Ultraproducts

As usual we work relative to a fixed but arbitrary first order language.

Let

$$\mathbb{A} = (\mathfrak{A}_i \mid i \in I)$$

be an indexed family of structures. We produce a structure

$$\mathfrak{A} = \prod \mathbb{A}/\mathcal{U} = \prod(\mathfrak{A}_i \mid i \in I)/\mathcal{U}$$

where the sentences true in $\mathfrak{A}$ are those true in 'almost all' $\mathfrak{A}_i$.

The gadget $\mathcal{U}$ makes sense of 'almost all'.

Filters on a set

Consider the set I. We may as well assume I is infinite. What follows does make sense when I is finite, but is mostly trivial.

A filter on I is a collection $\mathcal{F}$ of subsets of I with the following properties.

- No-empty $$I \in \mathcal{F}$$
- Upwards closed $$\left.\begin{array}{r} X \in \mathcal{F} \\ X \subseteq Y \end{array}\right\} \Longrightarrow Y \in \mathcal{F}$$
- $\cap$-closed $$X, Y \in \mathcal{F} \Longrightarrow X \cap Y \in \mathcal{F}$$

The filter is proper is $\emptyset \notin \mathcal{F}$.

Examples

- For each $i \in I$
  $$X \in \mathcal{F} \Longleftrightarrow i \in X$$
  gives the principal filter generated by i. These are not very useful.
- The Frechet filter is given by
  $$X \in \mathcal{F}\text{rechet} \Longleftrightarrow I - X \text{ is finite}$$
- A family $\mathcal{G}$ of subsets has f.i.p. [finite intersection property] if
  $$X_1, \ldots, X_n \in \mathcal{G} \Longrightarrow X_1 \cap \cdots \cap X_n \neq \emptyset$$
  The upwards closure of all these intersections is a proper filter.

Prime = Maximal = Ultrafilter

The following properties of a proper filter $\mathcal{F}$ are equivalent.

(i) $(\forall X, Y)[X \cup Y \in \mathcal{F} \Longrightarrow X \in \mathcal{F}$ or $X \in \mathcal{F}$

(ii) $(\forall X)[X \in \mathcal{F}$ or $I - X \in \mathcal{F}]$

(iii) $\mathcal{F}$ is maximal.

Such a filter is an ultrafilter.

Each principal filter is an ultrafilter (but not very useful).

By ZL each proper filter extends to an ultrafilter.

An ultrafilter $\mathcal{U}$ is non-principal precisely when $\mathcal{F}$rechet $\subseteq \mathcal{U}$.

Factoring out a filter $\mathcal{F}$

Let

$$\mathcal{A} = (A_i \mid i \in I)$$

be the indexed family of carrying sets of $\mathbb{A}$. Let

$$A = \prod \mathcal{A}$$

and recall that this is the set of choice functions for $\mathcal{A}$, those functions

$$a : I \longrightarrow \bigcup \mathcal{A} \quad \text{with} \quad a(i) \in A_i \quad \text{for all } i \in I$$

For $a, b \in A$ consider the 'measure of equality'

$$[\![a = b]\!] = \{i \in I \mid a(i) = b(i)\}$$

The induced equivalence relation

on A is given by

$$a \sim b \iff [\![a = b]\!] \in \mathcal{F}$$

Trivially this is reflexive and symmetric, and transitive since

$$[\![a = b]\!] \cap [\![b = c]\!] \subseteq [\![a = c]\!]$$

Let

$$A/\mathcal{F}$$

be the corresponding quotient set. We write

$$a/\mathcal{F}$$

for the block in which a lives.

Furnishing A as a structure

To produce a structure $\mathfrak{A}$ carried by A we need

$$\mathfrak{A}[\![K]\!] \quad \mathfrak{A}[\![R]\!] \quad \mathfrak{A}[\![O]\!]$$

for each symbol K, R, O of the language.

To produce these we take the '$\mathcal{F}$-consensus' of

$$\mathfrak{A}_i[\![K]\!] \quad \mathfrak{A}_i[\![R]\!] \quad \mathfrak{A}_i[\![O]\!]$$

as i varies through I.

Constant K

The choice function

$$K\hat{}: I \longrightarrow \bigcup \mathcal{A}$$

is given by

$$K\hat{}(i) = \mathfrak{A}_i[\![K]\!]$$

and we let

$$\mathfrak{A}[\![K]\!] = K\hat{}/\mathcal{F}$$

A more refined version of this idea gives us $\mathfrak{A}[\![R]\!]$ and $\mathfrak{A}[\![O]\!]$.

n-placed operation O or relation R

For $a_1, \ldots, i_n \in A$ and $i \in I$ let

$$O\hat{}(a_1, \ldots, a_n)(i) = \mathfrak{A}_i[\![O]\!](a_1(i), \ldots, a_n(i))$$

Now check that

$$a_1 \sim b_1, \ldots, a_n \sim b_n \Longrightarrow O\hat{}(a_1, \ldots, a_n) \sim O\hat{}(b_1, \ldots, b_n)$$

and hence can set

$$\mathfrak{A}[\![O]\!](a_1/\mathcal{F}, \ldots, a_n/\mathcal{F}) = O\hat{}(a_1, \ldots, a_n)/\mathcal{F}$$

A relation is dealt with in the same way (the target values are in $\{0, 1\}$, not A).

Łos's Theorem

Let $\mathfrak{A} = \prod \mathbb{A}/\mathcal{F}$ as just constructed.
For each $\neg$-free formula $\phi(v_1, \ldots, v_n)$ and $a_1, \ldots, a_n \in A$ we have

$$\mathfrak{A} \models \phi(a_1/\mathcal{F}, \ldots, a_n/\mathcal{F})$$

precisely when

$$\{i \in I \,|\, \mathfrak{A}_i \models \phi(a_1(i), \ldots, a_n(i))\} \in \mathcal{F}$$

Proof by induction over ϕ.

How do we handle negation? – Use an ultrafilter.

A proof of the compactness theorem

Let Σ be a set of sentences which is finitely satisfiable.

Take an enumeration of all finite subsets of Σ $\qquad (\Delta_i \mid i \in I)$

For each $i \in I$ let $\qquad \mathfrak{A}_i \models \Delta_i$

We produce an ultrafilter $\mathcal{U}$ on I with $\qquad \prod(\mathfrak{A}_i \mid i \in I)/\mathcal{U} \models \Sigma$

For each $i \in I$ let $\overline{i} \subseteq I$ be given by $\qquad j \in \overline{i} \iff \Delta_i \subseteq \Delta_j$

For $i(1), \dots, i(n) \in I$ let $\qquad \Delta_{i(1)} \cup \cdots \cup \Delta_{i(n)} = \Delta_j$

so that $\qquad j \in \overline{i(1)} \cap \cdots \cap \overline{i(n)}$

to show that $\qquad \{\overline{i} \mid i \in I\}$
has f.i.p. and hence extends to an ultrafilter $\mathcal{U}$.

The crunch

Consider any $\sigma \in \Sigma$ and let $$\{\sigma\} = \Delta_i$$

so that $$j \in \overline{i} \implies \sigma \in \Delta_j \implies \mathfrak{A}_j \models \sigma$$

and hence $$\{j \in I \mid \mathfrak{A}_j \models \sigma\} \in \mathcal{U}$$

to show $$\prod(\mathfrak{A}_j \mid j \in I) \models \sigma$$

as required.