A comparison of two systems of ordinal notations

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Abstract

The standard method of generating countable ordinals from uncountable ordinals can be replaced by a use of fixed point extractors available in the term calculus of Howard's system. This gives a notion of the intrinsic complexity of an ordinal analogous to the intrinsic complexity of a function described in Gödel's T.

Contents

1 Introduction

The primary purpose of a system of ordinal notations is to name each ordinal in a certain initial stretch of the countable ordinals. Usually this is done using a version of the Cantor normal form (to base ω) so the trick is to attach to each ordinal a collection of smaller exponents and multipliers. In general the multipliers do not cause problems, but the exponents can. There are some ordinals, the critical ordinals, for which there are no smaller exponents. These form an obstruction to this obvious method of naming ordinals.

Historically the first systems of ordinal notations were generated 'from below'. In [18] Veblen introduced what we now call the Veblen hierarchy and its iterated extensions. These are families of functions which enumerate sets of critical ordinals. The higher a function is in a hierarchy, the sparser is the enumerated set. New hierarchies are formed using previously named ordinals to index the construction. In [14] Schütte extended this method. In the general scheme of things the ordinals so named are not very big, and this method has not received much attention lately. However [1, 8] should be mentioned.

In [2] Bachmann used a different system. Over a period of many years this was turned into what is now the standard method of generating ordinals 'from above'. Let Ω be the first uncountable ordinal, and let Ω^+ be a suitable larger ordinal beyond Ω. We use a certain enumerating function

$$\psi : [0, \Omega^+) \longrightarrow [0, \Omega)$$

which takes only critical values. With these we generate other countable ordinals. This uses larger ordinals to index smaller ordinals. On cardinality grounds, the function must take many large ordinals into the same small ordinal. The trick is to design ψ so that it is as non-erratic as possible. A full description of this method can be found in [3], but [11, 12] contain relevant material. A short historical survey of this topic is given in [4].

In [10] Howard gave a system of constructive ordinals. The term algebra of that system can be seen as an applied λ-calculus $\boldsymbol{\lambda H}$. This can be used to name ordinals, ordinal functions, and other higher order gadgets. These ordinal are generated 'from below'.

In this paper I will show how these two methods can be matched.

I review the construction of the function ψ in section 5 with a briefer description in section 2. For what is done here we don't need the details of $\boldsymbol{\lambda H}$. The later part of section 2 contains the pertinent facts, much without proofs. I will indicate where full proofs can be found. Section 3 gathers together the standard information that we need.

The assignment $\xi \longmapsto \psi\xi$ can be viewed as a way of converting the complete decomposition of ξ (to base Ω) into an ordinal named by a term of $\boldsymbol{\lambda H}$ using a small family of constants. From this we see there is a direct correlation between the structure of ξ and the naming term in $\boldsymbol{\lambda H}$. These constants name certain higher order fixed point extractors. Here we can work directly with the functions, and the syntactic aspects of $\boldsymbol{\lambda H}$ can be hidden. The relevant facts are organized in section 4, again sometimes without proofs.

This paper is related to [15, 16]. In particular, the missing proofs can be found in [15]. A more leisurely account of this and related material can be found in [17].

A previous version of this paper contained a rather involved proof by induction. After reading it the referee indicated how this could be simplified considerably. I have done that and incorporated several of his other suggestions. I thank him for these comments.

2 Background material

Let Ω be the first uncountable ordinal, and let Ω^+ be a suitably larger ordinal. We say what this is shortly. We are interested in ordinals below Ω and ordinals below Ω^+. We use Greek letters for ordinals with

$$\alpha, \beta, \gamma, \delta, \ldots < \Omega \qquad \eta, \xi, \zeta, \chi, \ldots < \Omega^+ \qquad \Sigma, \Gamma, \Delta, \Pi, \ldots < \Omega^+$$

to indicate the range of variation. We often think of ordinals $\Sigma, \Gamma, \Delta, \ldots$ as fixed.

We say an ordinal ϵ is critical if $\epsilon = \omega^\epsilon$. Often we are concerned with 'more interesting criticals'. In section 5 we set up this notion as a parameter. The Cantor normal form of an ordinal uses exponentiation to base ω. We also use exponentiation to base Ω.

2.1 DEFINITION. For each ordinal Δ the stacking function

$$\beth(\Delta, \Omega, \cdot) : \mathbb{N} \longrightarrow [0, \Omega^+)$$

is generated by

$$\beth(\Delta, \Omega, 0) = \Delta \qquad \beth(\Delta, \Omega, r+1) = \Omega^{\beth(\Delta,\Omega,r)} = \beth(\Omega^\Delta, \Omega, r)$$

for $r < \omega$. ■

The strict upper bound for all the ordinals we use is

$$\Omega^+ = \bigvee \{\beth(0, \Omega, r) \mid r < \omega\} = \epsilon_{\Omega+1}$$

the next critical ordinal beyond Ω.

From above

We wish to produce a function

$$\psi : [0, \Omega^+) \longrightarrow [0, \Omega)$$

which enumerates those critical ordinals 'of interest'. There are several variants of such functions in the literature. Almost all are constructed in a similar way.

To produce a value $\psi\xi$ of ψ (for $\xi < \Omega^+$) we first attach to ξ a set Ξ of ordinals below Ω^+. This set Ξ is often called a Skolem hull (of a certain kind), and its construction proceeds by recursion using previous values of ψ (and sometimes earlier Skolem hulls). Once we have Ξ we take its minimum non-member to produce the required value.

2.2 DEFINITION. Suppose for $\xi < \Omega^+$ the value $\psi\eta$ is known for each $\eta < \xi$. Let Ξ be the smallest set included in $[0, \Omega^+)$ such that

$$(1)\quad 0, \Omega \in \Xi \qquad (2)\quad \begin{array}{l}\Xi \text{ is closed under } +, \omega^{\bullet} \\ \text{and perhaps other functions}\end{array} \qquad (3)\quad \left.\begin{array}{l}\eta < \xi \\ \eta \in \Xi \\ \eta \in \Psi\end{array}\right\} \Rightarrow \psi\eta \in \Xi$$

hold. Then $\psi\xi = \min\{\gamma < \Omega \mid \gamma \notin \Xi\}$. ■

The 'other functions' and the restriction '$\eta \in \Psi$' are the source of most of the variants.

Let ψ_A be the function where there are no other functions and where Ψ is as large as possible (so that '$\eta \in \Psi$' always holds). Then ψ_A is essentially the function ψ_0 of [3].

Let ψ_B be the function where the Veblen function ϕ is the only 'other function', and Ψ is as large as possible. Then ψ_B is function ψ of [11, 12].

Let ψ_C be the function where there are no 'other functions' but where Ψ is the earlier Skolem hull attached to η. Then ψ_C is function ψ of [13].

The choice between these variants seems to be a matter of taste. In fact ψ_A and ψ_C are the same function, and this agrees with ψ_B cofinally often.

In section 5 we simplify the construction of ψ. We produce a generating function ψ from a suitable ordinal function $\boldsymbol{N}$ and a starting ordinal ϵ. Once we have ψ we can combine it with $\sqsupset$ to generate a fundamental sequence $\boldsymbol{\nabla}[\cdot]$ of a reasonably large ordinal.

2.3 DEFINITION. For each $l < \omega$ let $\boldsymbol{\nabla}[l] = \psi(\sqsupset(0, \Omega, l))$. ■

Of course, for the most commonly used base pairs $(\boldsymbol{N}, \epsilon)$, we find that $\boldsymbol{\nabla}[\cdot]$ is a fundamental sequence for the Howard ordinal. One of the aims of this paper is to show how this sequence $\boldsymbol{\nabla}[\cdot]$ can be produced without the use of uncountable ordinals.

From below

The ordinal notations generated from below are obtained from an applied λ-calculus $\boldsymbol{\lambda H}$. In the first instance these notations are syntactic objects (and are more general than *ordinal* notations). However, for what we do here we can work directly with the set theoretic ordinals so named. A fuller discussion of $\boldsymbol{\lambda H}$ is given in [15, 17].

Let $\mathbb{O}\mathrm{rd}$ be the set of countable ordinals. Thus $\mathbb{O}\mathrm{rd}$ and $[0, \Omega)$ are the same set. The idea is to use various gadgets which live somewhere in the full type hierarchy over $\mathbb{O}\mathrm{rd}$.

2.4 DEFINITION. By iterating the function space construction, the chain $\mathbb{O}\mathrm{rd}^{(\cdot)}$ of spaces is generated by

$$\mathbb{O}\mathrm{rd}^{(0)} = \mathbb{O}\mathrm{rd} \qquad \mathbb{O}\mathrm{rd}^{(r+1)} = (\mathbb{O}\mathrm{rd}^{(r)} \to \mathbb{O}\mathrm{rd}^{(r)})$$

for each $r < \omega$. ■

Thus $\mathbb{O}\mathrm{rd}^{(0)}$ is just the space $\mathbb{O}\mathrm{rd}$ of ordinals, and $\mathbb{O}\mathrm{rd}^{(1)}$ is the space $(\mathbb{O}\mathrm{rd} \to \mathbb{O}\mathrm{rd})$ of ordinal functions. At higher levels we see that $\mathbb{O}\mathrm{rd}^{(l+2)}$ is a space of higher order ordinal functions. Almost every gadget we look at inhabits one of these spaces $\mathbb{O}\mathrm{rd}^{(l)}$.

Our first job is to construct the ordinal iterates g^α of a function $g : \mathbb{O}\mathrm{rd}^{(l+1)}$ (for an arbitrary level $l < \omega$) and an ordinal $\alpha < \Omega$. By decomposing the type

$$\mathbb{O}\mathrm{rd}^{(l+1)} = \mathbb{O}\mathrm{rd}^{(l)} \to \mathbb{O}\mathrm{rd}^{(l-1)} \to \cdots \to \mathbb{O}\mathrm{rd}^{(1)} \to \mathbb{O}\mathrm{rd} \to \mathbb{O}\mathrm{rd}$$

we see that each such function $g : \mathbb{O}\mathrm{rd}^{(l+1)}$ must receive successive arguments

$$g_l : \mathbb{O}\mathrm{rd}^{(l)},\ \ldots,\ g_1 : \mathbb{O}\mathrm{rd}^{(1)},\ \zeta : \mathbb{O}\mathrm{rd}$$

to return its eventual value $gg_l \cdots g_1\zeta$ which is an ordinal.

2.5 DEFINITION. For each $g : \mathbb{O}\mathrm{rd}^{(l+1)}$ the ordinal iterates of g are generated by

$$g^0 = id \quad g^{\alpha+1} = g \circ g^\alpha \quad g^\lambda g_l \cdots g_1\zeta = \bigvee\{g^\alpha g_l \cdots g_1\zeta \mid \alpha < \lambda\}$$

for each ordinal $\alpha < \Omega$ and limit ordinal $\lambda < \Omega$. Here id is the identity function on $\mathbb{O}\mathrm{rd}^{(l)}$, and $g_l : \mathbb{O}\mathrm{rd}^{(l)}, \ldots, g_1 : \mathbb{O}\mathrm{rd}', \zeta : \mathbb{O}\mathrm{rd}$ are arbitrary successive arguments. ■

We must be careful here for these ordinal iterates don't always behave as we want then to. Fortunately we can restrict our attention to a special subclass $\mathbb{H}^{(l+1)} \subseteq \mathbb{O}\mathrm{rd}^{(l+1)}$ of helpful functions on each level. We don't need a development, or even a definition, of these classes, but we do need some of their properties. Full details of Propositions 2.6, 2.9, and 2.13, are given in section 7 of [15], and section 11 of [17].

2.6 PROPOSITION. *For each level $l < \omega$, the following hold.*

(a) For each non-zero ordinal $\alpha < \Omega$ and each $h \in \mathbb{H}^{(l+1)}$, the iterate $h^\alpha \in \mathbb{H}^{(l+1)}$.

(b) For each pair of functions $H \in \mathbb{H}^{(l+2)}$ and $h \in \mathbb{H}^{(l+1)}$, the application $Hh \in \mathbb{H}^{(l+1)}$.

(c) For each selection $h \in \mathbb{H}^{(l+1)}$, $h_l : \mathbb{H}^{(l)}, \ldots, h_1 : \mathbb{H}^{(1)}$ the equality

$$h^\lambda h_l \cdots h_1 0 = h^\lambda h_l \cdots h_1\zeta$$

holds for all ordinals $\zeta < \lambda < \Omega$ with λ critical.

You may be wondering why, in part (c), we didn't write h_{l+1} for h. It is because the central components $h_l, \ldots, h_1$ play a passive role and there is no great need to display them, and there is a certain uniformity over levels l. To bring this out we introduce an abbreviation. Thus in such a situation we write

$$h^\lambda \boldsymbol{h}\zeta \quad \text{for} \quad h^\lambda h_l \cdots h_1\zeta$$

by condensing the central part $h_l \cdots h_1$ to $\boldsymbol{h}$. However, this kind of abbreviation must be used with care, for '$\boldsymbol{h}$' on its own would have a different meaning.

Part (c) of Proposition 2.6 has a simple, but crucial, consequence.

2.7 COROLLARY. *(1) For each* $h \in \mathbb{H}^{(1)}$

$$(\zeta < \nu = h^\nu 0) \Longleftrightarrow (0 < \nu = h^\nu \zeta)$$

holds for all ordinals $\zeta, \nu < \Omega$.

(>1) For each level l *and selection* $H \in \mathbb{H}^{(l+2)}, h \in \mathbb{H}^{(l+1)}, h_l \in \mathbb{H}^{(l)} H, \ldots, h_1 \in \mathbb{H}^{(1)}$

$$(\zeta < \nu = H^\nu h \boldsymbol{h} 0) \Longleftrightarrow (0 < \nu = H^\nu h \boldsymbol{h} \zeta)$$

holds for all ordinals $\zeta, \nu < \Omega$ *(where* $\boldsymbol{h}$ *abbreviates* $h_l \cdots h_1$*).*

We won't prove Proposition 2.6. In fact, we won't even define $\mathbb{H}^{(l+1)}$ except for $l = 0$.

2.8 DEFINITION. An ordinal function $h : \mathbb{O}\mathrm{rd}^{(1)}$ is in $\mathbb{H}^{(1)}$, or helpful on level 1, precisely when it is strictly inflationary, monotone, and takes only critical values, that is when

$$\alpha < h\alpha \qquad \alpha \le \beta \Longrightarrow h\alpha \le h\beta \qquad h\alpha \text{ is critical}$$

hold for all $\alpha, \beta < \Omega$. ■

It is not hard to prove Proposition 2.6(a,c) for the case $l = 0$. The construction of the higher order classes $\mathbb{H}^{(l+2)}$ is designed to make the full version go through.

Recall that a function $f : \mathbb{O}\mathrm{rd}^{(1)}$ is normal if it is strictly monotone and continuous. Here we consider only those normal functions f with $f\alpha \ge \omega^\alpha$ for each non-zero $\alpha < \Omega$.

2.9 PROPOSITION. *For each* $h \in \mathbb{H}^{(l+1)}, h_l : \mathbb{H}^{(l)}, \ldots, h_1 : \mathbb{H}^{(1)}$ *and ordinal* $\zeta \in \mathbb{O}\mathrm{rd}$*, the function* $(\lambda\alpha : \mathbb{O}\mathrm{rd} \,.\, h^\alpha \boldsymbol{h}\zeta)$ *is normal (where* $\boldsymbol{h}$ *abbreviates* $h_l \cdots h_1$*).*

In other words the classes $\mathbb{H}^{(l+1)}$ help us to produce many different normal functions. Strictly speaking, this will be the case once we have some examples of helpful functions.

2.10 DEFINITION. Let $\boldsymbol{Fix} : \mathbb{O}\mathrm{rd}^{(2)}$ be the higher order function given by

$$\boldsymbol{Fix} f \zeta = f^\omega(\zeta + 1)$$

for each $f : \mathbb{O}\mathrm{rd}^{(1)}$ and ordinal $\zeta : \mathbb{O}\mathrm{rd}$. ■

It is easy to see that when f is normal the value $\boldsymbol{Fix} f \zeta$ is the least fixed point of f beyond ζ, that is the least ordinal ν with $\zeta < \nu = f\nu$.

2.11 LEMMA. *For each normal function* $f : \mathbb{O}\mathrm{rd}^{(1)}$*, the function* $h = \boldsymbol{Fix} f$ *is helpful.*

Proof. Since we have a definition of the class $\mathbb{H}^{(1)}$, we can prove this result. For a given $\zeta \in \mathbb{O}\mathrm{rd}$ let $\nu = h\zeta$ so that we have $\zeta < \nu = f\nu$ with a certain minimality on ν. In particular, this shows that h is strictly inflationary. Consider any $\zeta \le \eta$ and let $\mu = h\eta$. Then $\zeta \le \eta < \mu = f\mu$ and hence $\nu \le \mu$ by the minimality of ν. This shows that h is monotone. Finally, since $\nu \ne 0$, we have $\omega^\nu \le f\nu = \nu$ to show that ν is critical. ■

Examples of higher level helpful functions can be obtained by fixed point extractions. The following functions are part of the novelty of this approach.

2.12 DEFINITION. For each $l < \omega$ let $[\,l\,] : \mathbb{O}\mathrm{rd}^{(l+2)}$ be the function given by

$$[\,l\,]h\boldsymbol{h} = \boldsymbol{Fix}(\lambda\alpha : \mathbb{O}\mathrm{rd}\,.\,h^{\alpha}\boldsymbol{h}0)$$

for each $h : \mathbb{O}\mathrm{rd}^{(l+1)}$, $h_l : \mathbb{O}\mathrm{rd}^{(l)}$, ..., $h_1 : \mathbb{O}\mathrm{rd}^{(1)}$ (where $\boldsymbol{h}$ abbreviates $h_l \cdots h_1$). ■

Each $[\,l\,]$ is defined to act on an arbitrary selection of function arguments but, in practice, we use only helpful arguments. Note also that for helpful $h, h_l, \ldots, h_1$ and ordinal $\zeta \in \mathbb{O}\mathrm{rd}$, the value $[\,l\,]h\boldsymbol{h}\zeta$ is critical and is the least ordinal ν such that $\zeta < \nu = h^{\nu}\boldsymbol{h}0$ or equivalently $0 < \nu = h^{\nu}\boldsymbol{h}\zeta$ (by Corollary 2.7).

2.13 PROPOSITION. *For each level $l < \omega$, the operator $[\,l\,] : \mathbb{O}\mathrm{rd}^{(l+2)}$ is helpful.*

With these and the closure properties of Proposition 2.6 we can generate many helpful functions. For each $m < l < \omega$ and (non-zero) ordinals $\alpha_m, \ldots, \alpha_l$, the compound

$$\Big(\cdots\big(([\,l\,]^{\alpha_l}[\,l-1\,])^{\alpha_{l-1}}\big)\cdots[\,m\,]\Big)^{\alpha_m}$$

is a member of $\mathbb{O}\mathrm{rd}^{(m+2)}$. By Lemma 2.11 the function

$$\boldsymbol{Next} = \boldsymbol{Fix}\omega^{\bullet}$$

is helpful. In fact, it is the slowest helpful function. For each ordinal ζ the ordinal $\boldsymbol{Next}\zeta$ is the next critical ordinal beyond ζ and so

$$\epsilon_{\alpha} = \boldsymbol{Next}^{1+\alpha}0 = \boldsymbol{Next}^{\alpha}\epsilon_0$$

(for each $\alpha < \Omega$). It can be shown that $[\,1\,][\,0\,]\boldsymbol{Next}\zeta$ is the next strongly critical ordinal beyond ζ. You might like to ponder what the ordinal $[\,2\,][\,1\,][\,0\,]\boldsymbol{Next}0$ is.

In section 5 we will parameterize the construction of an enumerating function ψ on an arbitrary $\boldsymbol{N} \in \mathbb{H}^{(1)}$ together with a starting critical ordinal ϵ. In the first instance it does no harm to assume this pair is just $(\boldsymbol{Next}, \epsilon_0)$.

Using the functions $[\,l\,]$ we generate another sequence of ordinals, this time from below.

2.14 DEFINITION. For and arbitrary function $\boldsymbol{N} \in \mathbb{H}^{(1)}$ and critical ordinal ϵ let

$$\boldsymbol{\Delta}[0] = \epsilon \qquad \boldsymbol{\Delta}[1] = \boldsymbol{N}\epsilon \qquad \boldsymbol{\Delta}[l+2] = [\,l\,]\cdots[\,0\,]\boldsymbol{N}\epsilon$$

for each $l < \omega$. ■

We will show the two sequence $\boldsymbol{\nabla}[\cdot]$ and $\boldsymbol{\Delta}[\cdot]$ are the same. More generally, we will show how each value $\psi\xi$ can be described explicitly in terms of the functions $[\,l\,]$.

3 The shape of an ordinal

Recall that each (non-zero) ordinal $\xi < \Omega^{+}$ can be written

$$\xi = \Omega^{\xi(s)} \cdot \alpha(s) + \cdots + \Omega^{\xi(0)} \cdot \alpha(0)$$

where the exponents descend, $\xi > \xi(s) \geq \cdots \geq \xi(0)$, and each multiplier $\alpha(\cdot)$ is countable. Only the first descent, $\xi > \xi(s)$, need be strict, and this is ensured by the choice of Ω^+. By combining appropriate components we convert such a slack decomposition into one with strictly descending exponents, the canonical decomposition. We meet several ordinals

$$\xi = \Sigma + \Omega^{\Delta} \cdot \delta$$

where $\Sigma, \Delta < \Omega^+$ and $\delta < \Omega$ (with $\delta \neq 0$). We always assume that such a sum is part of a slack decomposition; that Δ is no bigger than each exponent in the canonical decomposition of Σ. There will be times when we need this representation to be a part of the canonical decomposition of ξ but we always make this assumption clear.

There is a natural instinct to try to use only canonical decompositions. There are two reasons for not doing so here. As we will see, in some calculations it can be convenient to extract just a part of one of the components of a canonical decomposition. In such a case we deal with a slack decomposition. A more important reason appears when we use ordinals as iteration templates and embed them in a larger collection of gadgets. We then find that slack, and even non-descending decompositions occur quite naturally.

By iterating the canonical decomposition we obtain a tree-like description of each ordinal $\xi < \Omega^+$. We first take the canonical decomposition of ξ to obtain its exponents and related multipliers. We then take the canonical decomposition of each exponent to obtain a family of second level exponents and related multipliers, and so on. By continuing this process we eventually reduce all exponents to zero and so obtain a battery of multipliers (each of which is countable). This produces a finite tree $T(\xi)$ of multipliers which completely determines ξ. In the tree $T(\xi)$ there is a maximum occurring multiplier.

3.1 DEFINITION. For each ordinal $\xi < \Omega^+$ we generate the bound $|\xi|$ by

$$|0| = 0 \qquad |\Sigma + \Omega^{\Delta} \cdot \delta| = \max\{|\Sigma|, |\Delta|, \delta\}$$

where on the decomposition on the right is canonical. ■

A simple calculation gives $|\xi| \leq |\xi + 1| \leq |\xi| + 1$ to show that the function $|\cdot|$ grows quite slowly. After each limit ordinal it is constant for a while.

By inspecting its canonical decomposition we see that limit ordinal ξ has one of three shapes (1, 2, 3) as in Table 1. We will need to consider these three cases several times, and to help with that it is convenient to have some fixed notation. Thus we write

$$(1) \quad \xi = \Gamma + \Omega^{\Delta} \cdot \mu \qquad (2) \quad \xi = \Gamma + \Omega^{\Delta} \qquad (3) \quad \xi = \Gamma + \Omega^{\Pi+1}$$

where Γ is as in the Table. Notice that for shapes (2) and (3) this is a strict decomposition only when $\delta = 0$.

3.2 CONVENTION and DEFINITION. We attach to each limit ordinal $\xi < \Omega^+$ a length $\ell(\xi) \leq \Omega$ and a limiting sequence $\big(\xi(\alpha) \mid \alpha < \ell(\xi)\big)$ of ordinals as given in Table 1. ■

It is routine to check that

$$\alpha < \beta < \ell(\xi) \Longrightarrow \xi(\alpha) < \xi(\beta) < \xi \qquad \xi = \bigvee \big\{\xi(\alpha) \mid \alpha < \ell(\xi)\big\}$$

and the second property generalizes. Surveying the possible cases we obtain the following.

	ξ	where	using	we set	$\ell(\xi)$	$\xi(\alpha)$
(1)	$\Sigma + \Omega^{\Delta} \cdot \mu$	μ is a limit	$\Gamma = \Sigma$		μ	$\Gamma + \Omega^{\Delta} \cdot \alpha$
(2)	$\Sigma + \Omega^{\Delta} \cdot (\delta + 1)$	Δ is a limit	$\Gamma = \Sigma + \Omega^{\Delta} \cdot \delta$		$\ell(\Delta)$	$\Gamma + \Omega^{\Delta(\alpha)}$
(3)	$\Sigma + \Omega^{\Pi+1} \cdot (\delta + 1)$		$\Gamma = \Sigma + \Omega^{\Pi+1} \cdot \delta$		Ω	$\Gamma + \Omega^{\Pi} \cdot \alpha$

Table 1: Convention and notation for limit ordinals

3.3 LEMMA. *For each limit ordinal $\xi < \Omega^+$*

$$\xi(\nu) = \bigvee \{\xi(\alpha) \mid \alpha < \nu\}$$

holds for each limit ordinal $\nu < \ell(\xi)$.

As well as this we need a few more simple properties.

3.4 LEMMA. *For each limit ordinal ξ*

$$(a) \quad \alpha \le |\xi(\alpha)| \le \max\{|\xi|, \alpha\} \qquad (b) \quad \xi(\alpha) \le \eta < \xi \Longrightarrow |\xi(\alpha)| \le |\eta|$$

$$(c) \quad \ell(\xi) < \Omega \Longrightarrow \ell(\xi) \le |\xi| \qquad (d) \quad \ell(\xi) = \Omega \Longrightarrow \alpha \le |\xi(\alpha)|$$

where $\alpha < \ell(\xi)$ in each case.

Proof. Each of these is proved by surveying the possible shapes of ξ. Usually when case (2) (of Convention 3.2) occurs we invoke an induction hypothesis. We need not go through all the details, but a couple of cases are worth looking at.

(a) For the case (2) we have

$$\xi = \Sigma + \Omega^{\Delta} \cdot (\delta + 1) \qquad \xi(\alpha) = \Sigma + \Omega^{\Delta} \cdot \delta + \Omega^{\Delta(\alpha)}$$

where Δ is a limit ordinal and $\alpha < \ell(\xi) = \ell(\Delta)$. From these we have

$$|\xi| = \max\{|\Sigma|, |\Delta|, \delta + 1\} \qquad |\xi(\alpha)| = \max\{|\Sigma|, |\Delta|, \delta, |\Delta(\alpha)|\}$$

and the induction hypothesis gives $\alpha \le |\Delta(\alpha)| \le \max\{|\Delta|, \alpha\}$ for the required result.

(b) Only the case (2) is not straight forward. Consider a limit ordinal ξ and ordinal $\xi(\alpha)$ for some $\alpha < \ell(\xi) = \ell(\Delta)$. Consider any ordinal η with $\xi(\alpha) \le \eta < \xi$. We have

$$\eta = \Sigma + \Omega^{\Delta} \cdot \delta + \zeta$$

for some ordinal $\zeta < \Omega^{\Delta}$ (since $\eta < \xi$). Since $\xi(\alpha) \le \eta$, the canonical decomposition of ζ must contain at least one component Ω^{Λ} where $\Delta(\alpha) \le \Lambda$ (and the multiplier is non-zero). Also $\Lambda < \Delta$ (since $\zeta < \Omega^{\Delta}$). Thus the induction hypothesis gives $|\Delta(\alpha)| \le |\Lambda|$ and hence

$$|\xi(\alpha)| = \max\{|\Sigma|, |\Delta|, \delta, |\Delta(\alpha)|\} \le \max\{|\Sigma|, |\Delta|, \delta, |\Lambda|\} \le \max\{|\Sigma|, |\Delta|, \delta, |\zeta|\} = |\eta|$$

as required. ■

There are some hidden subtleties in the construction of Table 1. One of these is brought out by the following.

3.5 LEMMA. *Suppose $\xi < \Omega^+$ is a limit ordinal with $\ell(\xi) = \Omega$. Then*

$$\xi(\nu) \text{ is a limit ordinal with } \ell(\xi(\nu)) = \nu$$

for each limit ordinal $\nu < \Omega$, and

$$\xi(\nu)(\alpha) = \xi(\alpha)$$

for each $\alpha < \nu$.

Proof. We proceed by a progressive induction over ξ.

Since ξ is a limit ordinal it must have one of the three shapes (1, 2, 3) of Table 1. Since $\ell(\Omega) = \Omega$ the shape (1) does not occur. We look at the other two shapes in turn.

(2) We have

$$\xi = \Sigma + \Omega^{\Delta} \cdot (\delta + 1) = \Gamma + \Omega^{\Delta}$$

in the standard notation. Here Δ is a limit ordinal with

$$\ell(\Delta) = \ell(\xi) = \Omega$$

so we may apply the induction hypothesis to Δ. We have

$$\xi(\beta) = \Gamma + \Omega^{\Delta(\beta)}$$

for each $\beta < \Omega$. In particular, for the limit ordinal $\nu < \Omega$ we have

$$\xi(\nu) = \Gamma + \Omega^{\Delta(\nu)}$$

where, by the induction hypothesis, $\Delta(\nu)$ is a limit ordinal with $\ell(\Delta(\nu)) = \nu$ and $\Delta(\nu)(\alpha) = \Delta(\alpha)$ for each $\alpha < \nu$. In particular, $\xi(\nu)$ is a limit ordinal of shape (2). But now

$$\ell(\xi(\nu)) = \ell(\Delta(\nu)) = \nu$$

and

$$\xi(\nu)(\alpha) = \Gamma + \Omega^{\Omega(\nu)(\alpha)} = \Gamma + \Omega^{\Omega(\alpha)} = \xi(\alpha)$$

for each $\alpha < \nu$, as required.

(3) We have

$$\xi = \Sigma + \Omega^{\Pi+1} \cdot (\delta + 1) = \Gamma + \Omega^{\Pi+1}$$

in the standard notation. Also

$$\xi(\beta) = \Gamma + \Omega^{\Pi} \cdot \beta$$

for each $\beta < \Omega$. In particular, for the limit ordinal $\nu < \Omega$ we have

$$\xi(\nu) = \Gamma + \Omega^{\Pi} \cdot \nu$$

which is a limit ordinal of shape (1) with $\ell(\xi(\nu)) = \nu$, and with

$$\xi(\nu)(\alpha) = \Gamma + \Omega^{\Pi} \cdot \alpha = \xi(\alpha)$$

for each $\alpha < \nu$, to give the required result. ■

We can now begin the central development of this paper.

4 A battery of functions

From Definition 2.12 and Proposition 2.13, for each $l < \omega$ we have a helpful function $[\,l\,] : \mathbb{O}\mathrm{rd}^{(l+2)}$ (that is, a member of $\mathbb{H}^{(l+2)}$). The intention is that each such function should be used only on helpful arguments (and ordinals at level 0). We wish to use combinations of these functions to name the 'interesting criticals'. To do that we first have to make this notion precise.

4.1 CONTEXT and DEFINITION. Let $\boldsymbol{N} : \mathbb{O}\mathrm{rd}^{(1)}$ be an arbitrary helpful function, and let ϵ be an arbitrary critical ordinal. By definition, an $(\boldsymbol{N}, \epsilon)$-critical ordinal is one of the form $\boldsymbol{N}^{\alpha}\epsilon$ for some ordinal $\alpha < \Omega$. We often abbreviate this to $\boldsymbol{N}$-critical ordinal by tacitly assuming that ϵ is known.

We think of these as the 'interesting criticals'. Thus the selected start critical ϵ is the least 'interesting critical', and for each 'interesting critical' θ the next 'interesting critical' is $\boldsymbol{N}\theta$. In particular, the normal function

$$\alpha \longmapsto \boldsymbol{N}^{\alpha}\epsilon$$

is an enumeration of the 'interesting critical' ordinals. ■

We work relative to some fixed pair $(\boldsymbol{N}, \epsilon)$ chosen in any way we see fit. By Proposition 2.6 the functions $[\,l\,]$ and $\boldsymbol{N}$ can be combined to produce other helpful functions. With these Proposition 2.9 gives us many normal functions. In particular, with $D_1 = \boldsymbol{N}$ and $D_{l+2} = [\,l\,] \cdots [\,0\,]\boldsymbol{N}$ we have $D_{r+1} \in \mathbb{H}^{(r+1)}$, and

$$\boldsymbol{\Delta}[0] = \epsilon \qquad \boldsymbol{\Delta}[r+1] = D_{r+1}\epsilon$$

is a rephrasing of Definition 2.14.

We attach to each ordinal $\xi < \Omega^{+}$ a gadget $\langle\xi\rangle_l : \mathbb{O}\mathrm{rd}^{(l)}$ for each level $l < \omega$. The construction proceeds by recursion over the structure of ξ with variation of the level l.

4.2 DEFINITION. For each ordinal $\xi < \Omega^{+}$ let $\langle\xi\rangle_l : \mathbb{O}\mathrm{rd}^{(l)}$ be given by

$$\langle 0\rangle_0 = \epsilon \qquad \langle 0\rangle_1 = \boldsymbol{N} \qquad \langle 0\rangle_{l+2} = [\,l\,]$$

for $\xi = 0$; and for $\xi \neq 0$ with canonical decomposition, as to the left,

$$\xi = \Omega^{\xi(s)} \cdot \alpha(s) + \cdots + \Omega^{\xi(0)} \cdot \alpha(0) \qquad \langle\xi\rangle_l = \Big(\langle\xi(0)\rangle_{l+1}^{\alpha(0)} \circ \cdots \circ \langle\xi(s)\rangle_{l+1}^{\alpha(s)}\Big)\langle 0\rangle_l$$

and for each level $l < \omega$ we take $\langle\xi\rangle_l$, as to the right. ■

By Proposition 2.6(a), for each nonzero level l the associated function $\langle\xi\rangle_l$ is helpful.

Notice the order in which the exponents are used. The larger an exponent of ξ, the deeper (nearer to the argument) it is in the composite giving $\langle\xi\rangle_l$. This has a useful simple consequence which allows us to generate these associated functions in a different way. If $\Sigma + \Omega^{\Delta} \cdot \delta$ is part of a canonical decomposition then we have

$$\langle\Sigma + \Omega^{\Delta} \cdot \delta\rangle_l = \langle\Delta\rangle_{l+1}^{\delta}\langle\Sigma\rangle_l \qquad \langle\xi + 1\rangle_l = \langle 0\rangle_{l+1}\langle\xi\rangle_l \qquad \langle\xi + 1\rangle_{l+1} = [\,l\,]\langle\xi\rangle_{l+1}$$

for each level l. In fact, the left hand equality holds even when the given decomposition of ξ is slack, provided we take a bit of care combining the iterates, (This is a reason for using slack decompositions). For example, if ξ is a limit ordinal of shape (2) with

$$\xi = \Sigma + \Omega^{\Delta} \cdot (\delta + 1) = \Gamma + \Omega^{\Delta}$$

then

$$\langle \xi \rangle_l = \langle \Delta \rangle_{l+1}^{\delta+1} \langle \Sigma \rangle_l = \langle \Delta \rangle_{l+1} \langle \Gamma \rangle_l$$

for each level l. We use this in the proof of Lemma 4.4 below.

The proof of the following is an easy exercise in unravelling the various definitions.

4.3 THEOREM. *For each $r, s < \omega$ we have*

$$\langle \sqsupset(0, \Omega, r) \rangle_{s+2} = [\, r + s\,] \cdots [\, s\,]$$

and, in particular, we have

$$\boldsymbol{\Delta}[l] = \langle \sqsupset(0, \Omega, l) \rangle_0$$

for each level $l < \omega$.

We come now to the main result of this section. With a bit of care this could be stated (and proved) in a slightly more compact form, but it is convenient not to do so.

4.4 LEMMA. *For each limit ordinal $\xi < \Omega^+$ and $l < \omega$*

$$(<) \text{ if } \ell(\xi) < \Omega \text{ then } \quad \begin{cases} \langle \xi \rangle_0 & = \bigvee \{ \langle \xi(\alpha) \rangle_0 \mid \alpha < \ell(\xi) \} \\ \langle \xi \rangle_{l+1} \boldsymbol{h} \zeta & = \bigvee \{ \langle \xi(\alpha) \rangle_{l+1} \boldsymbol{h} \zeta \mid \alpha < \ell(\xi) \} \end{cases}$$

$$(=) \text{ if } \ell(\xi) = \Omega \text{ then } \quad \begin{cases} \langle \xi \rangle_0 & = \text{the least ordinal } \mu \text{ with } 0 < \mu = \langle \xi(\mu) \rangle_0 \\ \langle \xi \rangle_{l+1} \boldsymbol{h} \zeta & = \text{the least ordinal } \mu \text{ with } 0 < \mu = \langle \xi(\mu) \rangle_{l+1} \boldsymbol{h} \zeta \end{cases}$$

for each $h_l : \mathbb{H}^{(l)}, \ldots, h_1 : \mathbb{H}^{(1)}$ and $\zeta \in \mathbb{O}\mathrm{rd}$, where $\boldsymbol{h}$ abbreviates $h_l \cdots h_1$.

Proof. We proceed by a progressive induction over ξ with variation of the level l. We deal with the two cases (<) and (=) separately, but for each we must consider the possible shape of ξ. For this we use the decompositions and notation given in Convention 3.2.

(<) (1) For the case

$$\xi = \Sigma + \Omega^{\Delta} \cdot \mu$$

we have $\ell(\xi) = \mu$ where μ is a limit ordinal, and $\xi(\alpha) = \Sigma + \Omega^{\Delta} \cdot \alpha$ for each $\alpha < \ell(\xi)$. Since

$$\langle \xi \rangle_l = \langle \Delta \rangle_{l+1}^{\mu} \langle \Sigma \rangle_l \qquad \langle \xi(\alpha) \rangle_l = \langle \Delta \rangle_{l+1}^{\alpha} \langle \Sigma \rangle_l$$

we have

$$\langle \xi \rangle_{l+1} \boldsymbol{h} \zeta = \langle \Delta \rangle_{l+1}^{\mu} \langle \Sigma \rangle_l \boldsymbol{h} \zeta = \bigvee \{ \langle \Delta \rangle_{l+2}^{\alpha} \langle \Sigma \rangle_{l+1} \boldsymbol{h} \zeta \mid \alpha < \mu \} = \bigvee \{ \langle \xi(\alpha) \rangle_{l+1} \boldsymbol{h} \zeta \mid \alpha < \ell(\xi) \}$$

as required. A similar argument deals with $\langle \xi \rangle_0$.

(<) (2) For the case

$$\xi = \Sigma + \Omega^{\Delta} \cdot (\delta + 1) = \Gamma + \Omega^{\Delta}$$

we have $\ell(\xi) = \ell(\Delta)$ and $\xi(\alpha) = \Gamma + \Omega^{\Delta(\alpha)}$ for each $\alpha < \ell(\xi)$. Since

$$\langle\xi\rangle_l = \langle\Delta\rangle_{l+1}\langle\Gamma\rangle_l \qquad \langle\xi(\alpha)\rangle_l = \langle\Delta(\alpha)\rangle_{l+1}\langle\Gamma\rangle_l$$

the induction hypothesis gives

$$\begin{aligned}\langle\xi\rangle_{l+1}\boldsymbol{h}\zeta &= \langle\Delta\rangle_{l+2}\langle\Gamma\rangle_{l+1}\boldsymbol{h}\zeta \\ &= \bigvee\left\{\langle\Delta(\alpha)\rangle_{l+2}\langle\Gamma\rangle_{l+1}\boldsymbol{h}\zeta \mid \alpha < \ell(\Delta)\right\} \;=\; \bigvee\left\{\langle\xi(\alpha)\rangle_{l+1}\boldsymbol{h}\zeta \mid \alpha < \ell(\xi)\right\}\end{aligned}$$

as required. A similar argument deals with $\langle\xi\rangle_0$.

(=) (3) For the case

$$\xi = \Sigma + \Omega^{\Pi+1}\cdot(\delta+1) = \Gamma + \Omega^{\Pi+1}$$

we have $\xi(\alpha) = \Gamma + \Omega^{\Pi}\cdot\alpha$ for each $\alpha < \Omega$. Since

$$\langle\xi\rangle_{l+1} = \langle\Pi+1\rangle_{l+2}\langle\Gamma\rangle_{l+1} = [\,l+1\,]\langle\Pi\rangle_{l+2}\langle\Gamma\rangle_{l+1} \qquad \langle\xi(\mu)\rangle_{l+1} = \langle\Pi\rangle^{\mu}_{l+2}\langle\Gamma\rangle_{l+1}$$

we have

$$\begin{aligned}\langle\xi\rangle_{l+1}\boldsymbol{h}\zeta &= [\,l+1\,]\langle\Pi\rangle_{l+2}\langle\Gamma\rangle_{l+1}\boldsymbol{h}\zeta \\ &= \big(\text{least } \mu \text{ with } \zeta < \mu = \langle\Pi\rangle^{\mu}_{l+2}\langle\Gamma\rangle_{l+1}\boldsymbol{h}0\big) \\ &= \big(\text{least } \mu \text{ with } 0 < \mu = \langle\Pi\rangle^{\mu}_{l+2}\langle\Gamma\rangle_{l+1}\boldsymbol{h}\zeta\big) = \big(\text{least } \mu \text{ with } 0 < \mu = \langle\xi(\mu)\rangle_{l+1}\boldsymbol{h}\zeta\big)\end{aligned}$$

as required. The penultimate step uses Corollary 2.7(>1). A similar argument deals with $\langle\xi\rangle_0$. Notice that the induction hypothesis is not used in this part.

(=) (2) For the case

$$\xi = \Sigma + \Omega^{\Delta}\cdot(\delta+1) = \Gamma + \Omega^{\Delta}$$

where $\ell(\Delta) = \Omega$ we have $\langle\xi\rangle_{l+1}\boldsymbol{h}\zeta = \langle\Delta\rangle_{l+2}\langle\Gamma\rangle_{l+1}\boldsymbol{h}\zeta$, and this value is

$$\big(\text{least } \mu \text{ with } 0 < \mu = \langle\Delta(\mu)\rangle_{l+2}\langle\Gamma\rangle_{l+1}\boldsymbol{h}\zeta\big) = \big(\text{least } \mu \text{ with } 0 < \mu = \langle\xi(\mu)\rangle_{l+1}\boldsymbol{h}\zeta\big)$$

using the induction hypothesis, as required. A similar argument deals with $\langle\xi\rangle_0$. ■

We will need only the case $l = 0$ of this result (which is one reason for stating the case separately). However, this does not enable us to give a simpler proof, for in the two uses of the induction hypothesis we need a variation of the level l.

5 The generic enumerating function

We continue to work in the Context 4.1. That is we assume given a base pair $(\boldsymbol{N}, \epsilon)$ and an $\boldsymbol{N}$-critical is an ordinal $\theta = \boldsymbol{N}^{\alpha}\epsilon$ for some $\alpha < \Omega$. Notice that a supremum of $\boldsymbol{N}$-criticals is $\boldsymbol{N}$-critical.

5.1 DEFINITION. The enumerating function attached to the pair $(\boldsymbol{N}, \epsilon)$ is generated by recursion.

$$\psi : [0, \Omega^+) \longrightarrow [0, \Omega) \qquad \left.\begin{array}{r}\eta < \xi \\ |\eta| < \theta\end{array}\right\} \Longrightarrow \psi\eta < \theta$$

The value $\psi\xi$ is the least $\boldsymbol{N}$-critical ordinal θ such that implication above holds. ■

Vacuously $\psi 0 = \epsilon$, and a cofinality argument shows that ψ is defined for all $\xi < \Omega^+$. We may check that if $(\boldsymbol{N}, \epsilon) = (\boldsymbol{Next}, \epsilon_0)$ then $\psi = \psi_A = \psi_C$, and if $(\boldsymbol{N}, \epsilon) = ([\,1\,][\,0\,]\boldsymbol{Next}, \Gamma_0)$ then $\psi = \psi_B$.

5.2 LEMMA. *(a) The function ψ is monotone.*

(b) For each limit ordinal $\xi < \Omega^+$

$$\psi\xi = \bigvee \{\psi(\xi(\alpha)) \mid \alpha < \ell(\xi)\}$$

using the limiting sequence $\xi(\cdot)$.

Proof. (a) Consider any ordinals $\xi \leq \chi$. Then

$$\left.\begin{array}{r}\eta < \xi \\ |\eta| < \psi\chi\end{array}\right\} \Longrightarrow \left.\begin{array}{r}\eta < \chi \\ |\eta| < \psi\chi\end{array}\right\} \Longrightarrow \psi\eta < \psi\chi$$

and hence $\psi\xi \leq \psi\chi$ by the minimality of $\psi\xi$.

(b) For the limit ordinal ξ, as on the left

$$\xi = \bigvee \{\xi(\alpha) \mid \alpha < \ell(\xi)\} \qquad \theta = \bigvee \{\psi(\xi(\alpha)) \mid \alpha < \ell(\xi)\}$$

consider the $\boldsymbol{N}$-critical ordinal θ, as on the right. We have $\theta \leq \psi\xi$ by part (a), so we require $\psi\xi \leq \theta$. To the end consider $\eta < \xi$ with $|\eta| < \theta$, so that $\eta < \xi(\alpha)$ and $|\eta| < \psi(\xi(\alpha))$ for some $\alpha < \ell(\xi)$. But now $\psi\eta < \psi(\xi(\alpha)) \leq \theta$ to give the required result. ■

The function ψ is not strictly monotone (because of the cardinality restrictions). There are stretches where ψ is constant.

5.3 DEFINITION. An ordinal $\xi < \Omega$ is tame if $|\xi| < \psi\xi$, and wild if $\psi\xi \leq |\xi|$. ■

The next result shows that the behaviour of ψ is to repeatedly apply $\boldsymbol{N}$ to the previous value, except for periods of stagnation.

5.4 LEMMA. *Let $\theta = \psi\xi$ for some $\xi < \Omega^+$. Then*

$$\psi(\xi + 1) = \begin{cases} \boldsymbol{N}\theta & \text{if } \xi \text{ is tame} \\ \theta & \text{if } \xi \text{ is wild} \end{cases}$$

holds.

Proof. Let $\theta = \psi\xi$ and $\phi = \psi(\xi + 1)$. Thus ϕ is the smallest $\boldsymbol{N}$-critical such that

$$\left.\begin{array}{r}\eta \leq \xi \\ |\eta| < \phi\end{array}\right\} \Longrightarrow \psi\eta < \phi$$

holds. In particular, since ψ is monotone, we have $\theta \leq \phi \leq \boldsymbol{N}\theta$ and ϕ is one or other of these extremes. If $|\xi| < \theta$ then ξ itself is one η with $\eta \leq \xi$ and $|\eta| < \theta \leq \phi$, so that $\phi = \boldsymbol{N}\theta$. If $\theta \leq |\xi|$ then

$$\left.\begin{array}{r}\eta \leq \xi \\ |\eta| < \theta\end{array}\right\} \Longrightarrow \left.\begin{array}{r}\eta < \xi \\ |\eta| < \theta\end{array}\right\} \Longrightarrow \psi\eta < \theta$$

and hence $\phi = \theta$. ■

We know that $|\xi| \leq |\xi + 1| \leq |\xi| + 1$ for each ordinal ξ, and each value of ψ is at least a limit ordinal, so we have the following.

5.5 COROLLARY. *An ordinal ξ is tame if and only its successor $\xi+1$ is tame.*

Not every limit ordinal is wild. This feature is related to the behaviour of limiting sequences.

5.6 LEMMA. *Suppose $\xi < \Omega^+$ is a limit ordinal with $\theta = \psi\xi$. Suppose also there is some $\alpha < \ell(\xi)$ with $\xi(\alpha)$ wild. Then there is a unique $\gamma < \ell(\xi)$ such that*

$$(\downarrow) \quad \alpha < \gamma \Longrightarrow \xi(\alpha) \text{ is tame} \qquad\qquad (\uparrow) \quad \gamma \le \alpha \Longrightarrow \xi(\alpha) \text{ is wild}$$

hold for each $\alpha < \ell(\xi)$. Furthermore

$$\xi(\gamma) \le \eta < \xi \Longrightarrow \eta \text{ is wild with } \psi\eta = \theta$$

holds (for each $\eta < \Omega^+$).

Proof. Let $\gamma < \ell(\xi)$ be the least ordinal such that $\xi(\gamma)$ is wild, to ensure $(\downarrow)$.

Let $\phi = \psi(\xi(\gamma))$, so that $\phi \le \theta$. To improve this to an equality consider any $\eta < \xi$ with $|\eta| < \phi$ and observe where η must lie. If $\xi(\gamma) \le \eta < \xi$ then, since $\xi(\gamma)$ is wild, Lemma 3.4(b) gives $\phi \le |\xi(\gamma)| \le |\eta|$ which is not so. Thus $\eta < \xi(\gamma)$ with $|\eta| < \phi$ and hence $\psi\eta < \phi$, so that $\theta \le \phi$ by the minimality of θ.

As a consequence of this for each η with $\xi(\gamma) \le \eta < \xi$ we have

$$\theta = \psi(\xi(\gamma)) \le \psi\eta \le \psi\xi = \theta$$

and hence a second use of Lemma 3.4(b) gives $\theta \le |\xi(\gamma)| \le |\eta|$ to show that η is wild. ■

This $\gamma < \ell(\xi)$ is the flip ordinal where the sequence $\xi(\cdot)$ changes from being tame to being wild. In some cases $\gamma = 0$ occurs, and there is the other extreme.

5.7 COROLLARY. *If ξ is a tame limit ordinal with $\ell(\xi) < \Omega$, then each $\xi(\alpha)$ is tame.*

Proof. Suppose some $\xi(\alpha)$ is wild. Then, using the γ and θ of Lemma 5.6, we have $\theta \le |\xi(\gamma)| \le |\xi|$ by Lemma 3.4(c), to show that ξ is wild. ■

There is a converse to this. Suppose ξ is a limit ordinal where each $\xi(\alpha)$ is tame. By Lemma 5.4, we have $\psi(\xi(\alpha)) < \psi(\xi(\alpha)+1) \le \psi(\xi(\alpha+1))$ for each $\alpha < \ell(\xi)$. Thus

$$\psi(\xi(\cdot)) : [0, \ell(\xi)) \longrightarrow [0, \Omega)$$

is a strictly monotone, and hence $\ell(\xi) < \Omega$ (for otherwise the sequence exhausts $[0, \Omega)$).

In other words, if ξ is a limit ordinal with $\ell(\xi) = \Omega$, then there is at least one $\alpha < \ell(\xi)$ with $\xi(\alpha)$ wild. For this case we can pin down the flip ordinal.

5.8 LEMMA. *Suppose $\xi < \Omega^+$ is a tame limit ordinal with $\theta = \psi\xi$ and $\ell(\xi) = \Omega$. Then*

$$(\downarrow) \quad \alpha < \theta \Longrightarrow \xi(\alpha) \text{ is tame} \qquad\qquad (\uparrow) \quad \theta \le \alpha \Longrightarrow \xi(\alpha) \text{ is wild}$$

hold for each $\alpha < \Omega$. Furthermore, both

$$\xi(\theta) \le \eta < \xi \Longrightarrow \eta \text{ is wild with } \psi\eta = \theta \qquad\qquad \psi\eta = \theta \Longrightarrow \xi(\theta) \le \eta$$

holds for each ordinal $\eta < \Omega^+$.

Proof. As just explained, on cardinality grounds there is some $\alpha < \Omega$ with $\xi(\alpha)$ wild. Thus Lemma 5.6 gives us a flip ordinal γ with $\psi(\xi(\gamma)) = \theta$. If $\alpha < \gamma$ then

$$\alpha \leq |\xi(\alpha)| < \psi(\xi(\alpha)) \leq \theta$$

by Lemma 3.4(d) and the tameness of $\xi(\alpha)$. In particular, $\gamma \leq \theta$ (otherwise $\theta < \theta$). If $\gamma < \theta$ then Lemma 3.4(a) and the assumed tameness of ξ gives

$$\theta \leq |\xi(\gamma)| \leq \max\{|\xi|, \gamma\} < \theta$$

which is not so. Thus $\gamma = \theta$, which leads to most of the required result. Only the lower right hand implication needs to be checked. But, since θ is a limit ordinal we have

$$\xi(\theta) = \bigvee\{\xi(\alpha) \mid \alpha < \theta\}$$

by Lemma 3.3. If $\eta < \xi(\theta)$ then $\eta \leq \xi(\alpha)$ for some $\alpha < \theta$ and $\xi(\alpha)$ is tame, so that

$$\psi\eta \leq \psi(\xi(\alpha)) < \psi(\xi(\alpha + 1)) \leq \theta$$

which gives the contrapositive of the required implication. ■

This result gives us a fixed-point characterization of the flip value $\theta = \psi\xi$.

5.9 COROLLARY. *Suppose $\xi < \Omega^+$ is a tame limit ordinal with $\ell(\xi) = \Omega$. Then $\theta = \psi\xi$ is the least ordinal $\nu < \Omega$ such that $0 < \nu = \psi(\xi(\nu))$.*

Proof. By Lemma 5.8 we have $\theta = \psi(\xi(\theta))$. Conversely, if $0 < \nu = \psi(\xi(\nu))$ then Lemma 3.4(d) gives $\psi(\xi(\nu)) = \nu \leq |\xi(\nu)|$ to show that $\xi(\nu)$ is wild, and hence $\theta \leq \nu$. ■

This concludes the preliminaries.

6 The main result

We are now in a position to prove the main result of this paper.

6.1 THEOREM. *For each ordinal $\xi < \Omega^+$, if ξ is tame then $\psi\xi = \langle\xi\rangle_0$.*

Proof. We proceed by induction on ξ.

For the base case, $\xi = 0$, we have $\psi 0 = \epsilon = \langle 0\rangle_0$ by definition.

For the induction step, $\xi \mapsto \xi + 1$, suppose $\xi + 1$ is tame. Then ξ is tame, and hence

$$\psi(\xi + 1) = \boldsymbol{N}(\psi\xi) = \boldsymbol{N}(\langle\xi\rangle_0) = \langle 0\rangle_1\langle\xi\rangle_0 = \langle\xi + 1\rangle_0$$

by Lemma 5.4, the induction hypothesis, and the earlier calculation of $\langle\cdot\rangle_0$.

For the leap to a tame limit ordinal ξ we treat $\ell(\xi) < \Omega$ and $\ell(\xi) = \Omega$ separately.

If $\ell(\xi) < \Omega$ then by Corollary 5.7 we see that $\xi(\alpha)$ is tame for each $\alpha < \ell(\xi)$. Thus

$$\psi\xi = \bigvee\left\{\psi(\xi(\alpha)) \mid \alpha < \ell(\xi)\right\} = \bigvee\left\{\langle\xi(\alpha)\rangle_0 \mid \alpha < \ell(\xi)\right\} = \langle\xi\rangle_0$$

by Lemma 5.2(b), the induction hypothesis, and Lemma 4.4(<), as required.

It remains to deal with the case where ξ is a limit ordinal with $\ell(\xi) = \Omega$. By Lemma 4.4(=) and Corollary 5.9, for each such ordinal ξ we have

$$\langle \xi \rangle_0 = \text{least ordinal } \mu \text{ with } \mu = \langle \xi(\mu) \rangle_0 \qquad \psi\xi = \text{least ordinal } \nu \text{ with } \nu = \psi(\xi(\nu))$$

(where $\mu \neq 0 \neq \nu$). Also, by Lemma 5.8 with $\theta = \psi\xi = \nu$ we see that ν is wild by $\xi(\alpha)$ is tame for each $\alpha < \nu$. Thus we may apply the induction hypothesis to each such $\xi(\alpha)$.

By way of contradiction suppose $\mu < \nu$. Then

$$\psi(\xi(\mu)) = \langle \xi(\mu) \rangle_0 = \mu$$

by the induction hypothesis. Also, since $\ell(\xi) = \Omega$ and $\xi(\mu)$ is tame, Lemma 3.4(d) gives

$$\mu \leq |\xi(\mu)| < \psi(\xi(\mu)) = \mu$$

which is the contradiction.

This gives $\nu \leq \mu$, so it suffices to obtain a comparison $\mu \leq \nu$.

For convenience let $\eta = \xi(\nu)$. From Lemma 3.5 we see that η is a limit ordinal with $\ell(\eta) = \nu$ and $\eta(\alpha) = \xi(\alpha)$ for each $\alpha < \nu$. Thus, by the induction hypothesis we have

$$\psi(\eta(\alpha)) = \psi(\xi(\alpha)) = \langle \xi(\alpha) \rangle_0 = \langle \eta(\alpha) \rangle_0$$

for each $\alpha < \nu$. With this Lemmas 5.2(b) and 4.4(<) give

$$\nu = \psi\eta = \bigvee \{ \psi(\eta(\alpha)) \mid \alpha < \ell(\eta) \} = \bigvee \{ \langle \eta(\alpha) \rangle_0 \mid \alpha < \ell(\eta) \} = \langle \eta \rangle_0 = \langle \xi(\nu) \rangle_0$$

and hence $\mu \leq \nu$ by the minimality of μ, as required. ■

Since $|\Omega| = 1 < \theta$ for each critical $\theta < \omega$, we see that $\xi = \beth(0, \Omega, l)$ is tame for each $l < \omega$. With this we can match the two fundamental sequence $\mathbf{\nabla}[\cdot]$ and $\mathbf{\Delta}[\cdot]$.

6.2 COROLLARY. *For each $l < \omega$, we have $\mathbf{\nabla}[l] = \mathbf{\Delta}[l]$.*

Proof. Let $\xi = \beth(0, \Omega, l)$ so that ξ is tame and hence $\mathbf{\nabla}[l] = \psi\xi = \langle \xi \rangle_0 = \mathbf{\Delta}[l]$ by Definition 2.3, Theorem 6.1, and Theorem 4.3. ■

I will conclude with a few remarks on what might be done with this material.

Several number theoretic hierarchies can be generated by selecting a base function and jump operator

$$f : (\mathbb{N} \to \mathbb{N}) \qquad F : (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \to \mathbb{N})$$

and setting

$$f_\alpha = F^\alpha f$$

for some initial stretch of ordinals α. Usually both f and F are easily describable. Thus a description of f_α boils down to a description of α.

For instance, very little is describable in the simply typed λ-calculus, whereas all ordinals $\alpha < \epsilon_0$ are describable in Gödel's T. However, many more ordinals are describable in the second order λ-calculus $\boldsymbol{\lambda 2}$. Early versions of this kind of result can be found in [9, 19]. A more detailed analysis is undertaken in [5, 6, 7]. Much of this work is carried out directly within $\boldsymbol{\lambda 2}$ or a ramified (tiered) version of $\boldsymbol{\lambda 2}$. Another approach is to use the system $\boldsymbol{\lambda H}$. We first show that certain things can be done in a part of $\boldsymbol{\lambda H}$, and then we observe that this part can be simulated within some part of $\boldsymbol{\lambda 2}$. In some cases this can bring some uniformity to a whole family of results.

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