as required.

(c) The comparison

$$H^{\lambda}h0 \leq H^{\lambda}h\zeta$$

is immediate. For the converse consider any ordinal α with $\zeta < \alpha < \lambda$. Part (b) gives

$$H^{\alpha}h\zeta \leq H^{\alpha}h\alpha \leq H^{\alpha+1}h0 \leq H^{\lambda}h0$$

and hence taking the supremum over all such α gives the required result.

(d) Suppose

$$\nu = H^{\nu}h\eta$$

for some ordinal η. Since ν is a value of the helpful function $H^{\nu}h$, it is critical, and hence a limit ordinal. Two uses of part (c) now gives the required result. ■

Iterating a helpful function gives a normal function. Iterating a helpful operator gives many normal functions. The following result is a level 2 analogue of Lemma 2.9(hf).

3.5 LEMMA. *For each* $H \in \mathfrak{Help}, h \in \mathbb{H}$ *and* $\zeta \in \mathbb{O}\mathrm{rd}$, *the function* $f \in \mathbb{O}\mathrm{rd}^{\ddagger}$ *given by*

$$f\alpha = H^{\alpha}h\zeta$$

(for $\alpha \in \mathbb{O}\mathrm{rd}$*) is normal.*

Proof. We have

$$h\zeta < h^2\zeta \leq Hh\zeta$$

and hence

$$f\alpha = H^{\alpha}h\zeta < H(H^{\alpha}h\zeta) = H^{\alpha+1}h\zeta = f(\alpha+1)$$

using $H^{\alpha}h$ in place of h. By construction we have

$$f\lambda = \bigvee\{f\alpha \mid \alpha < \lambda\}$$

and hence f is strictly monotone.

A similar observation shows that f is continuous.

The function $H^{\alpha}h$ is helpful and $f\alpha$ is critical. Thus

$$\omega^{\alpha} \leq \omega^{f\alpha} = f\alpha$$

as required. ■

One of the minor problems with the standard approach to the Veblen hierarchy $\phi[f]$ is to show that an appropriate diagonal limit through the hierarchy does produce a new normal function. The helpful approach makes this easy. We have

$$^{*}f\alpha = \phi[f](1+\alpha)0 = [\,\circ\,]^{\alpha}h0$$

where $h = \boldsymbol{Fix}f$. Theorem 3.2 and Lemma 3.5 ensure that $^{*}f$ is normal.

The following construction is a level 2 analogue of that of Definition 2.10 (but without a corresponding notation in place of $\boldsymbol{Enm}$).

3.6 DEFINITION. For each $H \in \mathfrak{Help}$ and $h \in \mathbb{H}$ we set

$$[{\scriptstyle 1}]Hh = \boldsymbol{Fix} f \quad \text{where } f\alpha = H^{\alpha}h0$$

(for $\alpha \in \mathbb{O}\mathrm{rd}$) to produce a constructor $[{\scriptstyle 1}]$ on level 3. ■

By Lemma 3.5 the auxiliary function f is normal, and hence $\boldsymbol{Fix} f \in \mathbb{H}$ by Lemma 2.9(fh). This shows that

$$[{\scriptstyle 1}]H : \mathbb{H} \longrightarrow \mathbb{H}$$

which we can improve. The following result is the analogue of Theorem 3.2.

3.7 THEOREM. *For each $H \in \mathfrak{Help}$ and $h \in \mathbb{H}$ we have*

$$[{\scriptstyle 1}]Hh\zeta = (\textit{least } \nu \textit{ with } \zeta < \nu = H^{\nu}h0) = (\textit{least } \nu \textit{ with } 0 < \nu = H^{\nu}h\zeta)$$

for each input ordinal ζ. In particular, $H^{\epsilon_0} \le [{\scriptstyle 1}]H$ and $[{\scriptstyle 1}]H \in \mathfrak{Help}$.

Proof. By definition of $\boldsymbol{Fix}$ we have

$$[{\scriptstyle 1}]Hh\zeta = (\text{least } \nu \text{ with } \zeta < \nu = H^{\nu}h0)$$

for each input ordinal ζ. Thus Lemma 3.4(d) gives the required first part.

Consider any value

$$\nu = [{\scriptstyle 1}]Hh\zeta$$

of $[{\scriptstyle 1}]Hh$. Then

$$\nu = H^{\nu}h\zeta$$

and hence ν is critical. In particular, ν is non-zero, so that

$$Hh\zeta \le H^{\nu}h\zeta = [{\scriptstyle 1}]Hh\zeta$$

to give

$$H^{\epsilon_0}h \le [{\scriptstyle 1}]Hh$$

and hence

$$H^{\epsilon_0} \le [{\scriptstyle 1}]H$$

as required.

Using this we have

$$h^2 \le Hh \le H^{\epsilon_0}h \le [{\scriptstyle 1}]Hh$$

so it remains to show that

$$[{\scriptstyle 1}]Hf \le [{\scriptstyle 1}]hg$$

for $f, g \in \mathbb{H}$ with $f \le g$.

For such f and g and an arbitrary input ζ let

$$\zeta < \nu = [{\scriptstyle 1}]Hf\zeta = H^{\nu}f\zeta \qquad \zeta < \mu = [{\scriptstyle 1}]Hg\zeta = H^{\mu}g\zeta$$

and remember that ν has a certain minimality. The comparison $f \le g$ gives

$$H^{\mu} \le H^{\mu}g$$

so that

$$\mu \leq H^{\mu} f \zeta \leq H^{\mu} g \mu = \mu$$

(by a use of Lemma 3.4). The minimality of ν gives $\nu \leq \mu$ as required. ■

In most cases the comparison $H^{\epsilon_0} \leq [\,1\,]H$ can be improved to at least $H^{\Gamma_0} \leq [\,1\,]H$.

So far the only helpful operators that we have are $[\,0\,]$ and its iterates. The class $\mathfrak{Help}$ is smooth, and hence closed under iteration. Theorem 3.7 gives another method of producing members of $\mathfrak{Help}$, and this too can be iterated.

3.8 DEFINITION. For each $H \in \mathfrak{Help}$ we set

$$H_i = [\,1\,]^i H$$

for each ordinal i. In other words, we set

$$H_0 = H \qquad H_{i+1} = [\,1\,]H_i \qquad H_\theta = \bigvee\{H_i \mid i < \theta\}$$

for each ordinal i and limit ordinal θ. ■

This construction generates an ascending chain

$$H = H_0 \leq H_1 \leq \cdots \leq H_i \leq \cdots$$

through $\mathfrak{Help}$ which increases much faster than that generated by mere iteration of H. There will, of course, be ordinals δ with $H^\delta = H_\delta$, but that's another story.

3.9 THEOREM. *For each ordinal i the operator* $[\,1\,]^i[\,0\,]$ *is helpful.*

Each pair $h \in \mathbb{H}, \nabla \in \mathfrak{Help}$ gives us hierarchies of helpful and normal functions

$$\alpha \longmapsto \nabla^\alpha h \qquad \alpha \longmapsto \boldsymbol{Enm}(\nabla^\alpha h)$$

respectively. What has this got to do with the Veblen hierarchies?

For an arbitrary $f \in \mathbb{F}$ we can take $h = \boldsymbol{Fix} f$ for the base helpful function. For the helpful operators we start with $[\,0\,]$ and close off under composition, iteration, and use of $[\,1\,]$ (as in Definition 3.8). Let $\mathfrak{Veb}$ be the resulting family of helpful operators. It is not too hard to devise a method of indexing $\mathfrak{Veb}$, as in (1.j). We will see that Veblen's idea and the Schütte brackets pick out paths through the hierarchies associated with $\mathfrak{Veb}$.

Incidentally $\mathfrak{Veb}$ is only a small part of $\mathfrak{Help}$. There are other higher level functions $[\,2\,], [\,3\,], [\,4\,], \ldots$ which can be used to generate members of $\mathfrak{Help}$ and hierarchies that go far beyond the Veblen hierarchies, and produce ordinal notations up to the Howard ordinal. There is still much work to be done, but [6] and [5] make a start.

4 Derivatives

In this section we look at the idea of a derivative $\boldsymbol{D}$, the associated Veblen-like hierarchies (1.e, 1.f), and the notion of jumping a derivative.

4.1 DEFINITION. A derivative is an operator

$$\boldsymbol{D} : \mathbb{F} \longrightarrow \mathbb{F}$$

which is inflationary and monotone, that is

$$f \le \boldsymbol{D}f \qquad f \le g \Longrightarrow \boldsymbol{D}f \le \boldsymbol{D}g$$

for all $f, g \in \mathbb{F}$. Let $\mathfrak{Derv}$ be the class of derivatives. ■

For the record we have the following.

4.2 THEOREM. *The operator* $\boldsymbol{Veb}$ *is a derivative.*

Definition 4.1 gives the minimal properties a derivative should have. In practice the useful derivatives will have further properties.

As with $\mathbb{F}, \mathbb{H}$ and $\mathfrak{Help}$ we partially order $\mathfrak{Derv}$ using the pointwise comparison. Thus

$$\boldsymbol{D} \le \boldsymbol{E} \Longleftrightarrow (\forall f \in \mathbb{F})[\boldsymbol{D}f \le \boldsymbol{E}f] \Longleftrightarrow (\forall f \in \mathbb{F}, \zeta \in \mathbb{O}\mathrm{rd})[\boldsymbol{D}f\zeta \le \boldsymbol{E}f\zeta]$$

for $\boldsymbol{D}, \boldsymbol{E} \in \mathfrak{Derv}$. This is a lifting of the pointwise comparison on $\mathbb{F}$. For each non-empty, countable $\mathcal{D} \subseteq \mathfrak{Derv}$, the pointwise supremum

$$\bigvee \mathcal{D} : \mathbb{F} \longrightarrow \mathbb{F}$$

is given by

$$(\bigvee \mathcal{D})h = \bigvee\{\boldsymbol{D}f \mid \boldsymbol{D} \in \mathcal{D}\}$$

for each $F \in \mathbb{F}$, that is

$$(\bigvee \mathcal{D})f\zeta = \bigvee\{\boldsymbol{D}f\zeta \mid \boldsymbol{D} \in \mathbb{F}\}$$

for each $\zeta \in \mathbb{O}\mathrm{rd}$. This pointwise supremum is the actual supremum of $\mathcal{D}$ in $\mathfrak{Derv}$.

4.3 LEMMA. *The class* $\mathfrak{Derv}$ *is smooth. In other words* $\mathfrak{Derv}$ *is closed under composition and pointwise suprema of non-empty countable* $\mathcal{D} \subseteq \mathfrak{Derv}$.

For each family $\mathcal{D}$ of derivatives the pointwise supremum $\bigvee \mathcal{D}$ is a derivative. We use this to generate many derivatives. The following refinement of Lemma 2.11 will be useful.

4.4 LEMMA. *For each directed family* $\mathcal{D}$ *of derivatives*

$$\boldsymbol{Fix} \circ \bigvee \mathcal{D} = \bigvee\{\boldsymbol{Fix} \circ \boldsymbol{D} \mid \boldsymbol{D} \in \mathcal{D}\}$$

holds

Proof. Consider a fruitful function f. We have

$$\big(\bigvee \mathcal{D}\big)f = \bigvee\{\boldsymbol{D}f \mid \boldsymbol{D} \in \mathcal{D}\}$$

and the family $\{\boldsymbol{D}f \mid \boldsymbol{D} \in \mathcal{D}\}$ of functions is directed. Thus, using Lemma 2.11, we have

$$\begin{aligned}
\big(\boldsymbol{Fix} \circ \bigvee \mathcal{D}\big)f &= \boldsymbol{Fix}(\bigvee\{\boldsymbol{D}f \mid \boldsymbol{D} \in \mathcal{D}\}) \\
&= \bigvee\{\boldsymbol{Fix}(\boldsymbol{D}f) \mid \boldsymbol{D} \in \mathcal{D}\} \\
&= \bigvee\{(\boldsymbol{Fix} \circ \boldsymbol{D})f \mid \boldsymbol{D} \in \mathcal{D}\} = \big(\bigvee\{\boldsymbol{Fix} \circ \boldsymbol{D} \mid \boldsymbol{D} \in \mathcal{D}\}\big)f
\end{aligned}$$

as required. ■

As in (1.d) each derivative $\boldsymbol{D}$ can be iterated through the ordinals. For any $f \in \mathbb{F}$ this produces a hierarchy

$$f \le \boldsymbol{D}f \le \cdots \le \boldsymbol{D}^{\alpha} f \le \cdots$$

of fruitful functions. When $\boldsymbol{D} = \boldsymbol{Id}_{\mathbb{F}}$ there is nothing much happening here. To avoid such pathologies we usually assume that a derivative $\boldsymbol{D}$ is big, that is $\boldsymbol{Veb} \le \boldsymbol{D}$.

As in (1.e) each derivative $\boldsymbol{D}$ gives us a Veblen-like hierarchy

$$\Phi[\boldsymbol{D}, f]$$

on any base function $f \in \mathbb{F}$. This idea needs a derivative to get it going, and one that is not just a variant of $\boldsymbol{Veb}$. This is where helpful operators live up to their name.

4.5 DEFINITION. For each helpful operator ∇ the three operators

$$\flat\nabla \qquad \natural\nabla \qquad \sharp\nabla$$

are given by

$$\begin{array}{ll} \text{(Sharp)} & \sharp\nabla f\alpha = \nabla^{1+\alpha} h0 \\ \text{(Natural)} & \natural\nabla f\alpha = \nabla^{\alpha} h0 \\ \text{(Flat)} & \flat\nabla f\alpha = (\nabla h)^{1+\alpha} 0 \end{array}$$

for each fruitful function f with $h = \boldsymbol{Fix} f$ and each ordinal α. ■

Observe that

$$\flat\nabla = \boldsymbol{Enm} \circ \nabla \circ \boldsymbol{Fix}$$

and, in particular, $\flat\boldsymbol{Id}_{\mathbb{H}} = \boldsymbol{Veb}$.

For each fruitful function f the function $h = \boldsymbol{Fix} f$ is helpful, and hence so is $k = \nabla h$. By Lemmas 2.9(b) and 3.5, with $\kappa = k0$ the three functions

$$\begin{array}{lll} (\sharp\nabla f) & \alpha \longmapsto & \nabla^{\alpha} k0 \\ (\natural\nabla f) & \alpha \longmapsto & \nabla^{\alpha} h0 \\ (\flat\nabla f) & \alpha \longmapsto & (\nabla h)^{\alpha}\kappa \end{array}$$

are normal. Thus $\heartsuit\nabla \in \mathbb{F}^{\ddagger}$ for each $\heartsuit \in \{\flat, \natural, \sharp\}$. We show that

$$\heartsuit : \mathfrak{Help} \longrightarrow \mathfrak{Derv}$$

for each $\heartsuit$.

4.6 LEMMA. *For each helpful operator ∇ the three operators $\flat\nabla, \natural\nabla, \sharp\nabla$ are derivatives and*

$$\boldsymbol{Veb} \le \flat\nabla \le \sharp\nabla \qquad \boldsymbol{Veb} \le \natural\nabla \le \sharp\nabla \qquad \flat\nabla f\alpha \le \natural\nabla f\alpha$$

for each fruitful function f and ordinal $\alpha \ge 3$ (for the third comparison).

Proof. Consider $f \in \mathbb{F}$. We saw above that $\flat\nabla f, \natural\nabla f$, and $\sharp\nabla f$ are normal. We first verify the comparisons. This will show that $\flat\nabla, \natural\nabla, \sharp\nabla$ are inflationary (on fruitful functions).

Let $h = \boldsymbol{Fix} f$, so that $h \in \mathbb{H}$. For each ordinal α we have

$$\boldsymbol{Veb} f\alpha = h^{1+\alpha}0 \leq (\nabla h)^{1+\alpha}0 = \flat\nabla f\alpha$$

to give $\boldsymbol{Veb} \leq \flat\nabla$. Since $1+\alpha \leq 2^\alpha$, using Lemma 3.4(a) we have

$$\boldsymbol{Veb} f\alpha = h^{1+\alpha}0 \leq h^{2^\alpha}0 \leq \nabla^\alpha h0 = \natural\nabla f\alpha$$

to give $\boldsymbol{Veb} \leq \natural\nabla$. The comparison $\natural\nabla \leq \sharp\nabla$ is immediate. Finally, since $1+\alpha \leq 2^\alpha$ and $4+\alpha \leq 2^{2+\alpha}$, by two more uses of Lemma 3.4(a) we have

$$\begin{array}{rl@{\qquad\qquad}rl}
\flat\nabla f\alpha & = (\nabla h)^{1+\alpha}0 & \flat\nabla f(3+\alpha) & = (\nabla h)^{4+\alpha}0 \\
& \leq (\nabla h)^{2^\alpha}0 & & \leq (\nabla h)^{2^{2+\alpha}}0 \\
& \leq \nabla^\alpha(\nabla h)0 & & \leq \nabla^{2+\alpha}(\nabla h)0 \\
& = \nabla^{1+\alpha}h0 \ \ = \sharp\nabla f\alpha & & = \nabla^{1+\alpha}h0 \qquad = \sharp\nabla f(3+\alpha)
\end{array}$$

to give $\flat\nabla \leq \sharp\nabla$ and the required $\flat\nabla \sqsubseteq \natural\nabla$.

It remains to verify that the operations are monotone (on fruitful functions). Consider $f, g \in \mathbb{F}$ with $f \leq g$. Let $h = \boldsymbol{Fix} f, k = \boldsymbol{Fix} g$. By Lemma 2.11 we have $h \leq k$, so that $\nabla h \leq \nabla k$, and hence both $(\nabla h)^\eta \leq (\nabla k)^\eta$ and $\nabla^\eta h \leq \nabla^\eta k$ for all η. By selecting various instances of these we obtain the required monotonicity. ■

Notice how $\natural\nabla$ doesn't quite fit in with the notation. In fact, for $f \in \mathbb{F}$, the two functions $\natural\nabla f$ and $\sharp\nabla f$ agree on infinite arguments. In the main we will use only $\flat\nabla$ and $\sharp\nabla$. The reason for including $\natural\nabla$ here is that it crops up quite naturally later.

Each derivative $\boldsymbol{D}$ provides a hierarchy $\Phi[\boldsymbol{D}, f]$ on any $f \in \mathbb{F}$. This hierarchy has its barrier ordinals, those ordinals ν such that

$$\nu = \Phi[\boldsymbol{D}, f]\nu 0 = (\boldsymbol{Fix} \circ \boldsymbol{D}^\nu)f0$$

hold. To get beyond $\Phi[\boldsymbol{D}, f]$ we jump $\boldsymbol{D}$ to enumerate these barrier ordinals. We consider two possible jump operators.

4.7 DEFINITION. For each derivative $\boldsymbol{D}$ we set

$$(\Uparrow)\boldsymbol{D} f\alpha = (\boldsymbol{Fix} \circ \boldsymbol{D}^\alpha)f0 \qquad (\upuparrows)\boldsymbol{D} f\alpha = (\boldsymbol{Fix} \circ \boldsymbol{D}^{1+\alpha})f0$$

for each fruitful f and ordinal α to produce two operators $(\Uparrow)\boldsymbol{Der}$ and $(\upuparrows)\boldsymbol{Der}$. ■

The constructor $(\Uparrow)$ might seem more natural, but later we see that $(\upuparrows)$ leads to sharper calculations. The next result illustrate a minor irritation with $(\Uparrow)$. In the statement of the result we write

$$\boldsymbol{Veb}^2 \sqsubseteq (\Uparrow)\,\boldsymbol{Veb}$$

to indicate that

$$\boldsymbol{Veb}^2 f\alpha \leq (\Uparrow)\,\boldsymbol{Veb} f\alpha$$

for all $f \in \mathbb{F}$ and all sufficiently large $\alpha \in \mathbb{O}\mathrm{rd}$.

4.8 LEMMA. *Let* $\boldsymbol{D}$ *be a big derivative (with* $\boldsymbol{Veb} \le \boldsymbol{D}$*). Then both* $(\Uparrow)\boldsymbol{D}$ *and* $(\upuparrows)\boldsymbol{D}$ *are derivatives and the comparisons*

$$\boldsymbol{Veb}^2 \sqsubseteq (\Uparrow)\,\boldsymbol{Veb} \le (\Uparrow)\boldsymbol{D} \le (\upuparrows)\boldsymbol{D}$$

hold.

Proof. It can be checked by hand that $(\Uparrow)\,\boldsymbol{Veb}$ is a derivative with

$$\boldsymbol{Veb} \le (\Uparrow)\,\boldsymbol{Veb} \qquad \boldsymbol{Veb}^2 f\alpha \le (\Uparrow)\,\boldsymbol{Veb} f\alpha$$

for all $f \in \mathbb{F}$ and $3 \le \alpha \in \mathbb{O}\mathrm{rd}$. Later we give a quick proof of this. We use the first comparison in this proof.

Consider $f \in \mathbb{F}$ and let $g = (\Uparrow)\boldsymbol{D}f$. Our main problem is to show that g is fruitful.

Since $\boldsymbol{Veb} \le \boldsymbol{D}$ we have $\boldsymbol{Veb}^\alpha \le \boldsymbol{D}^\alpha$ and hence $\boldsymbol{Fix} \circ \boldsymbol{Veb}^\alpha \le \boldsymbol{Fix} \circ \boldsymbol{D}^\alpha$ for each $\alpha \in \mathbb{O}\mathrm{rd}$. Thus

$$f\alpha \le \boldsymbol{Veb} f\alpha \le (\Uparrow)\,\boldsymbol{Veb} f\alpha \le (\Uparrow)\boldsymbol{D} f\alpha = g\alpha$$

which is enough to show that g is inflationary. This also shows that $(\Uparrow)\,\boldsymbol{Veb} \le (\Uparrow)\boldsymbol{D}$.

For ordinals $\alpha \le \beta$ we have $\boldsymbol{D}^\alpha \le \boldsymbol{D}^\beta$ and hence $\boldsymbol{Fix} \circ \boldsymbol{D}^\alpha \le \boldsymbol{Fix} \circ \boldsymbol{D}^\beta$ which shows that g is monotone.

To verify continuity consider any limit ordinal λ. Using Lemma 4.4 we have

$$g\lambda = \big(\boldsymbol{Fix} \circ \boldsymbol{D}^\lambda\big) f0 = \bigvee \big\{ \big(\boldsymbol{Fix} \circ \boldsymbol{Der}^\alpha\big) f0 \,|\, \alpha < \lambda \big\} = \bigvee \{ g\alpha \,|\, \alpha < \lambda \}$$

as required.

This shows that g is fruitful with $f \le g$. The required monotonicity of $(\Uparrow)\boldsymbol{D}$ is straight forward, and hence $(\Uparrow)\boldsymbol{D}$ is a derivative with $(\Uparrow)\,\boldsymbol{Veb} \le (\Uparrow)\boldsymbol{D}$.

Finally, we have

$$(\upuparrows)\boldsymbol{D} f\alpha = (\boldsymbol{Fix} \circ \boldsymbol{D}^{1+\alpha}) f0 = (\boldsymbol{Fix} \circ \boldsymbol{D}^\alpha \circ \boldsymbol{D}) f0 = ((\Uparrow)\boldsymbol{D} \circ \boldsymbol{D}) f\alpha$$

so that $(\upuparrows)\boldsymbol{D}$ is a composite of two derivatives. ■

In the next section we how to rephrase these jumps in terms of helpful constructors. This will make it easier to iterate a jump.

5 The shuffle technique

Each pair

$$\boldsymbol{D} \in \mathfrak{Derv}, f \in \mathbb{F} \qquad \nabla \in \mathfrak{Help}, h \in \mathbb{H}$$

generates a hierarchy

$$\alpha \longmapsto \boldsymbol{D}^\alpha f \qquad \alpha \longmapsto \nabla^\alpha h$$

of fruitful and helpful functions, respectively. By hitting these with

$$\boldsymbol{Veb} \qquad \boldsymbol{Enm}$$

respectively, we obtain hierarchies of normal function. The left hand one is, of course, just $\Phi[\boldsymbol{D}, f]$, the generalized Veblen hierarchy obtained from $\boldsymbol{D}$ and f.

In this section we show how a Φ-hierarchy can be obtained from an associated helpful hierarchy. This makes it easier to jump the derivative and so move to the next hierarchy.

We generalize the step from the description (1.d) to that of Theorem 2.13.

5.1 DEFINITION. For a derivative $\boldsymbol{D}$ and a helpful operator ∇ we write

$$\boldsymbol{D} \rightleftharpoons \nabla$$

and say $\boldsymbol{D}$ is shuffled by ∇ if

$$\boldsymbol{Fix} \circ \boldsymbol{D} = \nabla \circ \boldsymbol{Fix}$$

holds. ■

We have already seen one example of a shuffle. By (2.b) we have the the following.

5.2 LEMMA. *We have* $\boldsymbol{Veb} \rightleftharpoons [\,0\,]$.

Each of the three operators of Definition 4.5 converts a helpful operator into a derivative. There are associated shuffles.

5.3 LEMMA. *For each helpful operator* ∇*, the shuffle equivalences*

$$\flat\nabla \rightleftharpoons [\,0\,] \circ \nabla \qquad \natural\nabla \rightleftharpoons [\,1\,]\nabla \qquad \sharp\nabla \rightleftharpoons [\,1\,]\nabla$$

hold.

Proof. For

$$\flat\nabla = \boldsymbol{Enm} \circ \nabla \circ \boldsymbol{Fix}$$

we have

$$\boldsymbol{Fix} \circ \flat\nabla = \boldsymbol{Fix} \circ \boldsymbol{Enm} \circ \nabla \circ \boldsymbol{Fix} = [\,0\,] \circ \nabla \circ \boldsymbol{Fix}$$

to deal with the left hand shuffle.

Consider any $f \in \mathbb{F}$ and set $h = \boldsymbol{Fix} f$. For each ordinal ζ we

$$\begin{aligned} (\boldsymbol{Fix} \circ \natural\nabla) f\zeta &= \text{least } \nu \text{ with } \zeta < \nu = (\natural\nabla) f\nu \\ &= \text{least } \nu \text{ with } \zeta < \nu = \nabla^{\nu} h0 \\ &= [\,1\,]\nabla h\zeta \qquad\qquad = \big([\,1\,]\nabla \circ \boldsymbol{Fix}\big) f\zeta \end{aligned}$$

to give the shuffle for $\natural\nabla$.

The proof for $\sharp\nabla$ is a minor variant of this argument. ■

Notice that a helpful operator can shuffle more than one derivative.

In Definition 4.7 we produced two jump constructors for derivatives. The shuffle trick gives us a better understanding of these.

5.4 LEMMA. *Suppose* $\boldsymbol{D} \rightleftharpoons \nabla$ *(where* $\boldsymbol{D}$ *is a derivative and* ∇ *is helpful). Then*

$$\boldsymbol{Veb} \circ \boldsymbol{D} = \flat\nabla \qquad (\Uparrow)\boldsymbol{D} = \natural\nabla \qquad (\upuparrows)\boldsymbol{D} = \sharp\nabla$$

hold.

Proof. For each fruitful f with $h = \boldsymbol{Fix} f$ and each ordinal α we have

$$(\boldsymbol{Veb} \circ \boldsymbol{D}) f \alpha = \big((\boldsymbol{Fix} \circ \boldsymbol{D}) f\big)^{1+\alpha} 0 = \big((\nabla \circ \boldsymbol{Fix}) f\big)^{1+\alpha} 0 = \big(\nabla h\big)^{1+\alpha} 0 = \flat \nabla f \alpha$$

to prove the first equality. The second step uses the given shuffle.

Similarly

$$(\Uparrow) \boldsymbol{D} f \alpha = \big(\boldsymbol{Fix} \circ \boldsymbol{D}^{\alpha}\big) f 0 = \big(\nabla^{\alpha} \circ \boldsymbol{Fix}\big) f 0 = \nabla^{\alpha} h 0 = \natural \nabla f \alpha$$

to prove the second equality. The second step uses the iterated shuffle.

Finally

$$(\upuparrows) \boldsymbol{D} f \alpha = \big(\boldsymbol{Fix} \circ \boldsymbol{D}^{1+\alpha}\big) f 0 = \big(\nabla^{1+\alpha} \circ \boldsymbol{Fix}\big) f 0 = \nabla^{1+\alpha} h 0 = \sharp \nabla f \alpha$$

to prove the third equality. ■

Earlier we omitted the proof that $(\Uparrow)\, \boldsymbol{Veb}$ is a derivative with $\boldsymbol{Veb}^2 \sqsubseteq (\Uparrow)\, \boldsymbol{Veb}$. We can now rectify this.

By Lemma 5.2 we have $\boldsymbol{Veb} \rightleftharpoons [\,{\scriptstyle 0}\,]$ and hence

$$\boldsymbol{Veb}^2 = \flat [\,{\scriptstyle 0}\,] \qquad (\Uparrow)\, \boldsymbol{Veb} = \natural [\,{\scriptstyle 0}\,]$$

by Lemma 5.4. With this Lemma 4.6 gives the required result.

Shuffles can be iterated quite a long way. The following is a crucial observation

5.5 LEMMA. *Suppose*

$$\boldsymbol{D} \rightleftharpoons \nabla$$

(where $\boldsymbol{D}$ is a derivative and ∇ is helpful). Then we have a shuffle equivalence

$$\boldsymbol{D}^{\alpha} \rightleftharpoons \nabla^{\alpha}$$

for each ordinal α, and shuffle equivalences

$$(\Uparrow)^i \boldsymbol{D} \rightleftharpoons [\,{\scriptstyle 1}\,]^i \nabla \qquad (\upuparrows)^i \boldsymbol{D} \rightleftharpoons [\,{\scriptstyle 1}\,]^i \nabla$$

for each ordinal i.

Proof. For the first part we proceed by induction on α. The base case, $\alpha = 0$ is trivial; and the case $\alpha = 1$ is just the given shuffle.

For the induction step, $\alpha \mapsto \alpha + 1$, we have

$$\begin{aligned} \boldsymbol{Fix} \circ \boldsymbol{D}^{\alpha+1} &= \boldsymbol{Fix} \circ \boldsymbol{D} \circ \boldsymbol{D}^{\alpha} \\ &= \nabla \circ \boldsymbol{Fix} \circ \boldsymbol{D}^{\alpha} \\ &= \nabla \circ \nabla^{\alpha} \circ \boldsymbol{Fix} \quad = \nabla^{\alpha+1} \circ \boldsymbol{Fix} \end{aligned}$$

where the second step uses the given shuffle, and the third uses the induction hypothesis.

For the induction leap to a limit ordinal λ we have

$$\begin{aligned} \boldsymbol{Fix} \circ \boldsymbol{D}^{\lambda} &= \boldsymbol{Fix} \circ \bigvee \{\boldsymbol{D}^{\alpha} \mid \alpha < \lambda\} \\ &= \bigvee \{\boldsymbol{Fix} \circ \boldsymbol{D}^{\alpha} \mid \alpha < \lambda\} \\ &= \bigvee \{\nabla^{\alpha} \circ \boldsymbol{Fix} \mid \alpha < \lambda\} \quad = \nabla^{\lambda} \circ \boldsymbol{Fix} \end{aligned}$$

where the second step uses Lemma 4.4 and the third uses the induction hypothesis.

For the second part we deal with $(\Uparrow)$. The proof for $(\upuparrows)$ is more or less the same. We have

$$\boldsymbol{D} \rightleftharpoons \nabla \quad \Longrightarrow \quad (\Uparrow)\boldsymbol{D} \rightleftharpoons [\,1\,]\nabla$$

by Lemmas 5.4 and 5.3. We use this to prove the result by induction over i. The base case, $i = 0$, is trivial; and the previous observation gives the induction step, $i \mapsto i + 1$.

For the induction leap to a limit ordinal θ, for each fruitful function f we have

$$(\Uparrow)^{\theta}\boldsymbol{D}f = \bigvee\{(\Uparrow)^{i}\boldsymbol{D}f \mid i < \theta\}$$

and hence

$$\boldsymbol{Fix}\big((\Uparrow)^{\theta}\boldsymbol{D}f\big) = \bigvee\{\boldsymbol{Fix}\big((\Uparrow)^{i}\boldsymbol{D}f\big) \mid i < \theta\}$$

by Lemma 2.11. With $h = \boldsymbol{Fix}f$ the induction hypothesis now gives

$$\boldsymbol{Fix}\big((\Uparrow)^{\theta}\boldsymbol{D}f\big) = \bigvee\{[\,1\,]^{i}\nabla h \mid i < \theta\} = [\,1\,]^{\theta}\nabla h$$

for the required result. ■

The first part of this result justifies (1.i) of Section 1. Using the second part we can produce a neat description of the spine of the super-hierarchy generated using $\boldsymbol{D}$ and f.

5.6 THEOREM. *Suppose*

$$\boldsymbol{D} \rightleftharpoons \nabla$$

for some derivative $\boldsymbol{Der}$ *and helpful operator* ∇. *Let* f *be an arbitrary fruitful function and let* $h = \boldsymbol{Fix}f$. *Then*

$$\Phi[(\Uparrow)^{i}\boldsymbol{D}, f]\alpha = \boldsymbol{Enm}\Big(\big([\,1\,]^{i}\nabla\big)^{\alpha}h\Big) = \Phi[(\upuparrows)^{i}\boldsymbol{D}, f]\alpha$$

for each pair i, α *of ordinals.*

Proof. The given shuffle equivalence $\boldsymbol{D} \rightleftharpoons \nabla$ ensures

$$\Phi[\boldsymbol{D}, f]\alpha = \boldsymbol{Enm}\Big(\nabla^{\alpha}h\Big)$$

by the calculation of (1.i). By Lemma 5.5 we also have

$$(\Uparrow)^{i}\boldsymbol{D} \rightleftharpoons [\,1\,]^{i}\nabla \qquad (\upuparrows)^{i}\boldsymbol{D} \rightleftharpoons [\,1\,]^{i}\nabla$$

for each ordinal i. Two instances of the previous observation gives the general result. ■

In Section 7 we look at a whole family of derivatives designed as the building blocks of the Schütte brackets. To help with that analysis we need a couple of preliminaries.

5.7 DEFINITION. A helpful operator ∇ is absorbent if $\nabla \circ [\,0\,] = \nabla$. ■

If ∇ is absorbent then $[\,0\,] \le \nabla$. No finite iterate of $[\,0\,]$ is absorbent, but $[\,0\,]^{\alpha}$ is for each infinite ordinals α. We will need other examples of absorbent operators.

5.8 LEMMA. *(a) If $\mathcal{D}$ is a (directed) family of absorbent operators, then $\bigvee\mathcal{D}$ is absorbent.*

(b) If the operator ∇ is absorbent, then so is $[\,1\,]\nabla$.

(c) For each non-zero ordinal i, the operator $[\,1\,]^i[\,0\,]$ is absorbent.

Proof. (a) We have

$$\left(\bigvee\mathcal{D}\right)\circ[\,0\,] = \bigvee\left\{\nabla\circ[\,0\,] \mid \nabla\in\mathcal{D}\right\} = \bigvee\left\{\nabla \mid \nabla\in\mathcal{D}\right\} = \bigvee\mathcal{D}$$

as required.

(b) Suppose ∇ is absorbent. By (a), each non-zero iterate of ∇ is absorbent, hence

$$\nabla^\lambda\circ[\,0\,] = \nabla^\lambda$$

for each limit ordinal λ. Consider any $h\in\mathbb{H}$, and $\zeta\in\mathbb{O}\mathrm{rd}$, and let

$$\nu = \big([\,1\,]\nabla\circ[\,0\,]\big)h\zeta \qquad \mu = [\,1\,]\nabla h\zeta$$

so that $\nu=\mu$ is required. But these are the least ordinals such that

$$\zeta<\nu=\nabla^\nu([\,0\,]h)0 \qquad \zeta<\mu=\nabla^\mu h0$$

respectively. Since ν is a limit ordinal, we have $\nu=\mu$, immediately.

(c) It suffices to show that $[\,1\,][\,0\,]$ is absorbent, for then (a, b) gives the full result by a simple induction.

Consider any $h\in\mathbb{H}$, and $\zeta\in\mathbb{O}\mathrm{rd}$, and let

$$\nu = \big([\,1\,][\,0\,]\circ[\,0\,]\big)h\zeta \qquad \mu = [\,1\,][\,0\,]h\zeta$$

so that $\nu=\mu$ is required. But these are the least ordinals such that

$$\zeta<\nu=[\,0\,]^\nu\big([\,0\,]h\big)0=[\,0\,]^{1+\nu}h0 \qquad \zeta<\mu=[\,0\,]^\mu h0$$

respectively. Since $1+\nu=\nu$, we have $\nu=\mu$, immediately. ■

This concludes the general development.

6 Schütte brackets

In the final three sections we analyse the Schütte brackets to show that the behaviour of each can be read off from the bracket without the need of an intricate recursion.

6.1 DEFINITION. A **Schütte bracket** - Klammersymbole in German - is an array of ordinals

$$\begin{pmatrix}\zeta & \alpha(1) & \cdots & \alpha(s)\\ r & i(1) & \cdots & i(s)\end{pmatrix} \tag{6.a}$$

where

$$r<i(1)<\cdots<i(s) \tag{6.b}$$

in the bottom row. ■

Each such bracket $(=)$ may be combined with an arbitrary normal function f (or even a fruitful function) to produce an ordinal

$$f(=) \tag{6.c}$$

which, in most cases, turns out to be a fixed point of f. (The base case merely outputs the values of f.) The evaluation of $f(=)$ proceeds by a rather intricate recursion given by the rules (2.1, 2.2, 2.3) of [4]. To describe this behaviour we decompose the bracket (6.a) and introduce some terminology.

6.2 DEFINITION. For each bracket (6.a) we call

ζ	the input	$\alpha(1),\dots,\alpha(s)$	the exponents
r	the rank	$i(1),\dots,i(s)$	the epochs

of the bracket. By removing the input and the rank we obtain the motor

$$\begin{bmatrix}\alpha(1) & \cdots & \alpha(s)\\ i(1) & \cdots & i(s)\end{bmatrix}$$

of the bracket. ■

We decompose a bracket because the motor, the rank, and the input play different roles, of decreasing importance. A motor may be empty, but this will not cause problems.

How do we evaluate (6.c)? Rules (2.1, 2.2, 2.3) of [4] are paraphrased as rules (A, B, C, D) of Table 1. Let's see how these match up.

Rule (2.1), the base case, says that for zero rank with empty motor we set

$$f\binom{\zeta}{0} = f\zeta$$

for each normal function f and input ζ. This is stated as rule (B) in Table 1.

Rule (2.2) is concerned with 'equal' brackets. Let us say a motor is canonical if each exponent is non-zero. Each motor has a unique canonical form obtained by omitting each column (exponent and epoch) where the exponent is zero. Two motors are deemed equal if they have the same canonical form. Two brackets are deemed equal if the have the same inputs, the same ranks, and equal motors.

Rule (2.2) says that we need deal only with canonical motors. This is stated as rule (A) in Table 1.

By (6.b) each epoch in a motor is non-zero. Each exponent in a canonical motor is non-zero. Thus, with a slight change of notation, each canonical motor has the form

$$\begin{bmatrix}1+\alpha(1) & \cdots & 1+\alpha(s)\\ 1+i(1) & \cdots & 1+i(s)\end{bmatrix}$$

for some list $\alpha(1),\dots,\alpha(s)$ of ordinals and matching list $i(1),\dots,i(s)$ of ordinals. We deal only with canonical motors in this standard form.

Of course,

$$1+\alpha=\alpha \qquad 1+i=i$$

if α is infinite or i is infinite. However, it is convenient to retain the '$1+\cdot$' in all cases.

We now come to rule (2.3), the crucial recursion step. It will take us some time to understand this and its consequences. It is stated as rule (C) in Table 1.

(A) If any exponent is zero, then that column (exponent and epoch) should be omitted.

(B) For rank 0 with an empty motor we set

$$f\begin{pmatrix} \cdot \\ 0 \end{pmatrix} = f$$

for each f.

(C) For rank 0 with a non-empty motor the function

$$f\begin{pmatrix} \cdot & 1+\alpha & - \\ 0 & 1+i & - \end{pmatrix}$$

enumerates the common fixed point of the family

$$\mathcal{G} = \left\{ f\begin{pmatrix} \cdot & \beta & - \\ r & 1+i & - \end{pmatrix} \,\middle|\, r < 1+i,\, \beta < 1+\alpha \right\}$$

of functions, some of which use brackets with non-zero rank.

(D) For non-zero rank $1+r$ we set

$$f\begin{pmatrix} 1+\eta & - \\ 1+r & - \end{pmatrix} = f\begin{pmatrix} 0 & 1+\eta & - \\ 0 & 1+r & - \end{pmatrix}$$

for each ordinal η. (The value $f\begin{pmatrix} 0 & - \\ 1+r & - \end{pmatrix}$ can be taken to be 1.)

Table 1: The condensed evaluation rules for brackets

The rule says the function

$$f\begin{pmatrix} \cdot & 1+\alpha(1) & \cdots & 1+\alpha(s) \\ 0 & 1+i(1) & \cdots & 1+i(s) \end{pmatrix}$$

enumerates the common fixed points of a whole family $\mathcal{G}$ of functions that have been generated 'earlier'. These common fixed points are precisely the fixed points of a certain function $\bigvee \mathcal{G}$ (the pointwise supremum and actual supremum of $\mathcal{G}$). Thus

$$f\begin{pmatrix} \cdot & 1+\alpha(1) & \cdots & 1+\alpha(s) \\ 0 & 1+i(1) & \cdots & 1+i(s) \end{pmatrix} = \boldsymbol{Veb}(\bigvee \mathcal{G})$$

is a concise statement of this rule. We look at the details in Section 8.

How do we deal with non-zero rank? The appropriate rule seems not to be stated explicitly in [4], but there are some clues. Because of rule (A) we have

$$f\begin{pmatrix} 1+\eta & - \\ 1+r & - \end{pmatrix} = f\begin{pmatrix} 0 & 1+\eta & - \\ 0 & 1+r & - \end{pmatrix}$$

for each pair η, r of ordinals. This is stated as rule (D) in Table 1.

There are some hidden difficulties in these rules. As an exercise try to evaluate

$$f\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

for $f = \omega^{\bullet}$. Or at least find a different description of this critical ordinal.

$$\begin{array}{lll}
\text{(epoch 1)} & \boldsymbol{S}\begin{bmatrix}\alpha\\ 1\end{bmatrix} & = \boldsymbol{Veb}^{\alpha}\\[2ex]
(*+) & \boldsymbol{S}\begin{bmatrix}*\\ i+1\end{bmatrix} f\eta & = \boldsymbol{S}\begin{bmatrix}\eta+1\\ i+1\end{bmatrix} f0\\[2ex]
(*\ell) & \boldsymbol{S}\begin{bmatrix}*\\ \theta\end{bmatrix} & = \bigvee\left\{\boldsymbol{S}\begin{bmatrix}*\\ i+1\end{bmatrix} \,\middle|\, i<\theta\right\}\\[2ex]
\text{(exp1)} & \boldsymbol{S}\begin{bmatrix}1\\ 1+i+1\end{bmatrix} & = \boldsymbol{Veb}\circ\boldsymbol{S}\begin{bmatrix}*\\ 1+i\end{bmatrix}\\[2ex]
\text{(exp+)} & \boldsymbol{S}\begin{bmatrix}\alpha+2\\ 1+i+1\end{bmatrix} & = \boldsymbol{Veb}\circ\boldsymbol{S}\begin{bmatrix}*\\ 1+i\end{bmatrix}\circ\boldsymbol{S}\begin{bmatrix}\alpha+1\\ 1+i+1\end{bmatrix}\\[2ex]
\text{(exp}\ell) & \boldsymbol{S}\begin{bmatrix}\lambda\\ 1+i+1\end{bmatrix} & = \bigvee\left\{\boldsymbol{S}\begin{bmatrix}\alpha+1\\ 1+i+1\end{bmatrix} \,\middle|\, \alpha<\lambda\right\}\\[2ex]
\text{(exp}\ell+) & \boldsymbol{S}\begin{bmatrix}\lambda+1\\ 1+i+1\end{bmatrix} & = \boldsymbol{Veb}\circ\boldsymbol{S}\begin{bmatrix}\lambda\\ 1+i+1\end{bmatrix}
\end{array}$$

In this table i and α are arbitrary ordinals, and θ and λ are limit ordinals.

Table 2: An erratic family of derivatives

Schütte proves that each function

$$f\begin{pmatrix}\cdot & -\\ r & -\end{pmatrix}$$

is normal. The proofs are rather intricate for the induction has to follow the recursion set out in (2.1, 2.2, 2.3). We use the general machinery of Sections 1 to 5 to give an explicit description of each such function. Once known this description makes the proof of normality almost routine. Of course, to verify the description we still have to go through an intricate induction, but once that has been done we can forget the rules of Table 1.

Schütte also proves that the function

$$f\begin{pmatrix}1\\ \cdot\end{pmatrix}$$

is normal, and shows that its fixed points (for the case $f=\omega^{\bullet}$) are Veblen's E-numbers. With a bit of give and take that result can be described here.

We first convert f into a helpful function.

$$h = \boldsymbol{Fix}\, f$$

For each ordinal $\zeta \in \mathbb{O}\mathrm{rd}$ the output $h\zeta$ is the next fixed point of f beyond ζ. Using the helpful operator $[\,\mathrm{0}\,]$ and helpful constructor $[\,\mathrm{1}\,]$, it turns out that

$$f\begin{pmatrix}1\\ 1+r\end{pmatrix} = [\,\mathrm{1}\,]^{r}[\,\mathrm{0}\,]h0$$

for each ordinal r. This description is quite suggestive, and the idea can be extended.

Let me explain what we are going to do.

Table 2 contains a doubly indexed family of derivatives. In fact, at this stage it is not clear that they are derivatives. That will be proved in Section 7. This table is generated in a rather erratic fashion. That is because the generation process follows the evaluation algorithm of a bracket with a few extra steps to deal with the behaviour at limit ordinals.

We use these particular derivatives to attach a derivative to each motor.

6.3 DEFINITION. For each canonical motor $\begin{bmatrix} - \\ - \end{bmatrix}$ the associated derivative

$$\boldsymbol{Sch}\begin{bmatrix} - \\ - \end{bmatrix}$$

is built up column by column. Thus we set

$$\boldsymbol{Sch}\begin{bmatrix} \end{bmatrix} = \boldsymbol{Id} \qquad \boldsymbol{Sch}\begin{bmatrix} 1+\alpha & - \\ 1+i & - \end{bmatrix} = \boldsymbol{S}\begin{bmatrix} \alpha+1 \\ i+1 \end{bmatrix} \circ \boldsymbol{Sch}\begin{bmatrix} - \\ - \end{bmatrix}$$

where here $\begin{bmatrix} \end{bmatrix}$ is the empty motor and $\boldsymbol{Id}$ is the identity operator on $\mathbb{F}$. ■

Thus

$$\boldsymbol{Sch}\begin{bmatrix} 1+\alpha(1) & \cdots & 1+\alpha(s) \\ 1+i(1) & \cdots & 1+i(s) \end{bmatrix} = \boldsymbol{S}\begin{bmatrix} \alpha(1)+1 \\ i(1)+1 \end{bmatrix} \circ \cdots \cdot \boldsymbol{S}\begin{bmatrix} \alpha(s)+1 \\ i(s)+1 \end{bmatrix}$$

by working through the motor column by column. Notice that each column has no influence on any of the other columns. Notice also that we have yet more flips

$$1 + \bullet \mapsto \bullet + 1$$

in the construction.

In Section 8 we obtain the following.

6.4 THEOREM. *For each canonical motor* $\begin{bmatrix} - \\ - \end{bmatrix}$ *we have*

$$f\begin{pmatrix} \cdot & - \\ 0 & - \end{pmatrix} = \boldsymbol{Sch}\begin{bmatrix} - \\ - \end{bmatrix} f$$

for each fruitful function f.

The proof of this is a bit messy. We first expand the evaluation rules of Table 1 and then match these against the construction of $\boldsymbol{Sch}\begin{bmatrix} - \\ - \end{bmatrix}$ via Table 2. What is more interesting is that we can give an explicit description of each derivative in Table 2. To do that we use the helpful gadgets $[\,0\,]$ and $[\,1\,]$ of Sections 2 and 3.

6.5 DEFINITION. For each pair α, i of ordinals let

$$\nabla\begin{bmatrix} \alpha+1 \\ i+1 \end{bmatrix} = \begin{cases} \big([\,1\,]^{i}[\,0\,]\big)^{1+\alpha} & \text{if } i \neq 0 \\ [\,0\,]^{\alpha} & \text{if } i = 0 \end{cases}$$

to produce a helpful operator. ■

For infinite α there is no difference between the cases $i = 0$ and $i \neq 0$. For finite α we have another one of those stutters, this one designed to match that in Table 2.

This definition is the reason for the terminology ‘exponent’ and ‘epoch’. A small increase in the exponent α produces a comparatively small increase in the power of the associated operator ∇. In contrast, a small increase in the epoch i produces a significant increase in the power of ∇.

With this notation we can state the main result of Section 7.

6.6 THEOREM. *We have*

$$\boldsymbol{S}\begin{bmatrix}\alpha+1\\ i+1\end{bmatrix} = \flat\nabla\begin{bmatrix}\alpha+1\\ i+1\end{bmatrix} = \boldsymbol{Enm}\circ\nabla\begin{bmatrix}\alpha+1\\ i+1\end{bmatrix}\circ\boldsymbol{Fix} \tag{6.d}$$

for each pair α, i *of ordinals.*

The definition of $[\,0\,]$ gives the following.

$$\boldsymbol{Fix}\circ\boldsymbol{S}\begin{bmatrix}\alpha+1\\ i+1\end{bmatrix} = [\,0\,]\circ\nabla\begin{bmatrix}\alpha+1\\ i+1\end{bmatrix}\circ\boldsymbol{Fix}$$

We wish to attach a helpful operator to each bracket.

6.7 DEFINITION. For each motor

$$\begin{bmatrix}1+\alpha & - & 1+\beta\\ 1+i & - & 1+j\end{bmatrix}$$

we set

$$\nabla\begin{bmatrix}\alpha+1 & - & \beta+1\\ i+1 & - & j+1\end{bmatrix} = \nabla\begin{bmatrix}\alpha+1\\ i+1\end{bmatrix}\circ[\,0\,]\circ\cdots\circ[\,0\,]\circ\nabla\begin{bmatrix}\beta+1\\ j+1\end{bmatrix}$$

to obtain a compound helpful operator. ■

In other words, for each column $\begin{bmatrix}1+\alpha\\ 1+\beta\end{bmatrix}$ we take the corresponding helpful operator $\nabla\begin{bmatrix}\alpha+1\\ i+1\end{bmatrix}$, separate the list of these by copies of $[\,0\,]$, and then take the functions composite of this extended list. For most cases these separating $[\,0\,]$ are absorbed by the following operator, but there are some cases when this doesn't happen.

This leads to an explicit description of the derivative attached to a non-empty motor.

6.8 THEOREM. *We have*

$$\boldsymbol{Sch}\begin{bmatrix}-\\ -\end{bmatrix} = \boldsymbol{Enm}\circ\nabla\begin{bmatrix}-\\ -\end{bmatrix}\circ\boldsymbol{Fix}$$

for each non-empty canonical motor $\begin{bmatrix}-\\ -\end{bmatrix}$.

Proof. We proceed by induction over the number of columns.

For a motor with just one column the result is (6.d).

For the induction step we have

$$\begin{aligned}\boldsymbol{Sch}\begin{bmatrix}\alpha+1 & -\\ i+1 & -\end{bmatrix} &= \boldsymbol{S}\begin{bmatrix}\alpha+1\\ i+1\end{bmatrix}\circ\boldsymbol{Sch}\begin{bmatrix}-\\ -\end{bmatrix}\\ &= \boldsymbol{Enm}\circ\nabla\begin{bmatrix}\alpha+1\\ i+1\end{bmatrix}\circ\boldsymbol{Fix}\circ\boldsymbol{Enm}\circ\nabla\begin{bmatrix}-\\ -\end{bmatrix}\circ\boldsymbol{Fix}\\ &= \boldsymbol{Enm}\circ\nabla\begin{bmatrix}\alpha+1\\ i+1\end{bmatrix}\circ[\,0\,]\circ\nabla\begin{bmatrix}-\\ -\end{bmatrix}\circ\boldsymbol{Fix}\\ &= \boldsymbol{Enm}\circ\nabla\begin{bmatrix}\alpha+1 & -\\ i+1 & -\end{bmatrix}\circ\boldsymbol{Fix}\end{aligned}$$

using (6.d) and the induction hypothesis at the second equality. ■

With this we can add to Theorem 6.4 and obtain an explicit description of the behaviour of each Schütte bracket of zero rank. Compare this result with Theorem 2.13.

6.9 THEOREM. *For each non-empty motor* $\begin{bmatrix}-\\-\end{bmatrix}$ *and fruitful function* f *with* $h = \boldsymbol{Fix}h$ *we have*

$$f\begin{pmatrix}\zeta & -\\ 0 & -\end{pmatrix} = \left(\nabla\begin{bmatrix}-\\-\end{bmatrix}h\right)^{1+\zeta}0$$

for each input ordinal ζ.

Proof. By Theorems 6.4 and 6.8 we have

$$f\begin{pmatrix}\zeta & -\\ 0 & -\end{pmatrix} = \boldsymbol{Sch}\begin{bmatrix}-\\-\end{bmatrix}f\zeta = \left(\boldsymbol{Enm}\circ\nabla\begin{bmatrix}-\\-\end{bmatrix}\circ\boldsymbol{Fix}\right)f\zeta = \left(\nabla\begin{bmatrix}-\\-\end{bmatrix}h\right)^{1+\zeta}0$$

where the last step follows by the definition of $\boldsymbol{Enm}$. ■

So far we have considered only brackets with zero rank. A use of rule (D) of Table 1 enables us to deal with non-zero ranks. An example will suffice to show how this is done. To avoid the stutter of Definition 6.5 let's consider rank $2+r$.

We have

$$f\begin{pmatrix}1+\eta\\ 2+r\end{pmatrix} = f\begin{pmatrix}0 & 1+\eta\\ 0 & 2+r\end{pmatrix} = \nabla\begin{bmatrix}\eta+1\\ r+2\end{bmatrix}h0 = \left([\,1\,]^{r+1}[\,0\,]\right)^{1+\eta}h0$$

for each ordinal η. In particular

$$f\begin{pmatrix}3\\2\end{pmatrix} = \left([\,1\,][\,0\,]\right)^{3}h0$$

is the answer to the exercise I set earlier.

All that remains now is to prove Theorems 6.4 and 6.6. This is done in Sections 7 and 8, respectively. These two section can be read independently.

7 An erratic family of derivatives

In this section we prove Theorem 6.6 and investigate the gadgets generated in Table 2. In particular we show that each one is a derivative, and produce an explicit non-recursive description of most of them. We also obtain some shuffle information.

We begin by inspecting the recursion of Table 2 to see exactly how it works.

For epoch 1 the operators $\boldsymbol{S}\begin{bmatrix}\cdot\\1\end{bmatrix}$ are defined by clause (epoch 1). These are derivatives.

7.1 LEMMA. *We have*

$$\boldsymbol{S}\begin{bmatrix}*\\1\end{bmatrix} = (\Uparrow)\,\boldsymbol{Veb}$$

and, in particular, $\boldsymbol{S}\begin{bmatrix}*\\1\end{bmatrix}$ *is a derivative.*

Proof. For each fruitful function f we have

$$\boldsymbol{Veb}f0 = \boldsymbol{Fix}f0$$

and hence

$$\begin{aligned}\boldsymbol{S}\begin{bmatrix}*\\1\end{bmatrix}f\zeta &= \boldsymbol{S}\begin{bmatrix}\zeta+1\\1\end{bmatrix}f0\\ &= \boldsymbol{Veb}^{\zeta+1}f0\\ &= \boldsymbol{Veb}(\boldsymbol{Veb}^{\zeta}f)0\\ &= (\boldsymbol{Fix}\circ\boldsymbol{Veb}^{\zeta})f0 = (\Uparrow)\,\boldsymbol{Veb}f\zeta\end{aligned}$$

by Definition 4.7. With this Lemma 4.8 shows that $\boldsymbol{S}\genfrac{[}{]}{0pt}{}{*}{1}$ is a derivative. ■

Suppose now for some ordinal i, we know that $\boldsymbol{S}\genfrac{[}{]}{0pt}{}{*}{1+i}$ is a derivative. We then use clauses (exp1, exp+, expℓ, expℓ+) to produce the derivatives $\boldsymbol{S}\genfrac{[}{]}{0pt}{}{\alpha}{1+i+1}$ for all non-zero ordinals α. Thus $\boldsymbol{S}\genfrac{[}{]}{0pt}{}{\alpha}{1+i+1}$ is a bit like an iterate

$$\Big(\boldsymbol{Veb}\circ\boldsymbol{S}\genfrac{[}{]}{0pt}{}{*}{1+i}\Big)^{\alpha}$$

of a known derivative. The construction doesn't produce these iterates, for clause (expℓ+) inserts a stutter after a limit exponent. We look at these 'symbolic iterates' in a moment.

Once these 'symbolic iterates' have been generated we set

$$\boldsymbol{S}\genfrac{[}{]}{0pt}{}{*}{1+i+1}f\zeta=\boldsymbol{S}\genfrac{[}{]}{0pt}{}{\zeta+1}{1+i+1}f0$$

using $(*+)$ at the next epoch. This looks a bit like the jump

$$(\Uparrow)\boldsymbol{S}\genfrac{[}{]}{0pt}{}{*}{1+i+i}$$

but the effect of the stutters at limit exponents is not so clear. In fact, it is not immediately obvious that $\boldsymbol{S}\genfrac{[}{]}{0pt}{}{*}{i+1}$ is a derivative, so this is something we have to prove.

Finally, for a limit ordinal θ we use $(*\ell)$ to produce $\boldsymbol{S}\genfrac{[}{]}{0pt}{}{*}{\theta}$. By that stage we know that $\boldsymbol{S}\genfrac{[}{]}{0pt}{}{*}{\theta}$ is a derivative since it is the pointwise supremum of a chain of derivatives. We then start to generate $\boldsymbol{S}\genfrac{[}{]}{0pt}{}{\beta}{\theta+1}$ using (exp1, exp+, expℓ, expℓ+).

There are no operators $\boldsymbol{S}\genfrac{[}{]}{0pt}{}{\beta}{\theta}$ for limit ordinals θ.

This construction repeatedly uses (exp1, exp+, expℓ, expℓ+) to produce many layers of derivatives. We need to understand what happens in each layer.

7.2 DEFINITION. Given a derivative $\boldsymbol{S}$ we generate the symbolic iterates $\boldsymbol{S}^{(\cdot)}$ by

$$\begin{array}{lll}
(\mathrm{exp1}) & \boldsymbol{S}^{(1)} & =\boldsymbol{Veb}\circ\boldsymbol{S}\\
(\mathrm{exp+}) & \boldsymbol{S}^{(\alpha+2)} & =\boldsymbol{Veb}\circ\boldsymbol{S}\circ\boldsymbol{S}^{(\alpha+1)}\\
(\mathrm{exp}\ell) & \boldsymbol{S}^{(\lambda)} & =\bigvee\{\boldsymbol{S}^{(\alpha+1)}\mid\alpha<\lambda\}\\
(\mathrm{exp}\ell+) & \boldsymbol{S}^{(\lambda+1)} & =\boldsymbol{Veb}\circ\boldsymbol{S}^{(\lambda)}
\end{array}$$

for each ordinal α and limit ordinal λ. ■

Almost trivially this iteration produces an ascending chain of derivatives.

$$\boldsymbol{S}\le\boldsymbol{S}^{(1)}\le\boldsymbol{S}^{(2)}\le\cdots\le\boldsymbol{S}^{(\omega)}\le\cdots\le\boldsymbol{S}^{(\alpha)}\le\cdots$$

The term $\boldsymbol{S}^{(\alpha)}$ is not quite

$$(\boldsymbol{Veb}\circ\boldsymbol{S})^{\alpha}$$

because of the stutter at limit ordinals. These iterates interlace with the symbolic iterates.

A comparison of this iteration with that of Table 2 gives the following.

7.3 LEMMA. *For an ordinal i suppose*

$$\boldsymbol{S} = \boldsymbol{S}\left[\begin{smallmatrix} * \\ 1+i \end{smallmatrix}\right]$$

is known to be a derivative. Then

$$\boldsymbol{S}\left[\begin{smallmatrix} \alpha \\ 1+i+1 \end{smallmatrix}\right] = \boldsymbol{S}^{(\alpha)}$$

for each non-zero ordinal α.

Proof. We proceed by induction on α.

For the base case, $\alpha = 1$, we have

$$\boldsymbol{S}\left[\begin{smallmatrix} 1 \\ 1+i+1 \end{smallmatrix}\right] = \boldsymbol{Veb} \circ \boldsymbol{S}\left[\begin{smallmatrix} * \\ 1+i \end{smallmatrix}\right] = \boldsymbol{Veb} \circ \boldsymbol{S} = \boldsymbol{S}^{(1)}$$

as required.

For the induction step, $\alpha \mapsto \alpha + 1$, there are two subcases.

For $\alpha = \beta + 1$ for some ordinal β we have

$$\begin{aligned} \boldsymbol{S}\left[\begin{smallmatrix} \alpha+1 \\ 1+i+1 \end{smallmatrix}\right] &= \boldsymbol{S}\left[\begin{smallmatrix} \beta+2 \\ 1+i+1 \end{smallmatrix}\right] \\ &= \boldsymbol{Veb} \circ \boldsymbol{S}\left[\begin{smallmatrix} * \\ 1+i \end{smallmatrix}\right] \circ \boldsymbol{S}\left[\begin{smallmatrix} \beta+1 \\ 1+i+1 \end{smallmatrix}\right] \\ &= \boldsymbol{Veb} \circ \boldsymbol{S} \circ \boldsymbol{S}^{(\beta+1)} = \boldsymbol{S}^{(\beta+2)} = \boldsymbol{S}^{(\alpha+1)} \end{aligned}$$

using the induction at the ante-penultimate step.

For $\alpha = \lambda$, a limit ordinal, we have

$$\boldsymbol{S}\left[\begin{smallmatrix} \lambda+1 \\ 1+i+1 \end{smallmatrix}\right] = \boldsymbol{Veb} \circ \boldsymbol{S}\left[\begin{smallmatrix} \lambda \\ 1+i+1 \end{smallmatrix}\right] = \boldsymbol{Veb} \circ \boldsymbol{S}^{(\lambda)} = \boldsymbol{S}^{(\lambda+1)}$$

as required.

For the induction leap to as limit ordinal λ, the induction hypothesis ensures

$$\left\{ \boldsymbol{S}^{(\beta+1)} \mid \beta < \lambda \right\}$$

is an ascending chain of derivatives. Thus

$$\boldsymbol{S}\left[\begin{smallmatrix} \lambda \\ 1+i+1 \end{smallmatrix}\right] = \bigvee \left\{ \boldsymbol{S}\left[\begin{smallmatrix} \beta+1 \\ 1+i+1 \end{smallmatrix}\right] \mid \beta < \lambda \right\} = \bigvee \left\{ \boldsymbol{S}^{(\beta+1)} \mid \beta < \lambda \right\} = \boldsymbol{S}^{(\lambda)}$$

as required. $\blacksquare$

The next layer after

$$\boldsymbol{S}\left[\begin{smallmatrix} * \\ 1+i \end{smallmatrix}\right], \boldsymbol{S}\left[\begin{smallmatrix} 1 \\ 1+i \end{smallmatrix}\right], \dots, \boldsymbol{S}\left[\begin{smallmatrix} \alpha \\ 1+i \end{smallmatrix}\right], \dots$$

begins with

$$\boldsymbol{S}\left[\begin{smallmatrix} * \\ 1+i+1 \end{smallmatrix}\right]$$

which is a kind of diagonal limit of the previous layer. Why is this a derivative?

7.4 DEFINITION. For each derivative $\boldsymbol{S}$ the symbolic jump $(\curlywedge)\boldsymbol{S}$ is given by

$$(\curlywedge)\boldsymbol{S}f\zeta = \boldsymbol{S}^{(\zeta+1)}f0$$

for each fruitful function f and ordinal ζ. $\blacksquare$

In other words we have

$$\boldsymbol{S}\Big[1+\overset{*}{i}+1\Big] = (\Yup)\boldsymbol{S}\Big[1+\overset{*}{i}\Big]$$

using the symbolic iterates and the symbolic jump. Of course, we don't yet know that this produces a derivative. This is where the shuffle technique comes into its own.

7.5 LEMMA. *Suppose*

$$\boldsymbol{S} \rightleftharpoons \nabla$$

for some derivative $\boldsymbol{S}$ *and helpful operator* ∇ *which is absorbent. Then*

$$\boldsymbol{Veb} \circ \boldsymbol{S} \circ \boldsymbol{Veb} = \boldsymbol{Veb} \circ \boldsymbol{S}$$

holds.

Proof. First of all we have

$$\boldsymbol{Fix} \circ \boldsymbol{S} \circ \boldsymbol{Veb} = \nabla \circ \boldsymbol{Fix} \circ \boldsymbol{Veb} = \nabla \circ [\,{\scriptstyle 0}\,] \circ \boldsymbol{Fix} = \nabla \circ \boldsymbol{Fix} = \boldsymbol{Fix} \circ \boldsymbol{S}$$

and hence, for each fruitful f and ordinal α

$$\big(\boldsymbol{Veb} \circ \boldsymbol{S} \circ \boldsymbol{Veb}\big)f\alpha = \big((\boldsymbol{Fix} \circ \boldsymbol{S} \circ \boldsymbol{Veb})f\big)^{1+\alpha}0 = \big((\boldsymbol{Fix} \circ \boldsymbol{S})f\big)^{1+\alpha}0 = (\boldsymbol{Veb} \circ \boldsymbol{S})f\alpha$$

as required. ■

This gives us quite a lot of information about symbolic iterates and the symbolic jump.

7.6 THEOREM. *Suppose*

$$\boldsymbol{S} \rightleftharpoons \nabla$$

for some derivative $\boldsymbol{S}$ *and helpful operator* ∇ *which is absorbent. Then*

$$\begin{array}{llll} (a) & \boldsymbol{S}^{(\alpha+1)} = \boldsymbol{Veb} \circ \boldsymbol{S}^{1+\alpha} = \flat(\nabla^{1+\alpha}) & \textit{and} & \boldsymbol{S}^{(\alpha+1)} \rightleftharpoons [\,{\scriptstyle 0}\,] \circ \nabla^{1+\alpha} \\ (b) & \boldsymbol{S}^{(\lambda)} = \boldsymbol{S}^{\lambda} & \textit{and} & \boldsymbol{S}^{(\lambda)} \rightleftharpoons \nabla^{\lambda} \end{array}$$

for each ordinal α *and limit ordinal* λ.

Proof. By Lemma 5.5 we have

$$\boldsymbol{S}^{\alpha} \rightleftharpoons \nabla^{\alpha}$$

for each ordinal α, including limit ordinals. We also have

$$\boldsymbol{Veb} \circ \boldsymbol{S}^{1+\alpha} = \flat(\nabla^{1+\alpha}) \rightleftharpoons [\,{\scriptstyle 0}\,] \circ \nabla^{1+\alpha}$$

by Lemmas 5.4 and 5.3. Thus it suffices to show

$$\boldsymbol{S}^{(\alpha+1)} = \boldsymbol{Veb} \circ \boldsymbol{S}^{1+\alpha} \qquad \boldsymbol{S}^{(\lambda)} = \boldsymbol{S}^{\lambda}$$

for each ordinal α and limit ordinal λ. We proceed by induction over the exponent.

The base case, $\alpha = 0$, holds by clause (exp1) of Definition 7.2.

For the induction step, $\alpha \mapsto \alpha + 1$, we have

$$\begin{aligned}\boldsymbol{S}^{(\alpha+2)} &= \boldsymbol{Veb} \circ \boldsymbol{S} \circ \boldsymbol{S}^{(\alpha+1)} \\ &= \boldsymbol{Veb} \circ \boldsymbol{S} \circ \boldsymbol{Veb} \circ \boldsymbol{S}^{1+\alpha} \\ &= \boldsymbol{Veb} \circ \boldsymbol{S} \circ \boldsymbol{S}^{1+\alpha} \qquad\qquad = \boldsymbol{Veb} \circ \boldsymbol{S}^{1+\alpha+1}\end{aligned}$$

using the induction hypothesis and Lemma 7.5.

For the induction leap to a limit ordinal λ, the induction hypothesis gives

$$\boldsymbol{S}^{1+\alpha} \le \boldsymbol{S}^{(\alpha+1)} \le \boldsymbol{S}^{1+\alpha+1}$$

so the obvious interlacing argument gives $\boldsymbol{S}^{(\lambda)} = \boldsymbol{S}^{\lambda}$. ■

With this we can deal with the symbolic jump.

7.7 COROLLARY. *Suppose*

$$\boldsymbol{S} \rightleftharpoons \nabla$$

for some derivative $\boldsymbol{S}$ *and helpful operator* ∇ *which is absorbent. Then*

$$(\Yup)\boldsymbol{S} = \sharp\nabla = (\upuparrows)\boldsymbol{S} \quad \textit{and} \quad (\Yup)\boldsymbol{S} \rightleftharpoons [\,{\scriptstyle 1}\,]\nabla$$

and, in particular, $(\Yup)\boldsymbol{S}$ *is a derivative.*

Proof. For $f \in \mathbb{F}$ with $h = \boldsymbol{Fix}\,f$, we have

$$(\Yup)\boldsymbol{S}f\alpha = \boldsymbol{S}^{(\alpha+1)}f0 = \flat(\nabla^{1+\alpha})f0 = \nabla^{1+\alpha}h0 = \sharp\nabla f\alpha$$

to show $(\Yup)\boldsymbol{S} = \sharp\nabla$, and hence $(\Yup)\boldsymbol{S}$ is a derivative by Lemma 4.6. With this Lemmas 5.4 and 5.3 complete the result. ■

This tells us most of what we need about the accumulation levels of the operators of Table 2. By Theorem 3.9 and Lemma 5.8(c), for each nonzero ordinal $1 + i$ the operator

$$\nabla = [\,{\scriptstyle 1}\,]^{1+i}[\,{\scriptstyle 0}\,]$$

is helpful and absorbent.

7.8 THEOREM. *For each ordinal* i *the operator*

$$\boldsymbol{S}\left[\begin{smallmatrix} * \\ 1+i \end{smallmatrix}\right]$$

is a derivative, and the shuffle equivalence

$$\boldsymbol{S}\left[\begin{smallmatrix} * \\ 1+i \end{smallmatrix}\right] \rightleftharpoons [\,{\scriptstyle 1}\,]^{1+i}[\,{\scriptstyle 0}\,]$$

holds.

Proof. We prove the two assertion in tandem by induction on i.

For the base case, $i = 0$, we have

$$\boldsymbol{S}\begin{bmatrix} * \\ 1 \end{bmatrix} = (\Uparrow)\,\boldsymbol{Veb} = \natural[\,0\,]$$

by Lemmas 7.1, 5.2, and 5.4. Thus Lemmas 5.3 complete this case.

For the induction step, $i \mapsto i + 1$, we have

$$\boldsymbol{S}\left[1 + \overset{*}{i} + 1\right] = (\curlywedge)\boldsymbol{S}\left[1 + \overset{*}{i}\right]$$

and

$$\boldsymbol{S}\left[1 + \overset{*}{i}\right] \rightleftharpoons \nabla = [\,1\,]^{1+i}[\,0\,]$$

by the induction hypothesis. Thus Corollary 7.7 and Lemma 4.6 complete this step.

For the induction leap to a limit ordinal θ, the induction hypothesis shows that

$$\boldsymbol{S}\begin{bmatrix} * \\ \theta \end{bmatrix} = \bigvee\left\{\boldsymbol{S}\left[\overset{*}{i+1}\right] \,\middle|\, i < \theta\right\}$$

is the pointwise supremum of an ascending chain of derivatives, and hence is a derivative.

For each $f \in \mathbb{F}$, with $h = \boldsymbol{Fix}f$, the continuity property of $\boldsymbol{Fix}$, Lemma 2.11, gives

$$\begin{aligned}
\left(\boldsymbol{Fix} \circ \boldsymbol{S}\begin{bmatrix} * \\ \theta \end{bmatrix}\right)f &= \boldsymbol{Fix}\left(\bigvee\left\{\boldsymbol{S}\left[\overset{*}{i+1}\right]f \mid i < \theta\right\}\right) \\
&= \bigvee\left\{\boldsymbol{Fix}\left(\boldsymbol{S}\left[\overset{*}{i+1}\right]f\right) \mid i < \theta\right\} \\
&= \bigvee\left\{\left(\boldsymbol{Fix} \circ \boldsymbol{S}\left[\overset{*}{i+1}\right]\right)f\right) \mid i < \theta\} \\
&= \bigvee\{[\,1\,]^{i+1}[\,0\,]h \mid i < \theta\} \\
&= [\,1\,]^{\theta}[\,0\,]h \qquad\qquad = ([\,1\,]^{\theta}[\,0\,] \circ \boldsymbol{Fix})f
\end{aligned}$$

for the required shuffle equivalence. ■

We now have the information we need to describe what is going on in Table 2. As we have seen, because of the erratic nature of the iterations there are several hiccoughs, especially near to limit ordinals. With a bit of subterfuge some of these can be hidden.

We use the column helpful operators $\nabla\begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$ of Definition 6.5.

7.9 THEOREM. *We have*

(a) $\boldsymbol{S}\begin{bmatrix} \alpha + 1 \\ i + 1 \end{bmatrix} = \flat\left(\nabla\begin{bmatrix} \alpha + 1 \\ i + 1 \end{bmatrix}\right)$

(b) $\boldsymbol{S}\begin{bmatrix} \alpha + 1 \\ i + 1 \end{bmatrix} \rightleftharpoons [\,0\,] \circ \nabla\begin{bmatrix} \alpha + 1 \\ i + 1 \end{bmatrix}$

(c) $\boldsymbol{S}\begin{bmatrix} \lambda \\ i + 1 \end{bmatrix} \rightleftharpoons \nabla\begin{bmatrix} \lambda + 1 \\ i + 1 \end{bmatrix}$

for all ordinals i, α and limit ordinals λ.

Proof. We look at (a, b, c) in turn.

(a) For $i = 0$ we have

$$\boldsymbol{S}\begin{bmatrix} \alpha \\ 1 \end{bmatrix} = \boldsymbol{Veb}^{\alpha}$$

for each ordinal α. In particular

$$\boldsymbol{S}\begin{bmatrix}\alpha+1\\1\end{bmatrix} = \boldsymbol{Veb}^{\alpha+1} = \flat\big([\,0\,]^{\alpha}\big) = \flat\Big(\nabla\begin{bmatrix}\alpha+1\\1\end{bmatrix}\Big)$$

By Lemmas 5.2, 5.5, and 5.4.

For $i \neq 0$ we have

$$\boldsymbol{S}\begin{bmatrix}*\\i\end{bmatrix} \rightleftharpoons [\,1\,]^{i}[\,0\,]$$

by Theorem 7.8, and hence

$$\boldsymbol{S}\begin{bmatrix}\alpha+1\\i+1\end{bmatrix} = \Big(\boldsymbol{S}\begin{bmatrix}*\\i\end{bmatrix}\Big)^{(\alpha+1)} = \flat\big(([\,1\,]^{i}[\,0\,])^{1+\alpha}\big) = \flat\Big(\nabla\begin{bmatrix}\alpha+1\\i+1\end{bmatrix}\Big)$$

by Theorem 7.6.

(b) This follows from (a) by Lemma 5.3.

(c) As in part (a) we have

$$\boldsymbol{S}\begin{bmatrix}\lambda\\1\end{bmatrix} = \boldsymbol{Veb}^{\lambda}$$

and hence

$$\boldsymbol{S}\begin{bmatrix}\lambda\\1\end{bmatrix} \rightleftharpoons [\,0\,]^{\lambda} = \nabla\begin{bmatrix}\lambda+1\\1\end{bmatrix}$$

by Lemmas 5.2 and 5.5.

For $i \neq 0$ we have

$$\boldsymbol{S}\begin{bmatrix}\lambda\\i+1\end{bmatrix} = \boldsymbol{S}\begin{bmatrix}*\\i\end{bmatrix}^{(\lambda)} \rightleftharpoons \big([\,1\,]^{i}[\,0\,]\big)^{\lambda} = \nabla\begin{bmatrix}\lambda+1\\i+1\end{bmatrix}$$

by Theorem 7.8 and Lemma 5.5. ■

Part (a) of this result gives us Theorem 6.6.

8 The intricate proof

In this section we prove Theorem 6.4 and the neat description of Theorem 6.9.

The derivative $\boldsymbol{Sch}\begin{bmatrix}-\\-\end{bmatrix}$ attached to a motor $\begin{bmatrix}-\\-\end{bmatrix}$ is built up column by column, as in Definition 6.3. The column derivatives $\boldsymbol{S}\begin{bmatrix}\cdot\\\cdot\end{bmatrix}$ are built up by the erratic rules of Table 2. These derivatives are analysed in Section 7 but for this section Table 2 tells us all we need to know. We carry out a double induction over the left-most epoch and exponent of a motor. The erratic nature of Table 2 means this proof is not as neat as we might expect.

To prove Theorem 6.4 we match the rules of Table 2 with the evaluation rules of Table 1. To do that we need a better understanding of these evaluation rules. We expand them into the rules of Table 3, so our first job here is to explain that table.

We deal only with canonical motors, so rule (A) is automatic. Rule (B) becomes (E0), and rule (D) becomes (E10). Rule (C), the important one, splits into nine cases.

Rule (C) says that the function

$$f\begin{pmatrix}\cdot & 1+\alpha & -\\ 0 & 1+i & -\end{pmatrix}$$

enumerates the common fixed point of the family $\mathcal{G}$ of functions

$$f\begin{pmatrix}\cdot & \beta & -\\ r & 1+i & -\end{pmatrix}$$

$$
\begin{array}{llll}
(E0) & f\left(\begin{smallmatrix} \cdot \\ 0 \end{smallmatrix}\right) & = f & \\
(E1) & f\left(\begin{smallmatrix} \cdot & 1 & - \\ 0 & 1 & - \end{smallmatrix}\right) & = \boldsymbol{Veb}g & g = f\left(\begin{smallmatrix} \cdot & - \\ 0 & - \end{smallmatrix}\right) \\
(E2) & f\left(\begin{smallmatrix} \cdot & 1+\alpha+1 & - \\ 0 & 1 & - \end{smallmatrix}\right) & = \boldsymbol{Veb}g & g = f\left(\begin{smallmatrix} \cdot & 1+\alpha & - \\ 0 & 1 & - \end{smallmatrix}\right) \\
(E3) & f\left(\begin{smallmatrix} \cdot & 1+\lambda & - \\ 0 & 1 & - \end{smallmatrix}\right) & = \boldsymbol{Veb}(\bigvee \mathcal{G}) & g_\alpha = f\left(\begin{smallmatrix} \cdot & 1+\alpha & - \\ 0 & 1 & - \end{smallmatrix}\right) \\
(E4) & f\left(\begin{smallmatrix} \cdot & 1 & - \\ 0 & 1+i+1 & - \end{smallmatrix}\right) & = \boldsymbol{Veb}g & g = f\left(\begin{smallmatrix} \cdot & - \\ 1+i & - \end{smallmatrix}\right) \\
(E5) & f\left(\begin{smallmatrix} \cdot & 1+\alpha+1 & - \\ 0 & 1+i+1 & - \end{smallmatrix}\right) & = \boldsymbol{Veb}g & g = f\left(\begin{smallmatrix} \cdot & 1+\alpha & - \\ 1+i & 1+i+1 & - \end{smallmatrix}\right) \\
(E6) & f\left(\begin{smallmatrix} \cdot & 1+\lambda & - \\ 0 & 1+i+1 & - \end{smallmatrix}\right) & = \boldsymbol{Veb}(\bigvee \mathcal{G}) & g_\alpha = f\left(\begin{smallmatrix} \cdot & 1+\alpha & - \\ 1+i & 1+i+1 & - \end{smallmatrix}\right) \\
(E7) & f\left(\begin{smallmatrix} \cdot & 1 & - \\ 0 & 1+\theta & - \end{smallmatrix}\right) & = \boldsymbol{Veb}(\bigvee \mathcal{G}) & g_i = f\left(\begin{smallmatrix} \cdot & - \\ 1+i & - \end{smallmatrix}\right) \\
(E8) & f\left(\begin{smallmatrix} \cdot & 1+\alpha+1 & - \\ 0 & 1+\theta & - \end{smallmatrix}\right) & = \boldsymbol{Veb}(\bigvee \mathcal{G}) & g_i = f\left(\begin{smallmatrix} \cdot & 1+\alpha & - \\ 1+i & 1+\theta & - \end{smallmatrix}\right) \\
(E9) & f\left(\begin{smallmatrix} \cdot & 1+\lambda & - \\ 0 & 1+\theta & - \end{smallmatrix}\right) & = \boldsymbol{Veb}(\bigvee \mathcal{G}) & g_{i,\alpha} = f\left(\begin{smallmatrix} \cdot & 1+\alpha & - \\ 1+i & 1+\theta & - \end{smallmatrix}\right) \\
(E10) & f\left(\begin{smallmatrix} 1+\eta & - \\ 1+r & - \end{smallmatrix}\right) & = f\left(\begin{smallmatrix} 0 & 1+\eta & - \\ 0 & 1+r & - \end{smallmatrix}\right) &
\end{array}
$$

Table 3: The expanded evaluation rules for brackets

---

for $r < 1+i$ and $\beta < 1+\alpha$. It turns out that $\mathcal{G}$ is directed so that

$$f\left(\begin{smallmatrix} \cdot & 1+\alpha & - \\ 0 & 1+i & - \end{smallmatrix}\right) = \boldsymbol{Veb}(\bigvee \mathcal{G})$$

using the single function $\bigvee \mathcal{G}$. We consider whether each of i and α is zero, a successor, or a limit ordinal, to give the nine cases. In four cases the family $\mathcal{G}$ has a maximum member.

In Table 3 the ordinals i, α, r are arbitrary, and θ, λ are limit ordinals. In rules (E4, E5, E6) the ordinal i is rigid, but in rules (E7, E8, E9) is varies over $i < \theta$. In rules (E2, E5, E8) the ordinal α is rigid, but in rules (E3, E6, E9) it varies over $\alpha < \lambda$.

In each case we start with a function of a particular shape as listed on the left of Table 4. These match the shapes of the left hand functions of Table 3 in rows (E1 - E9). In each case we use rules (C) to obtain a function or family of functions as in the central column of Table 4. Then for various different reasons we move to the function or family of functions in the right hand column. These match the right hand functions in Table 3.

(E1) After using (C) we obtain the single central function. We then use (A) to obtain the right hand function.

(E2) After using (C) we obtain the family of central functions for $\beta \leq \alpha$. It turns out that this family is a chain with a maximum member, the right hand function.

(E3) After using (C) we obtain the family of central functions for $\alpha < \lambda$. This family does not have a maximum member, and so is the family $\mathcal{G}$. This is the only case where there isn't a second step.

$(E1)$	$f\begin{pmatrix} \cdot & 1 & - \\ 0 & 1 & - \end{pmatrix}$	$f\begin{pmatrix} \cdot & 0 & - \\ 0 & 1 & - \end{pmatrix}$	$f\begin{pmatrix} \cdot & - \\ 0 & - \end{pmatrix}$
$(E2)$	$f\begin{pmatrix} \cdot & 1+\alpha+1 & - \\ 0 & 1 & \end{pmatrix}$	$f\begin{pmatrix} \cdot & 1+\beta & - \\ 0 & 1 & \end{pmatrix}$	$f\begin{pmatrix} \cdot & 1+\alpha & - \\ 0 & 1 & - \end{pmatrix}$
$(E3)$	$f\begin{pmatrix} \cdot & 1+\lambda & - \\ 0 & 1 & - \end{pmatrix}$	$f\begin{pmatrix} \cdot & 1+\alpha & - \\ 0 & 1 & - \end{pmatrix}$	$f\begin{pmatrix} \cdot & 1+\alpha & - \\ 0 & 1 & - \end{pmatrix}$
$(E4)$	$f\begin{pmatrix} \cdot & 1 & - \\ 0 & 1+i+1 & - \end{pmatrix}$	$f\begin{pmatrix} \cdot & 0 & - \\ r & 1+i+1 & - \end{pmatrix}$	$f\begin{pmatrix} \cdot & - \\ 1+i & - \end{pmatrix}$
$(E5)$	$f\begin{pmatrix} \cdot & 1+\alpha+1 & - \\ 0 & 1+i+1 & - \end{pmatrix}$	$f\begin{pmatrix} \cdot & 1+\beta & - \\ r & 1+i+1 & - \end{pmatrix}$	$f\begin{pmatrix} \cdot & 1+\alpha & - \\ 1+i & 1+i+1 & - \end{pmatrix}$
$(E6)$	$f\begin{pmatrix} \cdot & 1+\lambda & - \\ 0 & 1+i+1 & - \end{pmatrix}$	$f\begin{pmatrix} \cdot & 1+\alpha & - \\ r & 1+i+1 & - \end{pmatrix}$	$f\begin{pmatrix} \cdot & 1+\alpha & - \\ 1+i & 1+i+1 & - \end{pmatrix}$
$(E7)$	$f\begin{pmatrix} \cdot & 1 & - \\ 0 & 1+\theta & - \end{pmatrix}$	$f\begin{pmatrix} \cdot & 0 & - \\ r & 1+\theta & - \end{pmatrix}$	$f\begin{pmatrix} \cdot & - \\ 1+i & - \end{pmatrix}$
$(E8)$	$f\begin{pmatrix} \cdot & 1+\alpha+1 & - \\ 0 & 1+\theta & - \end{pmatrix}$	$f\begin{pmatrix} \cdot & 1+\beta & - \\ r & 1+\theta & - \end{pmatrix}$	$f\begin{pmatrix} \cdot & 1+\alpha & - \\ 1+i & 1+\theta & - \end{pmatrix}$
$(E9)$	$f\begin{pmatrix} \cdot & 1+\lambda & - \\ 0 & 1+\theta & - \end{pmatrix}$	$f\begin{pmatrix} \cdot & 1+\alpha & - \\ r & 1+\theta & - \end{pmatrix}$	$f\begin{pmatrix} \cdot & 1+\alpha & - \\ 1+i & 1+\theta & - \end{pmatrix}$

Table 4: A partial explanation of Table 3

(E4) After using (C) we obtain the family of central functions for $r \leq 1+i$. Rule (A) allows us to omit the left-most column of the motor, and then the resulting family has a maximum member as given on the right.

(E5) After using (C) we obtain the family of central functions for $r \leq 1+i$ and $\beta \leq \alpha$. This family has a maximum member as given on the right.

(E6) After using (C) we obtain the family of central functions for $r \leq 1+i$ and $\alpha < \lambda$. For each α the corresponding subfamily has a maximum member given by $r = 1+i$. Thus it suffices to use that family of functions on the right for $\alpha < \lambda$.

(E7) After using (C) we obtain the family of central functions for $r < 1+\theta$. Rule (A) allows us to omit the left-most column of the motor, so we may use the family of function on the right. You may wonder what has happened to the central function for $r = 0$. The resulting functions form a chain, and it is convenient to omit the bottom one given by $r = 0$.

(E8) After using (C) we obtain the family of central functions for $r < 1+\theta$ and $\beta \leq \alpha$. For each r this subfamily has a maximum member given by $\beta = \alpha$. Thus we may use the family of functions on the right for $i < \theta$. As in case (E7) we omit $r = 0$.

(E9) This, the most general case, gives a doubly indexded family of function. After using (C) we obtain the family of central functions for $r < 1+\theta$ and $\alpha < \lambda$. For each α we omit the function with $r = 0$, and use the family of functions on the right.

When written out as in Table 3 becomes clearer how the recursion works. It is clear that at each stage the recursion generates some kind of derivative, but it is not at all clear which one. Also, the influence of clause $(E10)$ still needs some thinking about.

To obtain Theorem 6.4 we have to prove something a bit stronger.

8.1 DEFINITION. Given an motor $\begin{bmatrix}-\\-\end{bmatrix}$ and a derivative $\boldsymbol{D}$ we write

$$\begin{bmatrix}-\\-\end{bmatrix} \Rrightarrow \boldsymbol{D}$$

and say $\begin{bmatrix}-\\-\end{bmatrix}$ drives $\boldsymbol{D}$ if

$$f\begin{pmatrix}\cdot & -\\ 0 & -\end{pmatrix} = \boldsymbol{D}f$$

for each fruitful function f. ■

Trivially

$$\begin{bmatrix}\,\\\,\end{bmatrix} \Rrightarrow \boldsymbol{Id}$$

the empty motor drives the identity derivative (which doesn't move any fruitful function).

We require

$$\begin{bmatrix}-\\-\end{bmatrix} \Rrightarrow \boldsymbol{Sch}\begin{bmatrix}-\\-\end{bmatrix}$$

for an arbitrary motor $\begin{bmatrix}-\\-\end{bmatrix}$. We build up each side column by column, but to do that we need to vary the epoch and the exponent separately.

8.2 DEFINITION. For each pair i, α of ordinals let

$$\langle i, \alpha\rangle \quad \text{abbreviate} \quad \left\{\begin{matrix}\text{For each motor } \begin{bmatrix}-\\-\end{bmatrix} \text{ and derivative } \boldsymbol{D}, \\ \text{if } \begin{bmatrix}-\\-\end{bmatrix} \Rrightarrow \boldsymbol{D} \text{ then } \begin{bmatrix}1+\alpha & -\\ 1+i & -\end{bmatrix} \Rrightarrow \boldsymbol{S}\begin{bmatrix}\alpha+1\\ i+1\end{bmatrix} \circ \boldsymbol{D}.\end{matrix}\right\}$$

to produce a double induction hypothesis. ■

In this notation we show

$$\langle i, \alpha\rangle$$

for each pair i, α of ordinals. Let

$$\langle i\rangle \quad \text{abbreviate} \quad (\forall\alpha)\langle i, \alpha\rangle$$

so that

$$(\forall i)\langle i\rangle$$

is out target. We proceed by a double induction, an outer induction over i and an inner induction over α. Observe that $\langle i, \alpha\rangle$ has hidden universal quantifications over fruitful functions, motors, and derivative. We make use of this in the induction proof.

We prove the nine implications (I1 - I9) as listed in Table 5. (The implication (I1) has an empty hypothesis.) With these, (I1, I2, I3) give (base), and (I4, I5, I6) give (step), and (I7, I8, I9) give (leap). Finally, (base, step, leap) give (target), as required.

We prove (I1 - I9) in turn, from simplest to most complicated.

8.3 LEMMA. *The three induction clauses (I1, I2, I3) hold.*

$(I1)$			$\Longrightarrow$	$\langle 0, 0\rangle$
$(I2)$		$\langle 0, \alpha\rangle$	$\Longrightarrow$	$\langle 0, \alpha+1\rangle$
$(I3)$		$(\forall \alpha < \lambda)\langle 0, \alpha\rangle$	$\Longrightarrow$	$\langle 0, \lambda\rangle$
(base)			$\Longrightarrow$	$\langle 0\rangle$
$(I4)$	$\langle i\rangle$		$\Longrightarrow$	$\langle i+1, 0\rangle$
$(I5)$	$\langle i\rangle$	$\langle i+1, \alpha\rangle$	$\Longrightarrow$	$\langle i+1, \alpha+1\rangle$
$(I6)$	$\langle i\rangle$	$(\forall \alpha < \lambda)\langle i+1, \alpha\rangle$	$\Longrightarrow$	$\langle i+1, \lambda\rangle$
(step)	$\langle i\rangle$		$\Longrightarrow$	$\langle i+1\rangle$
$(I7)$	$(\forall i < \theta)\langle i\rangle$		$\Longrightarrow$	$\langle \theta, 0\rangle$
$(I8)$	$(\forall i < \theta)\langle i\rangle$	$\langle \theta, \alpha\rangle$	$\Longrightarrow$	$\langle \theta, \alpha+1\rangle$
$(I9)$	$(\forall i < \theta)\langle i\rangle$	$(\forall \alpha < \lambda)\langle \theta, \alpha\rangle$	$\Longrightarrow$	$\langle \theta, \lambda\rangle$
(leap)	$(\forall i < \theta)\langle i\rangle$		$\Longrightarrow$	$\langle \theta\rangle$
(target)			$\Longrightarrow$	$(\forall i)\langle i\rangle$

Table 5: The induction clauses

Proof. By clause (epoch 1) of Table 2 we have

$$\boldsymbol{S}\begin{bmatrix}\alpha+1\\1\end{bmatrix} = \boldsymbol{Veb}^{\alpha+1}$$

for each ordinal α. We use this in each of the three parts. We also use

$$(!) \qquad \begin{bmatrix}-\\-\end{bmatrix} \Rrightarrow \boldsymbol{D}$$

as an hypothesis. Here $\begin{bmatrix}-\\-\end{bmatrix}$ is an arbitrary motor and $\boldsymbol{D}$ is an arbitrary derivative.

(I1) Assuming (!) we show that

$$\begin{bmatrix}1 & -\\1 & -\end{bmatrix} \Rrightarrow \boldsymbol{S}\begin{bmatrix}1\\1\end{bmatrix} \circ \boldsymbol{D}$$

holds. Thus we require

$$(?1) \qquad f\begin{pmatrix}\cdot & 1 & -\\0 & 1 & -\end{pmatrix} = (\boldsymbol{Veb} \circ \boldsymbol{D})f$$

for each $f \in \mathbb{F}$. By (E1) of Table 3 we have

$$f\begin{pmatrix}\cdot & 1 & -\\0 & 1 & -\end{pmatrix} = \boldsymbol{Veb}g$$

where

$$g = f\begin{pmatrix}\cdot & -\\0 & -\end{pmatrix}$$

is the auxiliary function. By (!) we have

$$g = \boldsymbol{D}f$$

for the required result.

(I2) Let α be a fixed ordinal. We assume

$$(!2) \qquad \langle 0, \alpha\rangle$$

and show that

$$\langle 0, \alpha+1\rangle$$

holds. To this end we suppose (!) and show

$$\left[\begin{smallmatrix}1+\alpha+1 & - \\ 1 & -\end{smallmatrix}\right] \Rrightarrow \boldsymbol{S}\left[\begin{smallmatrix}\alpha+2 \\ 1\end{smallmatrix}\right] \circ \boldsymbol{D}$$

that is

$$(?2) \qquad f\left(\begin{smallmatrix}\cdot & 1+\alpha+1 & - \\ 0 & 1 & -\end{smallmatrix}\right) = (\boldsymbol{Veb}^{\alpha+2} \circ \boldsymbol{D})f$$

for each $f \in \mathbb{F}$.

For such a function f let

$$g = f\left(\begin{smallmatrix}\cdot & 1+\alpha & - \\ 0 & 1 & -\end{smallmatrix}\right)$$

so that

$$g = (\boldsymbol{Veb}^{\alpha+1} \circ \boldsymbol{D})f$$

by (!2, !). With this (E2) of Table 3 gives

$$f\left(\begin{smallmatrix}\cdot & 1+\alpha+1 & - \\ 0 & 1 & -\end{smallmatrix}\right) = \boldsymbol{Veb}g = (\boldsymbol{Veb}^{\alpha+2} \circ \boldsymbol{D})f$$

as required.

(I3) Let λ be a fixed limit ordinal. We assume

$$(!3) \qquad (\forall \alpha < \lambda)\langle 0, \alpha\rangle$$

and show that

$$\langle 0, \lambda\rangle$$

holds. To this end we suppose (!) and show

$$\left[\begin{smallmatrix}1+\lambda & - \\ 1 & -\end{smallmatrix}\right] \Rrightarrow \boldsymbol{S}\left[\begin{smallmatrix}\lambda+1 \\ 1\end{smallmatrix}\right] \circ \boldsymbol{D}$$

that is

$$(?3) \qquad f\left(\begin{smallmatrix}\cdot & 1+\lambda & - \\ 0 & 1 & -\end{smallmatrix}\right) = (\boldsymbol{Veb}^{\lambda+1} \circ \boldsymbol{D})f$$

for each $f \in \mathbb{F}$.

For each $\alpha < \lambda$ let

$$g_\alpha = f\left(\begin{smallmatrix}\cdot & 1+\alpha & - \\ 0 & 1 & -\end{smallmatrix}\right)$$

and let

$$\mathcal{G} = \{g_\alpha \,|\, \alpha < \lambda\}$$

be the family of all such functions. By (!3) and (!) we have

$$g_\alpha = \left(\boldsymbol{S}\left[\begin{smallmatrix}\alpha+1 \\ 1\end{smallmatrix}\right] \circ \boldsymbol{D}\right)f = (\boldsymbol{Veb}^{\alpha+1} \circ \boldsymbol{D})f$$

and hence

$$\bigvee \mathcal{G} = (\boldsymbol{Veb}^{\lambda} \circ \boldsymbol{D})f$$

by the construction of the pointwise supremum.

A use of (E3) now gives

$$f\left(\begin{smallmatrix}\cdot & 1+\lambda & - \\ 0 & 1 & -\end{smallmatrix}\right) = \boldsymbol{Veb}(\bigvee \mathcal{G}) = (\boldsymbol{Veb}^{\lambda+1} \circ \boldsymbol{D})f$$

$$\begin{array}{lllll}
(4) & g & = f\left(\begin{smallmatrix} \cdot & - \\ 1+i & - \end{smallmatrix}\right) & h & = \left(\boldsymbol{S}\left[\begin{smallmatrix} * \\ i+1 \end{smallmatrix}\right] \circ \boldsymbol{Der}\right) f \\
(5) & g & = f\left(\begin{smallmatrix} \cdot & 1+\alpha & - \\ 1+i & 1+i+1 & - \end{smallmatrix}\right) & h & = \left(\boldsymbol{S}\left[\begin{smallmatrix} * \\ i+1 \end{smallmatrix}\right] \circ \boldsymbol{S}\left[\begin{smallmatrix} \alpha+1 \\ i+2 \end{smallmatrix}\right] \circ \boldsymbol{Der}\right) f \\
(6) & g_\alpha & = f\left(\begin{smallmatrix} \cdot & 1+\alpha & - \\ 1+i & 1+i+1 & - \end{smallmatrix}\right) & h_\alpha & = \left(\boldsymbol{S}\left[\begin{smallmatrix} * \\ i+1 \end{smallmatrix}\right] \circ \boldsymbol{S}\left[\begin{smallmatrix} \alpha+1 \\ i+2 \end{smallmatrix}\right] \circ \boldsymbol{Der}\right) f \\
(7) & g_i & = f\left(\begin{smallmatrix} \cdot & - \\ 1+i & - \end{smallmatrix}\right) & h_i & = \left(\boldsymbol{S}\left[\begin{smallmatrix} * \\ i+1 \end{smallmatrix}\right] \circ \boldsymbol{Der}\right) f \\
(8) & g_i & = f\left(\begin{smallmatrix} \cdot & 1+\alpha & - \\ 1+i & 1+\theta & - \end{smallmatrix}\right) & h_i & = \left(\boldsymbol{S}\left[\begin{smallmatrix} * \\ i+1 \end{smallmatrix}\right] \circ \boldsymbol{S}\left[\begin{smallmatrix} \alpha+1 \\ \theta+1 \end{smallmatrix}\right] \circ \boldsymbol{Der}\right) f \\
(9) & g_{i,\alpha} & = f\left(\begin{smallmatrix} \cdot & 1+\alpha & - \\ 1+i & 1+\theta & - \end{smallmatrix}\right) & h_{i,\alpha} & = \left(\boldsymbol{S}\left[\begin{smallmatrix} * \\ i+1 \end{smallmatrix}\right] \circ \boldsymbol{S}\left[\begin{smallmatrix} \alpha+1 \\ \theta+1 \end{smallmatrix}\right] \circ \boldsymbol{Der}\right) f
\end{array}$$

Table 6: The auxiliary functions

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for the required result. ■

The three clauses (I1, I2, I3) are straight forward for essentially they merely move up a Veblen hierarchy. To obtain the remaining six clauses we need a couple of extra tricks.

In each of these cases we have to describe a function

$$\boldsymbol{Veb}(\bigvee \mathcal{G})$$

for a family $\mathcal{G}$ of functions $g_\bullet$ indexed in a certain way. One trick is to replace $\mathcal{G}$ by a family $\mathcal{H}$ of functions $h_\bullet$ for which $\bigvee \mathcal{G}$ and $\bigvee \mathcal{H}$ have the same fixed points, and hence

$$\boldsymbol{Veb}(\bigvee \mathcal{G}) = \boldsymbol{Veb}(\bigvee \mathcal{H})$$

holds. These auxiliary functions $h_\bullet$ are given in Table 6.

Another trick is an interlacing argument. We use this in the third part of the following.

8.4 LEMMA. *The three induction clauses (I4, I5, I6) hold.*

Proof. We use

$$(!) \qquad \left[\begin{smallmatrix} - \\ - \end{smallmatrix}\right] \Rrightarrow \boldsymbol{D}$$

as an hypothesis in each of the three parts. Here $\left[\begin{smallmatrix} - \\ - \end{smallmatrix}\right]$ is an arbitrary motor and $\boldsymbol{D}$ is an arbitrary derivative.

(I4) Let i be a fixed ordinal. We assume

$$(!4) \qquad \langle i \rangle$$

and show that

$$\langle i+1, 0 \rangle$$

holds. Thus, assuming (!) we require

$$\left[\begin{smallmatrix} 1 & - \\ 1+i+1 & - \end{smallmatrix}\right] \Rrightarrow \boldsymbol{S}\left[\begin{smallmatrix} 1 \\ i+2 \end{smallmatrix}\right] \circ \boldsymbol{D}$$

that is

$$(?4) \qquad f\left(\begin{smallmatrix} \cdot & 1 & - \\ 0 & 1+i+1 & - \end{smallmatrix}\right) = \big(\boldsymbol{S}\left[\begin{smallmatrix} 1 \\ i+2 \end{smallmatrix}\right] \circ \boldsymbol{D}\big) f$$

for an arbitrary $f \in \mathbb{F}$.

For such a function f let

$$g = f\left(\begin{smallmatrix} \cdot & - \\ i+i & - \end{smallmatrix}\right) \qquad h = \big(\boldsymbol{S}\left[\begin{smallmatrix} * \\ i+1 \end{smallmatrix}\right] \circ \boldsymbol{D}\big) f$$

as in Tables 3 and 6.

Consider any ordinal η. We have

$$\left[\begin{smallmatrix} 1+\eta & - \\ 1+i & - \end{smallmatrix}\right] \Rrightarrow \boldsymbol{S}\left[\begin{smallmatrix} \eta+1 \\ i+1 \end{smallmatrix}\right] \circ \boldsymbol{D}$$

by (!) and the instance $\langle i, \eta \rangle$ of (!4). With (E10) this gives

$$\begin{aligned} g(1+\eta) &= f\left(\begin{smallmatrix} 1+\eta & - \\ 1+i & - \end{smallmatrix}\right) \\ &= f\left(\begin{smallmatrix} 0 & 1+\eta & - \\ 0 & 1+i & - \end{smallmatrix}\right) \\ &= \big(\boldsymbol{S}\left[\begin{smallmatrix} \eta+1 \\ i+1 \end{smallmatrix}\right] \circ \boldsymbol{D}\big) f0 \\ &= \big(\boldsymbol{S}\left[\begin{smallmatrix} * \\ i+1 \end{smallmatrix}\right] \circ \boldsymbol{D}\big) f\eta = h\eta \end{aligned}$$

using $(*+)$ of Table 2 at the penultimate step.

This shows that $g\eta = h\eta$ for all infinite η, and hence the two functions g and h have the same fixed points. Thus

$$f\left(\begin{smallmatrix} \cdot & 1 & - \\ 0 & 1+i+1 & - \end{smallmatrix}\right) = \boldsymbol{Veb} g = \boldsymbol{Veb} h$$

by (E4) of Table 3. Finally, (exp1) of Table 2 gives

$$\boldsymbol{Veb} h = \big(\boldsymbol{Veb} \circ \boldsymbol{S}\left[\begin{smallmatrix} * \\ i+1 \end{smallmatrix}\right] \circ \boldsymbol{D}\big) f = \big(\boldsymbol{S}\left[\begin{smallmatrix} 1 \\ i+2 \end{smallmatrix}\right] \circ \boldsymbol{D}\big) f$$

for the required result.

(I5) Let i and α be fixed ordinals. We assume

$$(!5) \quad \langle i \rangle \qquad\qquad (!!5) \quad \langle i+1, \alpha \rangle$$

and show that

$$\langle i+1, \alpha+1 \rangle$$

holds. Thus, assuming (!) we require

$$\left[\begin{smallmatrix} 1+\alpha+1 & - \\ 1+i+1 & - \end{smallmatrix}\right] \Rrightarrow \boldsymbol{S}\left[\begin{smallmatrix} \alpha+2 \\ i+2 \end{smallmatrix}\right] \circ \boldsymbol{D}$$

that is

$$(?5) \qquad f\left(\begin{smallmatrix} \cdot & 1+\alpha+1 & - \\ 0 & 1+i+1 & - \end{smallmatrix}\right) = \big(\boldsymbol{S}\left[\begin{smallmatrix} \alpha+2 \\ i+2 \end{smallmatrix}\right] \circ \boldsymbol{D}\big) f$$

for an arbitrary $f \in \mathbb{F}$.

For such a function f let

$$g = f\left(\begin{smallmatrix} \cdot & 1+\alpha & - \\ i+i & 1+i+1 & - \end{smallmatrix}\right) \qquad h = \big(\boldsymbol{S}\left[\begin{smallmatrix} * \\ i+1 \end{smallmatrix}\right] \circ \boldsymbol{S}\left[\begin{smallmatrix} \alpha+1 \\ i+2 \end{smallmatrix}\right] \circ \boldsymbol{D}\big) f$$

as in Tables 3 and 6.

Consider any ordinal η. We have

$$\left[\begin{smallmatrix} & 1+\alpha & - \\ & 1+i+1 & - \end{smallmatrix}\right] \Rrightarrow \boldsymbol{S}\left[\begin{smallmatrix}\alpha+1 \\ i+2\end{smallmatrix}\right] \circ \boldsymbol{D}$$

by (!) and (!!5), and hence

$$\left[\begin{smallmatrix} 1+\eta & 1+\alpha & - \\ 1+i & 1+i+1 & - \end{smallmatrix}\right] \Rrightarrow \boldsymbol{S}\left[\begin{smallmatrix}\eta+1 \\ i+1\end{smallmatrix}\right] \circ \boldsymbol{S}\left[\begin{smallmatrix}\alpha+1 \\ i+2\end{smallmatrix}\right] \circ \boldsymbol{D}$$

by the instance $\langle i, \eta\rangle$ of (!5). With (E10) this gives

$$\begin{aligned} g(1+\eta) &= f\left(\begin{smallmatrix} 1+\eta & 1+\alpha & - \\ 1+i & 1+i+1 & - \end{smallmatrix}\right) \\ &= f\left(\begin{smallmatrix} 0 & 1+\eta & 1+\alpha & - \\ 0 & 1+i & 1+i+1 & - \end{smallmatrix}\right) \\ &= \big(\boldsymbol{S}\left[\begin{smallmatrix}\eta+1 \\ i+1\end{smallmatrix}\right] \circ \boldsymbol{S}\left[\begin{smallmatrix}\alpha+1 \\ i+2\end{smallmatrix}\right] \circ \boldsymbol{D}\big) f0 \\ &= \big(\boldsymbol{S}\left[\begin{smallmatrix} * \\ i+1\end{smallmatrix}\right] \circ \boldsymbol{S}\left[\begin{smallmatrix}\alpha+1 \\ i+2\end{smallmatrix}\right] \circ \boldsymbol{D}\big) f\eta = h\eta \end{aligned}$$

using $(*+)$ of Table 2 at the penultimate step.

This shows that the two functions g and h have the same fixed points. Thus

$$f\left(\begin{smallmatrix} \cdot & 1+\alpha+1 & - \\ 0 & 1+i+1 & - \end{smallmatrix}\right) = \boldsymbol{Veb}g = \boldsymbol{Veb}h$$

by (E5) of Table 3. Finally, (exp+) of Table 2 gives

$$\boldsymbol{Veb}h = \big(\boldsymbol{Veb} \circ \boldsymbol{S}\left[\begin{smallmatrix} * \\ i+1\end{smallmatrix}\right] \circ \boldsymbol{S}\left[\begin{smallmatrix}\alpha+1 \\ i+2\end{smallmatrix}\right] \circ \boldsymbol{D}\big) f = \big(\boldsymbol{S}\left[\begin{smallmatrix}\alpha+2 \\ i+2\end{smallmatrix}\right] \circ \boldsymbol{D}\big) f$$

for the required result.

(I6) Let i and λ be fixed ordinals with λ a limit ordinal. We assume

$$(!6) \quad \langle i\rangle \qquad\qquad (!!6) \quad (\forall \alpha < \lambda)\langle i+1, \alpha\rangle$$

and show that

$$\langle i+1, \lambda\rangle$$

holds. Thus, assuming (!) we require

$$\left[\begin{smallmatrix} & 1+\lambda & - \\ & 1+i+1 & - \end{smallmatrix}\right] \Rrightarrow \boldsymbol{S}\left[\begin{smallmatrix}\lambda+1 \\ i+2\end{smallmatrix}\right] \circ \boldsymbol{D}$$

that is

$$(?6) \quad f\left(\begin{smallmatrix} \cdot & 1+\lambda & - \\ 0 & 1+i+1 & - \end{smallmatrix}\right) = \big(\boldsymbol{S}\left[\begin{smallmatrix}\lambda+1 \\ i+2\end{smallmatrix}\right] \circ \boldsymbol{D}\big) f$$

for an arbitrary $f \in \mathbb{F}$.

For such a function f let

$$g_\alpha = f\left(\begin{smallmatrix} \cdot & 1+\alpha & - \\ i+i & 1+i+1 & - \end{smallmatrix}\right) \qquad h_\alpha = \big(\boldsymbol{S}\left[\begin{smallmatrix} * \\ i+1\end{smallmatrix}\right] \circ \boldsymbol{S}\left[\begin{smallmatrix}\alpha+1 \\ i+2\end{smallmatrix}\right] \circ \boldsymbol{D}\big) f$$

for each $\alpha < \lambda$, as in Tables 3 and 6. Also let

$$\mathcal{G} = \{g_\alpha \mid \alpha < \lambda\} \qquad \mathcal{H} = \{h_\alpha \mid \alpha < \lambda\}$$

be the two families of these functions.

As in the case (I5), for each $\alpha < \lambda$ the two functions g_α and h_α have the same fixed points. Thus, by (E6) we have

$$f\begin{pmatrix} \cdot & 1+\lambda & - \\ 0 & 1+i+1 & - \end{pmatrix} = \boldsymbol{Veb}(\bigvee \mathcal{G}) = \boldsymbol{Veb}(\bigvee \mathcal{H})$$

and it remains to calculate $\boldsymbol{Veb}(\bigvee \mathcal{H})$.

Let

$$k_\alpha = \left(\boldsymbol{S}\begin{bmatrix} \alpha+1 \\ i+2 \end{bmatrix} \circ \boldsymbol{D}\right)f$$

for each $\alpha < \lambda$, and let

$$\mathcal{K} = \{k_\alpha \mid \alpha < \lambda\}$$

be the family of these functions. We have

$$\boldsymbol{S}\begin{bmatrix} \alpha+1 \\ i+2 \end{bmatrix} \le \boldsymbol{S}\begin{bmatrix} * \\ i+1 \end{bmatrix} \circ \boldsymbol{S}\begin{bmatrix} \alpha+1 \\ i+2 \end{bmatrix} \le \boldsymbol{Veb} \circ \boldsymbol{S}\begin{bmatrix} * \\ i+1 \end{bmatrix} \circ \boldsymbol{S}\begin{bmatrix} \alpha+1 \\ i+2 \end{bmatrix} = \boldsymbol{S}\begin{bmatrix} \alpha+2 \\ i+2 \end{bmatrix}$$

using (exp+) at the last step. Thus

$$k_\alpha \le h_\alpha \le k_{\alpha+1}$$

and hence

$$\bigvee \mathcal{H} = \bigvee \mathcal{K} = \left(\boldsymbol{S}\begin{bmatrix} \lambda \\ i+2 \end{bmatrix} \circ \boldsymbol{D}\right)f$$

by an interlacing argument and a use of (expℓ).

Finally, we have

$$\begin{aligned} f\begin{pmatrix} \cdot & 1+\lambda & - \\ 0 & 1+i+1 & - \end{pmatrix} &= \boldsymbol{Veb}(\bigvee \mathcal{G}) \\ &= \boldsymbol{Veb}(\bigvee \mathcal{H}) \\ &= \boldsymbol{Veb}(\bigvee \mathcal{K}) \\ &= \left(\boldsymbol{Veb} \circ \boldsymbol{S}\begin{bmatrix} \lambda \\ i+2 \end{bmatrix} \circ \boldsymbol{D}\right)f = \left(\boldsymbol{S}\begin{bmatrix} \lambda+1 \\ i+2 \end{bmatrix} \circ \boldsymbol{D}\right)f \end{aligned}$$

by a use of (expℓ+) at the last step, to give the required result. ■

These arguments have a common theme with various crucial differences at each case. As far as I can discern there doesn't seem to be a more uniform proof. This, I believe, is a consequence of the stutters in the constructions. This also illustrates the benefits of using helpful functions and operators, and the more uniform constructions these allow.

The proofs for the remaining three clauses are in the same mode. In each case much of the argument we have seen already, but each has a new twist.

8.5 LEMMA. *The three induction clauses (I7, I8, I9) hold.*

Proof. We use

$$(!) \qquad \begin{bmatrix} - \\ - \end{bmatrix} \Rrightarrow \boldsymbol{D}$$

as an hypothesis in each of the three parts. Here $\begin{bmatrix} - \\ - \end{bmatrix}$ is an arbitrary motor and $\boldsymbol{D}$ is an arbitrary derivative.

(I7) Let θ be a fixed limit ordinal. We assume

$$(!7) \qquad (\forall i < \theta)\langle i \rangle$$

and show that

$$\langle \theta, 0 \rangle$$

holds. Thus, assuming (!) we require

$$\begin{bmatrix} 1 & - \\ 1+\theta & - \end{bmatrix} \Rrightarrow \boldsymbol{S}\begin{bmatrix} 1 \\ \theta+1 \end{bmatrix} \circ \boldsymbol{D}$$

that is

$$(?7) \qquad f\begin{pmatrix} \cdot & 1 & - \\ 0 & 1+\theta & - \end{pmatrix} = \Big(\boldsymbol{S}\begin{bmatrix} 1 \\ \theta+1 \end{bmatrix} \circ \boldsymbol{D}\Big) f$$

for an arbitrary $f \in \mathbb{F}$.

For such a function f let

$$g_i = f\begin{pmatrix} \cdot & - \\ i+i & - \end{pmatrix} \qquad h_i = \Big(\boldsymbol{S}\begin{bmatrix} * \\ i+1 \end{bmatrix} \circ \boldsymbol{D}\Big) f$$

for each $i < \theta$, as in Tables 3 and 6. Also let

$$\mathcal{G} = \{ g_i \mid i < \theta \} \qquad \mathcal{H} = \{ h_i \mid i < \theta \}$$

be the two families of these functions.

As in the case (I4), for each $i < \theta$ the two functions g_i and h_i have the same fixed points. Also

$$\bigvee \mathcal{H} = \Big(\boldsymbol{S}\begin{bmatrix} * \\ \theta \end{bmatrix} \circ \boldsymbol{D}\Big) f$$

by $(*\ell)$ of Table 2. Thus a use of (E7) and (exp1) gives

$$\begin{aligned} f\begin{pmatrix} \cdot & 1 & - \\ 0 & 1+\theta & - \end{pmatrix} &= \boldsymbol{Veb}(\bigvee \mathcal{G}) \\ &= \boldsymbol{Veb}(\bigvee \mathcal{H}) \\ &= \Big(\boldsymbol{Veb} \circ \boldsymbol{S}\begin{bmatrix} * \\ \theta \end{bmatrix} \circ \boldsymbol{D}\Big) f = \Big(\boldsymbol{S}\begin{bmatrix} 1 \\ \theta+1 \end{bmatrix} \circ \boldsymbol{D}\Big) f \end{aligned}$$

for the required result.

(I8) Let θ and α be fixed ordinals with θ a limit ordinal. We assume

$$(!8) \qquad (\forall i < \theta)\langle i \rangle \qquad\qquad (!!8) \qquad \langle \theta, \alpha \rangle$$

and show that

$$\langle \theta, \alpha+1 \rangle$$

holds. Thus, assuming (!) we require

$$\begin{bmatrix} 1+\alpha+1 & - \\ 1+\theta & - \end{bmatrix} \Rrightarrow \boldsymbol{S}\begin{bmatrix} \alpha+2 \\ \theta+1 \end{bmatrix} \circ \boldsymbol{D}$$

that is

$$(?8) \qquad f\begin{pmatrix} \cdot & 1+\alpha+1 & - \\ 0 & 1+\theta & - \end{pmatrix} = \Big(\boldsymbol{S}\begin{bmatrix} \alpha+2 \\ \theta+1 \end{bmatrix} \circ \boldsymbol{D}\Big) f$$

for an arbitrary $f \in \mathbb{F}$.

For such a function f let

$$g_i = f\begin{pmatrix} \cdot & 1+\alpha & - \\ i+i & 1+\theta & - \end{pmatrix} \qquad h_i = \left(\boldsymbol{S}\begin{bmatrix} * \\ i+1 \end{bmatrix} \circ \boldsymbol{S}\begin{bmatrix} \alpha+1 \\ \theta+1 \end{bmatrix} \circ \boldsymbol{D}\right) f$$

for each $i < \theta$, as in Tables 3 and 6. Also let

$$\mathcal{G} = \{g_i \mid i < \theta\} \qquad \mathcal{H} = \{h_i \mid i < \theta\}$$

be the two families of these functions. As usual we show that for each $i < \theta$ the two functions g_i and h_i agree almost everywhere.

Consider any ordinal η. We have

$$\begin{bmatrix} 1+\alpha & - \\ 1+\theta & - \end{bmatrix} \Rrightarrow \boldsymbol{S}\begin{bmatrix} \alpha+1 \\ \theta+1 \end{bmatrix} \circ \boldsymbol{D}$$

by (!) and (!!8), and hence

$$\begin{bmatrix} 1+\eta & 1+\alpha & - \\ 1+i & 1+\theta & - \end{bmatrix} \Rrightarrow \boldsymbol{S}\begin{bmatrix} \eta+1 \\ i+1 \end{bmatrix} \circ \boldsymbol{S}\begin{bmatrix} \alpha+1 \\ \theta+1 \end{bmatrix} \circ \boldsymbol{D}$$

by the instance $\langle i, \eta \rangle$ of (!8). As in the cases (I4, I5) we find that

$$g_i(1+\eta) = h_i\eta$$

so that g_i and h_i have the same fixed point, and hence so do $\bigvee \mathcal{G}$ and $\bigvee \mathcal{H}$.

Finally, $(*\ell)$ gives

$$\bigvee \mathcal{H} = \left(\boldsymbol{S}\begin{bmatrix} * \\ \theta \end{bmatrix} \circ \boldsymbol{S}\begin{bmatrix} \alpha+1 \\ \theta+1 \end{bmatrix} \circ \boldsymbol{D}\right) f$$

and hence (E8) with (exp+) gives

$$\begin{aligned} f\begin{pmatrix} \cdot & 1+\alpha+1 & - \\ 0 & 1+\theta & - \end{pmatrix} &= \boldsymbol{Veb}(\textstyle\bigvee \mathcal{G}) \\ &= \boldsymbol{Veb}(\textstyle\bigvee \mathcal{H}) \\ &= \left(\boldsymbol{S}\begin{bmatrix} * \\ \theta \end{bmatrix} \circ \boldsymbol{S}\begin{bmatrix} \alpha+1 \\ \theta+1 \end{bmatrix} \circ \boldsymbol{D}\right) f = \left(\boldsymbol{S}\begin{bmatrix} 1 \\ \theta+1 \end{bmatrix} \circ \boldsymbol{D}\right) f \end{aligned}$$

for the required result.

(I9) Let θ and λ be fixed limit ordinals. We assume

$$\text{(!9)} \quad (\forall i < \theta)\langle i \rangle \qquad \text{(!!9)} \quad (\forall \alpha < \lambda)\langle \theta, \alpha \rangle$$

and show that

$$\langle \theta, \lambda \rangle$$

holds. Thus, assuming (!) we require

$$\begin{bmatrix} 1+\lambda & - \\ 1+\theta & - \end{bmatrix} \Rrightarrow \boldsymbol{S}\begin{bmatrix} \lambda+1 \\ \theta+1 \end{bmatrix} \circ \boldsymbol{D}$$

that is

$$\text{(?9)} \quad f\begin{pmatrix} \cdot & 1+\lambda & - \\ 0 & 1+\theta & - \end{pmatrix} = \left(\boldsymbol{S}\begin{bmatrix} \lambda+1 \\ \theta+1 \end{bmatrix} \circ \boldsymbol{D}\right) f$$

for an arbitrary $f \in \mathbb{F}$.

For such a function f let

$$g_{i,\alpha} = f\begin{pmatrix} \cdot & 1+\alpha & - \\ 1+i & 1+\theta & - \end{pmatrix} \qquad h_{i,\alpha} = \left(\boldsymbol{S}\begin{bmatrix} * \\ i+1 \end{bmatrix} \circ \boldsymbol{S}\begin{bmatrix} \alpha+1 \\ \theta+1 \end{bmatrix} \circ \boldsymbol{D}\right) f$$

for each $i < \theta$ and $\alpha < \lambda$, as in Tables 3 and 6. Also let

$$\mathcal{G} = \{g_{i,\alpha} \mid i < \theta, \alpha < \lambda\} \qquad \mathcal{H} = \{h_{i,\alpha} \mid i < \theta, \alpha < \lambda\}$$

be the two families of these functions.

As in cases (I4, I5) we find that for each $i < \theta$ and $\alpha < \lambda$ the two functions $g_{i,\alpha}$ and $h_{i,\alpha}$ have the same fixed points, and hence

$$f\begin{pmatrix} \cdot & 1+\lambda & - \\ - & i+\theta & - \end{pmatrix} = \boldsymbol{Veb}(\bigvee \mathcal{G}) = \boldsymbol{Veb}(\bigvee \mathcal{H})$$

by a use of rule (E9) of Table 3. Thus it remains to calculate $\bigvee \mathcal{H}$.

For each $\alpha < \lambda$ we have

$$\bigvee\{h_{i,\alpha} \mid i < \theta\} = \bigvee\{\big(\boldsymbol{S}\left[\begin{smallmatrix} * \\ i+1 \end{smallmatrix}\right] \circ \boldsymbol{S}\left[\begin{smallmatrix} \alpha+1 \\ \theta+1 \end{smallmatrix}\right] \circ \boldsymbol{D}\big) f \mid i < \theta\} = \bigvee\{\big(\boldsymbol{S}\left[\begin{smallmatrix} * \\ \theta \end{smallmatrix}\right] \circ \boldsymbol{S}\left[\begin{smallmatrix} \alpha+1 \\ \theta+1 \end{smallmatrix}\right] \circ \boldsymbol{D}\big) f$$

by $(*\ell)$ of Table 2. By (exp+) of that table this gives

$$\big(\boldsymbol{S}\left[\begin{smallmatrix} \alpha+1 \\ \theta+1 \end{smallmatrix}\right] \circ \boldsymbol{D}\big) f \leq \bigvee\{h_{i,\alpha} \mid i < \theta\} \leq \big(\boldsymbol{S}\left[\begin{smallmatrix} \alpha+2 \\ \theta+1 \end{smallmatrix}\right] \circ \boldsymbol{D}\big) f$$

and hence an interlacing argument gives

$$\bigvee \mathcal{H} = \bigvee\{\big(\boldsymbol{S}\left[\begin{smallmatrix} \alpha+1 \\ \theta+1 \end{smallmatrix}\right] \circ \boldsymbol{D}\big) f \mid \alpha < \lambda\} = \big(\boldsymbol{S}\left[\begin{smallmatrix} \lambda \\ \theta+1 \end{smallmatrix}\right] \circ \boldsymbol{D}\big) f$$

by clause $(\exp\ell)$ of Table 2.

Finally, this gives

$$\begin{aligned} f\begin{pmatrix} \cdot & 1+\lambda & - \\ 0 & 1+\theta & - \end{pmatrix} &= \boldsymbol{Veb}\big(\bigvee \mathcal{H}\big) \\ &= \big(\boldsymbol{Veb} \circ \boldsymbol{S}\left[\begin{smallmatrix} \lambda \\ \theta+1 \end{smallmatrix}\right] \circ \boldsymbol{D}\big) f = (\boldsymbol{S}\left[\begin{smallmatrix} \lambda+1 \\ \theta+1 \end{smallmatrix}\right] \circ \boldsymbol{D}) f \end{aligned}$$

by $(\exp\ell+)$ of Table 2, for the required result. ■

That concludes this example.

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