# The theory of graphs – Section 5.3

The theory of graphs has an $\aleph_0$-categorical model companion and this has $EQ$.

A graph (in this sense) consists of nodes (its elements) some of which are connected and some of which are not.

| connected | not connected |
|---|---|
| $a \text{ — } b$ | $a \mathrel{-\!\!\!|\!\!\!-} b$ |
| mates | not mates |

This is a symmetric irreflexive relation, so we have a rather simple first order language.

In this language $T(Gph)$ is $\forall_1$-axiomatizable.

# Solving ‘equations’ in graphs

On board

A graph is random is for each finite disjoint sets $L, R$ of nodes the problem

$$L \text{ --- } v \text{ -}\!\!|\!\!\text{- } R$$

is solvable.

The theory $T(Rdm)$ of random graphs is $\forall_2$-axiomatizable.

# Some random graphs

Each ec graph is random.

The set $V_\omega$ of hereditarly finite sets (in the Zermelo hierachy) gives a random graph.

In fact, for each limit ordinal $\lambda$ the level $V_\lambda$ of the ZF-hierachy gives a randon graph.

For each cardinal $\kappa$ the level $V_\kappa$ can solve infinite sets of equations up to a certain cardinality.

# Back-and-forth properties

For each pair $U, V$ of randon graphs the set of all finite partial isomorphisms form a b&f system.

A finite partial isomorphism is merely an isomorphism $f$ between two finite subgraphs.

$$\begin{tikzcd}
U & V \\
A \arrow[u, hook] \arrow[r, "f" description] & B \arrow[u, hook]
\end{tikzcd}$$

For random graphs $U, V$ we have

$$U \subseteq V \Longrightarrow U \prec V$$

so that

$$\text{random} = \text{ec}$$

# $T(Rdm)$ is model complete and $\aleph_0$-categorical

We have just seen that

$$\mathcal{E}(T(Gph)) = \mathcal{M}d(T(Rdm))$$

and hence

$$T(Rdm) = Th(\mathcal{E}(T(Gph)))$$

is model complete.

Trivially, $T(Gph)$ has $JEP$, so the b&f properties ensure that $T(Rdm)$ is $\aleph_0$-categorical.

Or: Let $U, V$ be two contable random graphs. Trivially there is a partial isomorphism between them (using 1-node graphs), so $U \equiv_p V$, hence $U \cong V$.

We may check that $T(Gph)$ has $AP$ hence $T(Rdm)$ has $EQ$.

# Existentially universal

$V_\omega$ is the unique countable random (ec) graph.

Since this ensure there are few $\exists_1$-types, this means there is a countable eu graph. This must be $V_\omega$.

Thus we can solve certain infinite graph ‘equations’ in $V_\omega$.

Is this surprising?